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Qualitative properties for abstract evolution equations in continuous and discrete time

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Trabajo para optar al grado de Doctor en Ciencia mención en Matemática

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Trabajo de Titulación presentado a la Facultad de Ciencia, en cumplimiento parcial de los requisitos exigidos para optar al grado de **Doctor en Ciencia mención en Matemática**. Este trabajo de graduación fue presentado bajo la supervisión del profesor guía Dr. Carlos Lizama Yáñez, y los profesores supervisores del trabajo de tesis

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Contents

Introduction

Evolution equations in continuous and discrete time have been studied for several decades due to they arise naturally in the mathematical modeling of phenomena in natural sciences. The major topics to be investigated in this field are existence, uniqueness and qualitative properties of their solutions.

Evolution equations in continuous time has great importance in concrete models from mathematical physics, viscoelasticity theory, mechanics [9, 75, 104, 108, 109], among others. A useful machinery for this study is the general theory of resolvent families [12, 46, 71, 92, 109, 112]. In 1980, Da Prato and Iannelli [46] introduced for first time the notion of resolvent family as an extension of the known concept of C_0-semigroups, and now plays a central role in the theory of abstract Volterra integral equations, as studied, for example in Prüss's book [109].

The qualitative properties of resolvent families such as the regularity, positivity, periodicity, approximation, uniform continuity and compactness have been studied by a number of authors. See e.g. [46, 70, 69, 90, 91, 109]. Many authors have applied the notion of resolvent family to abstract differential equations in Banach spaces to obtain variations of parameters formulae in order to define appropriate concepts of mild solutions. For instance, in 2010, Lizama and N'Guérékata [95] studied bounded mild solutions for semilinear integro differential equations in Banach spaces. The same year, Chen and Li [39] introduced fractional resolvent operator functions to study a fractional abstract Cauchy problem. For related work, see [40, 71, 86, 92] and references therein.

As a discrete counterpart of the classical theory of differential equations emerged the theory of difference equations. The study of difference equations has been subject of increasing interest in the last years due to sometimes continuous models need to be discretized in time for practical purposes, see e.g. [4, 14, 29, 33, 51, 85, 87, 89]. In [117] Xia established some sufficient criteria for the existence and, uniqueness and asymptotic behavior of solutions to Volterra difference equations of convolution type as well as to nonautonomous semilinear difference equations. Elaydi [53] obtained some of the fundamental results on the stability and asymptotic behavior of linear Volterra difference equations using the method of $z-$transform for equations of convolution type. The study of maximal regularity for discrete time abstract Cauchy problems in Banach spaces has been addressed in [6, 30, 31, 82, 83, 97]. In 2001, Blunck [31] established sufficient conditions for maximal regularity of an operator on vector-valued Lebesgue spaces. Kemmochi [82, 83] considered a discrete Cauchy problem in a Banach space and showed that continuous maximal regularity implies discrete maximal regularity for general time schemes of approximation in the case of UMD spaces. Lizama and Murillo [97] presented a method based on operator-valued Fourier multipliers to characterize the existence and uniqueness of l_p-solutions for some discrete time fractional models.

The main purpose of this thesis is to study conditions to guarantee the existence, uniqueness and qualitative properties of solutions for a distinguished class of models in continuous and discrete time.

The first problem consists of finding conditions that guarantee the existence of integrated solutions for the following second order problem with memory

$$u''(t) + Au(t) - (k * Au)(t) = f(t, u(t)), \quad t \in [0, b], \tag{0.0.1}$$

where X is a Banach space, $A : D(A) \subseteq X \to X$ generates a resolvent family with integrated kernel, and $f : [0, b] \times X \to X$.

We would like to address the following questions: What kind of conditions on the kernel k, and the operator A do we need to obtain existence of solutions for the semilinear problem (0.0.1) with nonlocal initial conditions? Can we consider weaker conditions than Lipschitz type conditions, on the external forcing term f?

The second problem is finding conditions that guarantee existence and uniqueness of mild solutions to the following class of abstract semilinear difference equations of Volterra type

$$u(n+1) = A\sum_{k=-\infty}^{n} a(n-k)u(k+1) + \sum_{k=-\infty}^{n} b(n-k)f(k,u(k)),\ n\in\mathbb{Z}, \tag{0.0.2}$$

where A is an unbounded operator on a Banach space X and $a(n), b(n)$ are appropriately chosen sequences.

Suppose that we know the behavior of the forcing sequence $f(k,x)$. What conditions do we need on the operator A and the kernels $a(n)$ and $b(n)$ in order to conclude that the solution u of (0.0.2) exists and has the same behavior as f? In this sense another questions naturally emerge: What is the appropriate definition of discrete resolvent family generated by the operator A in order to represent the solution of (0.0.2)? Is it possible to give an explicit representation in terms of bounded operators of the discrete resolvent family? Are difference equations of fractional type included in the general framework of nonlinear fractional difference equations of type (0.0.2)?

The third problem that we address in this thesis is to study the connection between monotonicity and convexity of sequences and the discrete version of the nonlocal extension of *time* differential operators, e.g. the Riemann-Liouville fractional differential operator, whose more studied definition is the following (see Gray and Zhang [68] or Atici and Eloe [2, 13, 15, 16])

$$\left(\Delta_a^\nu f\right)(t) := \frac{1}{\Gamma(-\nu)}\sum_{s=a}^{t+\nu}(t-s-1)^{\underline{-\nu-1}}f(s),\ t\in\mathbb{N}_{a+N-\nu}, \tag{0.0.3}$$

where $N\in\mathbb{N}$ is the unique integer satisfying $N-1<\nu<N$, and the map $t\mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$.

There are some interesting questions that are of recent interest, such as: Can we use the sign of discrete fractional difference operators to obtain positivity, monotonicity, and convexity type results for sequences $u:\mathbb{N}_a\to\mathbb{R}$? Do the properties of positivity, monotonicity, and convexity have a continuous transition when ν increases from 0 to 3?

The following review summarizes the state of the art of the three problems mentioned previously.

Concerning the first problem, second order partial differential equations with memory arise in several applied fields, like viscoelasticity or heat conduction with memory [21, 36, 77, 79]. The problem (0.0.1) was initially studied by Prüss. In the context of Hilbert spaces, Prüss obtained energy estimates and the optimal decay rate for the solutions of the linear problem, through frequency domain methods, see [108]. Afterwards, Cannarsa and Sforza [37] studied global existence and asymptotic behavior of solutions for (0.0.1) when $f(t, u(t)) = \nabla F(u(t)) + g(t)$, where ∇F denotes the gradient of a Gâteaux differentiable functional $F : D(A^1/2) \to \mathbb{R}$. Xu [119] studied decay properties of the numerical solutions for (0.0.1) with $f = 0$. It is worthwhile to note that in [108], [37] and [119] the authors considered the equation (0.0.1) in Hilbert spaces, with local initial conditions, and f a Lipschitz function. Recently, in the framework of Hilbert spaces, Luong [101] found mild solutions for (0.0.1) with nonlocal conditions $u(0) + g(u) = x_0$, and $u'(0) + h(u) = y_0$ using measure of noncompactness on the space of solutions, and proved the existence of a compact set containing decaying mild solutions, i.e. mild solutions such that $u(t) \to 0$ as $t \to +\infty$, for problem (0.0.1).

Although there exists a wide literature about the second order abstract Cauchy problem, the existence of solutions with nonlocal initial conditions for the equation (0.0.1) with damping by a convolution term, in Banach spaces, has not been studied in the literature.

We notice that nonlocal initial conditions are more practical than classical conditions when treating physical problems. For instance, the sum

$$u(x,0) + \sum_{k=1}^{n} \beta_k(x) u(x, T_k) \tag{0.0.4}$$

is more accurate to measurement of a state than $u(x, 0)$ alone. This approach was used by Deng in [50] to describe the diffusion phenomenon of a small amount of gas in a tube. If there is too little gas at the initial time, the measurement (0.0.4) of the sum of the amounts of the gas is more reliable than the measurement $u(x, 0)$ of the amount of the gas at the instant $t = 0$. For more information we refer the reader to the articles [7, 35, 114, 113] and references therein.

Regarding the state of art of the second problem, linear and nonlinear difference equations of Volterra type are often used in several applied fields like modeling of biological populations, see [41, 42, 43, 52, 54, 85, 103]. The theory of linear Volterra difference equations of both convolution and nonconvolution types have been studied, for example, by Elaydi, Gronek and Schmeidel in [54, 72], where the second author named proved the existence of bounded solutions via Darbo's fixed-point theorem using a measure of noncompactness in the space of bounded sequences.

Quite recently considerable attention has been paid to the nonlinear difference equations

$$\Delta^\alpha u(n) = Au(n+1) + f(n), n \in \mathbb{Z}, \tag{0.0.5}$$

for $0 < \alpha \leq 1$, where A is the generator of a resolvent sequence contained in the space of all bounded operators defined in a Banach space. Here Δ^α denotes fractional difference in Weyl-like sense and f satisfies Lipschitz conditions of global and local type. In [1, 10, 118] the authors studied existence, uniqueness of discrete weighted pseudo S-asymptotically ω-periodic mild solutions and asymptotic behavior for nonlinear fractional difference equations like (0.0.5).

A key observation is that equations of type (0.0.5) are subsumed under the general framework of nonlinear fractional difference equations of Volterra type (0.0.2). More precisely, equation (0.0.5) corresponds to the special case where $a(n) = b(n) = k^\alpha(n)$, with

$$k^\alpha(n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \ n \in \mathbb{N}_0.$$

We notice that this sequence appears in the definition of the fractional difference which has been the subject of much study in recent years (see e.g. [94] and the references therein).

We observe that the vast majority of research works related to the class of nonlinear discrete time evolution equations (0.0.2) are focused either in finite dimensional cases or are restricted to the case of a bounded operator A, see [43, 45, 52] and references therein. Moreover, the problem of existence and uniqueness of weighted pseudo asymptotically mild solutions to (0.0.2) appears not to have been considered in the literature. In this thesis we fill this gap by means of an operator theoretic approach.

Finally, with regard to the state of art of the third problem, we notice that time-discrete operators of fractional order appear in several areas of interest. For instance, in numerical analysis, as time-stepping schemes of approximation for fractional evolution equations [81, 98], and in the study of existence, uniqueness and qualitative properties of fractional difference equations [3, 14, 28, 38, 58, 74, 115, 120]. They also appear in the analysis of mixed partial difference-differential equations by means of operator theoretical methods [76, 88, 93, 97, 94].

A well-known geometrical fact in the difference calculus is the following characterization of monotonicity: $(\Delta f)(t) \geq 0$ if and only if f is increasing on $\mathbb{N}_0$. The question of whether such a monotonicity result holds in the discrete fractional setting. *The monotonicity conjecture* was posed in 2014 by Dahal and Goodrich [47, 48]. It turns out that the answer to this conjecture is not obvious and rather complicated due to the *nonlocal* character of the fractional difference operator (0.0.3). In fact, it was proved that if $1 < \nu < 2$ and $\big(\Delta_a^{\nu} f\big)(t) \geq 0$ for each $t \in \mathbb{N}_{a+2-\nu}$ one does not need to have f increasing [56, Example 2.4]. After the intense work of several authors [26, 47, 61, 80], the best answer to the monotonicity conjecture posed by Dahal and Goodrich was proved in [65, Theorem 6.3]. In such paper, it was also analyzed the case of compositions of discrete fractional operators, establishing many new results for all types of discrete fractional differences, and improving existing results in the literature.

Additionally, connections between (0.0.3) and the convexity of the map f was first investigated by Goodrich [60], proving that under certain hypotheses the positivity of $(\Delta_a^{\nu} f)(t)$, for $2 < \nu < 3$, implies the convexity of f, thereby associating some geometrical meaning to the fractional difference operator of order $\nu > 2$.

In 2017, Dahal and Goodrich [49] considered monotonicity-type results for sequences f satisfying the sequential fractional difference inequality $\Delta_{1+a-\mu}^{\nu} \Delta_a^{\mu} f(t) \geq 0$ for $t \in \mathbb{N}_{2+a-\mu-\nu}$, where $0 < \mu < 1, 0 < \nu < 1$, and $1 < \mu + \nu < 2$. Goodrich started the study of discrete sequential fractional boundary problems [59]. See also Sitthiwirattham [111]. Fractional operators are, in general, non commutative [78], this renders reduction of the order of fractional difference equations impossible. An interesting aspect of the sequential case is that the type of result obtained depends on the choice of ν and μ [63, 65] and therefore exhibits a complexity that appears to be absent in the non-sequential case.

On the other hand, in recent times Lizama [94] proposed an alternative definition to (0.0.3) by setting

$$\big(\Delta^{\alpha} f\big)(n) := \Delta^N \left[\sum_{j=0}^{n} k^{N-\alpha}(n-j) f(j) \right], \tag{0.0.6}$$

where $N-1 < \alpha < N$, $N \in \mathbb{N}$. This definition appears in several recent articles related to l_p-maximal regularity, existence and uniqueness of solutions of difference problems with fractional order [96, 93, 76]. Nevertheless, this definition was used for the first time to study either the monotonicity or convexity of a sequence in [65]. In [65][Theorem 4.3] the authors related (0.0.3) to Definition

(0.0.6) by means of the operator of translation. This property, named principle of transference has been developed and used in [65] to understand the connections between the sign of $\big(\Delta^\alpha f\big)(t)$ and either the positivity, monotonicity, or convexity of f. In such reference, many new results were proved, improving most, if not all, known existing results in the literature.

This thesis is organized in four chapters. In Chapter 1, we begin introducing some of the main concepts, notations and results that will be necessary in the development of this thesis. Mainly, we give the basic concepts related to resolvent families, fractional differences, vector-valued spaces, measure of noncompactness, and any related results of immediate use to us.

The following chapters are devoted to the detailed study of the three problems described above. We will give a brief description of each of them below.

Chapter 2 is concerned with the study of the first problem. Roughly speaking, we prove the existence of integrated solutions for the local and nonlocal initial value problem (0.0.1). In the local case, we use methods described in [57] to show the existence of integrated solutions under conditions of compactness of the resolvent generated by A. Concerning the nonlocal case, we follow ideas of Lizama and Pozo [99], using properties of the measure of noncompactness as the main tool. The concept of measure of noncompactness has been studied widely in [19, 20, 22, 23, 24, 25].

We observe that the approach based on the use of measure of noncompactness for abstract Cauchy problems allows us to remove stronger assumptions, like Lipschitz type conditions on the external forcing term f in (0.0.1) employed in the paper [108, Section 6] and so obtain more general results in comparison with other methods. Although this method has been employed in the last years by several authors for the study of existence of solutions to ordinary differential equations, their application to abstract evolution equations remains underdeveloped. Following ideas of [99] (see also the references therein), we will assume that the term f satisfies the following set of conditions.

(i) There exists a function $m \in L^1([0,b];\mathbb{R}^+)$ and a nondecreasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leq m(t)\phi(\|x\|+1), \text{ for all } x \in X \text{ and almost all } t \in [0,b].$$

(ii) There exists a function $H \in L^1([0,b];\mathbb{R}^+)$ such that for any bounded $S \subseteq X$

$$\xi(f(t,S)) \leq H(t)\xi(S), \tag{0.0.7}$$

for almost all $t \in [0,b]$, where ξ denote the Hausdorff measure of noncompactness defined in X.

Under the above assumptions, we prove our main result on the existence of integrated solutions to equation (0.0.1). Moreover, we include constructive examples to illustrate the feasibility of the given hypotheses. In the second example, we included a function f which satisfies the condition (0.0.7), but it is not Lipschitzian.

Chapter 3 deals with the second problem. We start proposing a new definition of discrete resolvent family $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ generated by the operator A, in order to represent the solution of (0.0.2). This new concept improves [1, 118, Definition 3.1] and [10, Definition 2.11] in the special case of (0.0.5). For the associated nonhomogeneous linear equation (0.0.2), we assume conditions in terms of generators of C_0-semigroups in order to prove the existence and summability of a discrete resolvent family.

Additionally, we prove that in case $a(n) = b(n), n \in \mathbb{Z}_+$, then the sequence the operators $S(n)$ has the following interesting representation

$$S(n)x = [\sum_{j=0}^{n-1} \frac{1}{a(0)^j}\phi_j(n)(T-I)^j]T^2x, \quad n \geq 2, \quad \text{for all} \qquad x \in X,$$

where $T := (I - a(0)A)^{-1}$, and $\phi_0(n) = a(n)$, $\quad \phi_1(n) = \sum_{k=1}^{n-1} a(n-k)a(k)$, and

$$\phi_j(n) = \sum_{k=j}^{n-1} a(n-k)\phi_{j-1}(k), \quad j \geq 2, \tag{0.0.8}$$

and for all $x \in X$ we have that $S(0)x = a(0)Tx$, $S(1) = a(1)T^2x$.

As regards to the asymptotic behavior and weighted pseudo S-asymptotic ω-periodic mild solutions to (0.0.2), we suppose that A is the generator of a summable discrete resolvent family $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X), f$ satisfies a $\theta-$Lipschitz condition, and using the Leray-Schauder Alternative Theorem, we show that there exist a sequence $(h(n))_{n\in\mathbb{Z}}$ and a mild solution u of (0.0.2) such that $u(n) = o(h(n))$, where the positive sequence $h : \mathbb{Z} \to \mathbb{R}^+$ satisfies appropriate convergence properties. The precise description of this result is in the context of Theorem 3.2.6. Additionally, we include a constructive example to illustrate the relevance and feasibility of the given hypotheses.

Finally, Chapter 4 is devoted to the third problem. Here, we show important new properties of the higher order differences Δ^l, for $l \in \mathbb{N}$ and of the α- th fractional difference operator Δ^α, for $\alpha > 0$. We provide a geometrical interpretation for α-increasing and α-convex sequences. We improve in a significant way the work done in [65]. For $2 \leq \alpha < 3$, we assume that $u \in s(\mathbb{N}_0; \mathbb{R})$ satisfies a suitable condition on $\Delta^\alpha u(n)$, and we conclude that u is positive, increasing and convex on $\mathbb{N}_0$. This new result allow us to deduce that the properties of positivity, monotonicity and convexity for a sequence u have a continuous transition as α increases from 0 to 3.

Furthermore, we refined the results related to the relationship between convexity and the sign of the composition of two operators. In the new theorems we include the border cases and new hypotheses that allow us to see how the hypotheses overlap in each of the regions. More precisely, our main result concerning the composition of two operators is Theorem 4.3.7. Using the transference principle obtained by Goodrich and Lizama in [65], we are able to transfer all our results to the operator (0.0.3) and thus fully understand the properties of positivity, monotonicity and convexity. Moreover, we present examples that demonstrate the sharpness of the hypothesis.

The results described in Chapters 2 and 3 have been published in mainstream international journals (ISI):

1. C. Lizama, S. Rueda. *Nonlocal integrated solutions for a class of abstract evolution equations.* Acta Applicandae Mathematicae, 164 (1) (2019), 165-183.
2. V. Keyantuo, C. Lizama, S. Rueda and M. Warma. *Asymptotic behavior of mild solutions for a class of abstract nonlinear difference equations of convolution type.* Advances in Difference Equations, 251 (1) (2019), 1-29.

The results of Chapter 3 can be found in the article submitted for publication: J. Bravo, C. Lizama, S. Rueda. *Analytical properties of nonlocal discrete operators: Convexity.*

Chapter 1

Preliminaries

The purpose of this chapter is to introduce certain notations, notions and theorems used throughout the present thesis.

Let $(X, \|\cdot\|)$ be a Banach space. We denote the space of all bounded linear operators from X into X by $\mathcal{B}(X)$. If A is a closed linear operator on X, we denote by $D(A)$ the domain of A equipped with the graph norm $|\cdot|_A$ of A, i.e. $|x|_A = |x| + |Ax|$, $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A defined for all $\lambda \in \rho(A)$.

Throughout this thesis, $\mathcal{C}([0,b];X)$, and $L^p(\mathbb{R}_+, X)$ for $1 \leq p < \infty$, denote the vector space of all continuous functions $f : [0,b] \to X$, and the vector valued space of all Bochner measurable functions $f : \mathbb{R}_+ \to X$ such that $\|f\|_p := \left(\int_0^\infty \|f(t)\|^p dt\right)^{1/p} < \infty$, respectively. $C^k(0,\infty)$, denotes the space of $k-$times continuously differentiable functions on $(0,\infty)$. For $\Omega \subset \mathbb{R}^n$, $W^{m,p}(\Omega; X)$ is the space of all functions $f : \Omega \to X$ having distributional derivatives $D^\alpha f \in L^p(\Omega; X)$ of order $|\alpha| \leq m$. The subscript 'loc' assigned to any of the above function spaces means membership to the corresponding space when restricted to compact subsets of its domain.

1.1 Resolvent families

The concept of resolvent family that we will define in this section was introduced by Prüss in [109], it corresponds to a class of families of operators that arise from the theory of linear Volterra equations.

In what follows we consider the integral equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \in [0, b], \tag{1.1.1}$$

where $f \in \mathcal{C}([0, b]; X)$. We recall the following definition.

Definition 1.1.1. [109, Definition 1.1] Let A be a closed linear operator with domain $D(A)$ defined in a Banach space X, $a \in L^1_{loc}(\mathbb{R}_+)$ be a scalar kernel and $f \in \mathcal{C}([0, b]; X)$. A function $u \in \mathcal{C}([0, b], X)$ is called

(i) strong solution of (1.1.1) on $[0, b]$ if $u \in \mathcal{C}([0, b], D(A))$ and (1.1.1) holds on $[0, b]$,

(ii) mild solution of (1.1.1) on $[0, b]$ if $a * u \in \mathcal{C}([0, b], D(A))$ and $u(t) = f(t) + A(a * u)(t)$ on $[0, b]$,

where the star indicates the finite convolution, i.e.

$$(k * u)(t) = \int_0^t k(t-s)u(s)ds, \quad t \geq 0.$$

The following concept of resolvent family will be fundamental in our considerations.

Definition 1.1.2. [109, Definition 1.3] Let A be a closed linear operator with domain $D(A)$ defined in a Banach space X, and $a \in L^1_{loc}(\mathbb{R}_+)$. A family $\{S(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ of bounded linear operators in X is called a resolvent family generated by A if the following conditions are satisfied.

(i) $S(t)$ is strongly continuous on $\mathbb{R}_+$, (i.e. $S(\cdot)x$ are continuous from $\mathbb{R}_+$ into X for every $x \in X$) and $S(0) = I$;

(ii) $S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(iii) the resolvent equation holds

$$S(t)x = x + \int_0^t a(t-s)AS(s)xds \quad \text{for all} \quad x \in D(A), t \geq 0.$$

Definition 1.1.3. [109, Definition 1.5] A resolvent family $\{S(t)\}_{t\geq 0}$ is called exponentially bounded if there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$, for all $t \geq 0$. The pair (M, ω) is called a type of $S(t)$.

Theorem 1.1.4. *[109, Proposition 1.2] Let A be a closed linear operator in a Banach space X with domain $D(A)$, and $a \in L^1_{loc}(\mathbb{R}_+)$. If $\{S(t)\}_{t\geq 0}$ is a resolvent family generated by A, and $f \in W^{1,1}([0,b]; D(A))$, then*

$$u(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds, \quad t \in [0,b],$$

is a strong solution of (1.1.1).

Let a be exponentially bounded of order ω. We denote the Laplace transform of a by

$$\hat{a}(\lambda) := \int_0^\infty e^{-\lambda t}a(t)dt, \ \mathrm{Re}(\lambda) > \omega.$$

Definition 1.1.5. [109, p. 90] A infinitely differentiable function $a : (0,\infty) \to \mathbb{R}$ is called completely monotonic if $(-1)^n a^{(n)}(t) \geq 0$ for all $t > 0, n \in \mathbb{N}_0$.

Definition 1.1.6. [107, p. 326] Let $a \in L^1_{loc}(\mathbb{R}_+)$ be such that a is Laplace transformable, a is called completely positive if and only if $\frac{1}{\lambda\hat{a}(\lambda)}$ and $\frac{-\hat{a}'(\lambda)}{[\hat{a}(\lambda)]^2}$, with $\lambda > 0$ are completely monotone functions.

We recall [109, Section 3.2 p.69] that a function $a \in L^1_{loc}(\mathbb{R}_+)$ of subexponential growth is called k-regular if there is a constant $c > 0$ such that

$$|\lambda^n \widehat{a}^{(n)}(\lambda)| \leq c|\widehat{a}(\lambda)| \text{ for all } \mathrm{Re}(\lambda) > 0,\ 0 \leq n \leq k.$$

Also a is said to be of positive type if $|\arg \widehat{a}(\lambda)| \leq \frac{\pi}{2}$ for all $\mathrm{Re}(\lambda) > 0$.

Definition 1.1.7. Let $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \geq 2$. We say that $a(t)$ is $k-$ monotone if $a \in C^{k-2}(0,\infty)$, $(-1)^n a^{(n)}(t) \geq 0$ for all $t > 0$, $0 \leq n \leq k-2$, and the function $(-1)^{k-2}a^{(k-2)}(t)$ is nonincreasing and convex.

1.2 Fractional differences

In this section, we introduce the notion of the fractional difference operator that will be used mainly in Chapters 3 and 4. In what follows, we denote $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$, for some $a \in \mathbb{R}$, and $\mathbb{N} \equiv \mathbb{N}_1$ as usual. We denote by $s(\mathbb{N}_a;\mathbb{R})$ the vectorial space that consists of all sequences $f : \mathbb{N}_a \to \mathbb{R}$. Recall that given a sequence $f \in s(\mathbb{N}_a;\mathbb{R})$ the first-order forward (or delta) difference of f at $t \in \mathbb{N}_a$, denoted $\big(\Delta_a f\big)(t)$, is defined by

$$\big(\Delta_a f\big)(t) := f(t+1) - f(t).$$

Then one may define iteratively the higher order differences Δ_a^n, for $n \in \mathbb{N}_1$, by writing

$$\big(\Delta_a^n f\big)(t) := \big(\Delta_a \circ \Delta_a^{n-1} f\big)(t).$$

We also denote $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_a;\mathbb{R})$ is the identity operator, $\Delta_a^1 \equiv \Delta_a$, and $\Delta^n \equiv \Delta_0^n$.

Remark 1.2.1. For any $f \in s(\mathbb{N}_0;\mathbb{R})$, $l \in \mathbb{N}_0$ we have

$$\Delta^l f(t) = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} f(t+j), \quad t \in \mathbb{N}_0.$$

We define

$$k^\alpha(n) := \begin{cases} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} & \text{if } \alpha > 0, \quad n \in \mathbb{N}_0, \\ \delta_0(n) & \text{if } \alpha = 0, \end{cases}$$

where $\delta_0(n)$ is the delta function,

$$\delta_0(n) := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The kernel k^α, introduced in [94], appears in many fields of research and has a number of important properties that concentrate all the information about the fractional difference operator which we will define next. For a review, see [65, Section 3].

The definition of α-th fractional sum on the set $\mathbb{N}_0$ is given by:

Definition 1.2.2. For each $\alpha > 0$ and $f \in s(\mathbb{N}_0; \mathbb{R})$, we define the fractional sum of order α as follows:

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^\alpha(n-j) f(j), \quad n \in \mathbb{N}_0.$$

The next concept was proposed in [94], it is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville, see [105].

Definition 1.2.3. Let $\alpha > 0$ be given. The $\alpha-$th fractional difference operator is defined by

$$\Delta^\alpha f(n) := \Delta^m \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha \leq m,\ m \in \mathbb{N}$.

Let u, v be sequences (defined on $\mathbb{Z}_+$). We define two convolution products $(u \circ v)(n)$ and $(u * v)(n)$ as follows,

$$(u \circ v)(n) = \sum_{k=-\infty}^{n} u(n-k) v(k), \tag{1.2.1}$$

and

$$(u * v)(n) = \sum_{k=0}^{n} u(n-k) v(k). \tag{1.2.2}$$

Note that for the first product we need conditions on the sequences in order to ensure the convergence of the series (1.2.2), while for the second product (1.2.1) no condition on the sequences is required.

A useful property that satisfies the kernel k^α is the semigroup property

$$(k^\alpha * k^\beta)(n) = k^{\alpha+\beta}(n), \qquad n \in \mathbb{N}_0, \quad \alpha, \beta > 0, \tag{1.2.3}$$

which is frequently used in the results and examples of Chapter 5.

We recall from [65, Lemma 3.2] the following result.

Lemma 1.2.4. *For any $\alpha > 0$ and $n \in \mathbb{N}_0$, the following identities hold:*

(i) $\Delta k^\alpha(n) = (\alpha - 1)\dfrac{k^\alpha(n)}{n+1}$.

(ii) $\Delta^2 k^\alpha(n) = (\alpha - 2)(\alpha - 1)\dfrac{k^\alpha(n)}{(n+1)(n+2)}$.

(iii) $\Delta^3 k^\alpha(n) = (\alpha - 3)(\alpha - 2)(\alpha - 1)\dfrac{k^\alpha(n)}{(n+1)(n+2)(n+3)}$.

Given $a, b \in \mathbb{R}$, we define the translation (by $a \in \mathbb{R}$) operator $\tau_a : s(\mathbb{N}_a; \mathbb{R}) \to s(\mathbb{N}_0; \mathbb{R})$ by

$$\tau_a f(n) := f(a+n), \quad n \in \mathbb{N}_0.$$

Note that $\tau_a^{-1} = \tau_{-a}$ and $\tau_{a+b} = \tau_a \circ \tau_b = \tau_b \circ \tau_a$.

Lemma 1.2.5. *[65, Lemma 2.3] Let $f, g \in s(\mathbb{N}_0; \mathbb{R})$ be sequences, then for each $p \in \mathbb{N}$ we have*

$$(f * \tau_p g)(n) = \tau_p (f * g)(n) - \sum_{j=0}^{p-1} \tau_p f(n-j) g(j).$$

We recall that the most commonly used fractional difference operator of order $\nu > 0$ was defined by Atici and Eloe [13, 15, 16]

$$\left(\Delta_a^{\nu} f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),\ t \in \mathbb{N}_{a+N-\nu}, \tag{1.2.4}$$

where $f \in s(\mathbb{N}_a;\mathbb{R})$, $N \in \mathbb{N}_1$ is the unique integer satisfying $N-1<\nu<N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. In the integer cases of $\nu = N$ we have

$$\Delta_a^N f(t) = \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} f(t+j), \quad t \in \mathbb{N}_a. \tag{1.2.5}$$

In [65, Theorem 4.3] the authors related (1.2.4) to Definition 1.2.3 by means of the operator of translation, which allowed to transfer the properties of a sequence u between both definitions and called a *transference principle.*

In the following, we have extended the formulation of the transference principle in order to include the integer cases $\alpha = N \in \mathbb{N}$, being the proof immediate taking into account (1.2.5).

Theorem 1.2.6. *(Transference Principle) Let $N-1<\alpha\le N$, $N\in\mathbb{N}$ and $a,\beta\in\mathbb{R}$. For each sequence $f\in s(\mathbb{N}_a;\mathbb{R})$ we have*

$$\tau_{a+N-\alpha} \circ \Delta_a^{\alpha} f = \Delta^{\alpha} \circ \tau_a f,$$

and for each $f \in s(\mathbb{N}_{a+N-\beta};\mathbb{R})$,

$$\tau_{N-\beta} \circ \Delta_{a+N-\beta}^{\alpha} f = \Delta_a^{\alpha} \circ \tau_{N-\beta} f.$$

In other words, the following diagrams are commutative:

$$\begin{array}{ccc} s(\mathbb{N}_a;\mathbb{R}) & \xrightarrow{\Delta_a^{\alpha}} & s(\mathbb{N}_{a+N-\alpha};\mathbb{R}) \\ \downarrow{\scriptstyle \tau_a} & & \downarrow{\scriptstyle \tau_{a+N-\alpha}} \\ s(\mathbb{N}_0;\mathbb{R}) & \xrightarrow{\Delta^{\alpha}} & s(\mathbb{N}_0;\mathbb{R}) \end{array} \qquad \begin{array}{ccc} s(\mathbb{N}_{a+N-\beta};\mathbb{R}) & \xrightarrow{\Delta_{a+N-\beta}^{\alpha}} & s(\mathbb{N}_{a+2N-\beta-\alpha};\mathbb{R}) \\ \downarrow{\scriptstyle \tau_{N-\beta}} & & \downarrow{\scriptstyle \tau_{N-\beta}} \\ s(\mathbb{N}_a;\mathbb{R}) & \xrightarrow{\Delta_a^{\alpha}} & s(\mathbb{N}_{a+N-\alpha};\mathbb{R}) \end{array}$$

1.3 Some vector-valued spaces

We denote the linear space consisting of all vector valued sequences $f : \mathbb{Z}_+ \to X$ by $s(\mathbb{Z}_+, X)$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $a : \mathbb{Z}_+ \to \mathbb{C}$ be given. If $\sum_{k=0}^{\infty} |a(k)| < \infty$, then we say that a is a summable sequence. We introduce some notation on the vector-valued spaces used in the sequel.

1. $l^\infty(\mathbb{Z}, X) := \{f : \mathbb{Z} \to X : \|f\|_\infty := \sup_{n\in\mathbb{Z}} \|f(n)\| < \infty\}$.
2. $l^p_\rho(\mathbb{Z}, X) := \{f : \mathbb{Z} \to X : \|f\|_{l^p_\rho} := \sum_{n=-\infty}^{\infty} \|f(n)\|^p \rho(n) < \infty\}$, where $\rho : \mathbb{Z} \to (0, \infty)$ is a positive sequence.
3. $C_0(\mathbb{Z}, X) := \{f \in l^\infty(\mathbb{Z}, X) : \lim_{n\to\infty} \|f(n)\| = 0\}$.
4. $C_\omega(\mathbb{Z}, X) := \{f \in l^\infty(\mathbb{Z}, X) : f$ is $\omega -$ periodic$\}$ where $\omega \in \mathbb{Z}_+ \setminus \{0\}$ is fixed.
5. $\mathcal{UC}(\mathbb{Z} \times X, X)$ is the set of all functions $f : \mathbb{Z} \times X \to X$ satisfying that for all $\epsilon > 0$ there exist $\delta > 0$ such that $\|f(k, x) - f(k, y)\| \leq \epsilon$ for all $k \in \mathbb{Z}$ and for all $x, y \in X$ with $\|x - y\| < \delta$.
6. $\overline{\mathcal{UC}}(\mathbb{Z} \times X, X)$ is the set of all functions $f : \mathbb{Z} \times X \to X$ satisfying that for all $\epsilon > 0$ there exist $\delta > 0$ such that $\|f(k, x) - f(k, y)\| \leq L_f(k)\epsilon$ for all $k \in \mathbb{Z}$ and $x, y \in X$ with $\|x - y\| \leq \delta$, where $L_f \in l^p(\mathbb{Z})$.

Let $h : \mathbb{Z} \to \mathbb{R}^+$ be a sequence such that $h(n) \geq 1$ for all $n \in \mathbb{Z}$, and $h(n) \to \infty$ as $|n| \to \infty$. Define

$$\mathcal{C}^0_h(\mathbb{Z}, X) = \{\xi : \mathbb{Z} \to X : \lim_{|n|\to\infty} \frac{\|\xi(n)\|}{h(n)} = 0\},$$

endowed with the norm $\|\xi\|_h = \sup_{n\in\mathbb{Z}} \frac{\|\xi(n)\|}{h(n)}$.

It is clear that $\mathcal{C}^0_h(\mathbb{Z}, X)$ is a Banach space isometrically isomorphic with the space $C_0(\mathbb{Z}, X)$ consisting of all sequences $\xi : \mathbb{Z} \to X$ that vanish at $\pm\infty$. Let U be the collection of positive sequences $\rho : \mathbb{Z} \to (0, \infty)$. For $\rho \in U$ and, for $n \in \mathbb{Z}_+$ we use the notation

$$\nu(n, \rho) = \sum_{k=-n}^{n} \rho(k),$$

$$U_\infty := \{\rho \in U : \lim_{|n|\to\infty} \nu(n,\rho) = \infty\},$$
$$U_b = \{\rho \in U_\infty :\ 0 < \inf_{k\in\mathbb{Z}} \rho(k) \le \sup_{k\in\mathbb{Z}} \rho(k) < \infty\} \subset U_\infty.$$

Hence, $\nu(n,\rho)$ are the symmetric partial sums, U_∞ consists of those positive sequences ρ over $\mathbb{Z}$ for which the sequence $(\nu(n,\rho))_{n\in\mathbb{N}}$ is unbounded, while U_b consists of the positive sequences ρ such that for some fixed $\tau > 0$, $\rho(n) \ge \tau$ for all $n \in \mathbb{Z}$.

Let $\rho_1, \rho_2 \in U_\infty$ be given. The sequence ρ_1 is said to be equivalent to ρ_2 (i.e. $\rho_1 \sim \rho_2$) if $\rho_1/\rho_2 \in U_b$. It can be proved that $U_\infty = \cup_{\rho\in U_\infty}\{\varrho \in U_\infty : \rho \sim \varrho\}$. For $\rho \in U_\infty$ and $m \in \mathbb{Z}$.

A sequence $f : \mathbb{Z} \to X$ is called almost automorphic if for every integer sequence $\{k'_n\}$, there exists a subsequence $\{k_n\}$ such that

$$\overline{f}(k) := \lim_{n\to\infty} f(k + k_n)$$

is well defined for each $k \in \mathbb{Z}$ and $\lim_{n\to\infty} \overline{f}(k - k_n) = f(k)$. The set of such sequences is denoted by $AA_d(\mathbb{Z}, X)$. It is well known that the set $AA_d(\mathbb{Z}, X)$ endowed with the norm $\|f\|_\infty := \sup_{k\in\mathbb{Z}} \|f(k)\|$ is a Banach space. (See [11]). A function $f : \mathbb{Z}\times X \to X$ is called almost automorphic if $f(k, x)$ is almost automorphic in $k \in \mathbb{Z}$ for any $x \in X$. We denote the space of all such functions by $AA_d(\mathbb{Z} \times X, X)$.

For $\rho_1, \rho_2 \in U_\infty$ [117], we define the space

$$PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2) := \{f \in l^\infty(\mathbb{Z}, X) : \lim_{n\to\infty} \frac{1}{\nu(n,\rho_1)} \sum_{k=-n}^{n} \|f(k)\|\rho_2(k) = 0\}.$$

Let $\rho_1, \rho_2 \in U_\infty$ be given. A sequence $f : \mathbb{Z} \to X$ is called discrete weighted pseudo almost automorphic if it can be represented as $f = g + \varphi$, where $g \in AA_d(\mathbb{Z}, X)$ and $\varphi \in PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$. The space of such functions is denoted by $WPAA_d(\mathbb{Z}, X)$. The space $WPAA_d(\mathbb{Z}, X)$ endowed with the norm $\|f\|_\infty := \sup_{k\in\mathbb{Z}} \|f(k)\|$ is a Banach space. (See [117, Lemma 10]). A function $f : \mathbb{Z}\times X \to X$ is called discrete weighted almost automorphic in $k \in \mathbb{Z}$ for each $x \in X$ if it can be expressed as $f = g + \varphi$, where $g \in AA_d(\mathbb{Z}\times X, X)$ and $\varphi \in PAA_0S(\mathbb{Z}\times X, X, \rho_1, \rho_2)$. The space of such functions is denoted by $WPAA_d(\mathbb{Z} \times X, X)$. In what follows, we denote by V_∞ the set of all the functions $\rho_1, \rho_2 \in U_\infty$ satisfying the following: there exists an unbounded set $\Omega \subset \mathbb{Z}$ such that for all $m \in \mathbb{Z}$,

$$\lim_{|k|\to\infty, k\in\Omega} \sup \frac{\rho_2(k+m)}{\rho_1(k)} < \infty \text{ and } \lim_{n\to\infty} \frac{\sum_{k\in([-n,n]\setminus\Omega)+m} \rho_2(k)}{\nu(n,\rho_1)} = 0.$$

A function $f : \mathbb{Z} \times X \to X$ is said to be locally Lipschitz-continuous with respect to the second variable if for each positive number r, for all $k \in \mathbb{Z}$ and for all $x, y \in X$ with $\|x\| \leq r$ and $\|y\| \leq r$, we have $\|f(k,x) - f(k,y)\| \leq L(r)\|x-y\|$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function.

A sequence $f \in l^\infty(\mathbb{Z}, X)$ is called discrete asymptotically ω-periodic if there exists $g \in C_\omega(\mathbb{Z}, X)$, $\varphi \in C_0(\mathbb{Z}, X)$ such that $f = g + \varphi$. The collection of such sequences is denoted by $AP_\omega(\mathbb{Z}, X)$. A sequence $f \in l^\infty(\mathbb{Z}, X)$ is called discrete S-asymptotically ω-periodic if there exist $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that $\lim_{n\to\infty}(f(n+\omega) - f(n)) = 0$. The collection of such sequences is denoted by $SAP_\omega(\mathbb{Z}, X)$. (See [117, Definition 5]).

Let $\rho \in U_\infty$ be given. A sequence $f \in l^\infty(\mathbb{Z}, X)$ is called discrete S-asymptotically ω-periodic if there exist $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that $\lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} \|f(k+\omega) - f(k)\| = 0$. The collection of such sequences is denoted by $PSAP_\omega(\mathbb{Z}, X)$. (See [117, Definition 6]).

Let $\rho_1, \rho_2 \in U_\infty$. A sequence $f \in l^\infty(\mathbb{Z}, X)$ is called discrete weighted pseudo S-asymptotically $\omega-$periodic if there exist $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{1}{\nu(n,\rho_1)} \sum_{k=-n}^{n} \rho_2(k)\|f(k+\omega) - f(k)\| = 0.$$

Denote by $WPSAP_\omega(\mathbb{Z}, X)$ the set of such sequences. (See [118, Definition 2.5]). Next, we will recall some properties of $WPSAP_\omega(\mathbb{Z}, X, \rho_1, \rho_2)$ proved in [118].

Lemma 1.3.1. *[118, Lemma 2.2] Let $\rho_1, \rho_2 \in V_\infty$ be given, then*

1. *For each $l \in \mathbb{Z}$, one has*
$$\limsup_{n\to\infty} \frac{\nu(n+l,\rho_2)}{\nu(n,\rho_1)} < \infty.$$
2. *$WPSAP_\omega(\mathbb{Z}, X, \rho_1, \rho_2)$, where $\omega \in \mathbb{Z}_+ \setminus \{0\}$, is translation invariant, that is $f(\cdot + l) \in WPSAP_\omega(\mathbb{Z}, X, \rho_1, \rho_2)$ for each $l \in \mathbb{Z}$, if $f \in WPSAP_\omega(\mathbb{Z}, X, \rho_1, \rho_2)$.*

3. *$WPSAP_\omega(\mathbb{Z}, X, \rho_1, \rho_2)$, where $\omega \in \mathbb{Z}_+ \setminus \{0\}$, is a closed subspace of $l^\infty(\mathbb{Z}, X)$.*

Remark 1.3.2. It is easy to see that the following inclusions hold: For $\omega \in \mathbb{Z}_+ \setminus \{0\}$

$$C_\omega(\mathbb{Z}, X) \subset AP_\omega(\mathbb{Z}, X) \subset SAP_\omega(\mathbb{Z}, X) \subset PSAP_\omega(\mathbb{Z}, X) \subset WPSAP_\omega(\mathbb{Z}, X) \subset l^\infty(\mathbb{Z}, X).$$

Let $\rho_1, \rho_2 \in U_\infty$ be given, and $\omega \in \mathbb{Z}_+ \setminus \{0\}$. In what follows, we will consider the sets $\mathcal{M}(\mathbb{Z}, X) := \{WPAA_d(\mathbb{Z}, X), WPSAP_\omega(\mathbb{Z}, X)\}$ and $\mathcal{M}(\mathbb{Z}\times X, X) := \{WPAA_d(\mathbb{Z}\times X, X), WPSAP_\omega(\mathbb{Z}\times X, X)\}$.

1.4 Measure of noncompactness and fixed point theorems

We will relate the notion of fixed point with the concept of measure of noncompactness. For this reason, we next recall properties of this concept. For general information, see [22].

Definition 1.4.1. Let B be a bounded subset of a normed space X. The Hausdorff measure of noncompactness of B is defined by

$$\chi_H(B) = \inf\{\epsilon > 0 : B \text{ has a finite cover by balls of radius } \epsilon\}.$$

The Hausdorff measure of noncompactness has some useful properties, now we list some of them that we will require in this thesis. See [22, 8, 23] for more details. Let B_1, B_2 be bounded subsets of a normed space X. Then

(i) $\chi_H(B_1) \leq \chi_H(B_2)$ if $B_1 \subseteq B_2$,

(ii) $\chi_H(B_1) = \chi_H(\overline{B_1})$, where $\overline{B_1}$ denotes the closure of B_1,

(iii) $\chi_H(B_1) = 0$ if and only if B_1 is totally bounded,

(iv) $\chi_H(\lambda B_1) = |\lambda|\chi_H(B_1)$ with $\lambda \in \mathbb{R}$,

(v) $\chi_H(B_1 \cup B_2) = \max\{\chi_H(B_1), \chi_H(B_2)\}$,

(vi) $\chi_H(B_1 + B_2) \leq \chi_H(B_1) + \chi_H(B_2)$, where $B_1 + B_2 = \{b_1 + b_2 : b_1 \in B_1, b_2 \in B_2\}$,

(vii) $\chi_H(B_1) = \chi_H(\overline{co}(B_1))$, where $(\overline{co}(B_1))$ is the closed convex hull of B_1.

In what follows, we denote by ξ the Hausdorff measure of noncompactness defined in X, by γ the Hausdorff measure of noncompactness on $\mathcal{C}([0, b]; X)$. The following lemma on measure of noncompactness will allow us to prove our main findings.

Lemma 1.4.2. *[23, Lemma 5.1] Let $G : X \to X$ be a Lipschitz continuous map with constant k. Then $\xi(G(B)) \leq k\xi(B)$ for any bounded subset B of X.*

Lemma 1.4.3. *[121, Property 1.1] Let $W \subseteq \mathcal{C}([0, b]; X)$ be a subset of continuous functions. If W is bounded and equicontinuous on $[0, b]$, then the set $\overline{co}(W)$ is also bounded and equicontinuous on $[0, b]$.*

Next, we set forth some lemmas that will play an important part in the proof of our main result in Chapter 2, Section 2.2.

Lemma 1.4.4. *[23, Lemma 5.3] Let $W \subseteq \mathcal{C}([0, b]; X)$ be a bounded set. Then $\xi(W(t)) \leq \gamma(W)$ for all $t \in [0, b]$. If W is equicontinuous on $[0, b]$, then $\xi(W(t))$ is continuous on $[0, b]$, and*

$$\gamma(W) = sup\{\xi(W(t)) : t \in [0, b]\},$$

where $W(t) = \{w(t) : w \in W\}$.

Lemma 1.4.5. *[23, Lemma 5.4] If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^1([0, b], X)$ is uniformly integrable, then for each*

$n \in \mathbb{N}$ the function $t \to \xi(\{u_n(t)\}_{n\in\mathbb{N}})$ is measurable and

$$\xi(\{\int_0^t u_n(s)ds\}_{n=1}^{\infty}) \leq 2\int_0^t \xi(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 1.4.6. *[32, Theorem 2] Let Y be a Banach space. If $W \subseteq Y$ is a bounded set, then for each $\epsilon > 0$, there exist a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W$ such that*

$$\xi(W) \leq 2\xi(\{u_n\}_{n=1}^{\infty}) + \epsilon. \tag{1.4.1}$$

Lemma 1.4.7. *[73, Theorem 3.1] For all $0 \leq m \leq n$, denote by $C_m^n = \frac{n!}{m!(n-m)!}$. If $0 < \epsilon < 1, h > 0$ and let*

$$S_n = \epsilon^n + C_1^n\epsilon^{n-1}h + C_2^n\epsilon^{n-2}\frac{h^2}{2!} + ... + \frac{h^n}{n!}, \quad n \in \mathbb{N}, \tag{1.4.2}$$

then $\lim_{n\to\infty} S_n = 0$.

Next, we recall important fixed-point theorems.

Lemma 1.4.8. *[73, Lemma 2.4] Let S be a closed and convex subset of a complex Banach space Y, $F : S \to S$ be a continuous operator such that $F(S)$ is a bounded set. Define*

$$F^1(S) = F(S)$$

and

$$F^n(S) = F(\overline{co}(F^{n-1}(S))), \quad n = 2, 3, ...$$

If there exist a constant $0 \leq r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\xi(F^{n_0}(S)) \leq r\xi(S),$$

then F has a fixed point in the set S.

Theorem 1.4.9. *[102, Matkowski Fixed Point Theorem] Let (X, d) be a complete metric space and let $\mathcal{F} : X \to X$ be a map such that $d(\mathcal{F}x, \mathcal{F}y) \leq \phi(d(x, y))$ for all $x, y \in X$, where $\phi : [0,\infty) \to [0,\infty)$*

is a nondecreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ *for all* $t > 0$, *where* ϕ^n *is the* n*-th iterate of* ϕ. *Then* $\mathcal{F}$ *has a unique fixed point* $z \in X$.

Theorem 1.4.10. *[67, Leray- Schauder Alternative Theorem] Let* Ω *be a closed convex subset of the Banach space* X *such that* $0 \in \Omega$. *Let* $\mathcal{F} : \Omega \to \Omega$ *be a completely continuous map. Then the set* $\{x \in \Omega : x = \lambda\mathcal{F}(x), 0 < \lambda < 1\}$ *is unbounded or the map* $\mathcal{F}$ *has a fixed point in* Ω.

Theorem 1.4.11 (*Schauder fixed point theorem*)**.** *Let* C *be a nonempty, closed, bounded, and convex subset of a Banach space* X. *Suppose that* $T : C \to C$ *is a compact operator. Then* T *has at least a fixed point in* C.

We recall a compactness criterion, the Leray-Schauder alternative theorem and Matkowski's fixed point theorem which will be useful in the future to prove existence and uniqueness of solution to (0.0.2).

Lemma 1.4.12. *[118, Lemma 2.1] Let* $h : \mathbb{Z} \to \mathbb{R}^+$ *be a function such that* $h(n) \geq 1$ *for all* $n \in \mathbb{Z}$, *and* $h(n) \to \infty$ *as* $|n| \to \infty$. *Let* S *be a subset of* $\mathcal{C}_h^0(\mathbb{Z}, X)$. *Suppose that the following conditions are satisfied:*

1. *The set* $\mathcal{H}(S) = \left\{\frac{u(n)}{h(n)} : u \in S\right\}$ *is relatively compact in* X *for all* $n \in \mathbb{Z}$.
2. S *is weighted equiconvergent at* $\pm\infty$, *that is for every* $\epsilon > 0$, *there is a* $T > 0$ *such that* $\|u(n)\| < \epsilon h(n)$ *for each* $|n| \geq T$ *for all* $u \in S$.

Then S *is relatively compact in* $\mathcal{C}_h^0(\mathbb{Z}, X)$.

Chapter 2

Nonlocal integrated solutions for a class of abstract evolution equations

In this chapter we study the following equation

$$\begin{aligned} u''(t) + Au(t) - (k * Au)(t) =& f(t, u(t)), \quad t \in [0, b], \\ u(0) = g(u), \quad u'(0) =& h(u), \end{aligned} \tag{2.0.1}$$

where X is a Banach space, $A : D(A) \subseteq X \to X$ the generator of a resolvent family $S(t)$ with integrated kernel $a(t) = \int_0^t (t-s)k(s)ds - t$, where $k \in L^1(\mathbb{R}_+)$. Here, $g, h : \mathcal{C}([0, b]; X) \to X$ are continuous maps, and $f : [0, b] \times X \to X$ satisfies Carathéodory type conditions, which will be described in Section 2.2. The objective is to establish a result concerning the existence of integrated solutions for the above problem, using the theory of measure of noncompactness and fixed point theorems.

2.1 A criteria for existence of integrated solutions

Let X be a Banach space, $A : D(A) \subseteq X \to X$ be a closed linear operator that generates a resolvent $\{S(t)\}_{t\geq 0}$ with kernel $a(t) = \int_0^t (t-s)k(s)ds - t$, where $k \in L^1(\mathbb{R}_+)$. In this section, we want to study the existence of solutions for the following semilinear problem with local conditions

$$\begin{aligned} u''(t) + Au(t) - (k * Au)(t) =& f(t, u(t)), \quad t \in [0, b], \\ u(0) = u_0, u'(0) =& u_1, \quad u_0, u_1 \in X. \end{aligned} \tag{2.1.1}$$

Here $f : \mathbb{R}_+ \times X \to X$ is locally integrable. Let us consider the associated linear problem

$$\begin{aligned} u''(t) + Au(t) - (k * Au)(t) =& f(t), \quad t \in [0, b], \\ u(0) = u_0, u'(0) =& u_1, \quad u_0, u_1 \in X. \end{aligned} \tag{2.1.2}$$

Observe that integrating (2.1.2) twice, we obtain the following equivalent representation,

$$u(t) - A(a * u)(t) = (g_2 * f)(t) + tu_1 + u_0, \tag{2.1.3}$$

where $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $\alpha > 0$ and $a(t) = (g_2 * k - g_2)(t)$. Here $\Gamma(\alpha)$ denotes the Gamma function. Then, by Theorem 1.1.4, we have that

$$u(t) = S(t)h(0) + \int_0^t S(t-s)h'(s)ds,$$

solves (2.1.3) with $h(t) := (g_2 * f)(t) + tu_1 + u_0$, thus

$$u(t) = S(t)u_0 + R(t)u_1 + \int_0^t R(t-s)f(s)ds, \tag{2.1.4}$$

where $R(t)x := \displaystyle\int_0^t S(\tau)x d\tau$, $x \in X$ solves (2.1.2) whenever $S(t)$ and the initial data are regular enough.

Motivated by this observation the following definition is meaningful.

Definition 2.1.1. Suppose that A is the generator of a resolvent family $\{S(t)\}_{t\geq 0}$ with kernel $a(t) = \int_0^t (t-s)k(s)ds - t$. Let $u_0, u_1 \in X$ be given. We say that $u \in \mathcal{C}([0, b]; X)$ is an integrated solution of (2.1.1) if u satisfy the integral equation

$$u(t) = S(t)u_0 + R(t)u_1 + \int_0^t R(t-s)f(s, u(s))ds, \quad t \in [0, b],$$

where $R(t)x := \int_0^t S(\tau)x d\tau$, $x \in X$.

Borrowing ideas from [57, Lemma 3.4 and Lemma 3.5] we obtain the following result.

Lemma 2.1.2. *Let $\{S(t)\}_{t\geq 0}$ be a resolvent family with generator A. Suppose that*

(i) $\{S(t)\}_{t\geq 0}$ is continuous in the uniform operator topology for all $t > 0$.

(ii) $S(t)$ is compact for each $t > 0$.

Then

(a) $\lim_{h\to 0} \|S(t+h) - S(h)S(t)\| = 0$ *for all* $t > 0$;

(b) $\lim_{h\to 0} \|S(t) - S(h)S(t-h)\| = 0$ *for all* $t > 0$.

Proof. We first prove (a). Let $x \in X$ with $\|x\| \leq 1$, $t > 0$ and $\epsilon > 0$ be given. From (ii) we deduce that the set $W_t := \{S(t)x : \|x\| \leq 1\}$ is also compact. Thus, there exists a finite family $\{S(t)x_1, S(t)x_2, ..., S(t)x_m\} \subset W_t$ such that for any x with $\|x\| \leq 1$, there exists $x_i (1 \leq i \leq m)$ such that

$$\|S(t)x - S(t)x_i\| \leq \frac{\epsilon}{3(M+1)}, \tag{2.1.5}$$

where $M = \sup_{t\in[0,b]} \|S(t)\| < \infty$. From the strong continuity of $S(t)$, there exists $0 < h_i < \min\{t, b\}$ such that

$$\|S(t)x_i - S(h)S(t)x_i\| \leq \frac{\epsilon}{3}, \tag{2.1.6}$$

for all $0 \leq h \leq h_i$ and $1 \leq i \leq m$. On the other hand, from (i), there exists $0 < h_2 < \min\{t, b\}$ such that

$$\|S(t+h)x - S(t)x\| \leq \frac{\epsilon}{3}, \tag{2.1.7}$$

for all $0 \leq h \leq h_2$ and $\|x\| \leq 1$. Thus, for $0 \leq h \leq \min\{h_1, h_2\}$ and $\|x\| \leq 1$, it follows from

(2.1.5)–(2.1.7) that

$$\begin{aligned}
&\|S(t+h)x - S(h)S(t)x\| \\
&\qquad \le \|S(t+h)x - S(t)x\| + \|S(t)x - S(t)x_i\| + \|S(t)x_i - S(h)S(t)x_i\| \\
&\qquad + \|S(h)S(t)x_i - S(h)S(t)x\| \\
&\qquad \le \|S(t+h)x - S(t)x\| + (M+1)\|S(t)x - S(t)x_i\| + \|S(t)x_i - S(h)S(t)x_i\| \le \epsilon,
\end{aligned}$$

which implies (a).

We prove (b). Let $t > 0$ and $0 < h < \min\{t, b\}$. Then, there exist $M > 0$ such that

$$\begin{aligned}
&\|S(t) - S(h)S(t-h)\| \\
&\le \|S(t) - S(t+h)\| + \|S(t+h) - S(h)S(t)\| + \|S(h)S(t) - S(h)S(t-h)\| \\
&\le \|S(t) - S(t+h)\| + \|S(t+h) - S(h)S(t)\| + M\|S(t) - S(t-h)\| \qquad (2.1.8)
\end{aligned}$$

which implies the desired result by (a) and (i). □

Remark 2.1.3. In contrast with the theory of C_0-semigroups, where compactness of the semigroup implies their continuity in the uniform operator topology for $t > 0$ [106, Theorem 3.2], the compactness of a resolvent family alone is not enough to guarantee their continuity in $\mathcal{B}(X)$, except in particular cases of the kernel $a(t)$. See [107, Corollary 2 and Theorem 7].

The following is the main result of this section.

Theorem 2.1.4. *Suppose that A is the generator of a resolvent $\{S(t)\}_{t\ge 0}$ exponentially bounded of type (M,ω) and kernel $a(t) = \int_0^t (t-s)k(s)ds - t$, that is in addition compact and continuous in the uniform operator topology for all $t > 0$. Let $f : [0,b]\times X \to X$ be continuous with respect to the second variable and assume that the function $f(\cdot, x)$ is measurable for all $x \in X$, and $\|f(t,x)\| \le \alpha(t)(\|x\|+1)$ for a.e. $t \in [0,b]$ and $x \in X$, with $\alpha \in L^1([0,b];X)$. Then there is at least one integrated solution of (2.1.1), provided that $Mbe^{\omega b}\|\alpha\|_{L^1} < 1$.*

Proof. Fix $u_0, u_1 \in X$ and define the operator $G : \mathcal{C}([0,b], X) \to \mathcal{C}([0,b], X)$ by

$$Gu(t) = S(t)u_0 + R(t)u_1 + \int_0^t R(t-s)f(s,u(s))ds, \qquad t \in [0,b],$$

where $R(t)x := \displaystyle\int_0^t S(\tau)xd\tau$, $x \in X$, and $\|S(t)\| \le Me^{\omega b}$ for all $t \in [0,b]$, where $M > 0$ and w.l.o.g. $\omega > 0$. Then G is clearly well defined, and u is an integrated solution of (2.1.1) if and only if it is a fixed point of operator G. Now we will show that the mapping G is continuous on $\mathcal{C}([0,b], X)$. Let $\{u_n\}_{n\ge 1}$ be a sequence in $\mathcal{C}([0,b], X)$ with $\lim_{n\to\infty} u_n = u$ in $\mathcal{C}([0,b], X)$. Then

$$\|(Gu_n)(t) - (Gu)(t)\| \le Mbe^{\omega b} \int_0^t \|f(s,u_n(s)) - f(s,u(s))\|ds, \quad t \in [0,b].$$

Since $f(s,u_n(s))$ converges to $f(s,u(s))$ in X for $s \in [0,b]$, and

$$\|f(s,u_n(s)\| \le \alpha(s)(\|u_n(s)\| + 1),$$

where $\alpha \in L^1([0,b],\mathbb{R})$ then, by the Lebesgue dominated convergence theorem, we obtain that $G(u_n) \to G(u)$, as $n \to \infty$. This proves the claim.

We denote

$$W_R = \{u \in \mathcal{C}([0,b], X) : \|u(t)\| \le R, \quad \text{for all} \quad t \in [0,b]\},$$

where $R > 0$ is given. We claim that there exists $r > 0$ such that G maps W_r into itself. We choose $r > 0$ such that

$$(Me^{\omega b}\|u_0\| + Mbe^{\omega b}\|u_1\| + Mbe^{\omega b}\|\alpha\|_{L^1})(1 - Mbe^{\omega b}\|\alpha\|_{L^1})^{-1} < r.$$

Note that the last inequality implies

$$Me^{\omega b}\|u_0\| + Mbe^{\omega b}\|u_1\| + Mbe^{\omega b}\|\alpha\|_{L^1}(r+1) < r.$$

Then by definition of G

$$\|G(u)\| \le Me^{\omega b}\|u_0\| + Mbe^{\omega b}\|u_1\| + Mbe^{\omega b}\|\alpha\|_{L^1}(r+1) < r.$$

This proves the claim.

Now, we will prove that $G : W_r \to W_r$ is a compact operator. Indeed, by the Arzela- Ascoli theorem, we have to show that the set $GW_r := \{Gu : u \in W_r\}$ is equicontinuous, and the set $\{Gu(t) : u \in W_r\}$ is relatively compact in X for each $t \in [0, b]$. Let $0 \le t_1 \le t_2 \le b$ and $u \in W_r$, we have

$$\begin{aligned}
\|(Gu)(t_2) - (Gu)(t_1)\| &\le \|S(t_2)u_0 - S(t_1)u_0\| + \|R(t_2)u_1 - R(t_1)u_1\| \\
&+ \|\int_0^{t_2} R(t_2 - s)f(s, u(s))ds - \int_0^{t_1} R(t_1 - s)f(s, u(s))ds\| \\
&\le \|S(t_2)u_0 - S(t_1)u_0\| + \|R(t_2)u_1 - R(t_1)u_1\| \\
&+ \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|\|f(s, u(s))ds\|ds \\
&+ \int_{t_1}^{t_2} \|R(t_2 - s)f(s, u(s))\|ds \\
&\le \|S(t_2)u_0 - S(t_1)u_0\| + \|R(t_2)u_1 - R(t_1)u_1\| \\
&+ \int_0^{t_1} \|R(t_2 - s) - R(t_1 - s)\|\alpha(s)rds + Mbe^{\omega b}(r + 1)\int_{t_1}^{t_2} \alpha(s)ds. \quad (2.1.9)
\end{aligned}$$

If $t_1 = 0$, then

$$\lim_{t_2\to 0} \|(Gu)(t_2) - (Gu)(t_1)\| = 0, \qquad \text{uniformly for} \quad u \in W_r.$$

If $0 < t_1 < b$. Note that $R(t)x = \int_0^t S(\tau)xd\tau$ and hence $R(t)$ is a norm continuous operator. Then, from (2.1.9) we obtain

$$\lim_{|t_1-t_2|\to 0} \|(Gu)(t_2) - (Gu)(t_1)\| = 0, \quad \text{uniformly for } u \in W_r.$$

Then, the set GW_r is equicontinuous on $\mathcal{C}([0, b], X)$, proving the first part of the claim.
Now, we will show that the set $M(t) := \{(Gu)(t) : u \in W_r\}$ is relatively compact in X for every $t \in [0, b]$. If $t = 0$ then the set $M(0)$ is clearly relatively compact in X. We denote $g(s, u) := \int_0^s f(t, u(t))dt$, $0 \le s \le b$, $u \in \mathcal{C}([0, b], X)$, and note that the hypothesis implies

$$\|g(s, u)\| \le \|\alpha\|_{L^1}(r + 1) \quad \text{ for all } 0 \le s \le b, \quad u \in W_r.$$

Moreover, integration by parts shows the identity

$$\int_0^t R(t - s)f(s, u(s))ds = \int_0^t S(t - s)\Big[\int_0^s f(\tau, u(\tau))d\tau\Big]ds = \int_0^t S(t - s)g(s, u)ds.$$

Let $0 < t \leq b$ be given and $0 < \epsilon < t$. We first observe that the set $\{\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds : u \in W_r\}$ is bounded. Indeed, for all $u \in W_r$

$$\|\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds\| \leq Mbe^{\omega b}\|\alpha\|_{L^1}(r+1).$$

Thus $\{S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds : u \in W_r\}$ is relatively compact, since $S(\epsilon)$ is compact, and the set $\{\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds : u \in W_r\}$ is bounded.

Moreover, for all $u \in W_r$

$$\begin{aligned}
&\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\| \\
\leq &\int_0^{t-\epsilon} \|S(\epsilon)S(t-s-\epsilon) - S(t-s)\|\|\alpha\|_{L^1}(r+1)ds
\end{aligned}$$

Moreover, since $S(t)$ is compact and continuous in the uniform operator topology for all $t > 0$. Then by part (b) of Lemma 2.1.2, we have that

$$S(\epsilon)S(t-s-\epsilon) - S(t-s) \to 0, \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad s \in [0, t-\epsilon]$$

and

$$\int_0^{t-\epsilon} \|S(\epsilon)S(t-s-\epsilon) - S(t-s)\|ds \leq ((Me^{\omega b})^2 + Me^{\omega b})b.$$

From the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\epsilon\to 0} \|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\| = 0.$$

Moreover,

$$\begin{aligned}
&\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t} S(t-s)g(s,u)ds\| \\
\leq &\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\| \\
&+ \|\int_0^{t-\epsilon} S(t-s)g(s,u)ds - \int_0^{t} S(t-s)g(s,u)ds\| \\
\leq &\|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t-\epsilon} S(t-s)g(s,u)ds\| + \int_{t-\epsilon}^{t} Me^{\omega b}\|\alpha\|_{L^1} r ds.
\end{aligned}$$

Thus,

$$\lim_{\epsilon\to 0} \|S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds - \int_0^{t} S(t-s)g(s,u)ds\| = 0.$$

Then, $\{\int_0^t R(t-s)f(s,u(s))ds : u \in W_r\} = \{\int_0^t S(t-s)g(s,u)ds : u \in W_r\}$ is relatively compact in X by using the relative compactness of the set $\{S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)g(s,u)ds : u \in W_r\}$. Hence, the set $M(t) = \{(Gu)(t) : u \in W_r\}$ is relatively compact in X for each $t \in [0,b]$, proving the claim.

Hence, by Schauder fixed point theorem, we conclude that (2.1.1) has an integrated solution. $\square$

Remark 2.1.5. The hypothesis $\|f(s,x)\| \leq \alpha(s)(\|x\|+1)$, $\alpha \in L^1([0,b],\mathbb{R})$, also has been considered previously in [8, Theorem 5.2.2] in order to prove the existence of mild solutions for a class of semilinear differential inclusions in a Banach space X.

2.2 Nonlocal initial conditions

Let X be a Banach space, $A : D(A) \subseteq X \to X$ be closed and linear operator and $k \in L^1(\mathbb{R}_+)$ be a scalar memory kernel. We consider the problem

$$\begin{aligned} u''(t) + Au(t) - (k * Au)(t) =& f(t,u(t)), \quad t \in [0,b], \\ u(0) = g(u), u'(0) =& h(u), \end{aligned} \tag{2.2.1}$$

where $g, h : \mathcal{C}([0,b];X) \to X$ are continuous maps and $f : [0,b] \times X \to X$. We set the following conditions.

(H1) A generates a exponentially bounded resolvent $\{S(t)\}_{t\geq 0}$ of type (M,ω) and kernel $a(t) = \int_0^t (t-s)k(s)ds - t$.

(H2) g, h are compact maps.

(H3) The function $f(\cdot,x)$ is measurable for all $x \in X$ and $f(t,\cdot)$ is continuous for almost all $t \in [0,b]$.

(H4) There exists a function $m \in L^1([0,b];\mathbb{R}^+)$ and a nondecreasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leq m(t)\phi(\|x\|),$$

for all $x \in X$ and almost all $t \in [0,b]$.

(H5) There exists a function $H \in L^1([0,b];\mathbb{R}^+)$ such that for any bounded $S \subseteq X$

$$\xi(f(t,S)) \leq H(t)\xi(S),$$

for almost all $t \in [0,b]$.

We introduce the following definition.

Definition 2.2.1. A function $u \in \mathcal{C}([0,b],X)$ is called a nonlocal integrated solution of the equation (2.2.1) if u satisfies

$$u(t) = R(t)h(u) + S(t)g(u) + \int_0^t R(t-s)f(s,u(s))ds, \quad t \in [0,b], \tag{2.2.2}$$

where $R(t)x := \displaystyle\int_0^t S(\tau)x d\tau$, $x \in X$.

The following is the main result of this section.

Theorem 2.2.2. *Suppose that A satisfies **(H1)**, $g,h : \mathcal{C}([0,b];X) \to X$ satisfies **(H2)**, $f : [0,b] \times X \to X$ satisfies **(H3)-(H5)**, and there exists a constant $R > 0$ such that*

$$Me^{\omega b}(h_R + bg_R + b\phi(R)\int_0^b m(s)ds) \leq R, \tag{2.2.3}$$

where $g_R := \sup\{\|g(u)\| : \|u\|_\infty \leq R\} < \infty$, and $h_R := \sup\{\|h(u)\| : \|u\|_\infty \leq R\} < \infty$. Then the problem (2.2.1) *has at least one nonlocal integrated solution.*

Proof. Define the operator $F : \mathcal{C}([0,b],X) \to \mathcal{C}([0,b],X)$ by

$$Fu(t) = S(t)h(u) + R(t)g(u) + \int_0^t R(t-s)f(s,u(s))ds, \quad t \in [0,b],$$

where $R(t)x := \displaystyle\int_0^t S(\tau)x d\tau$, $x \in X$. We will show that the mapping F is continuous on $\mathcal{C}([0,b],X)$. Indeed, let $\{u_n\}_{n\geq 1}$ be a sequence in $\mathcal{C}([0,b],X)$ with $\lim_{n\to\infty} u_n = u$, for the norm of uniform convergence. Then

$$\|(Fu_n)(t) - (Fu)(t)\| \leq Me^{\omega b}\|h(u_n) - h(u)\| + Mbe^{\omega b}\|g(u_n) - g(u)\|$$

$$+ Mbe^{\omega b} \int_0^t \|f(s,u_n(s)) - f(s,u(s))\| ds, \quad t \in [0,b],$$

where w.l.o.g. $\omega > 0$. Since g, h are continuous maps, we obtain $g(u_n) \to g(u)$, and $h(u_n) \to h(u)$ as $n \to \infty$. Moreover, since f satisfies hypotheses **(H3)**, by the Lebesgue dominated convergence theorem, we obtain that $F(u_n) \to F(u)$, as $n \to \infty$. This proves the claim.

Define the set

$$B_R = \{u \in \mathcal{C}([0,b], X) : \|u(t)\| \le R \quad \text{for all} \quad t \in [0,b]\}.$$

Then

$$\begin{aligned}\|Fu(t)\| \le & \|S(t)h(u)\| + \|R(t)g(u)\| + \int_0^t \|R(t-s)f(s,u(s))\| ds \\ \le & Me^{\omega b}(h_R + bg_R + b\phi(R) \int_0^b m(s)ds) \le R.\end{aligned}$$

Then, F maps B_R into itself, and $F(B_R)$ is a bounded set. On the other hand, as in the proof of Theorem 2.1.4, we get that the set $F(B_R)$ is an equicontinuous set of functions.

Now, we define the set $\mathfrak{A} := \overline{co}(F(B_R))$. By Lemma 1.4.3 such set $\mathfrak{A}$ is equicontinuous. Since $\mathfrak{A} \subseteq B_R$, we conclude that the map $F : \mathfrak{A} \to \mathfrak{A}$ is continuous and $F(\mathfrak{A})$ is a bounded set of functions.

Let $\epsilon > 0$ be fixed. By Lemma 4 there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq F(\mathfrak{A})$ such that $\xi(F(\mathfrak{A})) \le 2\xi(\{v_n(t)\}_{n=1}^{\infty}) + \epsilon \le 2\xi\big(\int_0^t \{R(t-s)f(s,u_n(s))\}_{n=1}^{\infty} ds\big) + \epsilon$, where in the second inequality we have used the compactness of h and g. By hypotheses **(H4)** and **(H5)**, we have that

$$\begin{aligned}\xi(F(\mathfrak{A})(t)) \le & 4Mbe^{\omega b} \int_0^t \xi(\{f(s,u_n(s))\}_{n=1}^{\infty}) ds + \epsilon \\ & \le 4Mbe^{\omega b} \int_0^t H(s)\xi(\{u_n(s)\}_{n=1}^{\infty}) ds + \epsilon \\ & \le 4Mbe^{\omega b}\gamma(\mathfrak{A}) \int_0^t H(s)ds + \epsilon.\end{aligned}$$

By the hypotheses **(H5)** we have $H \in L^1([0,b];\mathbb{R}^+)$. Then for $\alpha < \frac{1}{4Mbe^{\omega b}}$ there exist $\varphi \in \mathcal{C}([0,b],\mathbb{R}^+)$ such that $\int_0^b |H(s) - \varphi(s)| ds < \alpha$. Hence

$$\xi(F(\mathfrak{A})(t)) \le 4Mbe^{\omega b}\gamma(\mathfrak{A})\big[\int_0^t |H(s) - \varphi(s)| ds + \int_0^t \varphi(s)ds\big] + \epsilon$$

$$\leq 4Mbe^{\omega b}\gamma(\mathfrak{A})[\alpha + Nt] + \epsilon,$$

where $N = \|\varphi\|_\infty$. Then, we have

$$\xi(F(\mathfrak{A})(t)) \leq (a+ct)\gamma(\mathfrak{A}), \quad \text{where} \quad a = 4\alpha Mbe^{\omega b} \quad \text{and} \quad c = 4MNbe^{\omega b}. \tag{2.2.4}$$

Let $\epsilon > 0$ be given, then by Lemma 4 there exist a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \overline{co}(F(\mathfrak{A}))$ such that

$$\begin{aligned}
\xi(F^2(\mathfrak{A})(t)) \leq& 2\xi\big(\int_0^t \{R(t-s)f(s,w_n(s))\}_{n=1}^\infty ds\big) + \epsilon \\
\leq& 4Mbe^{\omega b}\int_0^t \xi\{f(s,w_n(s))\}_{n=1}^\infty ds + \epsilon \\
\leq& 4Mbe^{\omega b}\int_0^t H(s)\xi(\overline{co}(F^1(\mathfrak{A})(s))) + \epsilon \\
=& 4Mbe^{\omega b}\int_0^t H(s)\xi(F^1(\mathfrak{A})(s)) + \epsilon.
\end{aligned}$$

By (4.2) we have that

$$\begin{aligned}
\xi(F^2(\mathfrak{A})(t)) \leq& 4Mbe^{\omega b}\int_0^t [|H(s)-\varphi(s)| + |\varphi(s)|](a+cs)\gamma(\mathfrak{A})ds + \epsilon \\
&\leq 4Mbe^{\omega b}(a+ct)\gamma(\mathfrak{A})\int_0^t |H(s)-\varphi(s)|ds + 4MNbe^{\omega b}\gamma(\mathfrak{A})\big(at + \frac{ct^2}{2}\big) + \epsilon \\
&\leq (a(a+ct) + c(at + \frac{ct^2}{2}))\gamma(\mathfrak{A}) + \epsilon \\
&\leq (a^2 + 2act + \frac{(ct)^2}{2})\gamma(\mathfrak{A}).
\end{aligned}$$

By induction, for all $n \in \mathbb{N}$,

$$\xi(F^n(\mathfrak{A})(t)) \leq \big(a^n + C_1^n a^{n-1}ct + C_2^n a^{n-2}\frac{(ct)^2}{2!} + ... + \frac{(ct)^n}{n!}\big)\gamma(\mathfrak{A}),$$

where C_m^n denotes the binomial coefficient $\binom{m}{n}$, for $0 \leq m \leq n$. Moreover, $F^n(\mathfrak{A})$ is an equicontinuous set of functions for all $n \in \mathbb{N}$. Thus, by Lemma 2

$$\gamma(F^n(\mathfrak{A})) \leq \big(a^n + C_1^n a^{n-1}c + C_2^n a^{n-2}\frac{c^2}{2!} + ... + \frac{c^n}{n!}\big)\gamma(\mathfrak{A}).$$

Since $0 \leq a < 1$ and $c > 0$, by Lemma 3 there exist $n_0 \in \mathbb{N}$ such that

$$(a^{n_0} + C_1^{n_0} a^{n_0-1}ct + C_2^{n_0} a^{n_0-2}\frac{(ct)^2}{2!} + ... + \frac{(ct)^{n_0}}{n_0!}) = r < 1.$$

Therefore, $\gamma(F^{n_0}(\mathfrak{A})) \leq r\gamma(\mathfrak{A})$. Then F has a fixed point in $\mathfrak{A}$ by Lemma 1.4.8.

$\square$

2.3 Examples

Example 2.3.1. *We set $X = L^2([0,1])$, and we consider $Az(\xi) = \frac{d^2 z(\xi)}{d\xi^2}$ with domain $D(A) = \{z \in H^2[0,1] : z(0) = z(1) = 0\}$. It is well known that A generates a bounded analytic semigroup on $L^2[0,1]$, see [55, Example 4.8]. Let $a(t) = 1 + \frac{\beta}{\alpha} - \frac{\beta}{\alpha}e^{-\alpha t}$ where $-\alpha \leq \beta \leq 0 < \alpha$. We will prove that $a(t)$ is completely positive. See Definition 1.1.6.*

Indeed, we have

$$\hat{a}(\lambda) = \frac{\lambda + \alpha + \beta}{\lambda(\lambda + \alpha)},$$

then

$$\frac{1}{\lambda \hat{a}(\lambda)} = \frac{\lambda + \alpha}{\lambda + \alpha + \beta} \quad \text{and} \quad \frac{-\hat{a}'(\lambda)}{[\hat{a}(\lambda)]^2} = \frac{\lambda^2 + 2(\alpha+\beta)\lambda + \alpha\beta + \alpha^2}{(\lambda + \alpha + \beta)^2}.$$

If we denote $c_1(\lambda) = \frac{1}{\lambda \hat{a}(\lambda)}$ and $c_2(\lambda) = \frac{-\hat{a}'(\lambda)}{[\hat{a}(\lambda)]^2}$, then we obtain that

$$c_1^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta n!}{(\lambda + \alpha + \beta)^{n+1}} \quad \text{and} \quad c_2^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(\alpha+\beta)(n+1)!}{(\lambda + \alpha + \beta)^{n+2}} \quad \text{for} \quad n \in \mathbb{N}.$$

Since $-\alpha \leq \beta \leq 0 < \alpha$, c_1 and c_2 are completely monotone. We conclude that a is completely positive.

Let $p : [0,1] \times [0,1] \to \mathbb{R}$ be Hilbert Schmidt, i.e.

$$\int_0^1 \int_0^1 |p(x,y)|^2 dx dy = c < \infty. \tag{2.3.1}$$

Given $-\alpha \leq \beta \leq 0 < \alpha$, and $s > \frac{1}{2Mc}$, we consider the following problem

$$\begin{cases} \dfrac{\partial^2 u(t,\xi)}{\partial t^2} + \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + \displaystyle\int_0^t \alpha\beta e^{-\alpha(t-s)} \dfrac{\partial^2 u(s,\xi)}{\partial \xi^2} ds = t \int_0^1 \sin(u(t,s))ds, \;\; t, \xi \in [0,1], \\ u(t,0) = u(t,1) = 0, \quad t \in [0,1], \\ u(0,\xi) = \displaystyle\int_0^1 sp(\xi,y)u(1/2,y)dy, \quad \xi \in [0,1], \\ \dfrac{\partial u}{\partial t}(0,\xi) = \displaystyle\int_0^1 sp(\xi,y)u(1/2,y)dy, \quad \xi \in [0,1]. \end{cases} \tag{2.3.2}$$

The problem (2.3.2) can be rewritten as

$$\begin{cases} u''(t) + Au(t) - (k * Au)(t) = f(t, u(t)), \quad t \in [0,1] \\ u(0) = g(u), \\ u'(0) = h(u), \end{cases} \tag{2.3.3}$$

with $k(t) = -\alpha\beta e^{-\alpha t} \in L^1(\mathbb{R}_+)$*, and* $g, h : \mathcal{C}([0,1]; X) \to X$ *are explicitly given by*

$$g(u) = sk_g(u(1/2)), \quad h(u) = sk_h(u(1/2)), \tag{2.3.4}$$

with $(k_g v)(\xi) = (k_h v)(\xi) = \int_0^1 p(\xi, y)v(y)dy, \quad$ *for* $\quad v \in L^2[0,1], \quad \xi \in [0,1]$ *and* $f(t, \phi) = t\int_0^1 \sin(\phi(s))ds$.

*We will prove that the hypotheses **(H1) - (H5)** are satisfied.*

(H1) From the above, a *is completely positive. Thus, by [109, Theorem 4.2] the operator* A *generates a resolvent operator* $S(t)$*, which is exponentially bounded and of type* (M, ω).

(H2) By [110, Theorem 8.83] g *and* h *are compact maps.*

(H3) Is clear.

(H4) Note that,

$$\|f(t,\phi)\| \leq \|t\int_0^1 \sin(\phi(s))ds\| \leq m(t)\phi(\|x\|),$$

where $m(t) = |t|$*, and* $\phi(z) \equiv 1$.

(H5) Since

$$\|f(t,\phi_1) - f(t,\phi_2)\| = \|t\int_0^1 (\sin(\phi_1(s)) - \sin(\phi_2(s)))ds\| \leq |t|\|\phi_1 - \phi_2\| \leq \|\phi_1 - \phi_2\|.$$

Thus, by Lemma 1.4.2,

$$\xi(f(t,S)) \leq \xi(S) \leq H(t)\xi(S),$$

with $H(t) = 1$*, for all bounded* $S \subset X, t \in [0,1]$.

Then, we have that conditions (H1)- (H5) are satisfied. Now, we will prove that the inequality (2.2.3) *is satisfied. Since* $m(t) = |t|$*, and* $\phi(z) \equiv 1$*, then we have to find* $R > 0$ *such that*

$$M\Big(h_R + g_R + \phi(R)\int_0^1 m(s)ds\Big) = M(h_R + g_R + \frac{1}{2}) < R.$$

Indeed, by [110, Lemma 8.20], we have $\|k_g(v)\| \leq c\|v\|$, *for* $v \in X$, *where* c *is given in* (2.3.1). *Then*

$$g_R := \sup\{\|g(u)\| : \|u\|_\infty \leq R\} = \sup\{\|sk_g(u(1/2))\| : \|u\|_\infty \leq R\}$$
$$\leq \sup_{\|u\| \leq R} sc\|(u(1/2))\| \leq scR.$$

Thus, $g_R + h_R \leq 2Rcs$, *so* $M(h_R + g_R + \frac{1}{2}) \leq M(2Rcs + \frac{1}{2})$. *Since* $s < \frac{1}{2Mc}$, *we have that for all* $R > \frac{M}{2(1-2Mcs)}$ *the inequality* $M(h_R + g_R + \frac{1}{2}) < R$ *is fulfilled. Then, all the hypotheses of Theorem 2.2.2 are satisfied and we conclude that the problem* (2.3.2) *has at least one nonlocal integrated solution.*

Example 2.3.2. *We set* $X = c_0(\mathbb{N})$, *and we consider* $Az = M_q z = q \cdot z$ *where,* $q : \mathbb{N} \to \mathbb{C}$ *with real part bounded above, and domain* $D(A) = D(M_q) = \{z \in X : qz \in X\}$. *Then,* A *generates a strongly continuous semigroup* S *of type* $(M;\omega)$ *on* X, *see [55, Lemma, pag 65]. Let* $a(t) = 1 + \frac{\beta}{\alpha} - \frac{\beta}{\alpha}e^{-\alpha t}$ *where* $-\alpha \leq \beta \leq 0 < \alpha$. *Then* $a(t)$ *is completely positive by Example 2.3.1. Given* $-\alpha \leq \beta \leq 0 < \alpha$, *we consider the following problem*

$$\begin{cases} u''(t) + Au(t) - (k * Au)(t) = f(t, u(t)), \quad t \in [0,1] \\ u(0) = g(u), \\ u'(0) = h(u), \end{cases} \tag{2.3.5}$$

where $f : [0,1] \times X \to X$ *is given by*

$$f(t,x) = m(t)\Big\{ln(|x_k| + 1) + \frac{t}{k^2}\Big\}_{k=1}^{\infty}, \; for\, t \in [0,1], \; x = \{x_k\}_k \in c_0, \tag{2.3.6}$$

$k(t) = -\alpha\beta e^{-\alpha t} \in L^1(\mathbb{R}_+)$, $m \in L^1([0,1];\mathbb{R}^+)$, *such that* $\int_0^1 m(s)ds \neq \frac{1}{2M}$, *and* c_0 *represents the space of all sequences converging to zero, which is a Banach space with respect to the norm* $\|x\|_\infty = \sup_k |x_k|$.

Let $x = (\frac{1}{4M}, 0, 0, 0...)$. *Define* $g, h : \mathcal{C}([0,1];X) \to X$ *are explicitly given by*

$$g(u)_k = u(1/2)_k x_k, \quad h(u)_k = u(1/3)_k x_k, \quad k \in \mathbb{N}. \tag{2.3.7}$$

We will prove that the hypotheses ***(H1) - (H5)*** *are satisfied.*

(H1) For the above, a *is completely positive. Thus, by [109, Theorem 4.2] the operator* A *generates a resolvent operator* $S(t)$, *which is exponentially bounded of type* (M, ω).

(H2) Since g and h are bounded with finite rank, then g and h are compact maps.

(H3) Is clear.

(H4) Note that,

$$\begin{aligned}\|f(t,x)\|_\infty =& m(t)\|(\ln(|x_k|+1)+\frac{t}{k^2})_k\|_\infty \le m(t)(\sup_k |x_k|+t)\\ \le& m(t)(\|x\|_\infty+1) := m(t)\phi(\|x\|_\infty),\end{aligned}$$

where $\phi(z)=z+1$. This shows that (H4) holds.

(H5) The Hausdorff measure of noncompactness ξ in the space c_0 can be computed by means of the formula

$$\xi(B)=\lim_{n\to\infty}\sup_{x\in B}\|(I-P_n)x\|_\infty,$$

where B is a bounded subset in c_0 and P_n is the projection onto the linear span of the first n vectors in the standard basis. The reader can see [8, 1.1.9, p. 5]. Analogously to [23, Example 5.1, p. 227] we obtain

$$\xi(f(t,B)) \le m(t)\xi(B). \tag{2.3.8}$$

Then, we have that conditions (H1)- (H5) are satisfied. Now, we will prove that the inequality (2.2.3) *is satisfied. Since $\phi(z)=z+1$, then we have to find $R>0$ such that*

$$M\Big(h_R+g_R+\phi(R)\int_0^1 m(s)ds\Big)=M\Big(h_R+g_R+(R+1)\int_0^1 m(s)ds\Big)<R. \tag{2.3.9}$$

Note that,

$$g_R := \sup\{\|g(u)\| : \|u\|_\infty \le R\} \le \frac{R}{4M},\ h_R := \sup\{\|h(u)\| : \|u\|_\infty \le R\} \le \frac{R}{4M}.$$

Therefore, since that $\int_0^1 m(s)ds \neq \frac{1}{2M}$, then for all $R > \frac{2M\int_0^1 m(s)ds}{1-2M\int_0^1 m(s)ds}$ the inequality (2.3.9) *is fulfilled. Then, all the hypotheses of Theorem 2.2.2 are satisfied and we conclude that the problem* (2.3.5) *has at least one nonlocal integrated solution.*

Chapter 3

Asymptotic behavior of mild solutions for a class of abstract nonlinear difference equations of convolution type

In this chapter, we consider the following abstract difference equation of convolution type

$$u(n+1) = \sum_{k=-\infty}^{n} a(n-k)Au(k+1) + \sum_{k=-\infty}^{n} b(n-k)f(k,u(k)),\ n \in \mathbb{Z},$$

where $A : D(A) \subset X \to X$ is a closed linear operator on X, $f \in l^1(\mathbb{Z} \times X, X)$, and $a, b : \mathbb{Z}_+ \to \mathbb{R}_+$ are given bounded positive sequences. Note that the associated nonhomogeneous linear equation is given by

$$u(n+1) = \sum_{k=-\infty}^{n} a(n-k)Au(k+1) + \sum_{k=-\infty}^{n} b(n-k)f(k),\ n \in \mathbb{Z}. \tag{3.0.1}$$

3.1 Discrete resolvent families

In this section, we introduce the notion of discrete resolvent family of bounded and linear operators. This concept will be a crucial tool for the solution of equation (3.0.1). The understanding of the qualitive properties of this families provide insights on the qualitative behavior of the solutions of (3.0.1).

Definition 3.1.1. Let A be a closed linear operator with domain $D(A)$ defined on a Banach space X. Let a and b be scalar valued sequences. An operator-valued sequence $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called a discrete resolvent family generated by A if it satisfies the following conditions

1. $S(n)(X) \subset D(A)$, and $S(n)Ax = AS(n)x$ for all $x \in D(A)$, and $n \in \mathbb{N}_0$;
2. $S(n)x = b(n)x + A\sum_{k=0}^{n} a(n-k)S(k)x$ for $n \in \mathbb{N}_0$ and $x \in X$.

Remark 3.1.2. Note that Definition 3.1.1 corresponds to the resolvent sequence defined in [1, 10] when $b(n) = a(n) = k^{\alpha}(n) := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!}$, for $\alpha > 0$, $n \in \mathbb{N}_0$. Sequences of operators for abstract difference equations with the kernel $k^{\alpha}(n)$ were introduced by Lizama in [94] and [93] in connection with abstract difference equations of fractional order.

Remark 3.1.3. If $a(0) = b(0) = 1$, then by Definition 3.1.1 we have that

$$S(0)x = x + S(0)Ax,\ x \in D(A)$$

and

$$S(0)x = x + AS(0)x,\ x \in X.$$

Therefore $1 \in \rho(A)$ and

$$S(0)x = (I-A)^{-1}x.$$

Here, $\rho(A)$ denotes the resolvent set of A.

Proposition 3.1.4. *If $1/a(0) \in \rho(A)$, and there exists a discrete resolvent family corresponding to* *the kernels a and b, then it is unique.*

Proof. Suppose that $S(n)$ and $R(n)$ are resolvent families generated by A. Let $x \in X$ and define $\varphi(n) = S(n)x - R(n)x$, $n \in \mathbb{N}_0$. Note that $\varphi(n) \in D(A)$ for all $k \in \mathbb{N}_0$ and

$$\varphi(n) = A \sum_{k=0}^{n} a(n-k)\varphi(k). \tag{3.1.1}$$

Let us consider two cases according to whether $a(0) = 0$ or $a(0) \neq 0$. In the case $a(0) = 0$, expanding the sum in (3.1.1), we obtain $\varphi(n) = 0$ for all $n \in \mathbb{N}_0$. If $a(0) \neq 0$, then by Definition 3.1.1, and since $1/a(0) \in \rho(A)$, we obtain $S(0) = b(0)(I - a(0)A)^{-1} = R(0)$. Therefore $\varphi(0) = 0$. Using (3.1.1), we obtain $(I - a(0)A)\varphi(n) = 0$ for all $n \in \mathbb{N}_0$. Then, the invertibility of $(I - a(0)A)$ implies $S(n)x = R(n)x$ for all $n \in \mathbb{N}_0$ and $x \in X$. □

Remark 3.1.5. Note that if $b(0) \neq 0$, then $\frac{S(0)}{b(0)}(1 - Aa(0))x = x$ for all $x \in D(A)$. Thus combining with Definition 3.1.1 part (i), we have $1/a(0) \in \rho(A)$. Then the conclusion of the previous theorem holds.

Theorem 3.1.6. *If A is a closed linear operator defined on a Banach space X. Then, A is the generator of a discrete resolvent family $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ with $a(n) = b(n)$ if only if $\frac{1}{a(0)} \in \rho(A)$. Moreover, the discrete resolvent family $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is given by*

$$S(n)x = [\sum_{j=0}^{n-1} \frac{1}{a(0)^j}\phi_j(n)(T-I)^j]T^2x, \quad n \geq 2, \quad \textit{for all} \qquad x \in X, \tag{3.1.2}$$

where $T := (I - a(0)A)^{-1}$, and $\phi_0(n) = a(n)$, $\quad \phi_1(n) = \sum_{k=1}^{n-1} a(n-k)a(k)$,

and

$$\phi_j(n) = \sum_{k=j}^{n-1} a(n-k)\phi_{j-1}(k), \quad j \geq 2, \tag{3.1.3}$$

and for all $x \in X$ we have that $S(0)x = a(0)Tx$, $S(1) = a(1)T^2x$.

Proof. If A is the generator of a discrete resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$, then $\frac{1}{a(0)} \in \rho(A)$ by

Remark 3.1.5. On the other hand, we suppose that $\frac{1}{a(0)} \in \rho(A)$, and we will prove that $\{S(n)\}_{n\in\mathbb{N}_0}$ is a discrete resolvent family generated by A.

First, we will prove (1) in Definition 3.1.1. It is clear that, $S(n)X \subset D(A)$. Note that,

$$AT^j = (A - \frac{1}{a(0)}I + \frac{1}{a(0)}I)T^j = \frac{1}{a(0)}(T^j - T^{j-1}),$$

and

$$T^jA = T^j(A - \frac{1}{a(0)}I + \frac{1}{a(0)}I) = \frac{1}{a(0)}(T^j - T^{j-1}).$$

Therefore, for all $x \in X$, it is a straightforward consequence of the above representation of $S(n)$ that $S(n)Ax = AS(n)x$ for all $x \in D(A)$, and $n \in \mathbb{N}_0$.

Now, we will prove (2) in Definition 3.1.1. It is clear that $S(0)x = a(0)Tx$, satisfies the condition (2) in Definition 3.1.1 for $n = 0$. Now, let $x \in X$, we have that

$$S(1)x = a(1)T^2x,$$

then,

$$S(1)(I - a(0)A)x = a(1)Tx = a(1)(a(0)AT + I)x.$$

Thus,

$$S(1)x = a(1)x + A(a(1)S(0) + a(0)S(1))x,$$

where we have used that $T = a(0)AT + I$ and $S(0)x = a(0)Tx$. Therefore S satisfies the condition (2) in Definition 3.1.1 for $n = 1$. In what follows of the proof we write $\frac{1}{a(0)}(T - I)$ as AT. By induction, we suppose that

$$S(m)x = [\sum_{j=0}^{m-1} \phi_j(m)(AT)^j]T^2x,$$

satisfies the condition (2) in Definition 3.1.1 for $m \leq n-1$. For $x \in X$ we write, using the definition:

$$\begin{aligned} S(n)x =& [\sum_{j=0}^{n-1} \phi_j(n)(AT)^j]T^2x \\ =& a(n)T^2x + \phi_1(n)AT^3x + [\sum_{j=2}^{n-1} \phi_j(n)(AT)^j]T^2x. \end{aligned}$$

By the definition of $\phi_j(n)$ in (3.1.3), we obtain

$$S(n)x = a(n)T^2x + \phi_1(n)AT^3x + [\sum_{j=2}^{n-1}\sum_{m=j}^{n-1} a(n-m)\phi_{j-1}(k)(AT)^j]T^2x. \tag{3.1.4}$$

Now by Fubini's Theorem,

$$\begin{aligned} S(n)x =& a(n)T^2x + \phi_1(n)AT^3x + [\sum_{m=2}^{n-1}\sum_{j=2}^{m} a(n-m)\phi_{j-1}(m)(AT)^j]T^2x \\ =& a(n)T^2x + \phi_1(n)AT^3x + [\sum_{m=2}^{n-1}\sum_{j=1}^{m-1} a(n-m)\phi_j(m)(AT)^{j+1}]T^2x. \end{aligned}$$

Thus, by definition of $\phi_0(n)$, $\phi_1(n)$, and induction hypothesis,

$$\begin{aligned} S(n)x =& a(n)T^2x + \phi_1(n)AT^3x + \sum_{m=2}^{n-1} a(n-m)\sum_{j=1}^{m-1}\phi_j(m)A^{j+1}T^{j+3}x \\ =& a(n)T^2x + \sum_{m=1}^{n-1} a(n-m)a(m)AT^3x + \sum_{m=2}^{n-1} a(n-m)\sum_{j=1}^{m-1}\phi_j(m)A^{j+1}T^{j+3}x \\ =& a(n)T^2x + a(n-1)a(1)AT^3x + AT^3\sum_{m=2}^{n-1} a(n-m)a(m)x+ \\ & \sum_{m=2}^{n-1} a(n-m)\sum_{j=1}^{m-1}\phi_j(m)A^{j+1}T^{j+3}x \\ =& a(n)T^2x + a(n-1)a(1)AT^3x + \sum_{m=2}^{n-1} a(n-m)[a(m)AT^3x+ \\ & \sum_{j=1}^{m-1}\phi_j(m)A^{j+1}T^{j+3}]x \\ =& a(n)T^2x + a(n-1)a(1)AT^3x + \sum_{m=2}^{n-1} a(n-m)\sum_{j=0}^{m-1}\phi_j(m)A^{j+1}T^{j+3}x \\ =& a(n)T^2x + a(n-1)a(1)AT^3x + A\sum_{m=2}^{n-1} a(n-m)S(m)Tx. \end{aligned}$$

Thus, for $x \in X$,

$$S(n)x = a(n)T^2x + a(n-1)a(1)AT^3x + A\sum_{m=2}^{n-1} a(n-m)S(m)Tx. \tag{3.1.5}$$

Composing by the operator $(I - a(0)A)$ on both sides of the above identity we obtain,

$$(I - a(0)A)S(n)x = a(n)Tx + a(n-1)a(1)AT^2x + A\sum_{m=2}^{n-1} a(n-m)S(m)x.$$

Thus,

$$S(n) = a(n)Tx + a(n-1)a(1)AT^2x + a(0)AS(n)x + A\sum_{m=2}^{n-1} a(n-m)S(m)x.$$

Therefore, since that $T = I + a(0)AT$,

$$\begin{aligned} S(n) &= a(n)(I + a(0)AT)x + a(n-1)a(1)AT^2x + a(0)AS(n)x + A\sum_{m=2}^{n-1} a(n-m)S(m)x \\ &= a(n)x + a(n)a(0)ATx + a(n-1)a(1)AT^2x + a(0)AS(n)x + A\sum_{m=2}^{n-1} a(n-m)S(m)x. \end{aligned}$$

As $a(0)Tx = S(0)x$, and $a(1)T^2x = S(1)x$, we have

$$\begin{aligned} S(n)x =& a(n)x + a(n)AS(0)x + a(n-1)AS(1)x + a(0)AS(n)x + A\sum_{m=2}^{n-1} a(n-m)S(m)x \\ =& a(n)x + A\sum_{m=0}^{n} a(n-m)S(m)x. \end{aligned}$$

□

The next theorem gives necessary conditions in terms of C_0-semigroups in order to ensure the existence and summability of a discrete resolvent family. We will denote by $f * g := \int_0^t f(t-s)g(s)ds$, the Laplace convolution of the functions f and g, and $\rho_n(t)$ will be the function $\rho_n(t) = \frac{e^{-t}t^n}{n!}$.

Theorem 3.1.7. *Let A be the generator of a bounded analytic C_0-semigroup on a Banach space X. Let $k(t), g(t) \geq 0$ be given by $a(n) = \int_0^\infty \rho_n(t)k(t)dt$, $b(n) = \int_0^\infty \rho_n(t)g(t)dt$, where $k \in L^1_{loc}(\mathbb{R}_+)$ is 2-regular and of subexponential growth, of positive type, such that $\frac{1}{\lambda\widehat{k}(\lambda)}$ defined for $\lambda \neq 0$, has a locally analytic extension at $\lambda = 0$, $g(0) = 0$, $g \in W^{1,1}(\mathbb{R}_+)$, and $0 \in \rho(A)$. Then A generates a summable discrete resolvent family $\{R(n)\}_{n\in\mathbb{N}_0}$, with sequences $b(n)$ and $a(n)$.*

Proof. By [109, Corollary 3.1] $\widehat{k}(\lambda) \neq 0$, $\frac{1}{\widehat{k}(\lambda)} \in \rho(A)$ for all $Re\lambda > 0$, and there exists a constant $M \geq 1$ such that $H(\lambda) = (I - \widehat{k}(\lambda)A)^{-1}/\lambda$ satisfies $\|H(\lambda)\| \leq \frac{M}{|\lambda|}$, for all $Re\lambda > 0$. Then, by [109, Theorem 10.2] there exists a resolvent family $S(t)$ which is uniformly integrable. By [109, Definition 1.3,(S3)] for all $x \in X$ and $t > 0$, we have that,

$$S(t)x = x + A(k * S)(t)x. \tag{3.1.6}$$

Define $T(t)x := (g' * S)(t)x$, for all $x \in X$, and $t \geq 0$. Note that, $T(t) \in L^1(\mathbb{R}_+, X)$, since $S(t)$ uniformly integrable and $g' \in L^1(\mathbb{R}_+)$. Moreover, since $g(0) = 0$, it follows that $(g' * 1)(t) = \int_0^t g'(s)ds = g(t)$. Thus, for all $x \in X$, and $t \geq 0$, we obtain from (3.1.6):

$$\begin{aligned} T(t)x =&(g' * 1)(t)x + A(k * g' * S)(t)x, \\ =&g(t)x + A(k * T)(t)x. \end{aligned}$$

Define

$$R(n)x := \int_0^\infty \rho_n(t)T(t)xdt = \frac{(-1)^n}{n!}[\frac{\widehat{g}(\lambda)}{\widehat{k}(\lambda)}(\frac{1}{\widehat{k}(\lambda)} - A)^{-1}x]^{(n)}\Big|_{\lambda=1}, \text{ for all } n \in \mathbb{N}_0,\ x \in X,$$

then, $R(n)x \in D(A)$ for all $x \in X$. Now, from $a(n) = \int_0^\infty \rho_n(t)a(t)dt$, and $b(n) = \int_0^\infty \rho_n(t)g(t)dt$, and using [94, Theorem 3.4], we have that for all $x \in X$,

$$R(n)x = b(n)x + A\sum_{j=0}^{n} a(n-j)R(j)x.$$

Finally, we prove that $R(n)$ is summable. In fact, since that $T(t) \in L^1(\mathbb{R}_+, X)$, we have that,

$$\sum_{n=0}^\infty \|R(n)\| = \sum_{n=0}^\infty \|\int_0^\infty \rho_n(t)T(t)dt\| = \sum_{n=0}^\infty \|\int_0^\infty \frac{e^{-t}t^n}{n!}T(t)dt\| \leq \int_0^\infty \|T(t)\|dt < \infty.$$

□

Remark 3.1.8. We note that the family T defined in the previous theorem is the convolution of a function g' with a resolvent family S. The resolvent families have been studied extensively by Prüss in [109]. It is well known that under certain conditions on the function k, we can obtain resolvent families with various additional properties: analytic, differentiable, exponentially bounded, uniform integrable, among others, see [109]. Next, we will give conditions on g in order that the family T have the same properties of S.

- By Young's inequality, if $S(t)$ is uniformly integrable and $g' \in L^1(\mathbb{R}_+)$, then $T(t) \in L^1(\mathbb{R}_+, X)$.

- Suppose that S is differentiable, then by [109, Definition 1.4] we see that $S(\cdot)x \in W^{1,1}_{loc}(\mathbb{R}_+, X)$ for each $x \in D(A)$ and there is $\varphi \in L^1_{loc}(\mathbb{R}_+)$ such that $\|\dot{S}(t)x\| \le \varphi(t)\|x\|_A$ a.e on $\mathbb{R}_+$, for each $x \in D(A)$. If $g \in W^{1,1}_{loc}(\mathbb{R}_+)$, then $T(t) \in W^{1,1}_{loc}(\mathbb{R}_+)$. Moreover,

$$\begin{aligned}\|\frac{d}{dt}(g' * S)(t)x\| \le &\|S(0)g'(t)x\| + \|\int_0^t \dot{S}(t-s)g'(s)xds\| \le M|x|_A + M\int_0^t \|\dot{S}(t-s)x\|ds \\ &\le M|x|_A + M\int_0^t \varphi(t-s)\|x\|_A ds \le (M+K)\|x\|_A,\end{aligned}$$

  where $M = \sup\limits_{t\in\mathbb{R}_+} |g'(t)|$, and $K = M\int_0^t \varphi(t-s)ds$. Thus $T(t)$ is differentiable.

Remark 3.1.9. The conditions on the sequences $a(n)$ and $b(n)$ are sufficient but not necessary. In [1, Theorem 3.5] the authors proved that if $0 < \alpha < 1$ and A be the generator of an exponentially stable $C_0 - \mathit{semigroup}$ $\{T(t)\}_{t\ge 0}$, defined on a Banach space X, then A generates a summable discrete resolvent family $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ defined by

$$S_\alpha(n)x := \int_0^\infty \int_0^\infty e^{-t}\frac{t^n}{n!} f_{s,\alpha}(t)T(s)xdsdt,\ n \in \mathbb{N}_0,\ x \in X,$$

where $f_{t,\alpha}(\lambda)$ is a probability density function frequently called stable Lévy process. The latter is defined by [116]

$$f_{t,\alpha}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^\alpha}dz,\ \sigma > 0,\ t > 0,\ \lambda > 0,\ 0 < \alpha < 1,$$

where the branch of z^α is so taken that $Re(z^\alpha) > 0$ for $Re(z) > 0$.

Now we describe properties for the convolution products (1.2.2) and (1.2.1).

Theorem 3.1.10. *For f, g, h given sequences, the following properties hold:*

(i) $(f * g)(n) = (g * f)(n)$.

(ii) $((f * g) \circ h)(n) = (g \circ (f \circ h))(n)$.

(iii) $(g \circ (f \circ h))(n) = (f \circ (g \circ h))(n)$.

Proof. (i) is obvious. Now, we will prove (ii). By Fubini's Theorem,

$$\begin{aligned}((f*g)\circ h)(n) &= \sum_{j=-\infty}^{n}(f*g)(n-j)h(j) = \sum_{j=-\infty}^{n}\sum_{k=0}^{n-j} f(n-j-k)g(k)h(j)\\ &= \sum_{k=0}^{\infty} g(k)\sum_{j=-\infty}^{n-k} f(n-j-k)h(j) = \sum_{k=0}^{\infty} g(k)(f\circ h)(n-k)\\ &= \sum_{j=-\infty}^{n}(f\circ h)(j)g(n-j) = (g\circ(f\circ h))(n).\end{aligned}$$

This proves (ii). Now, (iii) follows from (i) and (ii). □

We next with the definition of strong solution for the equations (3.0.1).

Definition 3.1.11. Given $a, b \in s(\mathbb{Z}, X)$ bounded positive sequences, and $f \in l^1(\mathbb{Z}, X)$, a sequence $u : \mathbb{Z} \to [D(A)]$ is called a strong solution for equation (3.0.1) if $u \in l^1(\mathbb{Z}; [D(A)])$ and satisfies (3.0.1).

The following theorem gives conditions for better regularity.

Theorem 3.1.12. *Let $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ be a summable discrete resolvent family generated by A, and $f \in l^1(\mathbb{Z}, [D(A)])$, then*

$$u(n+1) := \sum_{k=-\infty}^{n} S(n-k)f(k)$$

is a strong solution of (3.0.1).

Proof. Note that u is clearly well defined and $u \in l^1(\mathbb{Z}, [D(A)])$. Now we will prove that u satisfies (3.0.1). Indeed, by Definition 3.1.1 and Theorem 3.1.10,

$$\begin{aligned}u(n+1) - \sum_{k=-\infty}^{n} b(n-k)f(k) &= (S\circ f)(n) - (b\circ f)(n) = \sum_{k=-\infty}^{n} S(n-k)f(k) - (b\circ f)(n)\\ &= \sum_{k=-\infty}^{n}[b(n-k) + A\sum_{l=0}^{n-k} a(n-k-l)S(l)]f(k) - (b\circ f)(n)\end{aligned}$$

$$
\begin{aligned}
&=(b \circ f)(n) + A(a * S \circ f)(n) - (b \circ f)(n) \\
&=A \sum_{k=-\infty}^{n} a(n-j)u(j+1).
\end{aligned}
$$

Thus,

$$u(n+1) = \sum_{k=-\infty}^{n} b(n-j)f(j) + A \sum_{k=-\infty}^{n} a(n-j)u(j+1).$$

This proves the claim. □

Example 3.1.13. *Consider the special case $a(n) = b(n) = k^{\alpha}(n), n \in \mathbb{Z}$ and $\{S(n)\}_{n\in\mathbb{N}}$ satisfying the Definition 3.1.1, then the nonlinear fractional difference equation*

$$\Delta^{\alpha} u(n) = Au(n+1) + f(n), n \in \mathbb{Z},$$

for $0 < \alpha < 1$, can be written in the form (3.0.1). Here, A is the generator of an α resolvent sequence $\{S(n)\}_{n\in\mathbb{N}_0}$ in $\mathcal{B}(X), \Delta^{\alpha}$ denote fractional difference in Weyl-like sense (see [1, 10]) and f satisfies Lipschitz conditions of global and local type.

Now if

$$\Delta^{\alpha} u(n) = Au(n+1) + f(n),$$

then,

$$\sum_{k=-\infty}^{n} k^{\alpha}(n-k)\Delta^{\alpha} u(k) = A \sum_{k=-\infty}^{n} k^{\alpha}(n-k)u(k+1) + \sum_{k=-\infty}^{n} k^{\alpha}(n-k)f(k).$$

Note that,

$$
\begin{aligned}
\sum_{k=-\infty}^{n} k^{\alpha}(n-k)(k^{1-\alpha} \circ u)(k+1) = \sum_{m=-\infty}^{n+1} k^{\alpha}(n-m+1)(k^{1-\alpha} \circ u)(m) &= (k^{\alpha} \circ (k^{1-\alpha} \circ u))(m+1) \\
&= ((k^{\alpha} * k^{1-\alpha}) \circ u)(m+1) = \sum_{k=-\infty}^{m+1} u(k).
\end{aligned}
$$

Analogously,

$$\sum_{k=-\infty}^{n} k^{\alpha}(n-k)(k^{1-\alpha} \circ u)(k) = \sum_{k=-\infty}^{m} u(k).$$

Thus,

$$\begin{aligned}\sum_{k=-\infty}^{n} k^{\alpha}(n-k)\Delta^{\alpha}u(k) &= \sum_{k=-\infty}^{n} k^{\alpha}(n-k)[(k^{1-\alpha}\circ u)(k+1)-(k^{1-\alpha}\circ u)(k)] \\ &= \sum_{k=-\infty}^{n+1} u(k) - \sum_{k=-\infty}^{n} u(k) = u(n+1).\end{aligned}$$

Then,

$$u(n+1) = A\sum_{k=-\infty}^{n} k^{\alpha}(n-k)u(k+1) + \sum_{k=-\infty}^{n} k^{\alpha}(n-k)f(k).$$

In the next result, borrowing ideas from [10], we prove regularity under convolution in the above mentioned spaces.

Theorem 3.1.14. *Let $\rho_1, \rho_2 \in V_\infty$ be given. Assume that A generates a summable discrete resolvent family $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$. If f belongs to one of the spaces $\Omega \in \mathcal{M}(\mathbb{Z}, X)$ then the sequence u defined by*

$$u(n+1) = \sum_{k=-\infty}^{n} S(n-k)f(k),\ n \in \mathbb{Z},$$

belongs to the same space Ω.

Proof. Note that $u(n+1)$ is well defined and $\|u(n+1)\| \leq \|S\|_1\|f\|_\infty$, for all $n \in \mathbb{Z}$. First, we consider $f \in WPAA_d(\mathbb{Z}, X)$. Let $f = f_1 + f_2$, where $f_1 \in AA_d(\mathbb{Z}, X)$ and $f_2 \in PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$ be the decomposition of f. Then

$$u(n) = \sum_{k=-\infty}^{n-1} S(n-1-k)f_1(k) + \sum_{k=-\infty}^{n-1} S(n-1-k)f_2(k) =: u_1(n) + u_2(n).$$

From [11, Theorem 2.12], $u_1 \in AA_d(\mathbb{Z}, X)$. Now, we will prove that $u_2 \in PAA_0S(\mathbb{Z}, X, \rho_1, \rho_2)$. Indeed,

$$\begin{aligned}\frac{1}{\nu(K,\rho_1)}\sum_{k=-K}^{K}\|u_2(k)\|\rho_2(k) &= \frac{1}{\nu(K,\rho_1)}\sum_{k=-K}^{K}\|\sum_{j=-\infty}^{k-1} S(k-1-j)f_2(j)\|\rho_2(k) \\ &\leq \sum_{m=0}^{\infty}\|S(m)\|\big(\frac{1}{\nu(K,\rho_1)}\sum_{k=-K}^{K}\|f_2(k-1-m)\|\rho_2(k)\big).\end{aligned}$$

By [117, Lemma 10] the space $PAA_0S(\mathbb{Z},X,\rho_1,\rho_2)$ is invariant under translations, then $f_2(\cdot - m) \in PAA_0S(\mathbb{Z},X,\rho_1,\rho_2)$. Thus, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{K\to\infty} \frac{1}{\nu(K,\rho_1)} \sum_{k=-K}^{K} \|u_2(k)\|\rho_2(k) = 0.$$

This proves the claim for such space. Now, let $f \in WPSAP_\omega(\mathbb{Z},X,\rho_1,\rho_2)$. Then, there exists $\omega \in \mathbb{Z}^+ \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{1}{\nu(n,\rho_1)} \sum_{k=-n}^{n} \rho_2(k)\|f(k+\omega) - f(k)\| = 0.$$

Now, we have

$$\begin{aligned}
&\frac{1}{\nu(m,\rho_1)} \sum_{n=-m}^{m} \|u(n+\omega) - u(n)\|\rho_2(n) \\
\leq &\frac{1}{\nu(m,\rho_1)} \sum_{n=-m}^{m} \sum_{k=-\infty}^{n-1} \|S(n-1-k)\|\|f(k+\omega) - f(k)\|\rho_2(n) \\
\leq &\sum_{k=0}^{\infty} \|S(k)\| \big(\frac{1}{\nu(m,\rho_1)} \sum_{n=-m}^{m} \|f(n-1-k+\omega) - f(n-1-k)\|\rho_2(n) \big).
\end{aligned}$$

By [118, Lemma 2.2] $WPSAP_\omega(\mathbb{Z},X)$ is invariant under translations. Thus, applying again the by Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{\nu(m,\rho_1)} \sum_{n=-m}^{m} \|u(n+\omega) - u(n)\|\rho_2(n) \\
\leq &\lim_{m\to\infty} \sum_{k=0}^{\infty} \|S(k)\| \Big(\frac{1}{\nu(m,\rho_1)} \sum_{n=-m}^{m} \|f(n-1-k+\omega) - f(n-1-k)\|\rho_2(n) \Big) = 0
\end{aligned}$$

□

3.2 Semilinear difference equations

In this section we use the above defined resolvent families to investigate the existence and uniqueness of solutions for the following class of abstract semilinear difference equations:

$$u(n+1) = A \sum_{k=-\infty}^{n} a(n-k)u(k+1) + \sum_{k=-\infty}^{n} b(n-k)f(k,u(k)),\ n \in \mathbb{Z}, \tag{3.2.1}$$

where A is the generator of a discrete resolvent family $\{S(n)\}_{n\in\mathbb{N}_0}$ in $\mathcal{B}(X)$, $f : \mathbb{Z} \times X \to X$ is a bounded function on bounded sets of X and a, b are given such that (3.2.1) makes sense.

We introduce the following conditions in order to prove our main results about asymptotic behavior of mild solutions.

(H2) A is the generator of a summable discrete resolvent sequence $\{S(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$.

(F1) f satisfies the Lipschitz condition:

$$\|f(k, h(k)u) - f(k, h(k)v)\| \le L_f\|u - v\|, \text{ for all } k \in \mathbb{Z}, u, v \in X,$$

where $L_f > 0$ is a constant and h is given in Lemma 1.4.12.

(F2) f satisfies the local Lipschitz condition, that is, for each positive number r, and all $u, v \in X$ with $\|u\| \le r$, $\|v\| \le r$, we have

$$\|f(k, h(k)u) - f(k, h(k)v)\| \le L_f(r)\|u - v\|, \text{ for all } k \in \mathbb{Z},$$

where $L_f : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function and h is given in Lemma 1.4.12.

(F3) f satisfies the following condition:

$$\|f(k, h(k)u) - f(k, h(k)v)\| \le L_f(k)\|u - v\|, \text{ for all } k \in \mathbb{Z}, u, v \in X,$$

where $L_f : \mathbb{Z} \to \mathbb{R}^+$ is a summable function and h is given in Lemma 1.4.12.

(F4) f satisfies

$$\|f(k, h(k)u) - f(k, h(k)v)\| \le \phi(\|u - v\|), \text{ for all } k \in \mathbb{Z}, u, v \in X,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function and h is given in Lemma 1.4.12.

(F5) f satisfies

$$\|f(k,u)-f(k,v)\| \le L_f(k)\|u-v\|^{\theta}, \text{ for all } k\in\mathbb{Z}, u,v\in X,$$

where $\theta\in(0,1), L_f:\mathbb{Z}\to\mathbb{R}^+$ is a sequence (depending on f) such that $\lim\limits_{|k|\to\infty} L_f(k)=0$.

Next we introduce the definition of solution for the semilinear difference equation.

Definition 3.2.1. Let A be the generator of a discrete resolvent family $\{S(n)\}_{n\in\mathbb{N}_0}\subset\mathcal{B}(X)$, and $f:\mathbb{Z}\times X\to X$. We say that a sequence $u:\mathbb{Z}\to\mathbb{X}$ is a mild solution of (3.2.1) if $k\to S(n-k)f(k,u(k))$ is summable on $\mathbb{N}_0$, for each $n\in\mathbb{Z}$ and u satisfies

$$u(n+1)=\sum_{k=-\infty}^{n} S(n-k)f(k,u(k)),\ n\in\mathbb{Z}.$$

In the next theorems 3.2.2, 3.2.3 and 3.2.4, we show existence, uniqueness and asymptotic behavior of discrete mild solutions of (3.2.1). We assume that f satisfies Lipschitz and locally Lipschitz conditions, the proofs are based on the Banach fixed point theorem.

Theorem 3.2.2. *Assume that (H2), (F1) hold and $L_f\|S\|_1<1$, then there is a unique mild solution $u(n)$ of (3.2.1) such that $\lim\limits_{|n|\to\infty}\dfrac{\|u(n)\|}{h(n)}=0$.*

Proof. Consider the operator $F:\mathcal{C}_h^0(\mathbb{Z},X)\to\mathcal{C}_h^0(\mathbb{Z},X)$ defined by

$$(Fu)(n):=\sum_{k=-\infty}^{n-1} S(n-1-k)f(k,u(k)),\ n\in\mathbb{Z}, \tag{3.2.2}$$

Note that F is well defined. Indeed,

$$\begin{aligned}\|(Fu)(n)\| \le& \sum_{k=-\infty}^{n-1}\|S(n-1-k)\|\|f(k,u(k))-f(k,0)\|+\sum_{k=-\infty}^{n-1}\|S(n-1-k)\|\|f(k,0)\|\\ \le& L_f\|S\|_1\|u\|_h+\|S\|_1\sup_{k\in\mathbb{Z}}\|f(k,0)\|,\end{aligned}$$

hence $\lim\limits_{|n|\to\infty} \dfrac{\|(Fu)(n)\|}{h(n)} = 0$, which implies that F is well defined. In addition, for $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X)$ and $n \in \mathbb{Z}$ the following inequality holds,

$$\|(Fu)(n) - (Fv)(n)\| \le \sum_{k=-\infty}^{n-1} \|S(n-1-k)\| \|f(k, u(k)) - f(k, v(k))\| \le L_f \|S\|_1 \|u - v\|_h,$$

therefore, $\|Fu - Fv\|_h \le L_f \|S\|_1 \|u - v\|_h$. From the assumption $L_f \|S\|_1 < 1$ we see that F is a contraction, and using the Banach fixed point Theorem we conclude that there exist a unique discrete mild solution of (3.2.1) such that $\lim\limits_{|n|\to\infty} \dfrac{\|u(n)\|}{h(n)} = 0$. □

In the following theorem, we assume a modified hypothesis on the previous Lipschitz condition, namely we consider a local condition instead of the global one.

Theorem 3.2.3. *Suppose (H2), (F2) and there exist* $r_0 > 0$ *such that* $\|S\|_1 \Big(L_f(r_0) + \frac{1}{r_0} \sup\limits_{k\in\mathbb{Z}} \|f(k, 0)\| \Big) < 1$, *then there is a unique mild solution* $u(n)$ *of* (3.2.1) *such that* $\lim\limits_{|n|\to\infty} \dfrac{\|u(n)\|}{h(n)} = 0$.

Proof. Define the operator F as in (3.2.2), then H is well defined. Let

$$B_{r_0} := \{u \in \mathcal{C}_h^0(\mathbb{Z}, X) : \|u\|_h \le r_0\}.$$

For $u \in B_{r_0}$,

$$\|(Fu)(n)\| \le \|S\|_1 \Big(L_f(r_0) + \frac{1}{r_0} \sup_{k\in\mathbb{Z}} \|f(k, 0)\| \Big) r_0 \le r_0.$$

Therefore, $\|Fu\| \le r_0$, that is $Fu \in B_{r_0}$. Moreover, for $u, v \in B_{r_0}$, we have

$$\|(Fu)(n) - (Fv)(n)\| \le L_f(r_0) \|S\|_1 \|u - v\|_h,$$

then there is a unique fixed point $u \in B_{r_0}$, so there is a unique mild solution $u(n)$ of (3.2.1) such that $\lim\limits_{|n|\to\infty} \dfrac{\|u(n)\|}{h(n)} = 0$. □

Theorem 3.2.4. *Assume that (H2), (F3) hold, then there is a unique mild solution* $u(n)$ *of* (3.0.1) *such that* $\lim\limits_{|n|\to\infty} \dfrac{\|u(n)\|}{h(n)} = 0$.

Proof. Define the operator F as in (3.2.2), then by the hypothesis $(F4)$, we have that

$$\begin{aligned}\|Fu)(n)\| \leq &\sum_{k=\infty}^{n-1} \|S(n-1-k)\| L_f(k) \frac{\|u(k)\|}{h(k)} + \|S\|_1 \sup_{k\in\mathbb{Z}} \|f(k,0)\| \\ \leq &\|S\|_\infty \|L_f\|_1 \|u\|_h + \|S\|_1 \sup_{k\in\mathbb{Z}} \|f(k,0)\|,\end{aligned}$$

where $\|L_f\|_1 := \sum_{k=-\infty}^{\infty} L_f(k)$, so F is well defined.
For $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X)$, one has

$$\|(Fu)(n) - (Fv)(n)\| \leq \|S\|_\infty \Big(\sum_{k=-\infty}^{n-1} L_f(k) \Big) \|u-v\|_h.$$

Similarly, by [44, Lemma 3.2],

$$\begin{aligned}\|(F^2u)(n) - (F^2v)(n)\| \leq &\sum_{k=-\infty}^{\infty} L_f(k) \|S(n-1-k)\| \frac{\|(Fu)(k) - (Fv)(k)\|}{h(k)} \\ \leq &(\|S\|_\infty)^2 \Big(\sum_{k=-\infty}^{n-1} L_f(k) \Big(\sum_{j=-\infty}^{k-1} L_f(j) \Big) \Big) \|u-v\|_h \\ \leq &\frac{(\|S\|_\infty)^2}{2!} \Big(\sum_{k=-\infty}^{\infty} L_f(k) \Big)^2 \|u-v\|_h.\end{aligned}$$

By induction, one easily see that

$$\|(F^n u)(n) - (F^n v)(n)\| \leq \frac{(\|S\|_\infty \|L_f\|_1)^n}{n!} \|u-v\|_h.$$

Therefore, $\|F^n u - F^n v\|_h \leq \frac{(\|S\|_\infty \|L_f\|_1)^n}{n!} \|u-v\|_h$. For sufficiently large n, we have $\frac{(\|S\|_\infty \|L_f\|_1)^n}{n!} < 1$. By the Banach contraction mapping principle, F has a unique fixed point in $\mathcal{C}_h^0(\mathbb{Z}, X)$, so there is a unique mild solution $u(n)$ of (3.0.1) such that $\lim_{|n|\to\infty} \frac{\|u(n)\|}{h(n)} = 0$. □

The following theorem establishes the existence of a mild solution of (3.2.1) in $\mathcal{C}_h^0(\mathbb{Z}, X)$, based on the fixed point theorem of Matkowski.

Theorem 3.2.5. *Let (H2), (F4) hold and assume further that* $(\|S\|_1\phi)^n(t) \to 0$ *as* $n \to \infty$ *for each* $t > 0$. *Then there is a unique mild solution* $u(n)$ *of* (3.2.1) *such that* $\lim_{|n|\to\infty} \frac{\|u(n)\|}{h(n)} = 0$.

Proof. Define the operator F as in (3.2.2), then by the hypothesis $(F3)$, we have that,

$$\begin{aligned}\|Fu)(n)\| \leq &\sum_{k=\infty}^{n-1} \|S(n-1-k)\|\phi\Big(\frac{\|u(n)\|}{h(n)}\Big) + \|S\|_1 \sup_{k\in\mathbb{Z}} \|f(k,0)\| \\ \leq &\|S\|_1\Big(\phi(\|u\|_h) + \sup_{k\in\mathbb{Z}} \|f(k,0)\|\Big).\end{aligned}$$

Thus, F is well defined. For $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X)$, we have

$$\|(Fu)(n) - (Fv)(n)\| \leq \sum_{k=-\infty}^{n-1} \|S(n-1-k)\|\phi\Big(\frac{u(k)-v(k)}{h(k)}\Big) \leq \|S\|_1\phi(\|u-v\|_h).$$

Therefore, $\|Fu - Fv\|_h \leq \|S\|_1\phi(\|u-v\|_h)$. Since $(\|S\|_1\phi)^n(t) \to 0$ as $n \to \infty$ for each $t > 0$, by Matkowski's fixed point Theorem, F has a unique fixed point $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$, so there is a unique mild solution $u(n)$ of (3.2.1) such that $\lim\limits_{|n|\to\infty} \dfrac{\|u(n)\|}{h(n)} = 0$. □

In the next theorem, the main tool used in proving existence of solutions of (3.2.1) is the classical Leray-Schauder Alternative Theorem combined with Lemma 1.4.12.

Theorem 3.2.6. *Suppose (H2), (F5). Then under the following additional condition:*
(A_1) For all $a, b \in \mathbb{Z}, a \leq b$ and $\sigma > 0$, the set $\{f(k,x) : a \leq k \leq b, \|x\| \leq \sigma\}$ is relatively compact in X; there exist a function $h : \mathbb{Z} \to [1,\infty)$ such that $h(n) \to \infty$ as $|n| \to \infty$ and a mild solution $u(n)$ of (3.2.1) such that $\lim\limits_{|n|\to\infty} \dfrac{\|u(n)\|}{h(n)} = 0$.

Proof. Define the function $h : \mathbb{Z} \to \mathbb{R}^+$ by $h(k) := \Big(\frac{L_\infty}{L_f(k)}\Big)^{1/\theta}$, where $L_\infty = \sup\limits_{k\in\mathbb{Z}} L_f(k)$, $k \in \mathbb{Z}$. Note that $h(k) \geq 1$, for all $k \in \mathbb{Z}$ and since $\lim_{|k|\to\infty} L_f(k) = 0$, we also have $\lim_{|k|\to\infty} h(k) = \infty$. Consider the operator F as defined in (3.2.2). Using the Leray-Schauder Alternative Theorem, we will show that F has a fixed point in $\mathcal{C}_h^0(\mathbb{Z}, X)$. The proof will be carried out in several steps.

(i) F is well defined. For $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$, one has

$$\|(Fu)(n)\| \leq \sum_{k=\infty}^{n-1} \|S(n-1-k)\|(L_f(k)\|u(k)\|^\theta + \|f(k,0)\|) \leq \|S\|_1(L_\infty\|u\|_h^\theta + \sup_{k\in\mathbb{Z}} \|f(k,0)\|),$$

whence $\lim_{|n|\to\infty} \dfrac{\|(Fu)(n)\|}{h(n)} = 0$. Thus F is $\mathcal{C}_h^0(\mathbb{Z}, X)$- valued.

(ii) $F : \mathcal{C}_h^0(\mathbb{Z}, X) \to \mathcal{C}_h^0(\mathbb{Z}, X)$ is a continuous map. In fact, for $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X)$, one has

$$\|(Fu)(n) - (Fv)(n)\| \leq \sum_{k=\infty}^{n-1} \|S(n-1-k)\| L_f(k) h(k)^\theta \Big(\frac{\|u(k) - v(k)\|}{h(k)}\Big)^\theta \leq L_\infty \|S\|_1 \|u - v\|_h^\theta.$$

Since that $h(k) = \left(\frac{L_\infty}{L_f(k)}\right)^{1/\theta} \geq 1$, for all $k \in \mathbb{Z}$, then

$$\sup_{n\in\mathbb{Z}} \frac{\|(Fu)(n) - (Fv)(n)\|}{h(n)} \leq L_\infty \|S\|_1 \|u - v\|_h^\theta.$$

So $\|Fu - Fv\|_h \leq L_\infty \|S\|_1 \|u - v\|_h^\theta$. Hence F is a continuous map.

(iii) F is a completely continuous map. Let r be a positive real number and $B_r(Z)$ be a closed ball with center at 0 and radius r in the space Z. We set $V = F(B_r(\mathcal{C}_h^0(\mathbb{Z}, X)))$ and $v = Fu$ for $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$. We prove that for each $n \in \mathbb{Z}$,

$$\Omega_n(V) := \Big\{ \frac{v(n)}{h(n)} : v \in V \Big\}$$

is relatively compact in X. Indeed, for $\epsilon > 0$, we choose $l \in \mathbb{Z}^+$ such that

$$\sum_{k=l}^{\infty} \|S(k)\| \Big(L_\infty r^\theta + \sup_{k\in\mathbb{Z}} \|f(k, 0)\|\Big) \leq \epsilon.$$

Since $v = Fu$ for $u \in B_r(\mathbb{C}_h^0(\mathbb{Z}, X))$,

$$v(n) = \sum_{k=0}^{l-1} S(k) f(n-1-k, u(n-1-k)) + \sum_{k=l}^{\infty} S(k) f(n-1-k, u(n-1-k)).$$

Thus,

$$\frac{v(n)}{h(n)} = \frac{l}{h(n)} \Big(\frac{1}{l} \sum_{k=0}^{l-1} S(k) f(n-1-k, u(n-1-k))\Big) + \frac{1}{h(n)} \sum_{k=l}^{\infty} S(k) f(n-1-k, u(n-1-k)).$$

Note that

$$\frac{1}{h(n)} \| \sum_{k=l}^{\infty} S(k) f(n-1-k, u(n-1-k))\| \leq \frac{1}{h(n)} \sum_{k=l}^{\infty} \|S(k)\| (L_\infty r^\theta + \sup_{k\in\mathbb{Z}} \|f(k, 0)\|) \leq \epsilon.$$

Therefore

$$\frac{v(n)}{h(n)} \in \frac{l}{h(n)} \overline{co(K)} + B_\epsilon(X), \tag{3.2.3}$$

where $\overline{co(K)}$ denotes the convex hull of K and

$$K = \cup_{k=0}^{l-1}\{S(k)f(\xi, x) : \xi \in [n-l, n-1] \cap \mathbb{Z}, \|x\| \leq R\},$$

where $R = r\max\{h(\xi) : \xi \in [n-l, n-1] \cap \mathbb{Z}\}$. Hence, K is relatively compact, since that $S(k) \in \mathcal{B}(X)$ for all $k \in \mathbb{N}_0$ and $\{f(k,x) : a \leq k \leq b, \|x\| \leq \sigma\}$ is relatively compact in X, for all $a, b \in \mathbb{Z}, a \leq b$ and $\sigma > 0$. In view of

$$\Omega_n(V) \subseteq \frac{l}{h(n)}\overline{co(K)} + B_\epsilon(X),$$

we deduce that $\Omega_n(V)$ is relatively compact in X.

Next, we prove that V is weighted equiconvergent at $\pm\infty$. Indeed proceeding as in (i), we have:

$$\frac{\|v(n)\|}{h(n)} \leq \frac{1}{h(n)}\sum_{k=-\infty}^{n-1}\|S(n-1-k)\|(L_\infty\|u\|_h^\theta + \|f(k,0)\|) \leq \frac{1}{h(n)}\|S\|_1(L_\infty r^\theta + \sup_{k\in\mathbb{Z}}\|f(k,0)\|),$$

therefore $\frac{\|v(n)\|}{h(n)} \to 0$ as $|n| \to \infty$ and this convergence is independent of $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$. Hence V satisfies the conditions of Lemma 1.4.12, so V is a relatively compact set in $\mathcal{C}_h^0(\mathbb{Z}, X)$.

(iv) Now, we will prove that the set $\mathcal{C} := \{u \in \mathcal{C}_h^0(\mathbb{Z}, X) : u = \lambda F u, \lambda \in (0,1)\}$ is bounded. In fact, if $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$ is a solution of $u = \lambda F u$ for $0 < \lambda < 1$, then

$$\frac{\|u(n)\|}{h(n)} \leq \sum_{k=-\infty}^{n-1}\|S(n-1-k)\|(L_\infty\|u\|_h^\theta + \sup_{k\in\mathbb{Z}}\|f(k,0)\|) \leq \|S\|_1(L_\infty\|u\|_h^\theta + \sup_{k\in\mathbb{Z}}\|f(k,0)\|).$$

Hence

$$\frac{\|u\|_h}{\|S\|_1(L_\infty\|u\|_h^\theta + \sup_{k\in\mathbb{Z}}\|f(k,0)\|)} \leq 1.$$

Observe that in view of the condition $\theta < 1$, it follows that $\mathcal{C}$ is bounded.

(v) There exists $r_0 > 0$ such that $F(B_{r_0}(\mathcal{C}_h^0(\mathbb{Z}, X))) \subset B_{r_0}(\mathcal{C}_h^0(\mathbb{Z}, X))$. Assume that the assertion is false, then for all $r > 0$, arguing similarly as in (iv), we deduce that $1 \leq \|S\|_1\Big(L_\infty r^{\theta-1} + \frac{1}{r}\sup_{k\in\mathbb{Z}}\|f(k,0)\|\Big)$, which is a contradiction because $\theta < 1$.

(vi) Finally, by Theorem 1.4.10, F has a fixed point $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$.

$\square$

In the next results, we study the existence uniqueness of $WPAA_d$, and $WPSAP_\omega$ mild solutions of (3.2.1).

Theorem 3.2.7. *Let $\rho_1, \rho_2 \in V_\infty$ be given. Assume that $(H2)$ holds, $f \in \Omega \in \mathcal{M}(\mathbb{Z} \times X, X)$ is globally Lipschitz in the following sense:*

$$\|f(n,x) - f(n,y)\| \leq L\|x - y\|, \text{ for all } n \in \mathbb{Z} \text{ and all } x, y \in X,$$

where $L < \frac{1}{\|S\|_1}$, then (3.2.1) has a unique mild solution u which belongs to the corresponding subset $\Omega \in \mathcal{M}(\mathbb{Z}, X)$.

Proof. Consider the operator $F : WPAA_d(\mathbb{Z}, X) \to WPAA_d(\mathbb{Z}, X)$ defined by

$$(Fu)(n) := \sum_{k=-\infty}^{n-1} S(n-1-k) f(k, u(k)), \; n \in \mathbb{Z}, \tag{3.2.4}$$

where $f \in WPAA_d(\mathbb{Z}, X)$. Note that F is well defined by [117, Theorem 16] and Theorem 3.1.14. In addition, for $u, v \in WPAA_d(\mathbb{Z}, X)$ and $n \in \mathbb{Z}$ the following inequality holds:

$$\|(Fu)(n) - (Fv)(n)\| = \| \sum_{k=-\infty}^{n-1} S(n-1-k)(f(k, u(k)) - f(k, v(k)))\| \leq L\|S\|_1 \|u - v\|_\infty.$$

Since $\|S\|_1 L < 1$ we conclude that F is a contraction, and using Banach's fixed point Theorem we get that there exists a unique discrete weighted pseudo almost automorphic solution of (3.2.1). The proof for the space of S-asymptotic $\omega-$periodic sequences is analogous, but in this case, we use [118, Theorem 2.4] and Theorem 3.1.14 in order to prove that F is well defined. □

In the following theorem, we show that with a local Lipschitz condition on f the conclusion of the previous theorem holds.

Theorem 3.2.8. *Let $\rho_1, \rho_2 \in V_\infty$ be given. Assume that $(H2)$ holds and let $f \in \mathcal{M}(\mathbb{Z} \times X, X)$ that satisfies a local Lipschitz condition, that is, for each positive number r, and all $u, v \in X$ with $\|u\| \leq r, \|v\| \leq r$, we have*

$$\|f(k, h(k)u) - f(k, h(k)v)\| \leq L_f(r)\|u - v\|, \text{ for all } k \in \mathbb{Z},$$

where $L_f : \mathbb{R}^+ \to \mathbb{R}^+$ *is a nondecreasing function. If there exists* $r_0 > 0$ *such that*

$$\|S\|_1\Big(L(r_0) + \frac{\sup\limits_{k\in\mathbb{Z}} \|f(k,0)\|}{r_0}\Big) < 1,$$

then the semilinear difference equation (3.2.1) *has a unique mild solution* u *which belongs to the same space as* f *with* $\|u\|_\infty \leq r_0$.

Proof. Consider $f \in WPAA_d(\mathbb{Z}\times X, X)$. Note that $F : WPAA_d(\mathbb{Z}, X) \to WPAA_d(\mathbb{Z}, X)$ given by (3.2.4) is well defined by Corollary [10, Corollary 2.4] and Theorem 3.1.14.
Let

$$B_{r_0}(0) := \{u \in WPAA_d(\mathbb{Z}, X) : \|u\|_\infty < r_0\}.$$

We show that $F(B_{r_0}(0)) \subset B_{r_0}(0)$. Indeed, let u be in $B_{r_0}(0)$. Since f is locally Lipschitz, we obtain

$$\|f(k, u(k))\| \leq \|f(k, u(k)) - f(k, 0)\| + \|f(k, 0)\| \leq L(r_0)\|u(k)\| + \|f(k, 0)\|, \text{ for } k \in \mathbb{Z}.$$

Moreover, we have the estimate

$$\begin{aligned}\|F(u)(n)\| \leq &\sum_{k=-\infty}^{n-1} \|S(n-1-k)\|\|(f(k,u(k)) - f(k,0)\| + \sum_{k=-\infty}^{n-1} \|S(n-1-k)f(k,0)\|\\
\leq &L(r_0)\sum_{k=-\infty}^{n-1} \|S(n-1-k)\|\|u(k)\| + \|S\|_1 \sup_k \|f(k,0)\|\\
\leq &\|S\|_1\big(L(r_0) + \frac{\sup_k \|f(k,0)\|}{r_0}\big)r_0 \leq r_0,\end{aligned}$$

proving the claim. On the other hand, for $u, v \in B_{r_0}(0)$ we have that

$$\begin{aligned}\|Fu(n) - Fv(n)\| \leq &\sum_{k=-\infty}^{n-1} \|S(n-1-k)\|\|(f(k,u(k)) - f(k,v(k)))\|\\
\leq &L(r_0)\sum_{k=-\infty}^{n-1} \|S(n-1-k)\|\|u(k) - v(k)\| \leq \|S\|_1 L(r_0)\|u - v\|_\infty.\end{aligned}$$

Observing that $\|S\|_1 L(r_0) < 1$, it follows that F is a contraction in $B_{r_0}(0)$. Then there is a unique $u \in B_{r_0}(0)$ such that $Fu = u$. Analogously, we can prove the Theorem for $f \in WPSAP_\omega(\mathbb{Z}\times X, X, \rho)$. For that purpose, we use [118, Theorem 2.4] and Theorem 3.1.14 in order to prove that F is well defined, and we just have to take the ball of radius r_0 in $WPSAP_\omega(\mathbb{Z}\times X, X, \rho)$. □

Theorem 3.2.9. *Let $\rho_1, \rho_2 \in V_\infty$ be given. Assume that $(H2)$ holds and $f \in \Omega \in \mathcal{M}(\mathbb{Z} \times X, X)$, satisfies the following condition:*

$$\|f(n,x) - f(n,y)\| \leq \phi(\|x-y\|), \text{ for all } n \in \mathbb{Z} \text{ and all } x, y \in X,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function. Then (3.2.1) *has a unique mild solution u which belongs to the corresponding subset $\Omega \in \mathcal{M}(\mathbb{Z}, X)$, provided $(\|S\|\phi)^n(t) \to 0$ as $n \to \infty$ for each $t > 0$.*

Proof. Consider the operator $F : WPAA_d(\mathbb{Z}, X) \to WPAA_d(\mathbb{Z}, X)$ defined by

$$(Fu)(n) := \sum_{k=-\infty}^{n-1} S(n-1-k) f(k, u(k)),\ n \in \mathbb{Z},$$

where $f \in WPAA_d(\mathbb{Z}, X)$. Note that F is well defined by [117, Theorem 16] and Theorem 3.1.14. In addition, for $u, v \in WPAA_d(\mathbb{Z}, X)$ and $n \in \mathbb{Z}$ the following inequality holds,

$$\|(Fu)(n) - (Fv)(n)\| = \| \sum_{k=-\infty}^{n-1} \|S(n-1-k)(f(k,u(k)) - f(k,v(k)))\| \leq L\|S\|_1 \phi(\|u-v\|).$$

Since $(\|S\|_1\phi)^n(t) \to 0$ as $n \to \infty$ for each $t > 0$, by Matkowski fixed point Theorem (Theorem 2.2), F has a unique fixed point $u \in WPAA_d(\mathbb{Z}, X)$, so there is a unique mild solution $u(n)$ of (3.2.1). The proof for the space of S-asymptotic $\omega-$periodic sequences is analogous, we just use Theorem [118, Theorem 2.4] and Theorem 3.1.14 in order to prove that F is well defined. □

In the following theorem, we study under certain non-Lipschitz conditions on the function f the existence of solutions in $WPSAP_\omega(Z \times X, X)$ of the equation (3.2.1). We consider functions $f : \mathbb{Z} \times X \to X$ to establish our result.

Remark 3.2.10. The hypothesis $(B1)$ below has also been considered previously in [5] and [118] in order to prove the existence of mild solutions for a class of semilinear difference equations in a Banach space X.

Theorem 3.2.11. *Assume that $f \in WPSAP_\omega(Z \times X, X) \cap \mathcal{UC}_k(\mathbb{Z} \times X, X)$, $\rho_1, \rho_2 \in V_\infty$ and that the assumptions (H1), (H2) and (A1) hold. Under the following additional conditions:*

(B1) There exists a nondecreasing function $W : \mathbb{R}^+ \to \mathbb{R}^+$ *and a function* $M : \mathbb{Z} \to \mathbb{R}^+$ *such that* $\|f(k,x)\| \leq M(k)W(\|x\|)$ *for all* $k \in \mathbb{Z}, x \in X$.

(B2) For each $\nu > 0$, $\lim_{|n|\to\infty} \dfrac{1}{h(n)} \sum_{k=-\infty}^{n-1} \|S(n-k-1)\|M(k)W(\nu h(k)) = 0$, *where* h *is given in Lemma 1.4.12.*

(B3) For each $\epsilon > 0$, *there exists* $\delta > 0$ *such that for* $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X)$, $\|u - v\|_h \leq \delta$ *implies that* $\sum_{k=-\infty}^{n} \|S(n-k)\|\|f(k, u(k) - f(k, v(k))\| \leq \epsilon$ *for all* $n \in \mathbb{Z}$.

(B4) $\liminf_{r\to\infty} \dfrac{r}{\beta(r)} > 1$, *where*

$$\beta(r) = \sup_{n\in\mathbb{Z}} \Big(\frac{1}{h(n+1)} \sum_{k=-\infty}^{n} \|S(n-k)\|M(k)W(rh(k)) \Big).$$

Then (3.2.1) *has a mild solution in* $WPSAP_\omega(Z \times X, X)$.

Proof. Consider the operator $F : \mathcal{C}_h^0(\mathbb{Z}, X) \to \mathcal{C}_h^0(\mathbb{Z}, X)$ defined by

$$(Fu)(n) := \sum_{k=-\infty}^{n-1} S(n-1-k)f(k, u(k)),\ n \in \mathbb{Z},$$

by Leray-Schauder Theorem, we will prove that F has a fixed point in $WPSAP_\omega(Z \times X, X)$. We divide the proof into several steps.

(i) F is well defined. For $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$, by $(B1)$, one has

$$\|(Fu)(n)\| \leq \sum_{k=-\infty}^{n-1} \|S(n-1-k)\|M(k)W(\|u(k)\|) \leq \sum_{k=-\infty}^{n-1} \|S(n-1-k)\|M(k)W(\|u\|_h h(k)),$$

whence $\frac{\|(Fu)(n)\|}{h(n)} \leq \frac{1}{h(n)} \sum_{k=-\infty}^{n-1} \|S(n-1-k)\|M(k)W(\|u\|_h h(k))$. It follows from $(B2)$ that F is $\mathcal{C}_h^0(\mathbb{Z}, X)-$valued.

(ii) F is continuous. In fact, for each $\epsilon > 0$, by $(B3)$ there exists $\delta > 0$ such that for $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X), \|u - v\|_h \leq \delta$, one has

$$\|(Fu)(n) - (Fv)(n)\| \leq \sum_{k=-\infty}^{n-1} \|S(n-1-k)\|\|f(k, u(k) - f(k, v(k))\|,$$

since $h(n) \geq 1$, by $(B3)$, one has $\|Fu - Fv\|_h \leq \epsilon$, hence F is continuous.

(iii) F is completely continuous. Let $V = F(B_r(\mathcal{C}_h^0(\mathbb{Z}, X)))$ and $v = Fu$ for $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$. First, we prove that $\Omega_n(V) := \left\{ \frac{v(n)}{h(n)} : v \in V \right\}$ is relatively compact in X for each $n \in \mathbb{Z}$. By $(B2)$, for $\epsilon > 0$, we can choose $l \in \mathbb{Z}^+$ such that

$$\sum_{k=l}^{\infty} \|S(k)\| M(n-1-k) W(rh(n-1-k)) \leq \epsilon.$$

Since $v = Fu$ for $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$, then

$$\frac{v(n)}{h(n)} = \frac{l}{h(n)} \Big(\frac{1}{l} \sum_{k=0}^{l-1} S(k) f(n-1-k, u(n-1-k)) \Big) + \frac{1}{h(n)} \sum_{k=l}^{\infty} S(k) f(n-1-k, u(n-1-k)).$$

Note that,

$$\begin{aligned} \frac{1}{h(n)} \| \sum_{k=l}^{\infty} S(k) f(n-1-k, u(n-1-k)) \| &\leq \frac{1}{h(n)} \sum_{k=l}^{\infty} \|S(k)\| M(n-1-k) W(rh(n-1-k)) \| \\ &\leq \epsilon. \end{aligned}$$

So (3.2.3) holds. Then $\Omega_n(V)$ is relatively compact in X for all $n \in \mathbb{Z}$. Next, we show that V is weighted equiconvergent at $\pm\infty$. In fact,

$$\frac{\|v(n)\|}{h(n)} \leq \frac{1}{h(n)} \sum_{k=-\infty}^{n-1} \|S(n-1-k)\| M(k) W(rh(k)),$$

hence $\frac{\|v(n)\|}{h(n)} \to 0$ as $|n| \to \infty$ by (B2), and this convergence is independent of $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$, V is a relatively compact set in $\mathcal{C}_h^0(\mathbb{Z}, X)$ by Lemma 1.4.12.

(iv) Now, we will prove that the set

$$\{u \in \mathcal{C}_h^0(\mathbb{Z}, X) : u = \lambda F u, \lambda \in (0,1)\} \tag{3.2.5}$$

is bounded. In fact, if $u \in \mathcal{C}_h^0(\mathbb{Z}, X)$ is a solution of $u = \lambda F u$ for $0 < \lambda < 1$, then

$$\|u(n)\| \leq \sum_{k=-\infty}^{n-1} \|S(n-1-k)\| M(k) W(\|u\|_h h(k)) \leq h(n) \beta(\|u\|_h).$$

Hence, $\frac{\|u\|_h}{\beta(\|u\|_h)} \leq 1$. We conclude using $(B4)$..

(v) There exists $r_0 > 0$ such that $F(B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X))) \subset B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X))$. Assume that the assertion is false, then for all $r > 0$, we can choose $u^r \in B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X))$ such that $\|Fu^r\|_h > r$. Similar as (iv), we deduce that $\frac{r}{\beta(r)} \leq 1$, then $\liminf_{\xi\to\infty} \dfrac{\xi}{\beta(\xi)} \leq 1$, which contradicts condition $(B4)$.

(vi) It follows from Theorem [118, Theorem 2.3] and Theorem 3.1.14, that the vector valued space $WPSAP_\omega(\mathbb{Z},X)$ is invariant under F. Consequently, combining this with step (iv), we have that

$$F(B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X)) \subseteq B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X).$$

Hence, we arrive at the following conclusion

$$\begin{aligned} F\Big(\overline{B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X)}^{\mathcal{C}_h^0(\mathbb{Z},X)}\Big) &\subseteq \overline{F(B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X))}^{\mathcal{C}_h^0,(\mathbb{Z},X)} \\ &\subseteq \overline{B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X)}^{\mathcal{C}_h^0,(\mathbb{Z},X)} \end{aligned}$$

where $\overline{B}^{\mathcal{C}_h^0(\mathbb{Z},X)}$ denotes the closure of a set B in the space $\mathcal{C}_h^0(\mathbb{Z},X)$. Thus, we can consider the following application

$$F : \overline{B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X)}^{\mathcal{C}_h^0(\mathbb{Z},X)} \to \overline{B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X)}^{\mathcal{C}_h^0(\mathbb{Z},X)}$$

By $(i)-(iii)$, we have that F is completely continuous. Applying (iv) and the Leray-Schauder Theorem (Theorem 1.4.10), F has a fixed point u which belongs to the space $\overline{B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X)}^{\mathcal{C}_h^0(\mathbb{Z},X)}$.

(vii) Finally, we prove that u (the fixed point of F given in (vi)) is discrete weighted pseudo S-asymptotically $\omega-$periodic. Indeed, let $(u_m)_m$ be a sequence in $B_{r_0}(\mathcal{C}_h^0(\mathbb{Z},X)) \cap WPSAP_\omega(\mathbb{Z},X)$ such that $u_m \to u$, as $m \to \infty$ in the norm $\mathcal{C}_h^0(\mathbb{Z},X)$. For $\epsilon > 0$, let $\delta > 0$ be the constant in $(B3)$, that is, there exist $m_0 \in \mathbb{Z}^+$ such that $\|u_m - u\|_h \leq \delta$ for all $m \geq m_0$. Note that for $m \geq m_0$,

$$\|Fu_m - Fu\|_\infty \leq \sup_{n\in\mathbb{Z}} \sum_{k=-\infty}^{n-1} \|S(n-1-k)\| \|f(k,u_m(k)) - f(k.u(k))\| \leq \epsilon,$$

which implies that $(Fu_m)_m$ converges to $Fu = u$ uniformly in $\mathbb{Z}$. Since Fu_m belongs to $WPSAP_\omega(\mathbb{Z},X)$, we get that $u \in WPSAP_\omega(\mathbb{Z},X)$.

□

Theorem 3.2.12. *Assume that $f \in WPSAP_\omega(Z \times X, X), \rho_1, \rho_2 \in V_\infty, (H2), (A1)$ hold, and satisfies the following conditions:*

(B5) There exists a nondecreasing and surjective function $\mathcal{W} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(k, h(k)u) - f(k, h(k)v\| \leq \mathcal{W}(\|u - v\|) \text{ for all } k \in \mathbb{Z}, u, v \in X,$$

where h is given in Lemma 1.4.12.

(B6) $\displaystyle\liminf_{\tau\to\infty} \frac{\tau}{\|S\|_1(\mathcal{W}(\tau) + \sup_{k\in\mathbb{Z}} \|f(k, 0)\|)} > 1.$

Then (3.0.1) *has a mild solution $u \in WPSAP_\omega(\mathbb{Z}, X)$.*

Proof. Consider the operator $F : \mathcal{C}_h^0(\mathbb{Z}, X) \to \mathcal{C}_h^0(\mathbb{Z}, X)$ defined by

$$(Fu)(n) := \sum_{k=-\infty}^{n-1} S(n - 1 - k)f(k, u(k)),\ n \in \mathbb{Z}.$$

For $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X)$, we have the following estimates:

$$\|Fu(n)\| \leq \sum_{k=-\infty}^{n-1} \|S(n - 1 - k)\|\|f(k, u(k))\| \leq \|S\|_1\Big(\mathcal{W}(\|u\|_h) + \sup_{k\in\mathbb{Z}} \|f(k, 0)\|\Big), \tag{3.2.6}$$

and $\|f(k, u) - f(k, v)\| \leq \mathcal{W}\Big(\frac{\|u-v\|}{h(k)}\Big) \leq \mathcal{W}(\|u - v\|)$, hence F is well defined and $f \in \mathcal{UC}(\mathbb{Z} \times X, X)$. For $u, v \in \mathcal{C}_h^0(\mathbb{Z}, X)$, we have

$$\|(Fu(n)) - (Fv(n))\| \leq \sum_{k=-\infty}^{n-1} \|S(n - 1 - k)\|\mathcal{W}\Big(\frac{\|u(k) - v(k)\|}{h(k)}\Big) \leq \|S\|_1\mathcal{W}(\|u - v\|_h),$$

which implies that F is continuous. Next, let $V = F(B_r(\mathcal{C}_h^0(\mathbb{Z}, X)))$ and $v = Fu$ for $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$. For $\epsilon > 0$, we choose $l \in \mathbb{Z}^+$ such that $\displaystyle\sum_{k=l}^{\infty} \|S(k)\|\Big(\mathcal{W}(r) + \sup_{k\in\mathbb{Z}} \|f(k, 0)\|\Big) \leq \epsilon$. Let $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$, we have

$$\frac{1}{h(n)}\|\sum_{k=l}^{\infty} S(k)f(n - 1 - k, u(n - 1 - k))\| \leq \sum_{k=l}^{\infty} \|S(k)\|\Big(\mathcal{W}(r) + \sup_{k\in\mathbb{Z}} \|f(k, 0)\|\Big) \leq \epsilon. \tag{3.2.7}$$

From (3.2.3),(3.2.7) and $(A1)$, we have that $\Omega_n(V)$ is relatively compact in X for all $n \in \mathbb{Z}$. For $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$, by (3.2.6), we have

$$\frac{\|v(n)\|}{h(n)} \leq \frac{\|S\|_1}{h(n)}\Big(\mathcal{W}(r) + \sup_{k\in\mathbb{Z}} \|f(k, 0)\|\Big).$$

Hence $\lim\limits_{|n|\to\infty} \dfrac{\|v(n)\|}{h(n)} = 0$ uniformly in $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$. By Lemma 1.4.12, F is completely continuous.

Finally, we prove the boundedness of the set defined in (3.2.5). If $u \in B_r(\mathcal{C}_h^0(\mathbb{Z}, X))$ is a solution of $u = \lambda F u$ for $0 < \lambda < 1$, then by (3.2.6),

$$\frac{\|u\|_h}{\|S\|_1(\mathcal{W}(\|u\|_h) + \sup\limits_{k\in\mathbb{Z}} \|f(k,0)\|)} \leq 1.$$

From $(B6)$, we conclude that the set (3.2.5) is bounded. Similar as the proof of Theorem 3.2.11, (3.2.1) has a mild solution $u \in WPSAP_\omega(\mathbb{Z}, X)$ by Theorem 1.4.10. □

3.3 An Example

Define $a(n) = \frac{1}{a}(1 - \frac{1}{(1+a)^{n+1}})$, and $b(n) = 1 - \frac{n-1}{2^{n+2}}$. Then one can verify that $a(n) = \int_0^\infty \rho_n(t)k(t)dt$, and $b(n) = \int_0^\infty \rho_n(t)g(t)dt$, where $k(t) = \int_0^t e^{-as}ds$, and $g(t) = 1 - e^{-t}(t+1)$ in a straightforward way. Denote $a_1(t) = e^{-at}, t \geq 0$, then $a_1 \in L^1(\mathbb{R}_+), a_1(t) \geq 0$, and $-a_1'(t) \geq 0$, for all $t > 0$, nonincreasing and convex, so a_1 is 3-monotone. Thus $a_1(t)$ is 2- regular and of positive type by [109, Proposition 3.3]. It is easy to see that $k(t)$ is of positive type and by the remarks following [109, Definition 3.3] it follows that $k(t)$ is 2-regular too.

Clearly $g(0) = 0$, and $g \in W^{1,1}(\mathbb{R}_+)$. Note that

$$\widehat{k}(\lambda) = \frac{a_0}{\lambda} + \frac{\widehat{a_1}(\lambda)}{\lambda}.$$

By [109, page 266] $\frac{1}{\lambda\widehat{k}(\lambda)}$ is locally analytic at $\lambda = 0$. Thus $k(t)$ satisfies the hypotheses of Theorem 3.1.7. Therefore, A generates a summable discrete resolvent family $\{R(n)\}_{n\in\mathbb{N}_0}$, with $a(n) = \frac{1}{a}(1 - \frac{1}{(1+a)^{n+1}})$, and $b(n) = 1 - \frac{n-1}{2^{n+2}}$, such that $\|R(n)\| \leq M, M > 0$, for all $n \in \mathbb{N}$. We set $X = L^2([0,1])$, and we consider the second order differential operator $Az(\xi) = \partial^2_{\xi\xi}z(\xi)$ with domain $D(A) = \{z \in H^2[0,1] : z(0) = z(1) = 0\}$. It is well known that A generates a bounded analytic semigroup on $L^2[0,1]$. (See [55, Example 4.8]). Let us consider the following differential-difference Volterra equation on $X = L^2([0,1])$,

$$u(n+1,x) = \sum_{k=-\infty}^{n} a(n-k)\partial^2_{xx}u(k+1,x) + \sum_{k=-\infty}^{n} b(n-k)f(k,u(k,x)), \quad n \in \mathbb{Z}, \quad x \in [0,1], \tag{3.3.1}$$

where $f(k,u) = \frac{\sin u}{e^{k^2}K}$, with $K > M$. Consider $h(n) = e^{n^2}$, then $h(n) \geq 1$ for all $n \in \mathbb{Z}$ and $\lim_{|n|\to\infty} h(n) = \infty$. On the other hand,

$$\|f(k,h(k)u) - f(k,h(k)v)\| = \|\frac{\sin(e^{k^2}u) - \sin(e^{k^2}v)}{Ke^{k^2}}\| \leq \frac{\|u-v\|}{K}.$$

Therefore, (H2), (F1) hold and $L_f\|R\|_1 < 1$, with $L_f = 1/K$. Thus by Theorem 3.2.2 there is a unique mild solution $u(n)$ of (3.3.1) such that $\lim_{|n|\to\infty} \frac{\|u(n)\|}{h(n)} = 0$.

Chapter 4

Analytical properties of nonlocal discrete operators: Convexity.

In this chapter, we study the connection between positivity, monotonicity and convexity of sequences and the sign of the discrete version of the Riemman-Liouville fractional difference operator proposed by Lizama in [94],

$$\big(\Delta^\alpha f\big)(n) := \Delta^N\bigg[\sum_{j=0}^{n} k^{N-\alpha}(n-j)f(j)\bigg], \tag{4.0.1}$$

where $N-1<\alpha<N$, $N\in\mathbb{N}$.

We extend the results obtained in [65] to an entire region that covers sequential orders in two parameters, exploring limit cases and providing examples that demonstrate the sharpness of the results. Furthemore, applying the Transference Principle, we transfer all the results to the operator (0.0.3) defined by Atici and Eloe in [2, 13, 15, 16].

4.1 Properties of the operator of fractional difference Δ^α

In this section we collect some important new properties of the higher order differences Δ^l, for $l \in \mathbb{N}$ and of the α-th fractional difference operator Δ^α, for $\alpha > 0$. These results generalize [65, Proposition 2.9] and [65, Theorems 5.7, 6.11, 6.16, 6.21, 7.8]. We begin with the following result.

Proposition 4.1.1. *For any $a, b \in s(\mathbb{N}_0;\mathbb{R})$ and $l \in \mathbb{N}$ we have*

$$\Delta^l(a*b)(n) = (\Delta^l a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}a(i)b(n+j-i).$$

Proof. Note that, by Remark 1.2.1 and Lemma 1.2.5 we get for $l \in \mathbb{N}$

$$\begin{aligned}(\Delta^l a * b)(n) &= \sum_{i=0}^{n} b(i)\Delta^l a(n-i) = \sum_{i=0}^{n} b(i)\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}a(n+j-i)\\ &= \sum_{i=0}^{n}\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}a(n+j-i)b(i) = \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}\sum_{i=0}^{n}\tau_j a(n-i)b(i)\\ &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n), \quad n \in \mathbb{N}_0.\end{aligned}$$

Thus, we have that for $n \in \mathbb{N}_0$,

$$\begin{aligned}\Delta^l(a*b)(n) &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(a*b)(n+j) = \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}\tau_j(a*b)(n) + \binom{l}{0}(-1)^{l-0}(a*b)(n)\\ &= \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}[(\tau_j a * b)(n) + \sum_{i=0}^{j-1}a(i)\tau_j b(n-i)] + (-1)^l(a*b)(n)\\ &= \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n) + (-1)^l(a*b)(n) + \sum_{j=1}^{l}\binom{l}{j}(-1)^{l-j}\sum_{i=0}^{j-1}a(i)\tau_j b(n-i)\\ &= \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}a(i)b(n-i+j)\\ &= (\Delta^l a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}a(i)b(n+j-i).\end{aligned}$$

which proves the result. $\square$

Remark 4.1.2. In particular, for $l-1<\alpha<l$, $l\in\mathbb{N}$, $a(n):=u(n)$ and $b(n):=k^{l-\alpha}(n)$ we have the following identity

$$\Delta^{\alpha}u(n)=(k^{l-\alpha}*\Delta^{l}u)(n)+\sum_{j=1}^{l}\sum_{i=0}^{j-1}\binom{l}{j}(-1)^{l-j}u(i)k^{l-\alpha}(n+j-i),\quad n\in\mathbb{N}_0.$$

Concerning the composition of two operators, we prove the following result.

Proposition 4.1.3. *The following properties hold:*

(*i*) *For any $\alpha>0$ and $u\in s(\mathbb{N}_0;\mathbb{R})$ we have*

$$\Delta\circ\Delta^{\alpha}u(n)=\Delta^{\alpha+1}u(n),$$

where $m-1\le\alpha\le m$, $m\in\mathbb{N}$.

(*ii*) *For any $\alpha>0$, $l\in\mathbb{N}_0$ and $u\in s(\mathbb{N}_0;\mathbb{R})$ we have*

$$\Delta^{l}\circ\Delta^{\alpha}u(n)=\Delta^{\alpha+l}u(n),$$

where $m-1\le\alpha\le m$, $m\in\mathbb{N}$.

(*iii*) *For any $\alpha>0$ and $u\in s(\mathbb{N}_0;\mathbb{R})$ we have*

$$\Delta^{\alpha}\circ\Delta u(n)=\Delta^{\alpha+1}u(n)-\Delta^{m}k^{m-\alpha}(n+1)u(0),$$

where $m-1\le\alpha\le m$, $m\in\mathbb{N}$.

(*iv*) *For any $\alpha>0$, $l\in\mathbb{N}$ and $u\in s(\mathbb{N}_0;\mathbb{R})$ we have*

$$\Delta^{\alpha}\circ\Delta^{l}u(n)=\Delta^{\alpha+l}u(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\alpha}(n+1)\Delta^{l-1-j}u(0),$$

where $m-1\le\alpha\le m$, $m\in\mathbb{N}$.

(*v*) *For any $\alpha,\beta>0$ and $u\in s(\mathbb{N}_0;\mathbb{R})$ we have*

$$\Delta^{\beta}\circ\Delta^{\alpha}u(n)=\Delta^{\alpha+\beta}u(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha}*u)(0),$$

where $m-1\le\beta\le m$, $l-1<\alpha\le l$, $m,l\in\mathbb{N}$, and $m+l-1<\alpha+\beta\le m+l$.

(vi) *For any* $\alpha, \beta > 0$ *and* $u \in s(\mathbb{N}_0; \mathbb{R})$ *we have*

$$\Delta^{\beta} \circ \Delta^{\alpha} u(n) = \Delta^{\alpha+\beta} u(n+1) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0),$$

where $m-1 \leq \beta \leq m$, $l-1 < \alpha \leq l$, $m, l \in \mathbb{N}$, *and* $m+l-2 < \alpha+\beta \leq m+l-1$.

Proof. (i) The cases $\alpha = m$ and $\alpha = m-1$ are trivial. If $m-1 < \alpha < m$, then the proof is immediate from the definition. Indeed,

$$\Delta \circ \Delta^{\alpha} u(n) = \Delta \circ \Delta^{m}(k^{m-\alpha} * u)(n) = \Delta^{m+1}(k^{m+1-(\alpha+1)} * u)(n) = \Delta^{\alpha+1} u(n).$$

(ii) By proceeding by induction on l. For $l = 0$ is trivial and $l = 1$ is the previous case (i). For $l \in \mathbb{N}_0$, we have $\Delta^{l} \circ \Delta^{\alpha} u(n) = \Delta^{\alpha+l} u(n)$, where $m-1 \leq \alpha \leq m$. Then, for $l+1$ we obtain

$$\Delta^{l+1} \circ \Delta^{\alpha} u(n) = \Delta \circ (\Delta^{l} \circ \Delta^{\alpha} u)(n) = \Delta \circ (\Delta^{\alpha+l} u)(n) = \Delta^{\alpha+l+1} u(n).$$

(iii) If $\alpha = m$, then $k^{m-\alpha}(n+1) = k^{0}(n+1) = 0$, hence we have the result. Note that, $\Delta^{m} k(n+1) = \Delta^{m}\mathbf{1} = 0$, where $\mathbf{1}(n) \equiv 1$, thus we get the case $\alpha = m-1$.

Suppose $m-1 < \alpha < m$. By Remark 1.2.1, Lemma 1.2.5 and the previous property, we obtain

$$\begin{aligned}
\Delta^{\alpha} \circ \Delta u(n) =& \Delta^{m}(k^{m-\alpha} * \Delta u)(n) = \Delta^{m}(k^{m-\alpha} * (\tau_1 u - u))(n) \\
=& \Delta^{m}(k^{m-\alpha} * \tau_1 u)(n) - \Delta^{m}(k^{m-\alpha} * u)(n) \\
=& \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} (k^{m-\alpha} * \tau_1 u)(n+j) - \Delta^{m}(k^{m-\alpha} * u)(n) \\
=& \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} [(k^{m-\alpha} * u)(n+j+1) - k^{m-\alpha}(n+j+1)u(0)] - \Delta^{m}(k^{m-\alpha} * u)(n) \\
=& \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} (k^{m-\alpha} * u)(n+j+1) - \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} k^{m-\alpha}(n+j+1)u(0) \\
& - \Delta^{m}(k^{m-\alpha} * u)(n) \\
=& \Delta^{m}(k^{m-\alpha} * u)(n+1) - \Delta^{m} k^{m-\alpha}(n+1)u(0) - \Delta^{m}(k^{m-\alpha} * u)(n) \\
=& \Delta \circ \Delta^{m}(k^{m-\alpha} * u)(n) - \Delta^{m} k^{m-\alpha}(n+1)u(0) \\
=& \Delta \circ \Delta^{\alpha} u(n) - \Delta^{m} k^{m-\alpha}(n+1)u(0) \\
=& \Delta^{\alpha+1} u(n) - \Delta^{m} k^{m-\alpha}(n+1)u(0).
\end{aligned}$$

(iv) By induction on l. For $l = 1$ is the previous case (iii). For $l \in \mathbb{N}$, we have

$$\Delta^{\alpha} \circ \Delta^{l} u(n) = \Delta^{\alpha+l} u(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\alpha}(n+1) \Delta^{l-1-j} u(0),$$

where $m - 1 \leq \alpha \leq m$. Then, for $l + 1$ we obtain

$$\begin{aligned}
\Delta^{\alpha} \circ \Delta^{l+1} u(n) &= \Delta^{\alpha} \circ \Delta^{l}(\Delta u)(n) = \Delta^{\alpha+l}(\Delta u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\alpha}(n+1) \Delta^{l-1-j}(\Delta u)(0) \\
&= \Delta^{l}(\Delta^{\alpha} \circ \Delta u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\alpha}(n+1) \Delta^{l-j} u(0) \\
&= \Delta^{l}[(\Delta^{\alpha+1} u)(n) - \Delta^{m} k^{m-\alpha}(n+1) u(0)] - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\alpha}(n+1) \Delta^{l-j} u(0) \\
&= \Delta^{\alpha+1+l} u(n) - \Delta^{m+l} k^{m-\alpha}(n+1) u(0) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\alpha}(n+1) \Delta^{l-j} u(0) \\
&= \Delta^{\alpha+1+l} u(n) - \sum_{j=0}^{l} \Delta^{m+j} k^{m-\alpha}(n+1) \Delta^{l-j} u(0).
\end{aligned}$$

(v) First we show the property when $m-1 < \beta < m$, $l-1 < \alpha < l$ and $m+l-1 < \alpha+\beta < m+l$. By definition and using the semigroup property of k^{α}, namely: $k^{\alpha} \circ k^{\beta} = k^{\alpha+\beta}$ for any $\alpha, \beta > 0$, as well as the properties (ii) and (iv), we have for $n \in \mathbb{N}_0$,

$$\begin{aligned}
\Delta^{\beta} \circ \Delta^{\alpha} u(n) &= \Delta^{\beta} \circ \Delta^{l}(k^{l-\alpha} * u)(n) \\
&= \Delta^{\beta+l}(k^{l-\alpha} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0) \\
&= \Delta^{l} \circ \Delta^{\beta}(k^{l-\alpha} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0) \\
&= \Delta^{l} \circ \Delta^{m}(k^{m-\beta} * k^{l-\alpha} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0) \\
&= \Delta^{m+l}(k^{m+l-(\alpha+\beta)} * u)(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0) \\
&= \Delta^{\alpha+\beta} u(n) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1) \Delta^{l-1-j}(k^{l-\alpha} * u)(0).
\end{aligned}$$

Thus, we obtain the first part of the (v). We observe that the following identity is true in general for $m-1<\beta<m$ and $l-1<\alpha<l$:

$$\Delta^\beta\circ\Delta^\alpha u(n)=\Delta^{m+l}(k^{m+l-(\alpha+\beta)}*u)(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha}*u)(0). \quad (4.1.1)$$

Now, we study the borderline cases. If $\beta=m$, by property (ii), we get $\Delta^m\circ\Delta^\alpha u(n)=\Delta^{\alpha+m}u(n)$ for any $l-1\le\alpha\le l$. Furthermore, $k^{m-m}(n+1)=k^0(n+1)=0$, thus

$$\Delta^m\circ\Delta^\alpha u(n)=\Delta^{\alpha+m}u(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-m}(n+1)\Delta^{l-1-j}(k^{l-\alpha}*u)(0)=\Delta^{\alpha+m}u(n).$$

If $\beta=m-1$, applying again property (ii), we obtain $\Delta^{m-1}\circ\Delta^\alpha u(n)=\Delta^{\alpha+m-1}u(n)$ for any $l-1\le\alpha\le l$. Also, $\Delta^{m+j}k^{m-(m-1)}(n+1)=\Delta^{m+j}k(n+1)=\Delta^{m+j}1=0$, thus we obtain the result analogously to the proof for $\beta=m$.

If $\alpha=l$, by property (iv), we have $\Delta^\beta\circ\Delta^l u(n)=\Delta^{\beta+l}u(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\beta}(n+1)\Delta^{l-1-j}u(0)$, for any $m-1\le\beta\le m$. Moreover, $\Delta^{l-1-j}u(0)=\Delta^{l-1-j}(k^0*u)(0)=\Delta^{l-1-j}(k^{l-\alpha}*u)(0)$, that shows the result. Moreover, since it is valid for the points $\alpha=l$ and $\beta=m$, we deduce that in particular it is true for $\alpha+\beta=m+l$.

(vi) We proceed analogously to the proof of property (v). We have, by (4.1.1), for $m-1\le\beta\le m$ and $l-1<\alpha\le l$,

$$\Delta^\beta\circ\Delta^\alpha u(n)=\Delta^{m+l}(k^{m+l-(\alpha+\beta)}*u)(n)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha}*u)(0).$$

Suppose $m+l-2<\alpha+\beta<m+l-1$. Using the fact that $\Delta(k*u)(n)=u(n+1)$, we have

$$\begin{aligned}\Delta^{m+l}(k^{m+l-(\alpha+\beta)}*u)(n)&=\Delta^{m+l-1}\circ\Delta(k*k^{m+l-1-(\alpha+\beta)}*u)(n)\\&=\Delta^{m+l-1}(k^{m+l-1-(\alpha+\beta)}*u)(n+1)\\&=\Delta^{\alpha+\beta}u(n+1),\quad n\in\mathbb{N}_0.\end{aligned}$$

Therefore,

$$\Delta^\beta\circ\Delta^\alpha u(n)=\Delta^{\alpha+\beta}u(n+1)-\sum_{j=0}^{l-1}\Delta^{m+j}k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha}*u)(0).$$

Finally, if $\alpha+\beta=m+l-1$, then $m-1<\beta=m+l-1-\alpha<m$ and $l-1<\alpha<l$. Hence, by (4.1.1), we have for all $n\in\mathbb{N}_0$

$$\Delta^{m+l}(k^{m+l-(\alpha+\beta)}*u)(n)=\Delta^{m+l}(k^{m+l-(m+l-1)}*u)(n)=\Delta^{m+l-1}u(n+1)=\Delta^{\alpha+\beta}u(n+1).$$

This finishes the proof. □

Remark 4.1.4. Examining the previous proof, we deduce that for any $\alpha, \beta > 0$ with $m-1 \leq \beta \leq m$, $l-1 < \alpha \leq l$, $m, l \in \mathbb{N}$, and $u \in s(\mathbb{N}_0;\mathbb{R})$, if $\alpha+\beta = m+l-1$, then

$$\Delta^\beta \circ \Delta^\alpha u(n) = \Delta^{\alpha+\beta}u(n+1) - \sum_{j=0}^{l-1} \Delta^{m+j} k^{m-\beta}(n+1)\Delta^{l-1-j}(k^{l-\alpha} * u)(0), \ n \in \mathbb{N}_0.$$

4.2 Positivity, α-monotonicity and α-convexity

In this section we recall the definitions of a α-monotone and α-convex sequence, and we provide a geometrical interpretation. We summarize, and in some cases improve, several theorems of [65, Sections 5 and 6]. Moreover, we give new examples showing the necessity of imposed conditions.

We begin recalling the following definition.

Definition 4.2.1. [17, Definitions 2.3 and 2.4] Let $\alpha \geq 0$ be given. We say that a sequence $u \in s(\mathbb{N}_0, \mathbb{R})$ is α-increasing (respectively, decreasing) if

$$u(n+1) \geq \alpha u(n) \tag{4.2.1}$$

(respectively, $u(n+1) \leq \alpha u(n)$) for all $\mathbb{N}_0$.

Remark 4.2.2. Iterating (4.2.1), we observe that each α-monotone increasing sequence must satisfy:

$$u(n) \geq \alpha^n u(0), \quad n \in \mathbb{N}_0.$$

We conclude that if a sequence u is $\alpha-$monotone increasing then their graph lies above the graph of the sequence $M_\alpha(n) := \alpha^n u(0)$. In Figure 4.1, assuming $u(0) = 1$, we have drawn the behavior of the sequence $M_\alpha(n)$ for different values of α. In particular, observe that the graph of each monotone increasing sequence lies above the graph of the constant sequence $M_1 \equiv 1$. Note that an α-monotone increasing sequence could be decreasing for $\alpha < 1$.

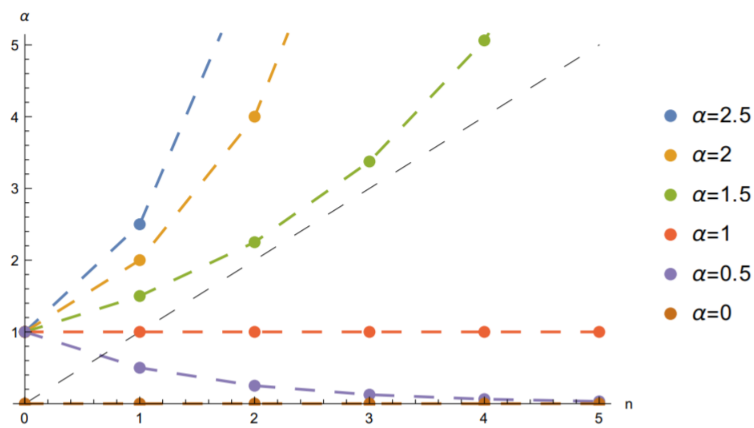

Figure 4.1: α- monotone increasing sequences

We now recall the following theorem that give us conditions to ensure positivity and α-monotonicity in the closed interval $0 \leq \alpha \leq 1$.

Theorem 4.2.3. *[65, Theorem 5.4] Let $0 \leq \alpha \leq 1$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be a given sequence. Suppose that*

(i) $(\Delta^\alpha u)(n) \geq 0$ *for all* $n \in \mathbb{N}_0$,

(ii) $u(0) \geq 0$.

Then u is positive and α-increasing on $\mathbb{N}_0$.

The following example shows that the condition $u(0) \geq 0$ is necessary for positivity.

Example 4.2.4. *Let $\gamma < \alpha < 1$ and define $u(n) := -\gamma^n,\ 0 < \gamma < 1,\ n \in \mathbb{N}_0$. Then*

- $(\Delta^\alpha u)(n) \geq 0$

- *u is negative.*

Indeed, by part (i) in Proposition 4.1.1 with $a := k^{1-\alpha}, b := u$, $l = 1$ and part (i) in Lemma 1.2.4, we obtain the identities

$$(\Delta^\alpha u)(n) = (\Delta k^{1-\alpha} * u)(n) + u(n+1) = -\alpha \sum_{j=0}^{n} \frac{k^{1-\alpha}(n-j)}{n-j+1} u(j) + u(n+1), \quad n \in \mathbb{N}_0.$$

Hence,

$$\begin{aligned}(\Delta^\alpha u)(n) &= -\alpha \sum_{j=0}^{n-1} \frac{k^{1-\alpha}(n-j)}{n-j+1} u(j) + (-\alpha)u(n) + u(n+1) \\ &= \alpha \sum_{j=0}^{n-1} \frac{k^{1-\alpha}(n-j)}{n-j+1} \gamma^j + \gamma^n(\alpha - \gamma) \geq 0,\end{aligned}$$

proving the claim.

Remark 4.2.5. Actually, a converse for Theorem 4.2.3 holds: If $u(0) \geq 0$ and u is increasing (hence positive), then $(\Delta^\alpha u)(n) \geq 0$ for all $n \in \mathbb{N}_0$. This follows immediately from Remark 4.1.2 with $l = 1$ which asserts

$$\Delta^\alpha u(n) = (k^{1-\alpha} * \Delta u)(n) + k^{1-\alpha}(n+1)u(0), \quad n \in \mathbb{N}_0, \quad 0 < \alpha < 1.$$

The following Theorem was proved in [65, Corollary 5.6] for the open sector $\{(\nu, \mu) : 0 < \nu < 1, \mu = 0\}$. We extend this result to the borderline cases.

Theorem 4.2.6. *[65, Corollary 5.6] Let $0 \leq \nu \leq 1$, $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a; \mathbb{R})$ be given a sequence. Suppose that*

(i) $\Delta_a^\nu v(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+1-\nu}$;

(ii) $v(a) \geq 0$.

Then v is positive and ν-increasing on $\mathbb{N}_a$.

Proof. In case $\nu = 0$, by hypothesis (i), we have $\Delta_a^0 v(t) = v(t) \geq 0$ on $\mathbb{N}_{a+1}$. Moreover, $v(a) \geq 0$ implies that v is positive. For $0 < \nu \leq 1$, the proof follows from [65, Corollary 5.6] and the transference principle (Theorem 1.2.6). $\square$

In the following theorem we show sufficient conditions to deduce the positivity and $(\alpha+\beta)-$monotonicity on $\mathbb{N}_0$ of a real sequence u in the region $\mathcal{R} := \{(\alpha, \beta) \in [0,1] \times [0,1] : 0 \leq \alpha+\beta \leq 1\}$. We remark that it was initially proved in [65, Theorem 5.8] for the open sector $\{(\alpha, \beta) \in (0,1) \times [0,1) : 0 < \alpha+\beta < 1\}$. We extend slightly this result to include the borders.

Theorem 4.2.7. *Let $(\alpha, \beta) \in \mathcal{R}$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be a given sequence. Assume that*

(i) $\big(\Delta^\beta \circ \Delta^\alpha u\big)(n) \geq \dfrac{\beta}{2}(1-\beta)u(0)$, *for all $n \in \mathbb{N}_0$;*

(ii) $u(1) \geq (\alpha+\beta)u(0)$;

(iii) $u(0) \geq 0$.

Then u is positive and $(\alpha+\beta)$-increasing on $\mathbb{N}_0$.

Proof. In case $\alpha + \beta = 1$, by Proposition 4.1.3 part (vi), we have the identity

$$(\Delta^{1-\alpha} \circ \Delta^\alpha u)(n) = \Delta u(n+1) - \Delta k^\alpha(n+1)u(0),$$

and then the proof follows analogously to [65, Theorem 5.8]. We observe that even the case $\alpha = 0$ is true by (i), because $\Delta^\beta \circ \Delta^0 u(n) = \Delta^\beta u(n) \geq 0$, and the proof follows from Theorem 4.2.3. In conclusion, Theorem 4.2.7 holds for $(\alpha, \beta) \in \mathcal{R}$. $\square$

The following example shows that the hypothesis (ii) in Theorem 4.2.7 is necessary in order to conclude positivity.

Example 4.2.8. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(0) = 0$ and $u(n) = -2^{-n}$, $n \in \mathbb{N}$. It is clear that $u(0) = 0$ and negative. For $\frac{1}{2} < \alpha + \beta < 1$, note that by Proposition 4.1.3, part (vi), with*

$l = m = 1$ we have the identity

$$(\Delta^\beta \circ \Delta^\alpha u)(n) = (\Delta^{\alpha+\beta}u)(n+1) - \Delta k^{1-\beta}(n+1)u(0) = (\Delta^{\alpha+\beta}u)(n+1),$$

where by Proposition 4.1.1 part (i) with $a := k^{1-(\alpha+\beta)}, b := u$, $l = 1$ and Lemma 1.2.4 part (i), we obtain

$$\begin{aligned}(\Delta^{\alpha+\beta}u)(n+1) &= (\Delta k^{1-(\alpha+\beta)} * u)(n+1) + u(n+2) \\ &= -(\alpha+\beta)\sum_{j=0}^{n} \frac{k^{1-(\alpha+\beta)}(n+1-j)}{n+2-j}u(j) - (\alpha+\beta)u(n+1) + u(n+2) \\ &= -(\alpha+\beta)\sum_{j=0}^{n} \frac{k^{1-(\alpha+\beta)}(n+1-j)}{n+2-j}u(j) + 2^{-(n+2)}\left[2(\alpha+\beta)-1\right] \geq 0.\end{aligned}$$

Therefore $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq 0$ and thus (i) in Theorem 4.2.7 is verified. However, we have $u(1) = -\frac{1}{2} < 0 = (\alpha+\beta)u(0)$.

Applying the Transference Principle (Theorem 1.2.6), we can slightly improve the statement of [65, Corollary 5.10].

Theorem 4.2.9. *Let $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given. Suppose that $(\mu,\nu) \in \mathcal{R}$ and*

(i) $\left(\Delta^{\nu}_{1+a-\mu} \circ \Delta^{\mu}_{a} v\right)(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+2-\mu-\nu}$;

(ii) $v(a+1) \geq (\mu+\nu)v(a)$;

(iii) $v(a) \geq 0$.

Then v is positive and $(\mu+\nu)$-increasing on $\mathbb{N}_a$.

Proof. Follows from application of the transference principle and Theorem 4.2.7. □

We next recall the following notion introduced in [65].

Definition 4.2.10. Let $\alpha \geq 1$. We say that a sequence $u \in s(\mathbb{N}_0, \mathbb{R})$ is α-convex (respectively, α-concave) if

$$u(n+2) - \alpha u(n+1) + (\alpha - 1)u(n) \geq 0, \quad n \in \mathbb{N}_0, \tag{4.2.2}$$

(respectively ≤ 0).

Note that, when $\alpha = 2$ we recover the geometrical notion of convexity for a sequence, and when $\alpha = 1$ the concept of monotonicity (increasing) on the set $\mathbb{N}$. It is interesting to observe the following counterpart of Remark 4.2.2, which is also new.

Remark 4.2.11. If $u \in s(\mathbb{N}_0, \mathbb{R})$ is α-convex then for each $\alpha \neq 2$

$$u(n) \geq \left[\frac{(\alpha-1)^n - 1}{\alpha - 2}\right](u(1) - u(0)) + u(0), \quad n \in \mathbb{N}_0,$$

and

$$u(n) \geq n(u(1) - u(0)) + u(0), \quad n \in \mathbb{N}_0,$$

in case $\alpha = 2$. Indeed, we note that u is α-convex if and only if $\Delta u(n+1) \geq (\alpha-1)\Delta u(n)$, $n \in \mathbb{N}_0$. Iterating, we obtain

$$u(n+1) \geq (\alpha-1)^n(u(1) - u(0)) + u(n),\ n \in \mathbb{N}_0. \tag{4.2.3}$$

Thus, iterating again we arrive at

$$u(n) \geq \left[\sum_{j=0}^{n-1} (\alpha-1)^j\right](u(1) - u(0)) + u(0),$$

and hence the conclusion follows.

Remark 4.2.12. If a sequence u is $\alpha-$convex, then their graph lies above the graph of the sequence $C_\alpha(n) := \left[\frac{1-(\alpha-1)^n}{2-\alpha}\right](u(1) - u(0)) + u(0)$ for $\alpha \neq 2$ and above of the graph of the sequence $C_2(n) = n(u(1) - u(0)) + u(0)$ in case $\alpha = 2$. Assuming $u(0) = 0, u(1) = 1$, the behavior of the sequence $C_\alpha(n)$ for different values of α is drawn in Figure 4.2. Since a sequence u is 2-convex iff u is convex, we observe that the graph of each convex sequence lies above the graph of the sequence $C_2(n) = n$. Also, we observe that an α-convex sequence could be geometrically concave for $1 < \alpha < 2$.

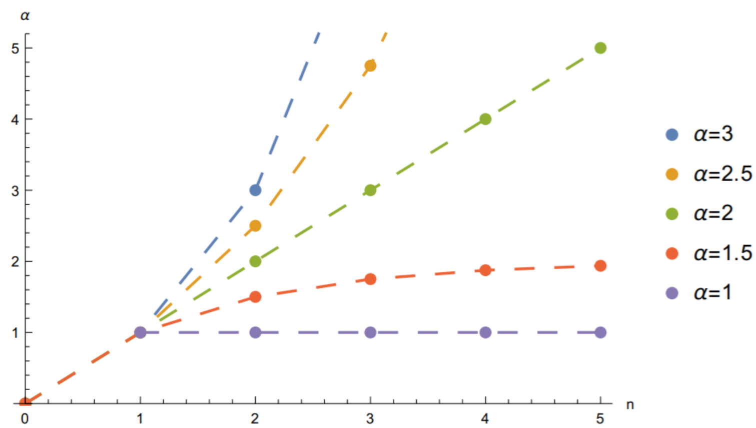

Figure 4.2: α-convex with $u(0) = 0$ and $u(1) = 1$

Conditions to obtain positivity, monotonicity and $\alpha-$convexity in the closed interval $1 \leq \alpha \leq 2$ are given in the following theorem.

Theorem 4.2.13. *[65, Theorem 6.3] Let $1 \leq \alpha \leq 2$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

(i) $\big(\Delta^\alpha u\big)(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;

(ii) $u(1) \geq \alpha u(0)$;

(iii) $u(0) \geq 0$.

Then u is positive, increasing and α-convex on $\mathbb{N}_0$.

Remark 4.2.14. A partial converse of the can be established by taking into account the following identity, valid for $1 \leq \alpha < 2$, and that follows from Remark 4.1.2 with $l = 2$:

$$\Delta^\alpha u(n) = (\Delta^2 k^{2-\alpha} * u)(n) + u(n+2) - \alpha u(n+1), \quad n \in \mathbb{N}_0,$$

which proves that if $u(0) \geq 0$ and u is α-increasing on $\mathbb{N}_0$ (and hence, positive) then $\Delta^\alpha u(n) \geq 0$.

Remark 4.2.15. A second partial converse of Theorem 4.2.13 can be established using now the following identity, that again follows from Remark 4.1.2 with $l = 2$:

$$\Delta^\alpha u(n) = (k^{2-\alpha} * \Delta^2 u)(n) + k^{2-\alpha}(n+1)(u(1) - u(0)) + k^{2-\alpha}(n+2)u(0), \quad n \in \mathbb{N}_0.$$

It proves that if $u(0) \geq 0$, $u(1) \geq \alpha u(0)$ (and therefore $u(1) \geq \alpha u(0) \geq u(0)$) and u is convex on $\mathbb{N}_0$ (i.e. 2-convex) then $\Delta^\alpha u(n) \geq 0$.

The following example shows that the condition $u(1) \geq \alpha u(0)$ in Theorem 4.2.13 is necessary for a sequence to be monotone increasing.

Example 4.2.16. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = \gamma^n$, $0 < \gamma < 1$. Assume that $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2} \leq \alpha < 2$. The following statements are true.*

- $(\Delta^\alpha u)(n) \geq 0$, *for all* $n \in \mathbb{N}_0$*;*
- $u(0) \geq 0$.
- u *is positive and decreasing.*

Indeed, it is clear that u is such that $u(0) \geq 0$ and is positive and decreasing. Next, observe that by Proposition 4.1.1 part (i), with $a := k^{2-\alpha}, b := u$, $l = 2$, we obtain for $n \in \mathbb{N}_0$:

$$\begin{aligned}
\Delta^\alpha u(n) &= (\Delta^2 k^{2-\alpha} * u)(n) + \sum_{j=1}^{2}\sum_{i=0}^{j-1} \binom{2}{j} (-1)^{2-j} k^{2-\alpha}(i) u(n+j-i) \\
&= (\Delta^2 k^{2-\alpha} * u)(n) + u(n+2) - \alpha u(n+1) \\
&= \sum_{j=0}^{n-1} \Delta^2 k^{2-\alpha}(n-j) u(j) + \Delta^2 k^{2-\alpha}(0) u(n) + u(n+2) - \alpha u(n+1) \\
&= \sum_{j=0}^{n-1} \Delta^2 k^{2-\alpha}(n-j)\gamma^j + \frac{\alpha(\alpha-1)}{2}\gamma^n + \gamma^{n+2} - \alpha\gamma^{n+1}.
\end{aligned}$$

By Lemma 1.2.4 part (ii), we have that $\Delta^2 k^{2-\alpha}(n) \geq 0$ for all $n \in \mathbb{N}_0$. Thus,

$$\Delta^\alpha u(n) \geq \frac{\alpha(\alpha-1)}{2}\gamma^n + \gamma^{n+2} - \alpha\gamma^{n+1} = \frac{\gamma^n}{2}[\alpha^2 - \alpha(1+2\gamma) + 2\gamma^2] \geq 0,$$

because $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2} \leq \alpha < 2$. *This proves the claim. On the other hand, we also have* $u(1) = \gamma < \alpha = \alpha u(0)$. *It shows that the condition* $u(1) \geq \alpha u(0)$ *in Theorem 4.2.13 is necessary.*

Observe that as γ goes from 0 to 1 the function $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2}$ goes from 1 to 2, respectively. We give a slight improvement to the borders of the following theorem.

Theorem 4.2.17. *[65, Corollary 6.9] Let* $1 \leq \nu \leq 2$, $a \in \mathbb{R}$ *and* $v \in s(\mathbb{N}_a; \mathbb{R})$ *be given a sequence. Suppose that*

(*i*) $\Delta_a^\nu v(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+2-\nu}$;

(*ii*) $v(a+1) \geq \nu v(a)$;

(*iii*) $v(a) \geq 0$.

Then v *is positive, increasing and* ν*-convex on* $\mathbb{N}_a$.

Proof. For $\nu = 1$, we obtain $\Delta_a v(t) = v(t+1) - v(t) \geq 0$, i.e., v is monotone increasing and positive, using hypotheses (*ii*) and (*iii*). For $\nu \in (1, 2]$, the conclusion follows from [65, Corollary 6.9]. □

Remark 4.2.18. Let $a \in \mathbb{R}$, and $v(n) := \tau_{-a} u(n)$, $n \in \mathbb{N}_0$ where $u(n) = \gamma^n$, $0 < \gamma < 1$. Assume that $\frac{1+2\gamma+\sqrt{1+4\gamma-4\gamma^2}}{2} \leq \alpha < 2$. By Theorem 1.2.6 and Example 4.2.22, we have

$$\Delta_a^\alpha v(t) = (\tau_{a+2-\alpha} \circ \Delta_a^\alpha v)(n) = (\tau_{a+2-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u)(n) = \Delta^\alpha u(n) \geq 0, \quad t := n + a + 2 - \alpha.$$

Therefore, we conclude that $\Delta_a^\alpha v(t) \geq 0$ for all $t \in \mathbb{N}_{a+2-\alpha}$, and $v(a) = u(0) \geq 0$. Also, v is decreasing if u is decreasing. Moreover $v(a+1) = u(1) = \gamma < 1 < \alpha = \alpha u(0) = \alpha v(a)$. It follows that the condition $v(a+1) \geq \alpha v(a)$ in Theorem 4.2.17 is necessary in order to have the conclusion.

Note that this example generalizes [56, Example 2.4] where the authors proved that for $v(t) = 2^{-t}$ and $\frac{2+\sqrt{2}}{2} < \alpha < 2$ they have $\Delta_0^\alpha v(t) \geq 0$ for all $t \in \mathbb{N}_{2-\alpha}$.

In the following theorem we summarize, improve and extend [65, Theorems 6.12, 6.17, 6.22]. Indeed, compared with [65], we have included the borders of the regions and added a new condition, namely $(\Delta^{\alpha+\beta}u)(0) \geq 0$, in order to ensure positivity, monotonicity and $(\alpha+\beta)-$convexity on $\mathbb{N}_0$ of a real valued sequence u in the region $\mathcal{M} := \{(\alpha,\beta) \in [0,2]\times[0,2] : 1 \leq \alpha+\beta \leq 2\}$. We will consider the following subsets of $\mathcal{M}$ (see also Figure 4.3):

$$\mathcal{M}_1 := \{(\alpha,\beta) \in [0,2]\times[0,2] : 0 \leq \alpha \leq 1, 1 \leq \beta \leq 2, 1 \leq \alpha+\beta \leq 2\},$$

$$\mathcal{M}_2 := \{(\alpha,\beta) \in [0,2]\times[0,2] : 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, 1 \leq \alpha+\beta \leq 2\},$$

$$\mathcal{M}_3 := \{(\alpha,\beta) \in [0,2]\times[0,2] : 1 \leq \alpha \leq 2, 0 \leq \beta \leq 1, 1 \leq \alpha+\beta \leq 2\}.$$

Theorem 4.2.19. *Suppose that $u \in s(\mathbb{N}_0;\mathbb{R})$, and*

(i)

$$(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \begin{cases} 0 & \text{if } (\alpha,\beta)\in\mathcal{M}_1, \\ \frac{\beta}{2}(1-\beta)u(0) & \text{if } (\alpha,\beta)\in\mathcal{M}_2, \\ \frac{\beta}{2}(1-\beta)[u(1)-(\alpha-1)u(0)] & \text{if } (\alpha,\beta)\in\mathcal{M}_3; \end{cases} \tag{4.2.4}$$

(ii) $u(2) \geq (\alpha+\beta)u(1) - \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0)$;

(iii) $u(1) \geq (\alpha+\beta)u(0)$;

(iv) $u(0) \geq 0$.

Then u is positive, increasing and $(\alpha+\beta)$-convex on $\mathbb{N}_0$.

Proof. We divide the proof in the following cases:

$\mathcal{M}_1$: $(\alpha,\beta)\in\mathcal{M}_1$. We have that the case $\alpha = 0$ is true by (i), because $\Delta^\beta \circ \Delta^0 u(n) = \Delta^\beta u(n) \geq 0$. Hence we can apply Theorem 4.2.13 for $\beta \in [1,2]$.

In other case, by Proposition 4.1.3 part (vi), with $l = 1$, $m = 2$, we obtain

$$(\Delta^{\alpha+\beta}u)(n+1) = (\Delta^\beta \circ \Delta^\alpha u)(n) + (\Delta^2 k^{2-\beta})(n+1)u(0). \tag{4.2.5}$$

Moreover, by Lemma 1.2.4 part (ii), we have $(\Delta^2 k^{2-\beta})(n+1) \geq 0$. Thus, by (iii), $(\Delta^2 k^{2-\beta})(n+1)u(0) \geq 0$. Then, by (i), we have that

$$(\Delta^{\alpha+\beta} u)(n) \geq 0, \text{ for all } n \in \mathbb{N}.$$

Therefore by hypotheses (ii), (iii), (iv) and Theorem 4.2.13, the conclusion follows.

$\mathcal{M}_2$: $(\alpha, \beta) \in \mathcal{M}_2$. Note that $k^\gamma(n) \geq 0$ and is decreasing for any $0 < \gamma < 1$ fixed. Then we have

$$\Delta k^\gamma(n+1) = \frac{\gamma-1}{n+2} k^\gamma(n+1) \geq \frac{\gamma-1}{2} k^\gamma(1) = \frac{\gamma-1}{2}\gamma. \tag{4.2.6}$$

Since $0 \leq 1-\beta \leq 1$, by hypothesis (iv) and (4.2.6), we obtain

$$\Delta k^{1-\beta}(n+1)u(0) \geq -\frac{\beta}{2}(1-\beta)u(0). \tag{4.2.7}$$

If $\alpha + \beta = 1$ then, by Remark 4.1.4 with $l = m = 1$, (4.2.7) and hypothesis (i), we get $\Delta u(n+1) = \Delta^\beta \circ \Delta^\alpha u(n) + \Delta k^{1-\beta}(n+1)u(0) \geq 0$, i.e., u is 1-convex = monotone increasing on $n \in \mathbb{N}$. But, by hypothesis (iii), it is also on $n \in \mathbb{N}_0$.

In other case, by Proposition 4.1.3 part (v), with $l = m = 1$, we obtain

$$(\Delta^{\alpha+\beta} u)(n) = (\Delta^\beta \circ \Delta^\alpha u)(n) + (\Delta k^{1-\beta})(n+1)u(0), \quad n \in \mathbb{N}_0. \tag{4.2.8}$$

Therefore, by hypothesis (i) and (4.2.6), we have $(\Delta^{\alpha+\beta} u)(n) \geq (\Delta^\beta \circ \Delta^\alpha u)(n) - \frac{\beta}{2}(1-\beta)u(0) \geq 0$. Then, $(\Delta^{\alpha+\beta} u)(n) \geq 0$ on $\mathbb{N}_0$. Using hypotheses (iii), (iv), and Theorem 4.2.13, the conclusion follows.

$\mathcal{M}_3$: $(\alpha, \beta) \in \mathcal{M}_3$. If $\alpha = 1$, by Proposition 4.1.3 part (iii), with $l = 2$, $m = 1$, hypothesis (i) and (4.2.6), we have $\Delta^{\beta+1} u(n) = \Delta^\beta \circ \Delta u(n) + \Delta k^{1-\beta}(n+1)u(0) \geq 0$. Hence we can apply Theorem 4.2.13 for $\beta + 1 \in [1, 2]$.

In other case, by Proposition 4.1.3 part (vi), with $l = 2$, $m = 1$, we have

$$(\Delta^{\alpha+\beta} u)(n+1) = (\Delta^\beta \circ \Delta^\alpha u)(n) + (\Delta^2 k^{1-\beta})(n+1)u(0) + (\Delta k^{1-\beta})(n+1)\Delta^{\alpha-1}u(0). \tag{4.2.9}$$

Since $\Delta^2 k^{1-\beta}(n+1) \geq 0$ and $\Delta k^{1-\beta}(n+1) \geq -\frac{\beta}{2}(1-\beta)$, by Lemma 1.2.4 part (ii) and (4.2.6) respectively. Then by the above identity and the hypotheses (i) and (iv), we obtain

$$\big(\Delta^{\alpha+\beta} u\big)(n+1) \geq \big(\Delta^\beta \circ \Delta^\alpha u\big)(n) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0),$$

$$\geq \left(\Delta^{\beta} \circ \Delta^{\alpha} u\right)(n) - \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0).$$

Hence $(\Delta^{\alpha+\beta}u)(n) \geq 0$ on $\mathbb{N}$. Using the hypotheses and Theorem 4.2.13 we obtain the conclusion.

□

Remark 4.2.20. In Theorem 4.2.19, for $1 < \alpha < 2$, we have $\Delta^{\alpha-1}u(0) = u(1) - (\alpha-1)u(0)$. Analogously, hypothesis (ii) can be rewritten only in terms of the positivity of $(\Delta^{\alpha+\beta}u)(0)$ because of the following identity

$$\frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0) - (\alpha+\beta)u(1) + u(2) = (\Delta^{\alpha+\beta}u)(0).$$

Remark 4.2.21. Note that if $\beta = 1$ the right hand side in (4.2.4) is exactly the same in the regions $\mathcal{M}_1$ and $\mathcal{M}_2$, and if $\alpha = 1$ then, again, the right hand side in (4.2.4) of Theorem 4.2.19 is the same in the regions $\mathcal{M}_2$ and $\mathcal{M}_3$. This means that the given conditions allows a continuous transition between a region and other.

The following example shows that the condition $u(1) \geq (\alpha+\beta)u(0)$ in Theorem 4.2.19 is necessary for a sequence to be positive and monotone increasing.

Example 4.2.22. *Define the sequence* $u : \mathbb{N}_0 \to \mathbb{R}$ *by* $u(n) = \gamma^n - 1$, $0 < \gamma < 1$. *Assume that* $\gamma + 1 < \alpha + \beta < 2$. *The following statements are true.*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq \begin{cases} 0 & \text{if } (\alpha,\beta) \in \mathcal{M}_1. \\ 0 & \text{if } (\alpha,\beta) \in \mathcal{M}_2. \\ \frac{\beta}{2}(1-\beta)u(1) & \text{if } (\alpha,\beta) \in \mathcal{M}_3. \end{cases}$

(ii) $u(2) \geq (\alpha+\beta)u(1)$

(iii) $u(0) = 0$.

(iv) u *is negative and decreasing.*

Indeed, it is clear that u is such that $u(0) = 0$ and is negative and decreasing. Also, $u(2) = (\gamma+1)(\gamma-1) \geq (\alpha+\beta)(\gamma-1) = (\alpha+\beta)u(1)$. This proves $(ii), (iii)$ and (iv). We will prove that (i) holds. In fact, observe that by Proposition 4.1.1 part (ii) with $a := u, b := k^{2-\alpha-\beta}$, $l_1 = l_2 = 1$, we obtain for $n \in \mathbb{N}_0$

$$\begin{aligned}
\Delta^{\alpha+\beta}u(n) &= (\Delta u * \Delta k^{2-\alpha-\beta})(n) - 2u(0)k^{2-\alpha-\beta}(n+1) + u(0)k^{2-\alpha-\beta}(n+2) \\
&\quad + u(1)k^{2-\alpha-\beta}(n+1) + \Delta u(n+1)k^{2-\alpha-\beta}(0) - \Delta u(0)k^{2-\alpha-\beta}(n+1) \\
&= (\Delta u * \Delta k^{2-\alpha-\beta})(n) - 2u(0)k^{2-\alpha-\beta}(n+1) + u(0)k^{2-\alpha-\beta}(n+2) \\
&\quad + u(1)k^{2-\alpha-\beta}(n+1) + \Delta u(n+1) - [u(1) - u(0)]k^{2-\alpha-\beta}(n+1) \\
&= \sum_{j=0}^{n} \Delta u(n-j)\Delta k^{2-\alpha-\beta}(j) + \Delta u(n+1) + \Delta k^{2-\alpha-\beta}(n+1)u(0) \\
&= \sum_{j=1}^{n} \Delta u(n-j)\Delta k^{2-\alpha-\beta}(j) + (1-\alpha-\beta)\Delta u(n) \\
&\quad + \Delta u(n+1) + \Delta k^{2-\alpha-\beta}(n+1)u(0).
\end{aligned}$$

By Lemma 1.2.4 part (i), we have that $\Delta k^{2-\alpha-\beta}(n) \leq 0$ for all $n \in \mathbb{N}_0$. Thus,

$$\begin{aligned}
\Delta^{\alpha+\beta}u(n) &\geq (1-\alpha-\beta)\Delta u(n) + \Delta u(n+1) = (1-\alpha-\beta)\gamma^n(\gamma-1) + \gamma^{n+1}(\gamma-1) \\
&= \gamma^n(\gamma-1)[(1-\alpha-\beta) + \gamma] \geq 0,
\end{aligned} \tag{4.2.10}$$

because $\gamma+1 < \alpha+\beta < 2$. On the other hand, by the identities (4.2.5), (4.2.8), (4.2.9) and $u(0) = 0$, we have

$$(\Delta^\beta \circ \Delta^\alpha u)(n) = \begin{cases} \Delta^{\alpha+\beta}u(n+1) & \text{if } (\alpha,\beta) \in \mathcal{M}_1. \\ \Delta^{\alpha+\beta}u(n) & \text{if } (\alpha,\beta) \in \mathcal{M}_2. \\ \Delta^{\alpha+\beta}u(n+1) - \Delta k^{1-\beta}(n+1)u(1) & \text{if } (\alpha,\beta) \in \mathcal{M}_3. \end{cases}$$

Moreover, by (4.2.6), we have the inequality $-\Delta k^{1-\beta}(n+1)u(1) \geq \frac{\beta}{2}(1-\beta)u(1)$. This, together with (4.2.10), proves (i). On the other hand, we also have $u(1) = \gamma - 1 < 0 = (\alpha+\beta)u(0)$. It shows that the condition $u(1) \geq (\alpha+\beta)u(0)$ in Theorem 4.2.19 is necessary.

In the next theorem, the result in $\mathcal{M}_1$ refines [65, Corollary 6.24]. The corresponding result to $\mathcal{M}_2$ is an improvement of [65, Corollary 6.14] with an extra assumption that was not previously considered. Finally, the result in $\mathcal{M}_3$ is a substantial improvement of [65, Corollary 6.19]. We note that a careful study of only monotonicity in the sectors $\mathcal{M}_1, \mathcal{M}_3$ and $\mathcal{M}_2$ was carry out by Goodrich (see [64] and [63], respectively). We remark that further advances in the sector $\mathcal{M}_2$ via homotopy methods have recently appeared [66].

Theorem 4.2.23. *Let $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a;\mathbb{R})$ be given. Suppose that,*

(i)

$$\begin{cases} (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_a v)(t) \geq 0, \ t \in \mathbb{N}_{a+3-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{M}_1, \\ (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_a v)(t) \geq \frac{\nu}{2}(1-\nu)v(a), \ t \in \mathbb{N}_{a+2-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{M}_2, \\ (\Delta^{\nu}_{a+2-\mu} \circ \Delta^{\mu}_a v)(t) \geq \frac{\nu}{2}(1-\nu)[v(a+1)-(\mu-1)v(a)], \ t \in \mathbb{N}_{a+3-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{M}_3. \end{cases}$$

(ii) $v(a+2) \geq (\nu+\mu)v(a+1) - \frac{(\nu+\mu)(\nu+\mu-1)}{2}v(a)$;

(iii) $v(a+1) \geq (\nu+\mu)v(a)$;

(iv) $v(a) \geq 0$.

Then v is positive, monotone increasing and $(\nu+\mu)$-convex on $\mathbb{N}_a$.

Proof. For $(\mu,\nu) = (0,1) \in \mathcal{M}_1$ we have $(\Delta^1_{a+1} \circ \Delta^0_a v)(t) = \Delta_{a+1}v(t) \geq 0$ and by hypotheses we arrive at the conclusion. For $(\mu,\nu) = (0,1) \in \mathcal{M}_2$ the reasoning is analogous. For $(\mu,\nu) = (1,0) \in \mathcal{M}_2$ we have $(\Delta^0_a \circ \Delta^1_a v)(t) = \Delta_a v(t) \geq 0$ and hence v is positive and monotone increasing. For $(\mu,\nu) = (1,0) \in \mathcal{M}_3$ we have $(\Delta^0_{a+1} \circ \Delta^1_a v)(t) = \Delta_{a+1}v(t) \geq 0$ and hence by hypotheses we obtain the conclusion. In other cases, the proof follows from the transference principle and Theorem 4.2.19. □

4.3 Monotonicity and convexity

In this section we improve several results from [65, Section 7] and show, in some cases, new conditions to ensure positivity, monotonicity and convexity.

First studies on convexity of difference operators were performed by Atici and Yaldiz [18] and Baoguo et.al. [27]. The next theorem is a substantial improvement of [65, Theorem 7.1] that now ensure the properties of positivity and monotonicity in the semi-closed interval $2 \leq \alpha < 3$ which were not considered in [65]. As already it was mentioned, it is important to observe that this new result, together with those proved in the previous section, allows us to conclude that the properties of positivity, monotonicity and convexity for a given sequence u have a continuous transition as α increase from 0 to 3. Our proof for convexity uses the new properties of the operator Δ^α established in Section 1.

Theorem 4.3.1. *Let $2 \leq \alpha < 3$ and $u \in s(\mathbb{N}_0;\mathbb{R})$ be given and assume that*

(i) $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;

(ii) $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$;

(iii) $u(1) \geq \alpha u(0)$;

(iv) $u(0) \geq 0$.

Then u is positive, increasing and convex on $\mathbb{N}_0$.

Proof. If $\alpha = 2$ then by hypothesis (i), $\Delta^2 u(n) \geq 0$, for all $n \in \mathbb{N}_0$, i.e. u is convex on $\mathbb{N}_0$. Now, using the fact that u is convex on $\mathbb{N}_0$, we get $\Delta u(n+1) \geq \Delta u(n)$. By hypotheses (iii) and (iv) we also have $u(1) \geq u(0)$, then $\Delta u(0) \geq 0$ and

$$\Delta u(n+1) \geq \Delta u(n) \geq ... \geq \Delta u(0) \geq 0.$$

Hence, u is monotone increasing and positive on $\mathbb{N}_0$. Now, we assume $2 < \alpha < 3$. By Remark 4.1.2 with $l = 3$, we have

$$\begin{aligned}(k^{3-\alpha} * \Delta^3 u)(n) &= \Delta^\alpha u(n) - \sum_{j=1}^{3}\sum_{i=0}^{j-1} \binom{3}{j} (-1)^{3-j} u(i) k^{3-\alpha}(n+j-i) \\ &= \Delta^\alpha u(n) - \sum_{j=1}^{3}\sum_{i=0}^{j-1} \binom{3}{j} (-1)^{3-j} u(i) \tau_{j-i} k^{3-\alpha}(n).\end{aligned}$$

Convolving with $k^{\alpha-2}$ we obtain

$$(k^{\alpha-2} * k^{3-\alpha} * \Delta^3 u)(n) = (k^{\alpha-2} * \Delta^\alpha u)(n) - \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)(k^{\alpha-2} * \tau_{j-i}k^{3-\alpha})(n).$$

Observe that, by Lemma 1.2.5 and the semigroup property of the kernel k^γ, we get

$$\begin{aligned}(k^{\alpha-2} * \tau_{j-i}k^{3-\alpha})(n) &= (k^{\alpha-2} * k^{3-\alpha})(n+j-i) - \sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l)\\ &= 1 - \sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l).\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^2 u(n+1) - \Delta^2 u(0) = (k^{\alpha-2} * \Delta^\alpha u)(n) &- \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\\ &+ \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l).\end{aligned} \tag{4.3.1}$$

Note that,

$$\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i) = 3u(0) - 3u(0) - 3u(1) + u(0) + u(1) + u(2) = \Delta^2 u(0). \tag{4.3.2}$$

Also, since for any $\gamma > 0$, $k^\gamma(0) = 1$, $k^\gamma(1) = \gamma$ and $k^\gamma(2) = \frac{\gamma(\gamma+1)}{2}$, we have

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l)\\ &= 3u(0)k^{\alpha-2}(n+1) - 3[u(0)(k^{\alpha-2}(n+2) + k^{\alpha-2}(n+1)(3-\alpha)) + u(1)k^{\alpha-2}(n+1)]\\ &+ [u(0)(k^{\alpha-2}(n+3) + k^{\alpha-2}(n+2)(3-\alpha) + k^{\alpha-2}(n+1)\frac{(3-\alpha)(4-\alpha)}{2})\\ &+ u(1)(k^{\alpha-2}(n+2) + k^{\alpha-2}(n+1)(3-\alpha)) + u(2)k^{\alpha-2}(n+1)].\end{aligned} \tag{4.3.3}$$

Replacing (4.3.2) and (4.3.3) in (4.3.1) we obtain that for $n \in \mathbb{N}_0$,

$$\begin{aligned}\Delta^2 u(n+1) = (k^{\alpha-2} * \Delta^\alpha u)(n) &+ k^{\alpha-2}(n+3)u(0) + k^{\alpha-2}(n+2)[u(1) - \alpha u(0)]\\ &+ k^{\alpha-2}(n+1)\left[u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0)\right].\end{aligned} \tag{4.3.4}$$

Using the hypotheses (ii), (iii) and (iv) we conclude from (4.3.4) that $\Delta^2 u(n) \geq 0$, for all $n \in \mathbb{N}$. On the other hand, using (ii), we have

$$u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0) = \Delta^2 u(0) - (\alpha-2)u(1) + \frac{(\alpha-2)(\alpha-1)}{2}u(0) \geq 0.$$

Hence, hypotheses (iii) and (iv) show that

$$\begin{aligned}\Delta^2 u(0) &\geq (\alpha-2)u(1) - \frac{(\alpha-2)(\alpha-1)}{2}u(0)\\ &\geq \left[(\alpha-2)\alpha - \frac{(\alpha-2)(\alpha-1)}{2}\right]u(0) = \frac{(\alpha-2)(\alpha+1)}{2}u(0) \geq 0.\end{aligned}$$

This proves that $\Delta^2 u(n) \geq 0$ for all $n \in \mathbb{N}_0$ – i.e., u is convex. □

Remark 4.3.2. Note that if a sequence u is convex on $\mathbb{N}_0$, and $u(1) > u(0) \geq 0$ then it is positive and increasing.

Remark 4.3.3. We can state the following converse for Theorem 4.3.1: Suppose that $u(0) \geq 0$, $u(1) \geq \alpha u(0)$ and u is α-convex. Then $\Delta^\alpha u(n) \geq 0$. In fact, we first observe that α-convexity implies $u(n+1)-u(n) \geq (\alpha-1)^n(u(1)-u(0))$ (see the proof of Remark 4.2.11). Since $(\alpha-1)^n(u(1)-u(0)) = (\alpha-1)^n(u(1)-\alpha u(0)) + (\alpha-1)^{n+1}u(0)$, we obtain in view of the given hypothesis that $\Delta u(n) \geq 0$. On the other hand, from Proposition 4.1.1 with $a := u, b := k^{3-\alpha}, l_1 = 1, l_2 = 2$ we have that the following identity holds:

$$\Delta^\alpha u(n) = (\Delta u * \Delta^2 k^{3-\alpha})(n) + \Delta^2 k^{3-\alpha}(n+1)u(0) + (\alpha-1)u(n+1) - \alpha u(n+2) + u(n+3).$$

Since $\Delta^2 k^{3-\alpha}(n) \geq 0$, it then follows from the given hypotheses and the previous identity that $\Delta^\alpha u(n) \geq 0$, as claimed.

The previous Remark, together with Remark 4.2.15, allows to conclude that the corresponding analogue for the fractional difference operator Δ^α of the well-known property

$$u \text{ convex } \implies \Delta^2 u(n) \geq 0,$$

for $\alpha \in (1,3)$, could be read as follows:

$$u(0) \geq 0,\ \ u(1) \geq \alpha u(0),\ \ u \text{ convex } \implies \Delta^\alpha u(n) \geq 0 \quad (1 < \alpha \leq 2)$$

and

$$u(0) \geq 0,\ \ u(1) \geq \alpha u(0),\ \ u\ \alpha\text{-convex } \implies \Delta^\alpha u(n) \geq 0 \quad (2 \leq \alpha < 3).$$

Indeed, taking into account that convex = 2-convex, we may conclude that as α increases from 1 to 3 then the geometrical property of convexity change continuously from convexity to α-convexity, which gives a reasonable converse for Theorems 4.2.13 and 4.3.1.

The following example shows that the condition $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$ in Theorem 4.3.1 is necessary for convexity.

Example 4.3.4. *Define* $u : \mathbb{N}_0 \to \mathbb{R}$ *by* $u(n) = \gamma - \frac{1}{\gamma^{n-1}}$ *where* $\gamma > 1$ *is fixed. Observe that* $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} < 3$ *and let* $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} \leq \alpha < 3$. *The following statements are true:*

- $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$*;*
- $u(1) \geq \alpha u(0)$*;*
- $u(0) \geq 0$*;*
- u *positive, monotone increasing and concave.*

Indeed, first observe that $u(0) = 0$, *and* $u(1) = \gamma - 1 > 0$. *Also, we have that* u *is positive and* $\Delta u(n) = u(n+1) - u(n) = \frac{\gamma-1}{\gamma^n} \geq 0$, *i.e.,* u *is monotone increasing on* $\mathbb{N}_0$.

Now, by Proposition 4.1.1 part (ii) with $a := k^{3-\alpha}$, $b := u$ *and* $l_1 = 2,\ l_2 = 1$, *we obtain for each* $n \in \mathbb{N}_0$

$$\begin{aligned}
\Delta^\alpha u(n) =&(\Delta^2 k^{3-\alpha} * \Delta u)(n) + 3k^{3-\alpha}(0)u(n+1) - 3[k^{3-\alpha}(0)u(n+2) + k^{3-\alpha}(1)u(n+1)] \\
&+ [k^{3-\alpha}(0)u(n+3) + k^{3-\alpha}(1)u(n+2) + k^{3-\alpha}(2)u(n+1)] + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
&- \Delta^2 k^{3-\alpha}(0)u(n+1) \\
=&(\Delta^2 k^{3-\alpha} * \Delta u)(n) + u(n+3) + u(n+2)[-3 + k^{3-\alpha}(1)] \\
&+ u(n+1)[3 - 3k^{3-\alpha}(1) + k^{3-\alpha}(2) - \Delta^2 k^{3-\alpha}(0)] + \Delta^2 k^{3-\alpha}(n+1)u(0) \\
=&\sum_{j=0}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + u(n+3) - \alpha u(n+2) + (\alpha-1)u(n+1) \\
&+ \Delta^2 k^{3-\alpha}(n+1)u(0) \\
=&\sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \Delta^2 k^{3-\alpha}(0)\Delta u(n) + u(n+3) - \alpha u(n+2)
\end{aligned}$$

$$
\begin{aligned}
&+(\alpha-1)u(n+1)+\Delta^2k^{3-\alpha}(n+1)u(0)\\
=&\sum_{j=1}^{n}\Delta^2k^{3-\alpha}(j)\Delta u(n-j)+\frac{(\alpha-1)(\alpha-2)}{2}[u(n+1)-u(n)]+u(n+3)-\alpha u(n+2)\\
&+(\alpha-1)u(n+1)+\Delta^2k^{3-\alpha}(n+1)u(0)\\
=&\sum_{j=1}^{n}\Delta^2k^{3-\alpha}(j)\Delta u(n-j)+u(n+3)-\alpha u(n+2)+\frac{\alpha(\alpha-1)}{2}u(n+1)\\
&-\frac{(\alpha-1)(\alpha-2)}{2}u(n)+\Delta^2k^{3-\alpha}(n+1)u(0)\\
=&\sum_{j=1}^{n}\Delta^2k^{3-\alpha}(j)\Delta u(n-j)+\frac{\gamma^{n+3}-1}{\gamma^{n+2}}-\alpha\frac{\gamma^{n+2}-1}{\gamma^{n+1}}+\frac{\alpha(\alpha-1)}{2}\frac{\gamma^{n+1}-1}{\gamma^{n}}\\
&-\frac{(\alpha-1)(\alpha-2)}{2}\frac{\gamma^{n}-1}{\gamma^{n-1}}.
\end{aligned}
$$

By Lemma 1.2.4, part (ii), and $\Delta u(n)\geq 0$, we have $\sum_{j=1}^{n}\Delta^2k^{3-\alpha}(j)\Delta u(n-j)\geq 0$. Thus, since $\alpha\in[\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma},3)$, and from the above, we obtain

$$
\begin{aligned}
\Delta^{\alpha}u(n)&\geq\frac{\gamma^{n+3}-1}{\gamma^{n+2}}-\alpha\frac{\gamma^{n+2}-1}{\gamma^{n+1}}+\frac{\alpha(\alpha-1)}{2}\frac{\gamma^{n+1}-1}{\gamma^{n}}-\frac{(\alpha-1)(\alpha-2)}{2}\frac{\gamma^{n}-1}{\gamma^{n-1}}\\
&=\frac{(\gamma^3-\gamma^2)\alpha^2+(2\gamma+\gamma^2-3\gamma^3)\alpha-2+2\gamma^3}{2\gamma^{n+2}}\geq 0.
\end{aligned}
$$

However u is concave, indeed,

$$
\Delta^2u(n)=u(n+2)-2u(n+1)+u(n)=\frac{\gamma^{n+2}-1}{\gamma^{n+1}}-2\frac{\gamma^{n+1}-1}{\gamma^{n}}+\frac{\gamma^{n}-1}{\gamma^{n-1}}=-\frac{(\gamma-1)^2}{\gamma^{n+1}}\leq 0.
$$

Moreover, $u(2)=\frac{(\gamma-1)(\gamma+1)}{\gamma}<2(\gamma-1)<\alpha(\gamma-1)=\alpha u(1)-\frac{\alpha(\alpha-1)}{2}u(0)$. It follows that the condition $u(2)\geq\alpha u(1)-\frac{\alpha(\alpha-1)}{2}u(0)$ in Theorem 4.3.1 is necessary in order to ensure the convexity of the sequence u.

Finally, observe that $\lim_{\gamma\to 1}\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma}=3$, and $\lim_{\gamma\to\infty}\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma}=2$. Consequently, the interval where α runs in the above example is better as γ increases.

From Theorem 4.3.1 and the Transference Principle we deduce the following consequence.

Theorem 4.3.5. *Let $2\leq\nu<3$, $a\in\mathbb{R}$ and $v\in s(\mathbb{N}_a;\mathbb{R})$ be given and assume that* (*i*) $\Delta_a^\nu v(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+3-\nu}$*;*

(*ii*) $v(a+2) \geq \nu v(a+1) - \frac{\nu(\nu-1)}{2} v(a)$*;*

(*iii*) $v(a+1) \geq \nu v(a)$*;*

(*iv*) $v(a) \geq 0$*.*

Then v *is positive, increasing and convex on* $\mathbb{N}_a$*.*

Proof. In case $\nu = 2$ the conclusion is clear from the hypotheses. Define $u := \tau_a v$. Using Theorem 1.2.6 we have

$$\Delta^\nu u(n) = \tau_{a+3-\nu} \circ \Delta_a^\nu \circ \tau_{-a} u(n) = \tau_{a+3-\nu} \circ \Delta_a^\nu \circ \tau_a v(n) = \Delta_a^\nu v(t) \geq 0,$$

for $t := n + a + 3 - \nu \in \mathbb{N}_{a+3-\nu}$. The conclusion follows from the transference principle. □

Remark 4.3.6. Let $a \in \mathbb{R}$, and $v(n) := \tau_{-a} u(n)$, $n \in \mathbb{N}_0$ where $u(n) = \frac{\gamma^n - 1}{\gamma^{n-1}}$, $\gamma > 1$. assume that $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} \leq \alpha < 3$, by Theorem 1.2.6, and Example 4.3.4, we have for $t := n + a + 3 - \alpha \in \mathbb{N}_{a+3-\alpha}$

$$\Delta_a^\alpha v(t) = (\tau_{a+3-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u)(n) = (\tau_{a+3-\alpha} \circ \Delta_a^\alpha v)(n) = \Delta^\alpha u(n) \geq 0.$$

Therefore, we conclude that $\Delta_a^\alpha v(t) \geq 0$ for all $t \in \mathbb{N}_{a+3-\alpha}$, $v(a) = u(0) \geq 0$, and $v(a+1) = u(1) \geq \alpha u(0) = \alpha v(a)$. But $v(a+2) = u(2) = \frac{(\gamma-1)(\gamma+1)}{\gamma} < \alpha(\gamma - 1) = \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0) = \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2} v(a)$. It follows that the condition $v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2} v(a)$ in Theorem 4.3.5 is necessary in order to ensure the convexity of the sequence v.

The following Theorem widely improves [65, Theorems 7.9, 7.11, 7.13, 7.15, 7.17]. We have included the borders of each region given and we have added a new hypotheses, namely: $(\Delta^{\alpha+\beta} u)(0) \geq 0$, in order to ensure positivity, monotonicity and convexity on $\mathbb{N}_0$ of a real sequence u in the set $\mathcal{C} := \{(\alpha, \beta) \in [0,3] \times [0,3] : 1 \leq \alpha + \beta \leq 2\}$. This allow us to see that all the conditions indicated in the theorem, below, overlap with all the conditions in Theorem 4.2.19. It implies that all the properties of a sequence u remain valid as the parameters (α, β) move and cross from one band $\mathcal{R}, \mathcal{M}$ or $\mathcal{C}$ to another. For the resulting plot, see Figure 4.3.

We will consider the following subregions of $\mathcal{C}$,

$$\mathcal{C}_1 := \{(\alpha,\beta) \in [0,3]\times[0,3] : 0 \le \alpha \le 1, 2 \le \beta < 3, 2 \le \alpha+\beta < 3\},$$

$$\mathcal{C}_2 := \{(\alpha,\beta) \in [0,3]\times[0,3] : 0 \le \alpha \le 1, 1 \le \beta \le 2, 2 \le \alpha+\beta < 3\},$$

$$\mathcal{C}_3 := \{(\alpha,\beta) \in [0,3]\times[0,3] : 1 \le \alpha \le 2, 1 \le \beta \le 2, 2 \le \alpha+\beta < 3\},$$

$$\mathcal{C}_4 := \{(\alpha,\beta) \in [0,3]\times[0,3] : 1 \le \alpha \le 2, 0 \le \beta \le 1, 2 \le \alpha+\beta < 3\},$$

$$\mathcal{C}_5 := \{(\alpha,\beta) \in [0,3]\times[0,3] : 2 \le \alpha < 3, 0 \le \beta \le 1, 2 \le \alpha+\beta < 3\}.$$

Theorem 4.3.7. *Suppose that*

(i)

$$(\Delta^\beta \circ \Delta^\alpha u)(n) \ge \begin{cases} \beta(\beta-1)(\beta-2)\frac{(3-\beta)}{24}u(0) & \text{if } (\alpha,\beta)\in\mathcal{C}_1. \\ 0 & \text{if } (\alpha,\beta)\in\mathcal{C}_2. \\ 0 & \text{if } (\alpha,\beta)\in\mathcal{C}_3. \\ \frac{\beta}{2}(1-\beta)[u(1)-(\alpha-1)u(0)] & \text{if } (\alpha,\beta)\in\mathcal{C}_4. \\ \frac{\beta}{2}(1-\beta)[u(2)-(\alpha-1)u(1)+\frac{(\alpha-1)(\alpha-2)}{2}u(0)] & \text{if } (\alpha,\beta)\in\mathcal{C}_5. \end{cases} \tag{4.3.5}$$

(ii) $u(3) \ge (\alpha+\beta)u(2) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1) + \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0);$

(iii) $u(2) \ge (\alpha+\beta)u(1) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(0);$

(iv) $u(1) \ge (\alpha+\beta)u(0);$

(v) $u(0) \ge 0.$

Then u is positive, monotone increasing and convex on $\mathbb{N}_0$.

Proof. We divide the proof in the following cases:

$\mathcal{C}_1$: Consider $(\alpha,\beta)\in\mathcal{C}_1$. The case $\alpha=0$ is true by (i), because $\Delta^\beta\circ\Delta^0 u(n)=\Delta^\beta u(n)\geq 0$. Hence we can apply Theorem 4.3.1 for $\beta\in[2,3)$. In other cases, by Proposition 4.1.3 part (vi), with $l=1$, $m=3$, we have

$$\Delta^{\alpha+\beta}u(n+1)=\Delta^\beta\circ\Delta^\alpha u(n)+\Delta^3 k^{3-\beta}(n+1)u(0). \tag{4.3.6}$$

By part (iii) of Lemma 1.2.4, we obtain

$$\Delta^3 k^{3-\beta}(n+1)=-\beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(n+1)}{(n+2)(n+3)(n+4)}.$$

Since that $0<3-\beta<1$, we deduce that

$$\Delta^3 k^{3-\beta}(n+1)\geq-\beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(1)}{24}.$$

Thus,

$$\Delta^{\alpha+\beta}u(n+1)\geq\Delta^\beta\circ\Delta^\alpha u(n)-\beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(1)}{24}.$$

Then, by hypothesis (i), we conclude that $\Delta^{\alpha+\beta}u(n+1)\geq 0$. Therefore the claim follows from hypothesis and Theorem 4.3.1.

$\mathcal{C}_2$: Suppose $(\alpha,\beta)\in\mathcal{C}_2$. If $\alpha+\beta=2$, then by Remark 4.1.4, with $l=1$, $m=2$ we have

$$\Delta^2 u(n+1)=(\Delta^{2-\alpha}\circ\Delta^\alpha)u(n)+\Delta^2 k^{1-\alpha}(n+1)u(0).$$

By Lemma 1.2.4 part (ii), we have $\Delta^2 k^{1-\alpha}(n+1)u(0)\geq 0$. Moreover, by hypothesis (i) we have $\Delta^2 u(n+1)\geq 0$. Thus, by hypothesis (iv) and Remark 4.3.2, we obtain the claimed conclusions.

In other cases, by Proposition 4.1.3 part (v), with $l=1$, $m=2$ we have

$$\Delta^\beta\circ\Delta^\alpha u(n)=\Delta^{\alpha+\beta}u(n)-\Delta^2 k^{2-\beta}(n+1)u(0). \tag{4.3.7}$$

But by Lemma 1.2.4 part (ii), $\Delta^2 k^{2-\beta}(n+1)\geq 0$. Then, by hypothesis (i)

$$\Delta^{\alpha+\beta}u(n)=\Delta^\beta\circ\Delta^\alpha u(n)+\Delta^2 k^{2-\beta}(n+1)u(0)\geq\Delta^\beta\circ\Delta^\alpha u(n)\geq 0.$$

Thus, $\Delta^{\alpha+\beta}u(n)\geq 0$ and the conclusion follows from hypothesis and Theorem 4.3.1.

$\mathcal{C}_3$: Assume $(\alpha,\beta)\in\mathcal{C}_3$. If $\alpha=1$ and $1\leq\beta\leq 2$ then, by Proposition 4.1.3 part (iii), with $m=2$, we have

$$\Delta^{\beta+1}u(n)=\Delta^\beta\circ\Delta u(n)+\Delta^2 k^{2-\beta}(n+1)u(0).$$

Thus by hypothesis (i), and since that $\Delta^2 k^{2-\beta}(n+1) \geq 0$, we obtain $\Delta^{\beta+1}u(n) \geq 0$. Hence, we can apply Theorem 4.3.1 for $\beta \in [1,2]$ and the conclusion follows. In other cases, by Proposition 4.1.3 part (vi), with $l = m = 2$, we have the identity

$$\Delta^{\alpha+\beta}u(n+1) = \Delta^{\beta} \circ \Delta^{\alpha}u(n) + \Delta^2 k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0) + \Delta^3 k^{2-\beta}(n+1)u(0). \tag{4.3.8}$$

Note that, $1+\beta < n+4$ for all $n \in \mathbb{N}_0$, thus $\frac{1+\beta}{n+4} - 1 < 0$, for all $n \in \mathbb{N}_0$. Then, $\beta + \alpha > \alpha + \frac{1+\beta}{n+4} - 1$ and we obtain

$$(\alpha+\beta)u(0) \geq [\alpha - 1 + \frac{1+\beta}{n+4}]u(0).$$

Since $u(1) \geq (\alpha+\beta)u(0)$, then

$$u(1) \geq [\alpha - 1 + \frac{1+\beta}{n+4}]u(0). \tag{4.3.9}$$

On the other hand,

$$\Delta^{\alpha-1}u(0) = u(1) - (\alpha-1)u(0). \tag{4.3.10}$$

And, by Lemma 1.2.4, part (ii) and (iii),

$$\Delta^2 k^{2-\beta}(n+1) = \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1), \tag{4.3.11}$$

as well as

$$\Delta^3 k^{2-\beta}(n+1) = -\beta(\beta-1)(1+\beta)\frac{k^{2-\beta}(n+1)}{(n+2)(n+3)(n+4)}. \tag{4.3.12}$$

Using (4.3.9), (4.3.10), (4.3.11) we obtain

$$\begin{aligned}
&\Delta^2 k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0) + \Delta^3 k^{2-\beta}(n+1)u(0) \\
&= \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1)[u(1) - (\alpha-1)u(0)] - \beta(\beta-1)(1+\beta)\frac{k^{2-\beta}(n+1)}{(n+2)(n+3)(n+4)}u(0) \\
&= \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1)[u(1) - (\alpha-1)u(0) - (1+\beta)\frac{1}{n+4}u(0)] \\
&= \frac{\beta(\beta-1)}{(n+2)(n+3)} k^{2-\beta}(n+1)\Big[u(1) - [(\alpha-1) + (1+\beta)\frac{1}{n+4}]u(0)\Big] \geq 0.
\end{aligned}$$

Then, by hypothesis (i) and the above inequality, we obtain from (4.3.8) that $\Delta^{\alpha+\beta}u(n+1) \geq 0$. Therefore the conclusion follow from hypothesis and Theorem 4.3.1.

$\mathcal{C}_4$: Suppose $(\alpha,\beta) \in \mathcal{C}_4$. If $\alpha + \beta = 2$, by Remark 4.1.4 with $l = 2$, $m = 1$, we have the identity

$$\Delta^2 u(n+1) = \Delta^{2-\alpha} \circ \Delta^{\alpha}u(n) + \Delta k^{\alpha-1}(n+1)\Delta^{\alpha-1}u(0) + \Delta^2 k^{\alpha-1}(n+1)u(0).$$

By part (ii) of Lemma 1.2.4 we have $\Delta^2 k^{1-\beta}(n+1) \geq 0$ and thus, by hypothesis (i), we conclude that $\Delta^2 u(n+1) \geq 0$. By hypothesis (iv) and Remark 4.3.2, we have proved the claim.

In other cases, by Proposition 4.1.3 part (v) with $l = 2$, $m = 1$, we obtain

$$\Delta^{\alpha+\beta}u(n) = \Delta^{\beta} \circ \Delta^{\alpha}u(n) + \Delta^2 k^{1-\beta}(n+1)u(0) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0). \tag{4.3.13}$$

By part (ii) of Lemma 1.2.4, we have $\Delta^2 k^{1-\beta}(n+1) \geq 0$. Since $0 < 1-\beta < 1$, we obtain

$$\Delta k^{1-\beta}(n+1) = -\beta\frac{k^{1-\beta}(n+1)}{(n+2)} \geq -\beta\frac{k^{1-\beta}(1)}{2}.$$

On the other had, by hypothesis (iv) and (v), we obtain

$$\Delta^{\alpha-1}u(0) = u(1) - (\alpha-1)u(0) \geq (\alpha+\beta)u(0) - (\alpha-1)u(0) = (\beta+1)u(0).$$

Thus,

$$\begin{aligned}\Delta^{\alpha+\beta}u(n) =& \Delta^{\beta} \circ \Delta^{\alpha}u(n) + \Delta^2 k^{1-\beta}(n+1)u(0) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0)\\ \geq& \Delta^{\beta} \circ \Delta^{\alpha}u(n) - \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0).\end{aligned}$$

Thus, if $(\alpha, \beta) \in \mathcal{C}_4$, we obtain $\Delta^{\alpha+\beta}u(n) \geq 0$, for all $n \in \mathbb{N}_0$. Therefore the conclusion follows from hypothesis and Theorem 4.3.1.

$\mathcal{C}_5$: Suppose $(\alpha, \beta) \in \mathcal{C}_5$. In case $\alpha = 2$ and $0 \leq \beta \leq 1$ by Proposition 4.1.3 part (iv), with $l = 2$, $m = 1$, we have

$$\Delta^{\beta+2}u(n) = \Delta^{\beta} \circ \Delta^2 u(n) + \Delta k^{1-\beta}(n+1)\Delta u(0) + \Delta^2 k^{1-\beta}(n+1)u(0).$$

Thus by hypothesis (i), (4.2.6), and since $\Delta^2 k^{2-\beta}(n+1) \geq 0$ we obtain $\Delta^{\beta+2}u(n) \geq 0$. Hence we can apply Theorem 4.3.1 for $\beta \in [1, 2]$ and obtain the claim. In other cases, by Proposition 4.1.3 part (vi), with $l = 2$, $m = 1$, we have

$$\begin{aligned}\Delta^{\alpha+\beta}u(n+1) =& \Delta^{\beta} \circ \Delta^{\alpha}u(n) + \Delta^3 k^{1-\beta}(n+1)u(0) + \Delta^2 k^{1-\beta}(n+1)\Delta^{\alpha-2}u(0)\\ &+ \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0).\end{aligned}$$

Moreover, by part (ii) Lemma 1.2.4,

$$\Delta^2 k^{1-\beta}(n+1) = \beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}. \tag{4.3.14}$$

Since $0 < \alpha - 2 < 1$, we also have

$$\begin{aligned}\Delta^{\alpha-2}u(0) &= \Delta(k^{3-\alpha} * u)(0) = (k^{3-\alpha} * u)(1) - (k^{3-\alpha} * u)(0) = (3-\alpha)u(0) + u(1) - u(0)\\ &= u(1) - (\alpha-2)u(0).\end{aligned} \tag{4.3.15}$$

Moreover, by part (iii) in Lemma 1.2.4,

$$\Delta^3 k^{1-\beta}(n+1) = -\,\beta(1+\beta)(\beta+2)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)(n+4)}. \tag{4.3.16}$$

And by part (i) in Lemma 1.2.4, since that $0 < 1-\beta < 1$, then

$$\Delta k^{1-\beta}(n+1) = -\beta\frac{k^{1-\beta}(n+1)}{n+2} \geq -\beta\frac{k^{1-\beta}(1)}{2}. \tag{4.3.17}$$

On the other hand, for each $n \in \mathbb{N}_0$, we have $(\beta+2)(n+4) \geq \beta+2$, and then

$$\alpha(n+4) + (\beta+2)(n+4) \geq (\beta+2) + \alpha(n+4),$$

as well as

$$(\alpha+\beta)(n+4) \geq \beta + 2 + (\alpha-2)(n+4).$$

Therefore

$$\alpha+\beta \geq \frac{\beta+2}{n+4} + \alpha - 2.$$

Thus, since that $u(1) \geq (\alpha+\beta)u(0)$ and $u(0) \geq 0$ we conclude that

$$u(1) \geq [\frac{\beta+2}{n+4} + \alpha - 2]u(0). \tag{4.3.18}$$

Therefore, using (4.3.18), (4.3.15), (4.3.14) and (4.3.16) we obtain

$$\begin{aligned}&\Delta^3 k^{1-\beta}(n+1)u(0) + \Delta^2 k^{1-\beta}(n+1)\Delta^{\alpha-2}u(0)\\ &= -\,\beta(1+\beta)(\beta+2)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)(n+4)}u(0) + \beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}[u(1) - (\alpha-2)u(0)]\\ &=\beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}[u(1) - (\alpha-2)u(0)) - (\beta+2)\frac{1}{(n+4)}u(0)]\\ &=\beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)}[u(1) - [(\alpha-2) + (\beta+2)\frac{1}{(n+4)}]u(0)] \geq 0.\end{aligned}$$

Thus, by hypothesis (i) and (4.3.17), we obtain from (**??**)

$$\Delta^{\alpha+\beta}u(n) \geq 0, \text{ for all } n \in \mathbb{N}_1. \tag{4.3.19}$$

Therefore the conclusion follows from hypothesis and Theorem 4.3.1.

□

Remark 4.3.8. Note that in the sector $\mathcal{C}_4$ we have $1 < \alpha < 2$ and an easy calculation shows that $\Delta^{\alpha-1}u(0) = u(1) - (\alpha - 1)u(0)$. In case of the sector $\mathcal{C}_5$, we have $2 < \alpha < 3$, and the identity $\Delta^{\alpha-1}u(0) = u(2) - (\alpha - 1)u(1) + \frac{(\alpha-1)(\alpha-2)}{2}u(0)$ holds.

Remark 4.3.9. Note that for $2 \leq \alpha + \beta < 3$ we obtain, after a calculation

$$u(3) - (\alpha + \beta)u(2) + \frac{(\alpha + \beta)(\alpha + \beta - 1)}{2}u(1) - \frac{(\alpha + \beta)(\alpha + \beta - 1)(\alpha + \beta - 2)}{6}u(0) = (\Delta^{\alpha+\beta}u)(0),$$

and hence the hypothesis (ii) in the previous theorem can be rewritten only in terms of the positivity of $(\Delta^{\alpha+\beta}u)(0)$.

Remark 4.3.10. The previous theorem improves the condition (i) on $(\Delta^{\beta} \circ \Delta^{\alpha}u)(n)$ in Theorems 7.13 and 7.17 of [65]. We recall that the condition (i) in [65, Theorem 7.13] is $(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) \geq \beta(1 + \beta)(\beta - 1)\frac{(2-\beta)}{24}u(0)$, whereas the condition (i) of [65, Theorem 7.17] is $(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) \geq \beta(1 + \beta)(2 + \beta)\frac{(1-\beta)}{24}u(0) + \beta\frac{(1-\beta)}{2}\Delta^{\alpha-1}u(0)$. In our findings, the condition on the right hand side of (4.3.5) is less restrictive. Moreover, our new hypotheses allows the conditions on the right hand side of (4.3.5) coincide when the pair (α, β) varies in the borders of each sector. This property was not posed in the results of [65].

Moreover, we have added the hypothesis $(\Delta^{\alpha+\beta}u)(0) \geq 0$, which allow us to improve the results in [65, Theorems 7.11, 7.13, 7.15, 7.17] ensuring not only convexity but also positivity and monotonicity of a sequence on the set $\mathbb{N}_0$. We also observe that for $(\alpha, \beta) \in \mathcal{C}_5$ we added the conditions (ii) and (iii), which were missing in [65, Theorem 7.17].

Remark 4.3.11. Note that if $\beta = 2$ the condition on the right hand side of (4.3.5) in Theorem 4.3.7 coincides in the regions $\mathcal{C}_1$ and $\mathcal{C}_2$. If $\alpha = 1$ then the condition on the right hand side of (4.3.5) coincides in the regions $\mathcal{C}_2$ and $\mathcal{C}_3$. If $\beta = 1$ then the conditions coincides in the regions $\mathcal{C}_3$ and $\mathcal{C}_4$ and, finally, if $\alpha = 2$ then the conditions coincide in the regions $\mathcal{C}_4$ and $\mathcal{C}_5$.

The following example shows that the condition (ii) in Theorem 4.3.7 is necessary for convexity in $\mathcal{C}_1$ and $\mathcal{C}_3$.

Example 4.3.12. *Define a sequence* $u : \mathbb{N}_0 \to \mathbb{R}$ *by* $u(0) = u(1) = 0$ *and* $u(n) := 2 - 2^{1-n}, n \in \mathbb{N}_2$. *Let* $\frac{4+\sqrt{2}}{2} < \alpha + \beta < 3$. *Then the following assertions hold:*

(i) $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \begin{cases} 0 & \text{if } (\alpha, \beta) \in \mathcal{C}_1. \\ 0 & \text{if } (\alpha, \beta) \in \mathcal{C}_3. \end{cases}$

(ii) $u(2) \geq (\alpha + \beta)u(1) - \frac{1}{2}(\alpha + \beta)(\alpha + \beta - 1)u(0)$;

(iii) $u(1) \geq (\alpha + \beta)u(0)$

(iv) $u(0) \geq 0$

(v) u *is positive, increasing and concave on* $\mathbb{N}_2$.

Indeed, it is clear that u *is positive, increasing and the items* $(ii), (iii), (iv)$ *and* (v) *are verified. Proceeding analogously to Example 4.3.4, using Proposition 4.1.1, part (ii), with* $a := k^{3-(\alpha+\beta)}, b := u, l_1 = 2, l_2 = 1$ *we obtain for any* $n \in \mathbb{N}_2$:

$$\begin{aligned}\Delta^{\alpha+\beta} u(n) &= \sum_{j=1}^{n} \Delta^2 k^{3-(\alpha+\beta)}(j) \Delta u(n-j) + u(n+3) - (\alpha+\beta)u(n+2) + \frac{(\alpha+\beta)((\alpha+\beta)-1)}{2} u(n+1) \\ &- \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{2} u(n) \\ &= \sum_{j=1}^{n} \Delta^2 k^{3-(\alpha+\beta)}(j) \Delta u(n-j) + \frac{2^{n+3}-1}{2^{n+2}} - (\alpha+\beta)\frac{2^{n+2}-1}{2^{n+1}} + \frac{(\alpha+\beta)(\alpha+\beta-1)}{2} \frac{2^{n+1}-1}{2^n} \\ &- \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{2} \frac{2^n-1}{2^{n-1}}.\end{aligned}$$

Since $\Delta u(n) \geq 0$, *by Lemma 1.2.4 part (ii), we have* $\sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j) \Delta u(n-j) \geq 0$. *Thus, since* $\alpha + \beta \in [\frac{4+\sqrt{2}}{2}, 3)$, *and from the previous identity, we obtain*

$$\Delta^{\alpha+\beta} u(n) \geq \frac{2(\alpha+\beta)^2 - 8(\alpha+\beta) + 7}{2^{n+2}} \geq 0, \quad n \in \mathbb{N}_2.$$

Note that $\frac{10+\sqrt{10}}{6} < \frac{4+\sqrt{2}}{2} < (\alpha+\beta) < 3$. *Therefore, we also have*

$$\begin{aligned}\Delta^{\alpha+\beta} u(1) &\geq \Delta^2 k^{3-(\alpha+\beta)}(1) \Delta u(0) + u(4) - (\alpha+\beta)u(3) + \frac{(\alpha+\beta)(\alpha+\beta-1)}{2} u(2) \\ &= \frac{1}{8}(6(\alpha+\beta)^2 - 20(\alpha+\beta) + 15) \geq 0.\end{aligned}$$

We conclude that $\Delta^{\alpha+\beta}u(n+1) \geq 0$ *on* $\mathbb{N}_0$. *On the other hand, by* (4.3.6), (4.3.8) *and taking into account that* $u(0) = u(1) = 0$, *we obtain*

$$(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) = \begin{cases} \Delta^{\alpha+\beta}u(n+1) & \text{if } (\alpha, \beta) \in \mathcal{C}_1, \\ \Delta^{\alpha+\beta}u(n+1) & \text{if } (\alpha, \beta) \in \mathcal{C}_3. \end{cases}$$

This proves (i). We now prove that u *is concave on* $\mathbb{N}_2$. *Indeed, by definition we obtain*

$$\Delta^2 u(n) = u(n+2) - 2u(n+1) + u(n) = \frac{2^{n+2}-1}{2^{n+1}} - 2\frac{2^{n+1}-1}{2^n} + \frac{2^n-1}{2^{n-1}} = -\frac{1}{2^{n+1}} \leq 0,$$

proving the claim. However, note that $u(3) = \frac{7}{4} < (\alpha+\beta)\frac{3}{2} = (\alpha+\beta)u(2)$. *It follows that the condition* (*ii*) *in Theorem 4.3.7 does not hold.*

The next example shows that the condition $u(2) \geq (\alpha+\beta)u(1) - \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0)$ in Theorem 4.3.7 is necessary for convexity in $\mathcal{C}_2$.

Example 4.3.13. *Define the sequence* $u : \mathbb{N}_0 \to \mathbb{R}$ *by* $u(n) := \gamma - \frac{1}{\gamma^{n-1}}$ *where* $\gamma > 1$ *is fixed. Let* $\frac{3\gamma+2+\sqrt{\gamma^2+4\gamma-4}}{2\gamma} \leq \alpha+\beta < 3$. *The following statements are true:*

- $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq 0$, *if* $(\alpha, \beta) \in \mathcal{C}_2$.
- $u(3) \geq (\alpha+\beta)u(2) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1) + \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0)$
- $u(1) \geq (\alpha+\beta)u(0)$
- $u(0) \geq 0$
- u *is positive, monotone increasing and concave on* $\mathbb{N}_0$.

In fact, we first observe that $u(0) = 0$, *and* $u(1) = \gamma - 1 > 0$. *Also, we have that* u *is positive and* $\Delta u(n) = u(n+1) - u(n) = \frac{\gamma-1}{\gamma^n} \geq 0$, *i.e.,* u *is monotone increasing on* $\mathbb{N}_0$. *Now, by Example 4.3.4,* u *is concave on* $\mathbb{N}_0$ *and, replacing* α *by* $\alpha+\beta$ *in Example 4.3.4, we have* $\Delta^{\alpha+\beta}u(n) \geq 0$. *Consequently, by* (4.3.7) *we obtain*

$$(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) = \Delta^{\alpha+\beta}u(n) - \Delta^2 k^{2-\beta}(n+1)u(0) \geq 0.$$

Notice that also have $u(3)-(\alpha+\beta)u(2)+\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1)-\frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0)\geq 0$. *Therefore all the assertions are verified, proving the claim. On the other hand, we have* $u(2) = \frac{(\gamma-1)(\gamma+1)}{\gamma} < 2(\gamma-1) < (\alpha+\beta)(\gamma-1) = (\alpha+\beta)u(1) - \frac{(\alpha+\beta)(\alpha+\beta-1)}{2}u(0)$. *It follows that the condition* (iii) *in Theorem 4.3.7 is necessary in order to ensure convexity.*

Now, in the following example we show that the condition (iii) in Theorem 4.3.7 is necessary for positivity, monotonicity and convexity in $\mathcal{C}_4$ and $\mathcal{C}_5$.

Example 4.3.14. *Define a sequence* $u : \mathbb{N}_0 \to \mathbb{R}$ *by* $u(0) = u(1) = 0$, $u(2) = -1$ *and* $u(n) := k^{\gamma}(n)$, $n \in \mathbb{N}_3$, $\gamma > 11$. *Let* $2 < \alpha+\beta < 3$. *Then the following assertions hold:*

(i) $(\Delta^{\beta}\circ\Delta^{\alpha}u)(n) \geq \begin{cases} 0 & \text{if } (\alpha,\beta)\in\mathcal{C}_4. \\ \frac{\beta}{2}(1-\beta)u(2) & \text{if } (\alpha,\beta)\in\mathcal{C}_5. \end{cases}$

(ii) $u(3) \geq (\alpha+\beta)u(2) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(1) + \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(0)$;

(iii) $u(1) \geq (\alpha+\beta)u(0)$

(iv) $u(0) \geq 0$

(v) u *is non positive, non increasing and non concave on* $\mathbb{N}_0$.

In fact, since $k^{\gamma}(n)\geq 0$, *it is clear that* $(ii),(iii)$ *and* (iv) *are verified. Moreover,* u *is non-positive and non-increasing because* $u(2) = -1$. *This property also implies* $\Delta^2u(0) = -1 < 0$ *and therefore* u *cannot be convex on* $\mathbb{N}_0$.

We check assertion (i). *By Definition 1.2.3 and the semigroup property of the kernel* k^{γ} *(see* (1.2.3)*) we obtain, by Definition and part (iii) of Lemma 1.2.4, the following identities:*

$$\begin{aligned}\Delta^{\alpha+\beta}u(n) &= \Delta^{\alpha+\beta}k^{\gamma}(n) = \Delta^{3}(k^{3-\alpha-\beta} * k^{\gamma})(n) = \Delta^{3}(k^{3-\alpha-\beta+\gamma})(n) \\ &= (\gamma-(\alpha+\beta))(1+\gamma-(\alpha+\beta))(2+\gamma-(\alpha+\beta))\frac{k^{3-\alpha-\beta-\gamma}}{(n+1)(n+2)(n+3)},\end{aligned}$$

for all $n\in\mathbb{N}_3$. *Therefore* $\Delta^{\alpha+\beta}u(n)\geq 0$, $n\in\mathbb{N}_3$.

We now prove that $\Delta^{\alpha+\beta}u(n) \geq 0$ *for* $n = 0, 1, 2$. *In fact, since* $2 < \alpha+\beta < 3$, *from Definition 1.2.3, we obtain* $\Delta^{\alpha+\beta}u(0) = u(3) - (\alpha+\beta)u(2) = k^{\gamma}(3) + (\alpha+\beta) \geq 0$.

On the other hand, since $2 < \alpha+\beta < 3$ *and* $\gamma > 11$, *we have* $\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1) < 3$ *and* $\gamma + 3 - 4(\alpha+\beta) > 1$. *Thus* $\frac{1}{4!}[\gamma+3-4(\alpha+\beta)](\gamma+2)(\gamma+1)\gamma > 3$ *and we obtain*

$$\begin{aligned}\Delta^{\alpha+\beta}u(1) &= u(4) - (\alpha+\beta)u(3) + \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(2)\\ &= k^{\gamma}(4) - (\alpha+\beta)k^{\gamma}(3) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)\\ &= \frac{1}{4!}[\gamma+3-4(\alpha+\beta)](\gamma+2)(\gamma+1)\gamma - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1) \geq 0.\end{aligned}$$

Finally, since $2 < \alpha+\beta < 3$ *and* $\gamma > 11$, *we have* $(\gamma+4) - 5(\alpha+\beta) > 0$. *Therefore,*

$$\begin{aligned}\Delta^{\alpha+\beta}u(2) &= u(5) - (\alpha+\beta)u(4) + \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(3) - \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)u(2)\\ &= k^{\gamma}(5) - (\alpha+\beta)k^{\gamma}(4) + \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1))k^{\gamma}(3) + \frac{1}{6}(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)\\ &\geq \frac{1}{5!}[(\gamma+3)(\gamma+4) - 5(\alpha+\beta)(\gamma+3) + 10(\alpha+\beta)(\alpha+\beta-1)](\gamma+2)(\gamma+1)\gamma\\ &\geq \frac{1}{5!}[(\gamma+3)[(\gamma+4) - 5(\alpha+\beta)] + 10(\alpha+\beta)(\alpha+\beta-1)](\gamma+2)(\gamma+1)\gamma \geq 0.\end{aligned}$$

We conclude that $\Delta^{\alpha+\beta}u(n) \geq 0$ *for all* $n \in \mathbb{N}_0$, *as claimed. Moreover, since* $u(0) = u(1) = 0$ *and* $u(2) = -1$, *by part* (v) *in Proposition 4.1.3, with* $l = 2$, $m = 1$ *we have in the sector* $\mathcal{C}_4$:

$$(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) = \Delta^{\alpha+\beta}u(n) \geq 0, \quad n \in \mathbb{N}_0,$$

and by part (vi) *in Proposition 4.1.3, for* $l = 3$, $m = 1$ *we obtain for the sector* $\mathcal{C}_5$:

$$(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) = \Delta^{\alpha+\beta}u(n) - \Delta k^{1-\beta}(n+1)u(2) \geq \frac{\beta}{2}(1-\beta)u(2), \quad n \in \mathbb{N}_0,$$

where we have used the inequality $-\Delta k^{1-\beta}(n+1)u(2) \geq \frac{\beta}{2}(1-\beta)u(2)$ *that follows from* (4.2.6). *This proves* (i). *However, note that* $u(2) = -1 < 0 = (\alpha+\beta)u(1)$. *It follows that the condition* (iii) *in Theorem 4.3.7 does not hold.*

Our final theorem provides our new results and insights on convexity, which seems to be the best possible. The result in the region $\mathcal{C}_1$ corresponds to [65, Theorem 7.9] after application of the transference principle and adding one missing hypothesis. The result in $\mathcal{C}_2$ corresponds to a substantial improvement of [65, Theorem 7.11] where not only one hypothesis was missing but also the conclusions on positivity and monotonicity. The result in $\mathcal{C}_3$ is an extension of [65, Theorem 7.13] where, even after application of the transference principle, both an additional hypothesis and the conclusions on positivity and monotonicity were absent. The result in $\mathcal{C}_4$ is a major improvement of [65, Theorem 7.15]. Finally, the conclusion about the sector $\mathcal{C}_5$ widely improves [65, Theorem 7.17]. We notice that our next theorem also improves those in the reference [62] for the sectors $\mathcal{C}_3, \mathcal{C}_2$ and $\mathcal{C}_4$.

Theorem 4.3.15. *Let $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a; \mathbb{R})$ be given. Suppose that,*

(i)

$$\begin{cases} (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_{a} v)(t) \geq \nu(\nu-1)(\nu-2)\frac{(3-\nu)}{24} v(a) & \text{if } (\mu,\nu) \in \mathcal{C}_1, \\ (\Delta^{\nu}_{a+1-\mu} \circ \Delta^{\mu}_{a} v)(t) \geq 0 & \text{if } (\mu,\nu) \in \mathcal{C}_2, \\ (\Delta^{\nu}_{a+2-\mu} \circ \Delta^{\mu}_{a} v)(t) \geq 0 & \text{if } (\mu,\nu) \in \mathcal{C}_3, \\ (\Delta^{\nu}_{a+2-\mu} \circ \Delta^{\mu}_{a} v)(t) \geq \frac{\nu}{2}(1-\nu)[v(a+1) - (\mu-1)v(a) & \text{if } (\mu,\nu) \in \mathcal{C}_4, \\ (\Delta^{\nu}_{a+3-\mu} \circ \Delta^{\mu}_{a} v)(t) \geq \frac{\nu}{2}(1-\nu)[v(a+2) - (\mu-1)v(a+1) + \frac{(\mu-1(\mu-2}{2} v(a) & \text{if } (\mu,\nu) \in \mathcal{C}_5, \end{cases} \tag{4.3.20}$$

where,

$$\begin{cases} t \in \mathbb{N}_{a+4-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{C}_1, \mathcal{C}_3, \mathcal{C}_5, \\ t \in \mathbb{N}_{a+3-\mu-\nu} & \text{if } (\mu,\nu) \in \mathcal{C}_2, \mathcal{C}_4. \end{cases}$$

(ii) $v(a+3) \geq (\mu+\nu)v(a+2) - \frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a+1) + \frac{1}{6}(\mu+\nu)(\mu+\nu-1)(\mu+\nu-2)v(a)$;

(iii) $v(a+2) \geq (\mu+\nu)v(a+1) - \frac{1}{2}(\mu+\nu)(\mu+\nu-1)v(a)$;

(iv) $v(a+1) \geq (\mu+\nu)v(a)$;

(v) $v(a) \geq 0$.

Then v is positive, monotone increasing and convex on $\mathbb{N}_a$.

Proof. Note that, when $(\mu,\nu)=(0,2)$, $(\mu,\nu)=(1,1)$ and $(\mu,\nu)=(2,0)$ the result is immediate in the respective regions. Define $u:=\tau_a v$. For $(\mu,\nu)\in\mathcal{C}_1$, using the transference principle we have,

$$\begin{aligned}\Delta^\nu\circ\Delta^\mu u(n)=&\tau_{a+3-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\Delta^\mu u(n)=\tau_{a+3-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\tau_{a+1-\mu}\circ\Delta_a^\mu\circ\tau_{-a}u(n)\\ =&\tau_{a+3-\nu}\circ\Delta_a^\nu\circ\tau_{1-\mu}\circ\Delta_a^\mu\circ\tau_{-a}u(n)=\tau_{a+3-\nu}\circ\tau_{1-\mu}\circ\Delta_{a+1-\mu}^\nu\circ\Delta_a^\mu\circ\tau_{-a}u(n)\\ =&\tau_{a+4-\mu-\nu}\circ\Delta_{a+1-\mu}^\nu\circ\Delta_a^\mu v(n),\end{aligned}$$

for each $n\in\mathbb{N}_0$. Therefore,

$$\Delta^\nu\circ\Delta^\mu u(n)=\Delta_{a+1-\mu}^\nu\circ\Delta_a^\mu v(t), \tag{4.3.21}$$

where $t:=n+a+4-\mu-\nu\in\mathbb{N}_{a+4-\mu-\nu}$. For $(\mu,\nu)\in\mathcal{C}_2$, using again the transference principle, we obtain

$$\begin{aligned}\Delta^\nu\circ\Delta^\mu u(n)=&\tau_{a+2-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\Delta^\mu u(n)=\tau_{a+2-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\tau_{a+1-\mu}\circ\Delta_a^\mu\circ\tau_{-a}u(n)\\ =&\tau_{a+2-\nu}\circ\Delta_a^\nu\circ\tau_{1-\mu}\circ\Delta_a^\mu\circ\tau_{-a}u(n)=\tau_{a+2-\nu}\circ\tau_{1-\mu}\circ\Delta_{a+1-\mu}^\nu\circ\Delta_a^\mu\circ\tau_{-a}u(n)\\ =&\tau_{a+3-\mu-\nu}\circ\Delta_{a+1-\mu}^\nu\circ\Delta_a^\mu v(n),\end{aligned}$$

for each $n\in\mathbb{N}_0$. Therefore, we conclude that

$$\Delta^\nu\circ\Delta^\mu u(n)=\Delta_{a+1-\mu}^\nu\circ\Delta_a^\mu v(t), \tag{4.3.22}$$

where $t:=n+a+3-\mu-\nu\in\mathbb{N}_{a+3-\mu-\nu}$. For $(\mu,\nu)\in\mathcal{C}_3$ we have for each $n\in\mathbb{N}_0$:

$$\begin{aligned}\Delta^\nu\circ\Delta^\mu u(n)=&\tau_{a+2-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\Delta^\mu u(n)=\tau_{a+2-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\tau_{a+2-\mu}\circ\Delta_a^\mu\circ\tau_{-a}u(n)\\ =&\tau_{a+2-\nu}\circ\Delta_a^\nu\circ\tau_{2-\mu}\circ\Delta_a^\mu\circ\tau_{-a}u(n)=\tau_{a+2-\nu}\circ\tau_{2-\mu}\circ\Delta_{a+2-\mu}^\nu\circ\Delta_a^\mu\circ\tau_{-a}u(n)\\ =&\tau_{a+4-\mu-\nu}\circ\Delta_{a+2-\mu}^\nu\circ\Delta_a^\mu v(n).\end{aligned}$$

We conclude that

$$\Delta^\nu\circ\Delta^\mu u(n)=\Delta_{a+2-\mu}^\nu\circ\Delta_a^\mu v(t), \tag{4.3.23}$$

where $t:=n+a+4-\mu-\nu\in\mathbb{N}_{a+4-\mu-\nu}$. Next, for $(\mu,\nu)\in\mathcal{C}_4$ we obtain for each $n\in\mathbb{N}_0$:

$$\Delta^\nu\circ\Delta^\mu u(n)=\tau_{a+1-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\Delta^\mu u(n)=\tau_{a+1-\nu}\circ\Delta_a^\nu\circ\tau_{-a}\circ\tau_{a+2-\mu}\circ\Delta_a^\mu\circ\tau_{-a}u(n)$$

$$
\begin{aligned}
&=\tau_{a+1-\nu} \circ \Delta_a^{\nu} \circ \tau_{2-\mu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n) = \tau_{a+1-\nu} \circ \tau_{2-\mu} \circ \Delta_{a+2-\mu}^{\nu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n) \\
&=\tau_{a+3-\mu-\nu} \circ \Delta_{a+2-\mu}^{\nu} \circ \Delta_a^{\mu} v(n).
\end{aligned}
$$

Therefore,

$$
\Delta^{\nu} \circ \Delta^{\mu} u(n) = \Delta_{a+2-\mu}^{\nu} \circ \Delta_a^{\mu} v(t), \tag{4.3.24}
$$

where $t := n + a + 3 - \mu - \nu \in \mathbb{N}_{a+3-\mu-\nu}$. Finally, for $(\mu, \nu) \in \mathcal{C}_5$

$$
\begin{aligned}
\Delta^{\nu} \circ \Delta^{\mu} u(n) &=\tau_{a+1-\nu} \circ \Delta_a^{\nu} \circ \tau_{-a} \circ \Delta^{\mu} u(n) = \tau_{a+1-\nu} \circ \Delta_a^{\nu} \circ \tau_{-a} \circ \tau_{a+3-\mu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n) \\
&=\tau_{a+1-\nu} \circ \Delta_a^{\nu} \circ \tau_{3-\mu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n) = \tau_{a+1-\nu} \circ \tau_{3-\mu} \circ \Delta_{a+3-\mu}^{\nu} \circ \Delta_a^{\mu} \circ \tau_{-a} u(n) \\
&=\tau_{a+4-\mu-\nu} \circ \Delta_{a+3-\mu}^{\nu} \circ \Delta_a^{\mu} v(n),
\end{aligned}
$$

for each $n \in \mathbb{N}_0$. Therefore,

$$
\Delta^{\nu} \circ \Delta^{\mu} u(n) = \Delta_{a+3-\mu}^{\nu} \circ \Delta_a^{\mu} v(t), \tag{4.3.25}
$$

where $t := n + a + 4 - \mu - \nu \in \mathbb{N}_{a+4-\mu-\nu}$. Moreover, if $(\mu, \nu) \in \mathcal{C}_4$, we have $\Delta^{\mu-1} u(0) = \tau_{a+2-\mu} \circ \Delta_a^{\mu-1} \circ \tau_{-a} u(0) = \Delta_a^{\mu-1} v(a + 2 - \mu)$, and if $(\mu, \nu) \in \mathcal{C}_5$, then $\Delta^{\mu-1} u(0) = \tau_{a+3-\mu} \circ \Delta_a^{\mu-1} \circ \tau_{-a} u(0) = \Delta_a^{\mu-1} v(a + 3 - \mu)$. Thus the conclusion follows of (4.3.21)-(4.3.25), hypotheses (i), $(ii), (iii), (iv)$, (v) and Theorem 4.3.7. $\square$

4.4 General conclusions

Geometric behavior for the composition of two operators $\Delta^{\nu} \circ \Delta^{\mu}$ is represented in the drawing below. The Theorem 4.2.7 represents the region $\mathcal{R}$, Theorem 4.2.19 represents the regions $\mathcal{M}_1 - \mathcal{M}_3$, and Theorem 4.3.7 represents the regions $\mathcal{C}_1 - \mathcal{C}_5$.

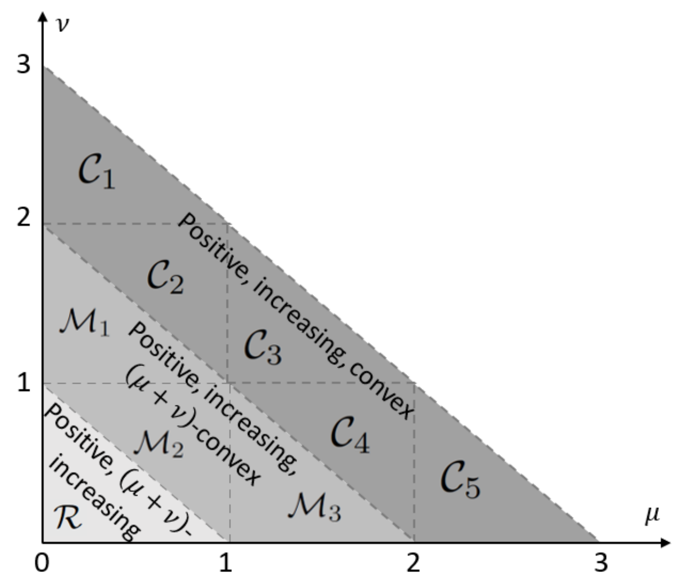

Figure 4.3: Geometry of the sequential operator $\Delta^{\nu} \circ \Delta^{\mu}$

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