UNIVERSIDAD DE SANTIAGO DE CHILE

FACULTAD DE CIENCIA

DEPARTAMENTO DE MATEMÁTICA Y CIENCIA DE LA COMPUTACIÓN

EXISTENCE OF POSITIVE SOLUTIONS FOR SOME QUASILINEAR PROBLEM

PATRICIO HUMBERTO CERDA LOYOLA

Profesor Tutor: Dr. Pedro Ubilla López
Profesor Cotutor: Dr. Leonelo Iturriaga Pastene

Trabajo de graduación presentado a la Facultad de Ciencia en cumplimiento parcial de los requisitos exigidos para optar al Grado de Doctor en Ciencia con mención en Matemática.

Santiago, Chile.
Agosto 2013.

EXISTENCE OF POSITIVE SOLUTIONS FOR SOME QUASILINEAR PROBLEM

Patricio Humberto Cerda Loyola

EXISTENCE OF POSITIVE SOLUTIONS FOR SOME QUASILINEAR PROBLEM

Trabajo de Graduación presentado a la Facultad de Ciencia, en cumplimiento parcial de los requisitos exigidos para optar al grado de Doctor en Ciencia en la Especialidad de Matemática.

UNIVERSIDAD DE SANTIAGO DE CHILE
SANTIAGO-CHILE
2013

EXISTENCE OF POSITIVE SOLUTIONS FOR SOME QUASILINEAR PROBLEM

Este trabajo de Graduación fue elaborado bajo la supervisión del profesor tutor Dr. Pedro Ubilla de la Universidad Santiago de Chile y cotutor Dr. Leonelo Iturriaga de la Universidad Técnica Federico Santa María.

Ha sido aprobado por los miembros de la Comisión Calificadora, compuesta por:

PROFESOR TUTOR
Dr. Pedro Ubilla López.

PROFESOR CO-TUTOR
Dr. Leonelo Iturriaga Pastene.

PROFESOR INFORMANTE
Dr. Djairo Guedes de Figueiredo.

PROFESOR INFORMANTE
Dr. Alexander Quaas.

PROFESOR INFORMANTE
Dr. Sebastián Lorca.

PROFESOR INFORMANTE
Dr. Humberto Ramos.

DIRECTOR
Fredi Palominos Villavicencio.

Agradecimientos

Mi más profundo agradecimiento a mi familia, en especial a mi abuela Juana Salas Gonzalez y a mi madre Viviana Loyola, sin ellas nada de esto hubiese sido posible.

A Pedro Ubilla por su tutela y amistad, sus consejos e incentivo para ingresar al doctorado. También agradezco a Leonelo Iturriaga por su apoyo y ayuda incondicional.

A los profesores del programa de doctorado en matemática, en especial al profesor Victor Guiñez.

A los profesores Bernhard Ruf y João Marcos do Ó por su hospitalidad y ayuda durante mis estadías en Milán y João Pessoa respectivamente.

Al profesor Jorge García Melián por sus valiosos consejos para el desarrollo de este trabajo.

A Evelyn Aguilar por su paciencia y gran disposición hacia los alumnos del postgrado.

A mis compañeros de doctorado Gonzalo Castro, Jarnishs Beltran, Solange Aranzubia, Lida Mendoza y Rodrigo Ponce por su amistad y compañia en este camino.

Por último, quiero agradecer el apoyo económico entregado por la Comisión Nacional de Investigación Científica y Tecnológica de Chile (CONICYT), mediante la beca para estudios doctorales en Chile.

Contents

Introduction

In this tesis, we are concerned with quasilinear elliptic equations of the type

$$\begin{aligned} -\operatorname{div}(a(x,u)\nabla u) &= f(x,u,\nabla u), \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{0.1}$$

where Ω is open bounded in $\mathbb{R}^n$ with $n \geq 1$. Let us observe that under some conditions on a and f, the existence, uniqueness and regularity of solutions of (0.1) have been widely studied. These issues are usually treated by nonlinear functional analysis techniques such as variational and non–variational methods, see e.g. [1], [4], [8], [9], [10], [12], [14], [15], [18] and [19].

This thesis is composed by the study of three problems, which belong to the field of nonlinear elliptic differential equations. The main techniques used here are the Blow up technique, to obtain a priori bounds, and tools from the Topological Degree Theory.

We have divided our work in three chapters. In chapter 1, we start our investigation by considering a one–dimensional problem. More specifically, we prove the existence of a positive solution for the Sturm–Liouville problem

$$\begin{aligned} -(p(s,u)\,u')' &= \hat{q}(s)\,u^p\,h(s,u,u') \text{ in } (0,1), \\ u(0) &= 0 = u(1). \end{aligned} \tag{0.2}$$

Here $p\,(s,u) = 1/(a(s) + c\,g(u))$, where g is a positive continuous function $\hat{q}$ is a nonnegative continuous function, a is a positive and continuous function, $c \geq 0$, $p > 1$, and the function h is sub–quadratic function with respect to u', i.e. given a compact set K in $[0,1]\times\mathbb{R}$, there exist positive constants A and B such that, for all $(s,u,\psi) \in K \times \mathbb{R}$, we have

$$h(s,u,\psi) \leq A + B\,\psi^2. \tag{0.3}$$

By *solutions* we understand classical solutions, that is, $u \in C^1(0,1) \cap C^0[0,1]$ and $\dfrac{u'}{a(s) + c\,g(u)} \in C^1(0,1)$ verifying equation (0.2).

We observe that one of the difficulties to prove our result lies in the fact that the coefficient $p(s,u) = 1/(a(s) + c\,g(u))$ is not bounded from below by a positive constant which is independent of u. In order to overcome this difficulty, we use a truncation $g_n(u)$ of the function $g(u)$, so that the new coefficient $p_n(s,u) = 1/(a(s) + c\,g_n(u))$ is bounded below by a positive constant. This allows us to use results from the topological degree theory to prove that the truncated problem has at least one positive solution. Finally, we show that for n large enough, the solutions of the truncated problem are solutions of the initial problem. Another difficulty of the problem is the dependence of h on u', that leads us towards the problem of establishing a priori bounds for the derivative. Note that this feature provides a non-variational structure to the problem.

For a further discussion on problems modeled by equations of the type $-(q(s,u)u')' = f(s,u,u')$, see e.g. [2], [16], [28], [32], [39], [47], [48], [49], [50], [51]. Existence of a solution when the coefficient $q(s,u)$ does not depend on u is considered in [2], [28], [39], [47], [48], [50]. For Sturm-Liouville's problems where the coefficient $q(s,u)$ depends explicitly on the variable u, see for example [16], [32], [49], [51].

In this thesis we study problems whose structure differs from those mentioned above. For instance, in [32], [49], [51] the problems are modeled by a nonlinearity $f(s,u,u')$ which is negative, whereas in [16] the nonlinearity is bounded with respect to the variable u and u'. Note that in our work $f(s,u,u')$ is nonnegative and unbounded with respect to the variables u, u'.

The second step in our work is chapter 2, where we study the existence of positive solutions of the system of ordinary differential equations

$$\begin{aligned} -(p_1(t,u,v)u')' &= h_1(t)f_1(t,u,v) \text{ in } (0,1)\,, \\ -(p_2(t,u,v)v')' &= h_2(t)f_2(t,u,v) \text{ in } (0,1)\,, \\ u(0)=u(1) &= v(0)=v(1)=0. \end{aligned} \tag{0.4}$$

Here $p_1(t,u,v) = 1/(a_1(t) + c_1\,g_1(u,v))$ and $p_2(t,u,v) = 1/(a_2(t) + c_2\,g_2(u,v))$. We assume that g_1, g_2 are nonnegative continuous functions, a_1, a_2 are positive continuous functions, $c_1, c_2 \geq 0$, $h_1\,, h_2 \in L^1(0,1)$ and the nonlinearities f_1, f_2 are super-linear at *zero* and $+\infty$, i.e. we have

$$\lim_{u+v\to 0} \frac{f_1(t,u,v)}{u+v} = 0 \quad \text{and} \quad \lim_{u+v\to 0} \frac{f_2(t,u,v)}{u+v} = 0\,,$$

uniformly for all $t \in [0,1]$, and there exist p, $q > 1$, $\eta_i > 0$ and $0 < \alpha_i < \beta_i < 1$ for $i = 1, 2$, such that

$$f_1(t,u,v) \geq \eta_1\, u^p \quad \text{for all} \quad u \geq 0 \quad \text{and} \quad t \in (\alpha_1, \beta_1)$$

and

$$f_2(t,u,v) \geq \eta_2\, v^q \quad \text{for all} \quad v \geq 0 \quad \text{and} \quad t \in (\alpha_2, \beta_2).$$

Once again, the difficulty we have to confront to prove our result is the fact that the coefficients of the differential operator of the problem are nonlinear and not necessarily bounded from below by a positive constant independent of u and v. We prove the existence of a positive solution of the truncated problem (as in problem (0.2)), and we show that for some n sufficiently large the solution of the truncated problem is a solution of Problem (0.4). Note that, in general, this problem has a non–variational estructure.

Problems as (0.4) with $a_1(s) = a_2(s) = 1$ and $g_1(u, v) = g_2(u, v) = 0$ have been widely studied under different conditions on the nonlinearity. For instance, assuming superlinear hypothesis, many authors have obtained multiplicity of solutions with applications to elliptic systems in annular domains. For homogenous Dirichlet boundary conditions, see [23], [17], [29] and [30]. For non-homogenous Dirichlet boundary conditions, see [27] and [38].

The third part of the tesis (chapter 3) contains our main result: we study existence and nonexistence of radial positive solutions for some nonlinear elliptic equations of the form

$$\begin{aligned} -\text{div}(a(x,u)\nabla u) &= b(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{0.5}$$

where $\Omega = B_R(0)$, is the ball of radius $R > 0$ in $\mathbb{R}^n$, $a(x, u) = \dfrac{|x|^\alpha}{(1 + g(u))^\gamma}$, and $b(x, u) = |x|^\beta u^p$. We assume that g is a nonnegative increasing continuous function, $\alpha, \beta \in \mathbb{R}$, $\gamma \in (0, 1)$, and $p > 1$.

Our main results are:

(i) For $\alpha, \beta \in \mathbb{R}$ such that $N + \alpha - 2 > 0$, $N + \beta > 0$, $\beta - \alpha + 1 > 0$ and $1 < p < (1-\gamma)\frac{N+2\beta-\alpha+2}{N+\alpha-2}$, the problem (3.1) has at least one positive solution.

(ii) For $\alpha, \beta \in \mathbb{R}$ such that $N + \alpha - 2 > 0$, $N + \beta > 0$ and $p \geq \frac{N+2\beta-\alpha+2}{N+\alpha-2}$, the problem (3.1) does not have positive solutions. Furthermore, for $\beta - \alpha + 2 \leq 0$ we show the nonexistence of positive solutions.

As for the one-dimensional problem, we obtain a priori bounds for the truncated problem. Nonetheless, the discussion is more complex and we have to study the *limiting problem*, including Liouville type theorems. Then, by means of fixed-point arguments, we prove the existence of a positive solution for the truncated problem. Finally, we make use of Liouville type theorems to prove that for n large enough , the solution of the truncated problem is a solution of the initial problem.

On the other hand, for $p \geq \frac{N+2\beta-\alpha+2}{N+\alpha-2}$, via an adaptation of the Derrick-Pohozaev inequality (see [31]) we prove the nonexistence of positive solutions of problem (3.1). Finally, assuming existence of (3.1), we use the properties of the solutions to prove nonexistence of positive solutions if $\beta - \alpha + 2 \leq 0$.

We observe that this result extends one of the results introduced by [19] (see Theorem 3.2) to a different class of differential operators. Problems as (3.1) have been widely studied, see e.g. [1], [3], [8], [9], [10], [11], [12], [13], [14]. Note that in [1], [8], [9], [10], [12], [14], the nonlinearity does not depend on u and the existence results hold for solutions in the viscosity sense. In [8], the regularity of solutions is studied for a nonlinearity which is independent of u. In [11], existence and uniqueness of solutions is studied for a differential operator having a different structure from the one introduced here. In [3], [13], the existence and nonexistence of solutions is studied for a nonlinearity with singularities.

Chapter 1

Positive solutions of a nonlinear Sturm–Liouville boundary value problem

In this chapter we establish the existence of positive solutions of the Sturm–Liouville problem

$$\begin{aligned} -(p(s,u)\,u')' &= \hat{q}(s)\,u^p\,h(s,u,u') \quad \text{in } (0,1)\,, \\ u(0) &= 0 = u(1) \end{aligned}$$

where $p\,(s,u) = 1/(a(s) + c\,g(u))$. We assume g and $\hat{q}$ non–negative, continuous functions, $a(s)$ a positive continuous function, $c \geq 0$, $p > 1$, and the function h sub–quadratic with respect to u'. We combine a priori estimates with a fixed–point result of Krasnosel'skii to obtain the existence of a positive solution.

1.1 Introduction

We consider the second–order Sturm–Liouville problem

$$\begin{aligned} -\left(\frac{u'}{a(s) + c\,g(u)}\right)' &= \hat{q}(s)\,u^p\,h(s,u,u') \quad \text{in } (0,1)\,, \\ u(0) &= 0 = u(1) \end{aligned} \tag{1.1}$$

where $p > 1\,,$ $c \geq 0,$ $a : [0,1] \to]0,+\infty[$ is a continuous function, and
$\hat{q} : [0,1] \to [0,+\infty[$ is a non–trivial, continuous function. We will assume that the function

g is continuous and increasing, and is such that $g(0) \geq 0$ and that

$$\lim_{u \to +\infty} \frac{g(u)}{u^{\frac{p-1}{p+1}}} = 0\,. \tag{1.2}$$

In addition, we will assume that the nonlinearity h is continuous, as well as bounded from below, in other words

$$c_h \leq h(t,u,\psi)\,, \quad \text{for all } (t,u,\psi) \in [0,1] \times [0,\infty[\times\mathbb{R} \tag{1.3}$$

where c_h is a positive constant. We will further assume the following quadratic growth condition on the function h with respect to the derivative: Given a compact set K in $[0,1]\times\mathbb{R}$, there exist positive constants A and B such that, for all $(s,u,\psi) \in K \times \mathbb{R}$, we have

$$h(s,u,\psi) \leq A + B\,\psi^2. \tag{1.4}$$

By *solutions* will be meant classical solutions, that is, $u \in C^1(0,1) \cap C^0[0,1]$ and $\dfrac{u'}{a(s) + c\,g(u)} \in C^1(0,1)$ satisfying equation (1.1).

Our main result can be stated as follows:

Theorem 1.1. *The Problem* (1.1) *has at least one positive solution.*

Certain difficulties which we may encounter while proving our main result are that the coefficient $p(s,u) = 1/(a(s) + c\,g(u))$ is nonlinear and that it may not necessarily be bounded from below by a positive bound which is independent of u. In order to overcome these difficulties, we introduce a truncation $g_n(u)$ of the function $g(u)$ so that the new coefficient $p_n(s,u) = 1/(a(s) + c\,g_n(u))$ becomes bounded from below by a uniformly positive constant. (See (1.5).) This allows us to use a fixed–point argument for the truncated problem. Finally, we show the main result proving that, for n sufficiently large, the solutions of the truncated problem are solutions of Problem (1.1). A further difficulty in this argument is the dependence of the function h on the derivative, which leads us to the problem of establishing a priori bounds for the derivative. Observe that this dependence gives the problem a non–variational structure.

For further discussion on problems modeled by equations of the type $-(q(s,u)u')' = f(s,u,u')$, see for example [2], [16], [28], [32], [39], [47], [48], [49], [50], [51]. For a study of existence of solutions when the coefficient $q(s,u)$ is constant on the variable u, see for example [2], [28], [39], [47], [48], [50]. For Sturm–Liouville problems where the coefficient $q(s,u)$ depends explicitly on the variable u, see for example [16], [32], [49], [51]. Note that the problems studied in the preceding papers do not have the same structure as ours. For example, the phenomena of [32], [49], [51] are modeled by a negative nonlinearity $f(s,u,u')$, while in [16] the nonlinearity is bounded with respect to the variables u and u'. Observe that, in this work, $f(s,u,u')$ is both non–negative and unbounded with respect to the variables u, u'. As a model example, consider the equation

$$\begin{aligned} -(\frac{u'}{a(s)\,+\,c\,u^q})' &= \hat{q}(s)\,u^p\,(c_0+c_1|u'|^{\,\theta}) \quad \text{in } (0,1)\,, \\ u(0) &= 0 \,=\, u(1) \end{aligned}$$

where $1<p$, $0<q<\dfrac{p-1}{p+1}$, $0\le\theta\le 2$, $c\ge 0$, $c_0>0$ and $c_1\ge 0$.

Our study is organized as follows. In Section 2, we show the existence of positive solutions of the truncation problem. In Section 3, we show that, for n sufficiently large, the solutions of the truncation problem are solutions of Problem (1.1), which proves our main result: Theorem 1.1.

1.2 The truncation problem

Given $n\in\mathbb{N}$, we consider the function

$$T_n(s)=\max\{-n,min\{n,s\}\}\,,$$

and we define the truncation $g_n(u)$ of the function $g(u)$ defined by

$$g_n(u)=(g\circ T_n)(u)\,. \tag{1.5}$$

Consider the truncation problem

$$\begin{aligned} -\left(\frac{u'}{a(s)\,+\,c\,g_n(u)}\right)' = \hat{q}(s)\,u^p\,h(s,u,u') \quad \text{in } (0,1)\,, \qquad (1.6)_n \\ u(0)=0\,=\,u(1)\,. \end{aligned}$$

The following is an existence result for the truncation problem.

Theorem 1.2. *Suppose hypotheses* (1.3) *and* (1.4). *Then Problem* $(1.6)_n$ *has at least one positive solution.*

The proof of Theorem 1.2 is based on the well known fixed–point result due to Krasnosel'skii (see Theorem 4.2).

In order to apply the Krasnosesl'skii result, we need to establish a priori bounds for the solutions of a family of problems parameterized by $\lambda \geq 0$.

In fact, consider the family

$$-\left(\frac{u'}{a(s) \,+\, c\, g_n(u)}\right)' = \hat{q}(s)\, u^p\, h(s,u,u') \,+\, \lambda \quad \text{in } (0,1)\,, \qquad (1.7)_n$$
$$u(0) = 0 \,=\, u(1)\,.$$

We now present three lemmas which lead to the proof of Theorem 1.2. We begin with a result concerned with a priori bounds for the positive solutions of Problem $(1.7)_n$.

Lemma 1.1. *Suppose hypothesis* (1.3). *Then there exists a positive constant* B_1*, which does not depend on* λ*, such that for every positive solution* u *of Problem* $(1.7)_n$*, we have*

$$||u||_\infty \leq B_1\,. \qquad (1.8)$$

Proof. It is not difficult to show that every positive solution u of Problem $(1.7)_n$ satisfies

$$u(t) \;=\; \int_0^1 K_n(t,s)(\hat{q}(s)\, u^p\, h(s,u,u') \,+\, \lambda)\; ds \qquad (1.9)$$

where $K_n(t,s)$ is the associated Green's function

$$K_n(t,s) \;=\; \begin{cases} \frac{1}{\rho}\int_0^t (a(\tau)+c\, g_n(u(\tau))\int_s^1 (a(\tau)+c\, g_n(u(\tau)) & \text{if} \quad 0 \leq t \leq s \leq 1, \\ \frac{1}{\rho}\int_0^s (a(\tau)+c\, g_n(u(\tau))\int_t^1 (a(\tau)+c\, g_n(u(\tau)) & \text{if} \quad 0 \leq s \leq t \leq 1\,. \end{cases} \qquad (1.10)$$

Here ρ is given by $\rho = \int_0^1 (a(\tau)+c\, g_n(u(\tau))$. Simple computations show that every solution u satisfies

$$u(s) \;\geq\; q(s)\, ||u||_\infty\,, \quad \text{for all } \; s \in [0,1] \qquad (1.11)$$

where $q(s) = \dfrac{1}{\rho}\,\min\left\{ \displaystyle\int_0^s (a(\tau)+c\, g_n(u(\tau)))\,,\; \int_s^1 (a(\tau)+c\, g_n(u(\tau)))\right\}$.

Hence

$$q(s) \;\geq\; \frac{\min a}{||a||_\infty \,+\, c\, g(n)}\, s\,(1-s)\,, \qquad \text{for all } \; s \in [0,1] \qquad (1.12)$$

and

$$K_n(t,s) \;\geq\; \begin{cases} \dfrac{(\min a)^2}{||a||_\infty \,+\, c\,g(n)}\; t\,(1-s) & \text{if} \quad 0 \leq t \leq s \leq 1, \\ \dfrac{(\min a)^2}{||a||_\infty \,+\, c\,g(n)}\; s\,(1-t) & \text{if} \quad 0 \leq s \leq t \leq 1. \end{cases} \tag{1.13}$$

Therefore, for all $t \in [0,1]$, every solution u of Problem $(1.7)_n$ satisfies

$$u(t) \;\geq\; \frac{c_h\,(\min a)^{p+2}}{(||a||_\infty \,+\, c\,g(n))^{p+1}}\; ||u||_\infty^p \int_\alpha^\beta G(t,s)\,\hat{q}(s)\,s^p(1-s)^p\,ds \tag{1.14}$$

where

$$G(t,s) = \begin{cases} t\,(1-s) & \text{if} \quad 0 \leq t \leq s \leq 1, \\ s\,(1-t) & \text{if} \quad 0 \leq s \leq t \leq 1. \end{cases} \tag{1.15}$$

The existence of a priori bounds B_1 for the solutions u now follows. □

The following shows the existence of a priori bounds for the derivatives of the solutions.

Lemma 1.2. *Suppose hypotheses* (1.3) *and* (1.4). *Then, for all* $\bar{\lambda}$ *positive, there exists a constant* B_2 *such that, for* $\lambda \in [0,\bar{\lambda}]$, *every solution of Problem* $(1.7)_n$ *satisfies*

$$||u'||_\infty \;\leq\; B_2\,. \tag{1.16}$$

Proof. By Lemma 1.1 and hypothesis (1.4) we know that there exist positive constants A and B so that, for all $s \in [0,1]$, every solution u of Problem $(1.7)_n$ satisfies

$$h(s,u(s),u'(s)) \;\leq\; A + B\,u'(s)^2. \tag{1.17}$$

Therefore,

$$h(s,u(s),u'(s)) \;\leq\; A + C_n \left(\frac{u'(s)}{a(s) \,+\, c\,g_n(u)} \right)^2 \tag{1.18}$$

where $C_n = B\,(||a||_\infty \,+\, c\,g(n))^2$.

Let u be a positive solution of Problem $(1.7)_n$. Note that

$$\frac{d}{ds}\,\ln(\lambda +\, B_1^p\,||\hat{q}||_\infty\,A +\, B_1^p\,||\hat{q}||_\infty\,C_n\,(\frac{u'(s)}{a(s)\,+\,c\,g_n(u)})^2)$$
$$= \frac{-2\,B_1^p\,||\hat{q}||_\infty\,C_n\,\frac{u'(s)}{a(s)+c\,g_n(u)}\,(\hat{q}(s)\,u^p\,h(s,u,u')\,+\,\lambda)}{\lambda +\, B_1^p\,||\hat{q}||_\infty\,A +\, B_1^p\,||\hat{q}||_\infty\,C_n\,(\frac{u'(s)}{a(s)+c\,g_n(u)})^2}\,.$$

According to inequality (1.18), if $u'(s) < 0$, then

$$\frac{d}{ds} \ln \left(\lambda + B_1^p \, ||\hat{q}||_\infty \, A + B_1^p \, ||\hat{q}||_\infty \, C_n \, \left(\frac{u'(s)}{a(s) \, + \, c \, g_n(u)} \right)^2 \right) \leq -2 \, \frac{C_n \, B_1^p \, ||\hat{q}||_\infty}{\min a} \, u'(s) \, , \quad (1.19)$$

and if $u'(s) > 0$, then

$$\frac{d}{ds} \ln \left(\lambda + B_1^p \, ||\hat{q}||_\infty \, A + B_1^p \, ||\hat{q}||_\infty \, C_n \, \left(\frac{u'(s)}{a(s) \, + \, c \, g_n(u)} \right)^2 \right) \geq -2 \, \frac{C_n \, B_1^p \, ||\hat{q}||_\infty}{\min a} \, u'(s) \, . \quad (1.20)$$

On the other hand, observe that there exists an $s_0 \in]0,1[$ so that $u(s_0) = ||u||_\infty$, and so that u is increasing on $[0, s_0]$, while decreasing on $[s_0, 1]$. Integration on the intervals $[s, s_0]$ and $[s_0, s]$ yields

$$\begin{aligned} &\ln(\lambda + B_1^p \, ||\hat{q}||_\infty \, A + B_1^p \, ||\hat{q}||_\infty \, C_n \, (\frac{u'(s)}{a(s) \, + \, c \, g_n(u)})^2) \\ &\qquad \leq 2 \, \frac{C_n \, B_1^p \, ||\hat{q}||_\infty}{\min a} \, (||u||_\infty - u(s)) \, + \ln(\bar{\lambda} + B_1^p \, ||\hat{q}||_\infty \, A) \, . \end{aligned} \quad (1.21)$$

Therefore,

$$\begin{aligned} &\ln(\lambda + B_1^p \, ||\hat{q}||_\infty \, A + B_1^p \, ||\hat{q}||_\infty \, B \, u'(s)^2) \\ &\qquad \leq 2 \, \frac{C_n \, B_1^{p+1} \, ||\hat{q}||_\infty}{\min a} \, + \ln(\bar{\lambda} + B_1^p \, ||\hat{q}||_\infty \, A) \, . \end{aligned} \quad (1.22)$$

The Lemma clearly results from this inequality. □

We need the following step. Consider the Banach space

$$X = \mathcal{C}^1([0,1], \mathbb{R})$$

endowed with the norm $||u||_1 = ||u||_\infty + ||u'||_\infty$, where $||u||_\infty = \max_{t \in [0,1]} |u(t)|$.
Define the cone C by

$$C = \{ u \in X \ : \ u \geq 0 \ \text{and} \ u(0) = u(1) = 0 \}$$

and the operator $\mathcal{F}_\lambda : X \to X$ by

$$\mathcal{F}_\lambda(u)(s) \quad = \quad \int_0^1 K_n(s, \tau) \, (\hat{q}(\tau) \, u(\tau)^p \, h(\tau, u(\tau), u'(\tau)) \, + \, \lambda) \, d\tau.$$

Lemma 1.3. *The operator $\mathcal{F}_\lambda : X \to X$ is compact, and the cone C is invariant under $\mathcal{F}_\lambda$.*

Proof Outline. The compactness of $\mathcal{F}_\lambda$ follows from the well known Arzelá–Ascoli Theorem. The invariance of the cone C is a consequence of the fact that the nonlinearities are non–negative.

***Proof of Theorem* 1.2.–** To prove Theorem 1.2, it suffices to show that $\mathcal{F}_0$ has a fixed point. For this, we will check that the four conditions of Theorem 4.2 are satisfied. Consider the homotopy $H : [0,1] \times C \longrightarrow C$ given by

$$H(t,u)(s) \,=\, \mathcal{F}_{\lambda\, t}(u)(s)\,.$$

Note that $H(t,u)$ is a compact homotopy, and since $H(0,u) = \mathcal{F}_0(u)$, we have that condition (b) is satisfied.

Concerning condition (a), by continuity, there exists a $M > 0$ such that, if $||u||_1 \le 1$, then

$$|K_n(s,\tau)h(\tau,u(\tau),u'(\tau))| \le M\,. \tag{1.23}$$

Now if $||u||_1 = \delta,$ with $\delta > 0$, then

$$||\mathcal{F}_0(u)||_1 = ||\mathcal{F}_0(u)||_\infty + ||\mathcal{F}_0(u)'||_\infty \le C\,\delta^{p-1}||u||_1$$

where C is a positive constant. Taking δ sufficiently small, we have

$$||\mathcal{F}_0(u)||_1 < ||u||_1\,. \tag{1.24}$$

From Lemmas 1.1 and 1.2 we conclude that there exists a η sufficiently large so that Condition (c) is satisfied.

Checking that Condition (d) is satisfied is based on the following subsidiary lemma.

Lemma 1.4. *The Problem* $(1.7)_n$ *has no solutions for* λ *large.*

Proof. Let u be a solution of Problem $(1.7)_n$, or in other words

$$u(t) = \int_0^1 K_n(s,t)\,(\hat{q}(s)\,u(s)^p\,h(s,u(s),u'(s)) \,+\, \lambda)\,ds.$$

Then

$$||u||_\infty \ge \lambda \int_0^1 K_n(s,\frac{1}{2})\,ds.$$

By Lemma 1.1 we know that $||u||_\infty \le B_1$, and hence

$$\lambda \le \frac{B_1}{\int_0^1 K_n(s,\frac{1}{2})\,ds}.$$

Therefore, for

$$\lambda > \frac{B_1}{\int_0^1 K_n(s, \frac{1}{2})\, ds}$$

there are no solutions of Problem $(1.7)_n$. □

So choosing λ sufficiently large in the homotopy $H(t,u)$, we see that Condition (d) is satisfied by Lemma 1.4.

Thus all of Krasnosel'skii's conditions are satisfied.

■

1.3 Proof of Theorem 1.1

The proof of Theorem 1.1 is direct consequence of the following.

Lemma 1.5. *There exists an* $n_0 \in \mathbb{N}$ *such that every solution* u *of Problem* $(1.6)_n$ *satisfies*

$$||u||_\infty < n_0\,. \tag{1.25}$$

Proof. For otherwise, there would exist a sequence of solutions $\{u_n\}_n$ of Problem $((1.6)_n$ such that $||u_n||_\infty \geq n$, for all $n \in \mathbb{N}$. Using the same argument as in Lemma 1.1, we would obtain the estimate

$$\begin{aligned} 1 &\geq \frac{(\min a)^2}{||a||_\infty + c\,g(n)} \left(\frac{\min a}{||a||_\infty + c\,g(n)}\right)^p c_h n^{p-1} \max_{t\in[0,1]} \int_0^1 G(t,s)\,\hat{q}(s)\,s^p(1-s)^p\,ds \\ &\geq (\min a)^{p+2} c_h \left(\frac{n^{\frac{p-1}{p+1}}}{||a||_\infty + c\,g(n)}\right)^{p+1} \max_{t\in[0,1]} \int_0^1 G(t,s)\,\hat{q}(s)\,s^p(1-s)^p\,ds\,. \end{aligned}$$

But this is impossible, since $\lim_{n\to+\infty} \frac{n^{\frac{p-1}{p+1}}}{||a||_\infty + c\,g(n)} = +\infty$ by hypothesis (1.2).

□

Chapter 2

Nonlinear Systems of Second–Order ODE'S

In this chapter we study existence of positive solutions of the system

$$\begin{aligned} -(p_1(t,u,v)u')' &= h_1(t)f_1(t,u,v) \ \text{ in } \ (0,1)\,, \\ -(p_2(t,u,v)v')' &= h_2(t)f_2(t,u,v) \ \text{ in } \ (0,1)\,, \\ u(0)=u(1) &= v(0)=v(1)=0 \end{aligned}$$

where $p_1(t,u,v) = 1/(a_1(t) + c_1\, g_1(u,v))$ and $p_2(t,u,v) = 1/(a_2(t) + c_2\, g_2(u,v))$. Here is assumed that g_1, g_2 are non-negative continuous functions, $a_1(t)$, $a_2(t)$ are positive continuous functions, $c_1, c_2 \geq 0$, $h_1\,, h_2 \in L^1(0,1)$ and that the nonlinearities f_1, f_2 satisfy super-linear hypotheses at *zero* and $+\infty$. The existence of solutions will be obtained using a combination among the method of truncation, a priori bounded and the Krasnosel'skii well known result on fixed–point indices in cones.

2.1 Introduction

We study existence of positive solutions for the following nonlinear system of second–order ordinary differential equations

$$\begin{aligned} -\Big(\frac{u'}{a_1(s) + c_1\, g_1(u,v)}\Big)' &= h_1(t)f_1(t,u,v) \quad \text{in} \ (0,1)\,, \\ -\Big(\frac{v'}{a_2(s) + c_2\, g_2(u,v)}\Big)' &= h_2(t)f_2(t,u,v) \quad \text{in} \ (0,1)\,, \\ u(0)=u(1)=v(0) &= v(1)=0 \end{aligned} \tag{2.1}$$

where c_1, c_2 are non–negatives constants, the functions $a_1,\, a_2 : [0,1] \longrightarrow]0,+\infty[$ are continuous, the functions $f_1, f_2 : [0,1]\times[0,+\infty[^2 \longrightarrow [0,+\infty[$ are continuous and $h_1\,, h_2 \in L^1(0,1)$. We will suppose the following four hypotheses:

(H_1) We have

$$\lim_{u+v\to 0}\frac{f_1(t,u,v)}{u+v} = 0 \quad \text{and} \quad \lim_{u+v\to 0}\frac{f_2(t,u,v)}{u+v} = 0\,,$$

uniformly for all $t \in [0,1]$.

(H_2) There exist $p\,,\ q > 1$, $\eta_i > 0$ and $0 < \alpha_i < \beta_i < 1$ for $i = 1,2$, such that

$$f_1(t,u,v) \geq \eta_1\, u^p \quad \text{for all} \quad u \geq 0 \quad \text{and} \quad t \in (\alpha_1,\beta_1)$$

and

$$f_2(t,u,v) \geq \eta_2\, v^q \quad \text{for all} \quad v \geq 0 \quad \text{and} \quad t \in (\alpha_2,\beta_2).$$

(H_3) The functions $g_1,\, g_2 : [0,+\infty[^2 \longrightarrow [0,+\infty[$ are continuous and

$$\lim_{u\to+\infty} g_i(u,u) = +\infty\,, \quad \text{for} \quad i = 1,2\,.$$

We suppose that there exists a $n_* \in \mathbb{N}$ such that $g_1,\, g_2$ are non–decreasing for all $u^2 + v^2 \geq n_*^2$. Here $g_1,\, g_2$ non–decreasing, means that

$$g_i(u_1,v_1) \leq g_i(u_2,v_2)\,, \quad \text{for} \quad i = 1,2$$

whenever $(u_1,v_1) \leq (u_2,v_2)$, where the inequality is understood inside every component.

(H_4) We have

$$\lim_{n\to+\infty}\frac{g_1(n,n)}{n^{\frac{r}{p+1}}} = 0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{g_2(n,n)}{n^{\frac{r}{q+1}}} = 0$$

where $r = \min\{p-1, q-1\}$.

Here are some comments on the above hypotheses. Hypothesis (H_1) is a superlinear condition at 0 and Hypothesis (H_2) is a local superlinear assumption at $+\,\infty$. About hypothesis (H_3) the fact that g_1, g_2 are unbounded leads us to use the strategy of considering a truncation problem. Note that if g_1, g_2 are bounded we would not need to use that problem. Hypothesis (H_4) allows us to have a control on the nonlinear operator in Problem 2.1.

We remark that, the case when $a_1(s) = a_2(s) = 1$ and $g_1(u,v) = g_2(u,v) = 0$, problems of type (2.1) have been extensively studied in the literature under different sets of conditions on the nonlinearities. For instance, assuming superlinear hypothesis, many authors have obtained multiplicity of solutions with applications to elliptic systems in annular domains. For homogeneous Dirichlet boundary conditions, see de Figueiredo–Ubilla [23], Conti–Merizzi–Terracini [17], Dunninger–Wang [29] and [30]. For non–homogeneous Dirichlet boundary conditions, see Lee [38] and do Ó–Lorca–Ubilla [27].

Our main goal is to study problems of type (2.1) by considering local superlinear assumptions at $+\infty$ and global superlinear at zero.

The main result is the following

Theorem 2.1. *Assume hypotheses* (H_1) *through* (H_4). *Then Problem (2.1) has at least one positive solution.*

Our study is organized as follows. In Section 2, we show the a priori bounds for a truncation problem. In Section 3, we show that the a priori bounds imply a nonexistence result for problem $(2.4)_n$. In Section 4, we enunciated to the well-known result of Krasnosel'skii of fixed point in cones. In Section 5, we show the existence of positive solutions of the truncation problem. In Section 6, we prove the existence of solutions of the system of ordinary differential equations (2.1) .

2.2 A priori bounds for a truncation problem

In this section we establish a priori bounds for a truncation problem. The hypothesis (H_3) allows us to find a $n_{**} \in \mathbb{N}$ so that $n \geq n_{**}$ implies

$$g_1(u,v) \leq g_1(n,n) \quad \text{and} \quad g_2(u,v) \leq g_2(n,n),$$

for all $u^2 + v^2 \leq n^2$.

Now for every $n \in \mathbb{N}$ such that $n \geq n_{**}$ we define

$$g_{i,n}(u,v) = \begin{cases} g_i(u,v) & \text{if} \quad u^2+v^2 \leq n^2\,, \\ g_i(\frac{nu}{\sqrt{u^2+v^2}}, \frac{nv}{\sqrt{u^2+v^2}}) & \text{if} \quad u^2+v^2 \geq n^2\,. \end{cases} \tag{2.2}$$

for $i = 1, 2$.

Our goal is to prove the existence of a positive solution for the following truncation problem

$$
\begin{aligned}
-\left(\frac{u'}{a_1(s)\,+\,c_1\,g_{1,n}(u,v)}\right)' &= h_1(t)f_1(t,u,v) \quad \text{in } (0,1)\,,\\
-\left(\frac{v'}{a_2(s)\,+\,c_2\,g_{2,n}(u,v)}\right)' &= h_2(t)f_2(t,u,v) \quad \text{in } (0,1)\,,\\
u(0)=u(1)=v(0) &= v(1)=0\,.
\end{aligned}
\tag{2.3$_n$}
$$

For our purpose we need to establish a priori bounds for solutions of a family of problems parameterized by $\lambda \geq 0$. In fact, consider the family

$$
\begin{aligned}
-\left(\frac{u'}{a_1(s)\,+\,c_1\,g_{1,n}(u,v)}\right)' &= h_1(t)f_1(t,u,v)+\lambda \quad \text{in } (0,1)\,,\\
-\left(\frac{v'}{a_2(s)\,+\,c_2\,g_{2,n}(u,v)}\right)' &= h_2(t)f_2(t,u,v)+\lambda \quad \text{in } (0,1)\,,\\
u(0)=u(1)=v(0) &= v(1)=0\,.
\end{aligned}
\tag{2.4$_n$}
$$

It is not difficult to prove that every solution of Problem $(2.4)_n$ satisfies

$$
\begin{aligned}
u(t) &= \int_0^1 K_{1,n}(t,s)(h_1(s)f_1(s,u(s),v(s))\,+\,\lambda)\;ds\\
v(t) &= \int_0^1 K_{2,n}(t,s)(h_2(s)f_2(s,u(s),v(s))\,+\,\lambda)\;ds\,.
\end{aligned}
\tag{2.5}
$$

Here $K_{i,n}(t,s)$, $i=1,2$ are the de Green's functions given by

$$
K_{i,n}(t,s) \;=\; \left\{\begin{array}{lll} \frac{1}{\rho_i}\int_0^t(a_i(\tau)+c_i\,g_{i,n}(u(\tau))\int_s^1(a_i(\tau)+c_i\,g_{i,n}(u(\tau)) & \text{if} & 0\leq t\leq s\leq 1,\\ \frac{1}{\rho_i}\int_0^s(a_i(\tau)+c_i\,g_{i,n}(u(\tau))\int_t^1(a_i(\tau)+c_i\,g_{i,n}(u(\tau)) & \text{if} & 0\leq s\leq t\leq 1\,.\end{array}\right.
\tag{2.6}
$$

where ρ_i denotes $\rho_i=\int_0^1(a_i(\tau)+c_i\,g_{i,n}(u(\tau))$.

In order to establish the a priori bound result we need the following two lemmata

Lemma 2.1. *Assume hypotheses (H_1) and (H_3). Then every solution of Problem $(2.4)_n$ satisfies*

$$
u(t)\geq q_1(t)||u||_\infty \quad \textit{and} \quad v(t)\geq q_2(t)||v||_\infty\,, \quad \textit{for all}\;\; s\in[0,1]
\tag{2.7}
$$

where $q_i(t)=\frac{(\min a_i)t(1-t)}{||a_i||_\infty+c_ig_i(n,n)}$ *with* $i=1,2$.

Proof. A simple computation shows that every solution (u,v) satisfies

$$u(s) \geq \hat{q}_1(s,u,v)\,||u||_\infty \quad \text{and} \quad v(s) \geq \hat{q}_2(s,u,v)\,||v||_\infty\,, \quad \text{for all } s\in[0,1] \tag{2.8}$$

where $\hat{q}_i(s,u,v) = \dfrac{1}{\rho_i}\,\min\{\int_0^s (a_i(\tau)+c_i\,g_{i,n}(u(\tau),\,v(\tau)))\,,\,\int_s^1 (a_i(\tau)+c_i\,g_{i,n}(u(\tau),\,v(\tau)))\}\,.$

Since

$$\hat{q}_i(s,u,v) \geq \frac{(\min a_i)s(1-s)}{||a_i||_\infty + c_i g_i(n,n)} \quad \text{for} \quad i=1,2 \tag{2.9}$$

we have that (2.7) is proved. □

Lemma 2.2. *Assume hypothesis (H_3). Then the Green's functions satisfy*

$$K_{i,n}(t,s) \geq \frac{(\min a_i)^2}{||a_i||_\infty + c_i g_i(n,n)} G(t,s)\,, \qquad i=1,2\,.$$

where

$$G(t,s) = \begin{cases} (1-t)s, & 0\leq s<t\leq 1 \\ (1-s)t, & 0\leq t\leq s\leq 1. \end{cases}$$

Proof. The inequality follows from (2.6) and using that $g_{i,n}$ is uniformly bounded by $g_i(n,n)$ for $i=1,2$. □

Theorem 2.2. *Assume hypothesis $(H_1)-(H_3)$. Then there is a positive constant B_1 which does not depend on λ, such that, for every solution (u,v) of Problem $(2.4)_n$, we have*

$$||(u,v)||_1 \leq B_1\,. \tag{2.10}$$

Proof. By Lemma 2.1 and 2.2 every solution (u,v) of Problem $(2.4)_n$ satisfies

$$\begin{aligned} ||(u,v)||_1 \quad &\geq \quad \frac{(\min a_1)^2\eta_1}{||a_1||_\infty + c_1 g_1(n,n)} \int_{\alpha_1}^{\beta_1} h_1(s)u^p(s)\,ds + \frac{(\min a_2)^2\eta_2}{||a_2||_\infty + c_2 g_2(n,n)} \int_{\alpha_2}^{\beta_2} h_2(s)v^q(s)\,ds \\ &\geq \quad \hat{c}\,(||u||_\infty^p + ||v||_\infty^q)\,, \end{aligned}$$

where $\hat{c} = \min\left\{\frac{(\min a_1)^{p+2}\alpha_1^p(1-\beta_1)^p\eta_1}{(||a_1||_\infty+c_1g_1(n,n))^{p+1}}\int_{\alpha_1}^{\beta_1} h_1(s)\,ds,\, \frac{(\min a_2)^{q+2}\alpha_2^q(1-\beta_2)^q\eta_2}{(||a_2||_\infty+c_2g_2(n,n))^{q+1}}\int_{\alpha_2}^{\beta_2} h_2(s)\,ds\right\}.$

Thus

$$1 \geq \hat{c}\frac{||u||_\infty^p + ||v||_\infty^q}{||u||_\infty + ||v||_\infty} \tag{2.11}$$

which proves (2.10). □

2.3 A Nonexistence Result

In this section we see that the a priori bounds imply a nonexistence result for Problem $(2.4)_n$.

Theorem 2.3. *Problem* $(2.4)_n$ *has no solution for all* λ *sufficiently large.*

Proof. Let (u,v) be a solution of Problem $(2.4)_n$, in other words,

$$\begin{aligned} u(t) &= \int_0^1 K_{1,n}(t,s)(h_1(s)f_1(s,u(s),v(s)) + \lambda)\, ds \\ v(t) &= \int_0^1 K_{2,n}(t,s)(h_2(s)f_2(s,u(s),v(s)) + \lambda)\, ds\,. \end{aligned} \tag{2.12}$$

Then,

$$||(u,v)||_1 \geq \lambda \left(\int_0^1 K_{1,n}(s,\frac{1}{2})\, ds + \int_0^1 K_{2,n}(s,\frac{1}{2})\, ds \right).$$

By Theorem 2.2 we know that $||(u,v)||_1 \leq B_1$, thus

$$\lambda \leq \frac{B_1}{\int_0^1 K_{1,n}(s,\frac{1}{2})\, ds + \int_0^1 K_{2,n}(s,\frac{1}{2})\, ds},$$

which proves Theorem 2.3. □

2.4 Fixed Point Operators

Consider the following Banach space

$$X = \mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R})$$

endowed with the norm $||(u,v)|| = ||u||_\infty + ||v||_\infty$, where $||u||_\infty = \max_{t\in[0,1]} |u(t)|$.

Define the cone C by

$$C = \{(u,v) \in X \ : \ (u,v)(0) = (u,v)(1) = 0\,, \text{ y } (u,v) \geq 0\}.$$

Define also the operator $\mathcal{F}_\lambda : X \longrightarrow X$ by

$$\mathcal{F}_\lambda(u,v)(s) = (\mathcal{A}_\lambda(u,v)(s), \mathcal{B}_\lambda(u,v)(s)), \quad \text{for } s \in [0,1]$$

where

$$\begin{array}{rcl}\mathcal{A}_\lambda(u,v)(s) & = & \int_0^1 K_{1,n}(s,\tau)\,(h_1(\tau)f_1(\tau,u(\tau),v(\tau))+\lambda)\,d\tau \quad \text{and} \\ \mathcal{B}_\lambda(u,v)(s) & = & \int_0^1 K_{2,n}(s,\tau)\,(h_2(\tau)f_2(\tau,u(\tau),v(\tau))+\lambda)\,d\tau.\end{array}$$

Note that a simple calculation shows us that the fixed points of the operator $\mathcal{F}_\lambda$ are the positive solutions of System $(2.4)_n$.

Lemma 2.3. *The operator* $\mathcal{F}_\lambda : X \to X$ *is compact, and the cone* C *is invariant under* $\mathcal{F}_\lambda$*.*

Proof Outline. The compactness of $\mathcal{F}_\lambda$ follows from the well known Arzelá–Ascoli Theorem. The invariance of the cone C is a consequence of the fact that the nonlinearities are non–negative.

In section 5 we will give an existence result of the truncation Problem $(2.3)_n$. The proof will be based on the well known fixed–point result due to Krasnosel'skii (see Theorem 4.2).

2.5 Existence result of Truncation Problem $(2.3)_n$

The following is an existence result of the truncation problem.

Theorem 2.4. *Assume hipotheses* (H_1) *through* (H_3)*. Then there exists a positive solution of Problem* $(2.3)_n$*.*

Proof. We will verify the hypotheses of Theorem 4.2. Define the homotopy $\mathcal{H} : [0,1]\times C^2 \longrightarrow C^2$ by

$$\mathcal{H}(t,(u,v))(s) = (\mathcal{A}_\lambda(t,u,v)(s), \mathcal{B}_\lambda(t,u,v)(s)), \quad \text{for } s,t \in [0,1]$$

where λ is a sufficiently large parameter, and where

$$\begin{array}{rcl}\mathcal{A}_\lambda(t,u,v)(s) & = & \int_0^1 K_{1,n}(s,\tau)\,(h_1(\tau)f_1(\tau,u(\tau),v(\tau))+t\lambda)\,d\tau \quad \text{and} \\ \mathcal{B}_\lambda(t,u,v)(s) & = & \int_0^1 K_{2,n}(s,\tau)\,(h_2(\tau)f_2(\tau,u(\tau),v(\tau))+t\lambda)\,d\tau.\end{array}$$

Note that $\mathcal{H}(t,u,v)$ is a compact homotopy and that $\mathcal{H}(0,u,v) = \mathcal{F}_0(u,v)$, which verifies (b).

On the other hand, we have

$$\begin{array}{rcl}||\mathcal{F}_0(u,v)||_1 & \le & (||a_1||_\infty + c_1 g_1(n,n)) \displaystyle\int_0^1 h_1(\tau)\frac{f_1(\tau,u(\tau),v(\tau))}{u(\tau)+v(\tau)}\,d\tau ||(u,v)||_1 \\ & + & (||a_2||_\infty + c_2 g_2(n,n)) \displaystyle\int_0^1 h_2(\tau)\frac{f_2(\tau,u(\tau),v(\tau))}{u(\tau)+v(\tau)}\,d\tau ||(u,v)||_1 \,.\end{array}$$

Taking $||(u,v)||_1=\delta$ with $\delta>0$ sufficiently small, from hypothesis we have

$$||\mathcal{F}_0(u,v)||_1<||(u,v)||_1, \tag{2.13}$$

which verifies (a) of Theorem 4.2. By Theorem 2.2 we clearly have (c).
Finally, choosing λ sufficiently large in the homotopy $\mathcal{H}(t,u)$, we see that Condition (d) of Theorem 4.2 is satisfied by Theorem 2.3. The proof of Theorem 2.4 is now complete.

□

2.6 Proof of main result: Theorem 2.1

The proof of Theorem 2.1 is direct consequence of the following.

Theorem 2.5. *There exists an $n_0\in\mathbb{N}$ such that every solution (u,v) of Problem $(2.4)_n$ which $n>n_{**}$ satisfies*

$$||(u,v)||_1<n_0^2\,. \tag{2.14}$$

Proof. For otherwise, there would exist a sequence of solutions $\{(u_n,v_n)\}_n$ of Problem $(2.4)_n$ such that $||(u_n,v_n)||_1\geq n^2$, for all $n\in\mathbb{N}$ which $n>n_{**}$. Using the same argument as in Theorem 2.2, we would obtain the estimate

$$1\geq\min\left\{\frac{(\min a_1)^{p+2}\alpha_1^p(1-\beta_1)^p\eta_1}{(||a_1||_\infty+c_1g_1(n,n))^{p+1}}\int_{\alpha_1}^{\beta_1}h_1(s)\,ds,\frac{(\min a_2)^{q+2}\alpha_2^q(1-\beta_2)^q\eta_2}{(||a_2||_\infty+c_2g_2(n,n))^{q+1}}\int_{\alpha_2}^{\beta_2}h_2(s)\,ds\right\}\frac{||u||_\infty^p+||v||_\infty^q}{||u||_\infty+||v||_\infty} \tag{2.15}$$

We have $||u_n||_\infty=\sqrt{||u_n||_\infty^2+||v_n||_\infty^2}\sin\theta_n$ and $||v_n||_\infty=\sqrt{||u_n||_\infty^2+||v_n||_\infty^2}\cos\theta_n$ with $\theta_n\in[0,\frac{1}{2}]$. Moreover, there exists a constant $c>0$ such that $\sin^p\theta_n+\cos^q\theta_n>c$. Then

$$\frac{1}{n^{\min\{p-1,q-1\}}}\geq\min\left\{\frac{(\min a_1)^{p+2}\alpha^p(1-\beta)^p\eta_1c}{(||a_1||_\infty+c_1g_1(n,n))^{p+1}}\int_{\alpha_1}^{\beta_1}h_1(s)\,ds,\frac{(\min a_2)^{q+2}\alpha^q(1-\beta)^q\eta_2c}{(||a_2||_\infty+c_2g_2(n,n))^{q+1}}\int_{\alpha_2}^{\beta_2}h_2(s)\,ds\right\}, \tag{2.16}$$

which is impossible, since $\lim_{n\to+\infty}\frac{n^{\frac{r}{p+1}}}{||a_1||_\infty+c_1\,g_1(n,n)}=+\infty$ and $\lim_{n\to+\infty}\frac{n^{\frac{r}{q+1}}}{||a_2||_\infty+c_2\,g_2(n,n)}=+\infty$ by hypothesis (H_4).

□

2.7 Remarks

(*i*) We note that the solutions of nonlinear Problem (2.1) are C^1 class in $[0,1]$ and C^2 class almost every where, in $(0,1)$. Note also that when $h_1(t)$, $h_2(t)$ are continuous functions the solutions of Problem (2.1) are classic.

(*ii*) A little modification of our argument may be done to obtain an existence result of the following more general system

$$\begin{aligned} -\Big(\frac{u'}{a_1(s)\,+\,c_1\,g_1(u,v)}\Big)' &= k_1(t,u,v) \quad \text{in} \;\; (0,1)\,, \\ -\Big(\frac{v'}{a_2(s)\,+\,c_2\,g_2(u,v)}\Big)' &= k_2(t,u,v) \quad \text{in} \;\; (0,1)\,, \\ u(0)=u(1)=v(0) &= v(1)=0 \end{aligned} \tag{2.17}$$

where k_1, k_2 satisfy (H_2) and the property that there exist continuous functions $\hat{f}_1, \hat{f}_2 : [0,1]\times[0,+\infty)^2 \longrightarrow [0,+\infty)$ satisfying (H_1) and (H_2), and that there exist $h_1\,, h_2 \in L^1(0,1)$, such that

$$k_1(t,u,v) \le h_1(t)\hat{f}_1(t,u,v) \quad \text{and} \quad k_2(t,u,v) \le h_2(t)\hat{f}_2(t,u,v)\,,$$

for all $t\in[0,1]$.

Chapter 3

Existence and Nonexistence of positive radial solution of a nonlinear boundary value problem

In this chapter we study the existence and nonexistence of radial positive solution for some nonlinear elliptic equations of the form

$$\begin{aligned} -\mathrm{div}(a(x,u)\nabla u) &= b(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{3.1}$$

where $\Omega = B_R(0)$, $R > 0$, is the ball of radius R in $\mathbb{R}^n$ and the functions $a(x,u) = \dfrac{|x|^\alpha}{(1+g(u))^\gamma}$, $b(x,u) = |x|^\beta u^p$, we assume g nonnegative increasing continuous function, the constants $\alpha, \beta \in \mathbb{R}$, $\gamma \in (0,1)$, $p > 1$. We combine blow-up techniques and a priori estimates with a fixed-point result of Krasnosel'skii.

3.1 Introduction

We study existence and nonexistence of positive radial solutions for the Problem (3.1), that is, we study the existence and nonexistence of positive solutions for the following problem:

$$\begin{aligned} -\left(\frac{r^{N+\alpha-1}v'}{(1+g(v))^\gamma}\right)' &= r^{N+\beta-1}v^p \quad \text{for} \quad r < R \\ v'(0) &= 0\,, \\ v(R) &= 0\,. \end{aligned} \tag{3.2}$$

where $p > 1$, α, $\beta \in \mathbb{R}$, $g :]0, \infty[\to]0, \infty[$ is continuous and increasing.

Arguing as in the one-dimensional case, for $N + \alpha - 2 > 0$, $N + \beta > 0$, $\beta - \alpha + 1 > 0$ and $1 < p < (1 - \gamma)\frac{N+2\beta-\alpha+2}{N+\alpha-2}$ we obtain a priori bounds for the truncated problem. Nonetheless, the discussion is more complex and we have to study the *limiting problem*, including Liouville type theorems. Then, by means of fixed-point arguments, we prove the existence of positive solution for the truncated problem. Finally, we make use of Liouville type theorems to prove that for n large enough , the solution of the truncated problem is a solution of the initial problem.

On the other hand, for $p \geq \frac{N+2\beta-\alpha+2}{N+\alpha-2}$, via an adaptation of the Derrick-Pohozaev inequality (see [31]) we prove the nonexistence of positive solutions of problem (3.1). Finally, we use the properties of the solutions of (3.1), to prove nonexistence of positive solutions if $\beta - \alpha + 2 \leq 0$.

We observe that this result extends one of the results introduced by [19] (see Theorem 3.2), to a different class of differential operators. Problems of the form (3.1) have been widely studied, see, for example, [1], [3], [8], [9], [10], [11], [12], [13], [14]. When we observe that in [1], [8], [9], [10], [12], [14], nonlinearity does not depend on u, the results of existence of solutions are solutions in the viscosity sense. In [8], regularity of solutions are studied where nonlinearity is independent from u. In [11] the existence and the uniqueness of solution is studied, where the differential operator has a different structure from the one we introduce here. In [3], [13], the existence and nonexistence of solution is studied where the nonlinearity has singularities.

3.2 Existence of positive solution

We study existence of positive solutions for the problem (3.2), under the following hypotheses:

(H_0) The constants $\gamma \in (0, 1)$ and α, $\beta \in \mathbb{R}$ such that $N + \alpha - 2 > 0$, $N + \beta > 0$ and $\beta - \alpha + 1 > 0$.

(H_1) We suppose that $1 < p < (1 - \gamma)\dfrac{N + 2\beta - \alpha + 2}{N + \alpha - 2}$.

(H_2) The function $g : [0, +\infty[\to]0, +\infty[$ is increasing, such that

$$\lim_{v \to +\infty} \frac{g(v)}{v} = 1\,.$$

The main result is the following:

Theorem 3.1. *Assume hypotheses (H_0), (H_1) and (H_2). The Problem (3.2) has at least one positive solution.*

One of the difficulties here is that the coefficient of the operator, that is, $\frac{1}{(1+g(u))^\gamma}$ is not bounded from below. This implies that the clasical methods used in order to prove the existence of a solution for problem (3.2) cannot be applied in general. In order to overcome these difficulties, we introduce a truncation problem (see $(3.3)_k$) depending on k so that the new coefficient of the truncation problem becomes bounded from below by a uniformly positive constant.

3.2.1 Truncation Problem

With the aim to prove the existence of positive solutions of the problem, for $k > 0$, we consider the truncation function

$$T_k(s) = \max\{-k, min\{k, s\}\}\,,$$

and we define the function $g_k(s) = (g \circ T_k)(s)$.

Now, for $k \in \mathbb{N}$, we consider the following truncated problem

$$\begin{aligned} -\left(\frac{r^{N+\alpha-1}v'}{(1+g_k(v))^\gamma}\right)' &= r^{N+\beta-1}v^p \text{ for } r < R \\ v'(0) &= 0\,, \\ v(R) &= 0\,. \end{aligned} \tag{$(3.3)_k$}$$

3.2.2 A priori bounds for the truncated problem $(3.3)_k$

Theorem 3.2. *Assume hypoteses (H_0), (H_1) and (H_2). Then there is a positive constant C which depend on k, for every solution v of Problem $(3.3)_k$, we have*

$$||v||_\infty \leq C\,.$$

Proof. Let $k \in \mathbb{N}$, by contradiction argument assume that there is a sequence of positive solutions $\{v_n\}_n$ of the equation $(3.3)_k$, so that $||v_n||_\infty \to +\infty$ when $n \to +\infty$.
Now, consider the following changes of variables

$$\begin{aligned} y &= \frac{z_n}{t_n} r\,, \\ w_n(y) &= \frac{v_n(r)}{t_n} \end{aligned} \tag{3.4}$$

where $t_n := ||v_n||_\infty$ and $z_n = (1+g(k))^{\frac{\gamma}{\beta-\alpha+2}} {t_n}^{\frac{\beta-\alpha+1+p}{\beta-\alpha+2}}$.

Then, the function w_n is a solution of the following problem:

$$\begin{aligned} -\left(\frac{y^{N+\alpha-1}w_n'}{(1+g_k(t_n w_n))^\gamma}\right)' &= \tfrac{t_n^{\beta-\alpha+1+p}}{z_n^{\beta-\alpha+2}} y^{N+\beta-1} w_n^p \text{ for } y < \tfrac{z_n R}{t_n} \\ w_n'(0) &= 0\,, \\ w_n(0) &= 1\,, \\ w_n(\tfrac{Rz_n}{t_n}) &= 0\,. \end{aligned} \tag{3.5}$$

Besides, observe that the functions w_n verify the equation

$$-(N+\alpha-1)y^{N+\alpha-2}\left(\frac{w_n'(y)}{(1+g_k(t_n w_n)(y))^\gamma}\right) - y^{N+\alpha-1}\left(\frac{w_n'(y)}{(1+g_k(t_n w_n)(y))^\gamma}\right)' = \frac{t_n^{\beta-\alpha+1+p}}{z_n^{\beta-\alpha+2}} y^{N+\beta-1} w_n^p(y)\,,$$

Since w_n is a decreasing function, we see that

$$-(N+\alpha-1)y^{N+\alpha-2}\left(\frac{w_n'(y)}{(1+g_k(t_n w_n)(y))^\gamma}\right) > 0\,,$$

then, we obtain the inequality

$$-\left(\frac{w_n'(y)}{(1+g_k(t_n w_n)(y))^\gamma}\right)' \le \frac{t_n^{\beta-\alpha+1+p}}{z_n^{\beta-\alpha+2}} y^{\beta-\alpha} w_n^p(y)$$

Replacing z_n, the inequality adopts the shape

$$-\left(\frac{w_n'(y)}{(1+g_k(t_n w_n)(y))^\gamma}\right)' \le \frac{1}{(1+g(k))^\gamma} y^{\beta-\alpha} w_n^p(y)$$

Integrating from 0 to y,

$$-w_n'(u) \le \frac{(1+g_k(t_n w_n)(y))^\gamma}{(1+g(k))^\gamma} \int_0^y \tau^{\beta-\alpha} w_n^p(\tau)\, d\tau\,,$$

Since $w_n'(y) < 0$ for all $y \in (0, R\frac{z_n}{t_n})$, we have

$$|w_n'(y)| \le \int_0^y \tau^{\beta-\alpha}\, d\tau\,. \tag{3.6}$$

By (3.6), we get that w_n' is uniformly bounded in compact intervals, this is, for each $M \in \mathbb{R}$, there is a constant $C(M) > 0$ so that

$$w_n'(y) \le C(M)\,, \quad \text{for all } n \in \mathbb{N}\,, \quad \text{and for all } y \in [0,M]\,. \tag{3.7}$$

Thus the sequence $\{w_n\}_n$ is equicontinuous in $[0,M]$. Since this sequence is uniformly bounded, applying Ascoli Arzèla's theorem, we obtain that $\{w_n\}_n$ contains a convergent subsequence, which we still denote by $\{w_n\}_n$, say $w_n \to w$ in $C[0,M])$ when $n \to +\infty$.

Since every function w_n, is solution of the equation

$$\begin{aligned} -\left(\frac{y^{N+\alpha-1}w_n'(y)}{(1+g_k(t_nw_n(y)))^\gamma}\right)' &= \frac{y^{N+\beta-1}}{(1+g(k))^\gamma}w_n^p(y) \text{ for } r<R \\ w_n'(0) &= 0\,, \\ w_n(0) &= 1\,, \\ w_n(\tfrac{Rz_n}{t_n}) &= 0\,. \end{aligned} \tag{3.8}$$

From the properties of the solutions of (3.8) we can observe that $t_nw_n(y) \to +\infty$ when $n \to +\infty$ for each $y \in [0,M]$. If we suppose that exists $y_0 \in]0,M[$ so that $t_nw_n(y_0) \to c_0$, and so $w_n(y_0) \to 0$ when $n \to +\infty$, thus $w(y) = 0$ for all $y \ge y_0$.

Integrating equation in (3.8) on $[0,y] \subset [0,M]$, we find that

$$-w_n'(y) = \frac{(1+g_k(t_nw_n(y)))^\gamma}{(1+g(k))^\gamma}\frac{1}{y^{N+\alpha-1}}\int_0^y \tau^{N+\beta-1}w_n^p(\tau)\,d\tau\,. \tag{3.9}$$

Using Lebesgue's dominated convergence theorem, and letting $n \to +\infty$ we get

$$\frac{(1+g_k(t_nw_n(y)))^\gamma}{y^{N+\alpha-1}}\int_0^y \tau^{N+\beta-1}w_n^p(\tau)\,d\tau \to \frac{(1+g(c_0))^\gamma}{y^{N+\alpha-1}}\int_0^y \tau^{N+\beta-1}w^p(\tau)\,d\tau\,,$$

integrating again in $[y_0,y] \subset [0,M]$, we have

$$-(w_n(y)-w_n(y_0)) = \int_{y_0}^y \frac{(1+g_k(t_nw_n(s)))^\gamma}{(1+g(k))^\gamma}\frac{1}{s^{N+\alpha-1}}\int_0^s \tau^{N+\beta-1}w_n^p(\tau)\,d\tau\,ds\,,$$

Using again Lebesgue's dominated convergence theorem, letting $n \to +\infty$, we obtain

$$0 = \int_{y_0}^y \frac{(1+g(c_0))^\gamma}{(1+g(k))^\gamma}\frac{1}{s^{N+\alpha-1}}\int_0^s \tau^{N+\beta-1}w^p(\tau)\,d\tau\,ds\,.$$

Wich is absurd, because $w \in C[0,M]$ with $w(0) = 1$.

Thus, we conclude that $t_nw_n(y) \to +\infty$ when $n \to +\infty$, for each $y \in [0,M]$

Then, integrating again (3.9) in $[0,y]\subset[0,M]$, we have

$$-(w_n(y)-1)=\int_0^y \frac{(1+g_k(t_nw_n(s)))^\gamma}{(1+g(k))^\gamma}\frac{1}{s^{N+\alpha-1}}\int_0^s \tau^{N+\beta-1}w_n^p(\tau)\,d\tau\,ds\,,$$

Using again Lebesgue's dominated convergence theorem, letting $n\to+\infty$, we find that

$$-w(y)+1=\int_0^y \frac{1}{s^{N+\alpha-1}}\int_0^s \tau^{N+\beta-1}w^p(\tau)\,d\tau\,ds\,.$$

Thus w is a nonnegative nontrivial solution in $[0,M]$ to the initial value problem

$$\begin{aligned} -(y^{N+\alpha-1}w')' &= y^{N+\beta-1}w^p \\ w(0)=1\,,\ \ & w'(0)=0\,. \end{aligned} \tag{3.10}$$

By using a diagonal iterative scheme, as for example the last part of the proof of [[19], Proposition 4.1], w can be extended to all $\mathbb{R}_+$, as a nonnegative solution of (3.10). Furthermore, using the argument of [19], it can be shown that w is indeed a positive solution of class $C^2(0,+\infty)$ of (3.10). □

Theorem 3.3. *Assume the hypoteses (H_0) and (H_1). Then Problem (3.10) does not have nonnegative solutions.*

Before the demonstration of the previous theorem, let's consider the next proposition:

Proposition 3.1. *Let $u\in C^2(0,+\infty)$ be a nonnegative solution of (3.10), where α,β and p satisfy (H_0) and (H_1) respectively. Then, the function $y\,w'(y)+\rho\,w(y)$ is nonnegative and nonincreasing for $\rho=N+\alpha-2$.*
In particular, the function $y^\rho\,w(y)$ is nondecreasing on $]0,+\infty[$.

Proof. We know that $-\left(y^{N+\alpha-1}w'(y)\right)'\geq 0$ for all $y\in]0,+\infty[$ then

$$-\left[(N+\alpha-1)y^{N+\alpha-2}w'(y)+y^{N+\alpha-1}w''(y)\right]\geq 0$$

thus

$$y^{N+\alpha-2}\left[(N+\alpha-1)w'(y)+yw''(y)\right]\leq 0$$

for $y \in]0,+\infty[$.
Let's observe that:

$$\begin{aligned}\left[y\,w'(y)+\rho\,w(y)\right]' &= (y\,w'(y))'+\rho\,w'(y)\\ &= w'(y)+y\,w''(y)+\rho\,w'(y)\\ &= (1+\rho)\,w'(y)+yw''(y)\\ &= (N+\alpha-1)\,w'(y)+y\,w''(y)\leq 0\end{aligned}$$

this means that, $y\,w'(y)+\rho\,w(y)$ is a nonincreasing function.

On the other side, if we suppose that exists $y_0>0$ such that $y_0\,w'(y_0)+\rho\,w(y_0)<m_0$, with $m_0<0$. then, $y\,w'(y)+\rho\,w(y)<m_0$ for all $y\geq y_0$. Then $w'(y)<m_0 y^{-1}$, and integrating from y_0 to y, we have

$$w(y)-w(y_0)<m_0\ln(\frac{y}{y_0})\,.$$

Thus

$$\lim_{y\to\infty} w(y)=-\infty\,,$$

which is a contradiction. Thus we have proved that function $y\,w'(y)+\rho\,w(y)$ is nonnegative and nonincreasing.

On the other hand, we have

$$\begin{aligned}(y^{\rho}\,w(y))' &= \rho\,y^{\rho-1}w(y)+y^{\rho}w'(y)\\ &= y^{\rho-1}(y\,w'(y)+\rho\,w(y))\,.\end{aligned}$$

Since function $y\,w'(y+\rho w(y))$ is nonnegative, we have that $(y^{\rho}\,w(y))'\geq 0$. □

Proof Theorem 3.3. Assume that w is a nonnegative solution of Problem (3.10)

Integrating equation (3.10) from y to t, we obtain

$$-[t^{N+\alpha-1}w'(t)-y^{N+\alpha-1}w'(y)]=\int_y^t s^{N+\beta-1}w^p(s)\,ds\,.$$

Thus, using Proposition 3.1 and $w'(y)\leq 0$, we have

$$
\begin{aligned}
t^{N+\alpha-1}|w'(t)| &\geq \int_y^t s^{N+\beta-1} w^p(s)\, ds \\
&\geq \int_y^t s^{N+\beta-1-\rho p} (s^\rho\, w(s))^p\, ds \\
&\geq (y^\rho\, w(y))^p \int_y^t s^{N+\beta-1-\rho p}\, ds \\
&= y^{\rho\, p} w^p(y) \frac{t^{N+\beta-\rho\, p} - y^{N+\beta-\rho\, p}}{N+\beta-\rho\, p}
\end{aligned}
$$

Since $\rho w(t) \geq -t w'(t)$, we obtain

$$
t^{N+\alpha-2} w(t) \geq y^{\rho\, p} w^p(y) \frac{t^{N+\beta-\rho\, p} - y^{N+\beta-\rho\, p}}{N+\beta-\rho\, p}
$$

Taking $t = 2y$ and using that w is decreasing, we have

$$
\begin{aligned}
(2y)^{N+\alpha-2} w(y) &\geq y^{\rho\, p} w^p(y) \frac{(2y)^{N+\beta-\rho\, p} - y^{N+\beta-\rho\, p}}{N+\beta-\rho\, p} \\
&= y^{N+\beta} w^p(y) \frac{2^{N+\beta-\rho\, p} - 1}{N+\beta-\rho\, p}\,,
\end{aligned}
$$

and so, from the last inequality, we have

$$
w^{p-1}(y) \leq y^{-(\beta-\alpha+2)} 2^{N+\alpha-2} \left[\frac{2^{N+\beta-\rho\, p} - 1}{N+\beta-\rho\, p} \right]. \tag{3.11}
$$

On another side, multiplying the equation of the initial value problem (3.10) by $y w'(y)$ and integrating from 0 to t, we obtain

$$
\begin{aligned}
\int_0^t s^{N+\beta} w^p(y) w'(y)\, ds &= -\int_0^t (s^{N+\alpha-1} w'(s))' s w'(s)\, ds \\
&= -t^{N+\alpha} w'(t)^2 + \int_0^t s^{N+\alpha-1} w'(s)^2\, ds + \int_0^t s^{N+\alpha} w''(s) w'(s)\, ds\,.
\end{aligned}
$$

Observe that

$$
\int_0^t s^{N+\alpha} w''(s) w'(s)\, ds = \frac{t^{N+\alpha}}{2} w'(t)^2 - \frac{N+\alpha}{2} \int_0^t s^{N+\alpha-1} w'(s)^2\, ds\,.
$$

Thus, combining the last two equations, we obtain

$$\int_0^t s^{N+\beta} w^p(y) w'(y)\, ds = -\frac{t^{N+\alpha}}{2} w'(t)^2 - \frac{N+\alpha-2}{2} \int_0^t s^{N+\alpha-1} w'(s)^2\, ds\,. \tag{3.12}$$

On another side, we have

$$\int_0^t s^{N+\beta} w^p(y) w'(y)\, ds = \frac{1}{p+1}\left[t^{N+\beta} w^{p+1}(t) - (N+\beta)\int_0^t s^{N+\beta-1} w^{p+1}(s)\, ds\right]. \tag{3.13}$$

From equations (3.12) and (3.13), we have

$$-\frac{t^{N+\alpha}}{2} w'(t)^2 - \frac{N+\alpha-2}{2} \int_0^t s^{N+\alpha-1} w'(s)^2\, ds = \frac{1}{p+1}\left[t^{N+\beta} w^{p+1}(t) - (N+\beta)\int_0^t s^{N+\beta-1} w^{p+1}(s)\, ds\right].$$

Multipying (3.10) by w and integrating from 0 to t, we obtain

$$\int_0^t s^{N+\alpha-1} w'(s)^2\, ds = t^{N+\alpha-1} w'(t) w(t) + \int_0^t s^{N+\beta-1} w^{p+1}(s)\, ds,$$

then,

$$-\frac{t^{N+\alpha}}{2} w'(t)^2 - \frac{N+\alpha-2}{2}\left[t^{N+\alpha-1} w'(t) w(t) + \int_0^t s^{N+\beta-1} w^{p+1}(s)\, d\right] = \frac{1}{p+1}\left[t^{N+\beta} w^{p+1}(t) - (N+\beta)\int_0^t s^{N+\beta-1} w^{p+1}(s)\, ds\right]$$

hence,

$$\left(-\frac{N+\alpha-2}{2} + \frac{N+\beta}{p+1}\right)\int_0^t s^{N+\beta-1} w^{p+1}(s)\, ds = \frac{1}{p+1} t^{N+\beta} w^{p+1}(t) + w'(t)\frac{t^{N+\alpha-1}}{2}\left[t w'(t) + (N+\alpha-2) w(t)\right]. \tag{3.14}$$

From the inequality (3.11), we have that

$$t^{N+\beta} w^{p+1}(t) \le C(N,\alpha,\beta) t^{\frac{p(N+\alpha-2)-(N+2\beta-\alpha+2)}{p-1}}\,, \tag{3.15}$$

where $C(N,\alpha,\beta) > 0$.

By hypotesis (H_1), we have $N+2\beta-\alpha+2-p(N+\alpha-2) > 0$, which implies

$$\lim_{t\to\infty} t^{N+\beta} w^{p+1}(t) = 0\,. \tag{3.16}$$

We also know that

$$w'(t)\frac{t^{N+\alpha-1}}{2}\left[tw'(t)+(N+\alpha-2)w(t)\right] \le 0\,, \tag{3.17}$$

in addition, we have

$$-\frac{N+\alpha-2}{2}+\frac{N+\beta}{p+1} = \frac{N+2\beta-\alpha+2-p(N+\alpha-2)}{2(p+1)} > 0\,.$$

Then,

$$\left(-\frac{N+\alpha-2}{2}+\frac{N+\beta}{p+1}\right)\int_0^t s^{N+\beta-1} w^{p+1}(s)\,ds > 0. \tag{3.18}$$

Observing the inequalities (3.16), (3.17) and (3.18), of the equality (3.14), we have that

$$\lim_{t\to+\infty}\int_0^t s^{N+\beta-1} w^{p+1}\,ds \le 0\,. \tag{3.19}$$

Which contradicts (3.18). Then the initial value problem (3.10), does not have nonnegative solutions in $\mathbb{R}_+$. □

Finally, for $k>0$ fixed we conclude that there exists a constant $C>0$ such that for each function v solution of the problem $(3.3)_k$

$$||v||_\infty \le C\,.$$

The existence of positve solution of the truncation Problem $(3.3)_k$, will be based on the well known fixed-point result due to Krasnosel'skii (See Theorem 4.2).

Consider the following Banach space

$$X = C([0,1],\mathbb{R})$$

endowed whith the norm $||v||_\infty = \sup_{t\in[0,1]}|v(t)|$.

Define the cone C by

$$C = \{v \in C[0,R];\ \ v\ge 0\,,\ v(R)=0\}.$$

Define also the operator $F : X \to X$ by

$$(Fv)(r) = \int_r^R \frac{(1 + g_k(v)(s))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau)\, d\tau\, ds. \tag{3.20}$$

Note that a simple calculation shows that the fixed point of the operator F are the positive solutions of Problem (3.2).

Lemma 3.1. *The operator $F : X \to X$ defined by (3.20) is compact, and the cone C is invariant under F.*

Proof Outline. The compactness of F follows from the well known Ascoli Arzèla's theorem. The invariance of the cone C is a consequence of the fact that the nonlinearities are nonnegative.

3.2.3 Existence result of Truncation Problem $(3.3)_k$

We will give an existence result of the Truncation Problem $(3.3)_k$. The proof will be based on the well known fixed-point result due to Krasnosel'skii (See Theorem 4.2).

Theorem 3.4. *Assume hipotheses $(H_0), (H_1)$ and (H_2). Then there exists a positive solution of Problem $(3.3)_k$.*

Proof. To prove the existence of positive solution for the truncated problem, it sufficies to show that F has a fixed point. For this we will check the conditions of Theorem 4.2. Define the homotopy $H : [0,1] \times C \to C$ by

$$H(t,v)(r) = \int_r^R \frac{(1 + g_k(v)(s))^{(1-t)\gamma + t\bar{c}}}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau)\, d\tau\, ds,$$

where $\bar{c}$ is a parameter sufficiently large.

Note that $H(t,v)$ is a compact homotopy and that $H(0,v)(r) = F(v)(r)$, which verifies (b).

On the other hand, we have

$$\begin{aligned} tF(v)(r) &\le (1 + g(k))^\gamma ||v||_\infty^p \int_r^R \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1}\, d\tau\, ds \\ &= \frac{(1 + g(k))^\gamma}{(N+\beta)(\beta-\alpha+2)} R^{\beta-\alpha+1} ||v||_\infty^p. \end{aligned}$$

Taking $\delta = ||v||_\infty$ sufficiently small, such that

$$\frac{(1+g(k))^\gamma}{(N+\beta)(\beta-\alpha+2)} R^{\beta-\alpha+1}\delta^{p-1} < 1,$$

we have $|tF(v)| < ||v||_\infty$, that is, $tF(v) \neq v$ for all $||v||_\infty = \delta$ y $t \in [0,1]$.

If we consider η large such that $\eta > C_0$, with C_0 given by Theorem 3.2, we have that $H(t,v) \neq v$ for all $||v||_\infty = \eta$ y $t \in [0,1]$.

Then, taking $\bar{c}$ large enough, we have that:

$$\begin{aligned} H(1,v)(r) &= \lambda \int_r^R \frac{(1+g_k(v)(s))^{\bar{c}}}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau)\, d\tau\, ds \\ &> C_0 , \end{aligned}$$

thus, $H(1,v) \neq v$ for all $||v||_\infty < \eta$.

Hence using Theorem 4.2, we have that the operator (3.20) has a fixed–point v such that $\delta < ||v||_\infty < \eta$, which is solution of equation $(3.3)_k$.

□

3.2.4 Proof of existence result

Proof. Theorem 1. Now our objective is to prove that the problem (3.2) has a nontrivial solution. For this, we will prove there is $k_0 \in \mathbb{N}$, such that the solution v_{k_0} of problem $(3.3)_{k_0}$verifies $||v_{k_0}||_\infty \le k_0$.

We suppose by contradiction that $||v_k||_\infty > k$ for all $k \in \mathbb{N}$, where v_k is a solution of $(3.3)_k$.

Let us consider the change of variables

$$\begin{aligned} y &= \frac{z_k}{t_k} r\,, \\ w_k(y) &= \frac{v_k(r)}{t_k} \end{aligned} \qquad (3.21)$$

where $t_k := ||v_k||_\infty$ and $z_k = (1+g(k))^{\frac{\gamma}{\beta-\alpha+2}} t_k^{\frac{\beta-\alpha+1+p}{\beta-\alpha+2}}$.

From (3.21), we have that w_k satisfies

$$
\begin{aligned}
-\left(\frac{y^{N+\alpha-1}w_k'}{(1+g_k(t_kw_k))^\gamma}\right)' &= \tfrac{t_k^{\beta-\alpha+1+p}}{z_k^{\beta-\alpha+2}}y^{N+\beta-1}w_k^p\ , \\
w_k'(0) &= 0\,, \\
w_k(0) &= 1\,, \\
w_k(\tfrac{Rz_k}{t_k}) &= 0\,.
\end{aligned}
\tag{3.22}
$$

We note that $\dfrac{Rz_k}{t_k} \to +\infty$ when $k \to +\infty$.

Since $w_k'(y) < 0$ for all $y \in (0, \frac{z_k}{t_k}R)$, we have

$$
-y^{N+\alpha-1}\left(\frac{w_k'(y)}{(1+g_k(t_kw_k(y)))^\gamma}\right)' \leq \frac{t_k^{\beta-\alpha+1+p}}{z_k^{\beta-\alpha+2}}y^{N+\beta-1}w_k^p(y)
$$

then,

$$
-\left(\frac{w_k'(y)}{(1+g_k(t_kw_k(y)))^\gamma}\right)' \leq \frac{t_k^{\beta-\alpha+1+p}}{z_k^{\beta-\alpha+2}}y^{\beta-\alpha}w_k^p(y)\,.
$$

Integrating from 0 to y,

$$
-w_k'(y) \leq (1+g(k))^\gamma \frac{t_k^{\beta-\alpha+1+p}}{z_n^{\beta-\alpha+2}} \int_0^y s^{\beta-\alpha}w_k^p(s)\,ds\,,
$$

we have

$$
-w_k'(y) \leq \int_0^y s^{\beta-\alpha}w_k^p(s)\,ds\,,
$$

which implies

$$
|w_k'(y)| \leq \int_0^y s^{\beta-\alpha}
$$

for all $k \in \mathbb{N}$.

Therefore, there exists a constant $C_1(M) > 0$ such that

$$
|w_k'(y)| < C_1(M) \quad \text{for all } \ k \in \mathbb{N} \ \text{ and all } \ y \in [0, M]\,.
$$

Which means that the sequence $\{w_k\}_k$ is equicontinuous. Since it is also uniformly bounded, an application of Ascoli Arzèla's theorem yields that $\{w_k\}_k$ contains a convergent subsequence, which we denote again by $\{w_k\}_k$, verifying

$$
w_k \to w \ \text{ in } \ C[0, M] \ \text{ when } \ k \to +\infty\,. \tag{3.23}
$$

Now, we will study the *limiting problem* associated with (3.22).

Since $t_k > k$ for all $k \in \mathbb{N}$, we have $0 < \frac{k}{t_k} < 1$. Then there is a subsequence, that we again refer to by $\{\frac{k}{t_k}\}_k$ and $l \in [0,1]$ such that $\frac{k}{t_k} \to l$.

Since $\{w_k\}_k$ is a sequence of continuous functions, for each $k \in \mathbb{N}$ there is $s_k \in \left]0, \dfrac{z_k R}{t_k}\right[$ such that

$$w_k(s_k) = \frac{k}{t_k}\,. \tag{3.24}$$

1. **Suppose** $l = 0$

   In this case, we have $\dfrac{k}{t_k} \to 0$, but $s_k \leq \dfrac{z_k R}{t_k}$ with $s_k \to +\infty$ when $k \to +\infty$.

   Then, for all $M > 0$ there is $k_M \in \mathbb{N}$ such that $s_k > M$ for all $k \geq k_M$. Then we have $g_k(t_k w_k(y)) = g(k)$ for all $y \in [0, M]$ y $k \geq k_M$.

   From equation (3.22), is easy to see that w_k satisfies

   $$-w_k(y) + 1 = \int_0^y \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w_k^p(\tau)\, d\tau\, ds\,, \tag{3.25}$$

   for each $y \in [0, M]$.

   From (3.23), Lebesgue's dominated theorem and by letting $k \to +\infty$ in (3.25), we obtain

   $$1 - w(y) = \int_0^y \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w^p(\tau)\, d\tau\, ds\,. \tag{3.26}$$

   Then, by differentianting (3.26) we obtain

   $$-w'(y) = \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w^p(\tau)\, d\tau\,.$$

   Thus w is a nontrivial solution in $[0, M]$ of the initial value problem

   $$\begin{aligned} -(y^{N+\alpha-1} w')' &= y^{N+\beta-1} w^p \\ w(0) = 1\,, &\quad w'(0) = 0\,. \end{aligned} \tag{3.27}$$

   Since in the initial value problem (3.10), in (3.27), using a diagonal iterative scheme, w can be extended to all $\mathbb{R}_+$, as a nonnegative solution of (3.27), and using [19], it can be shown that w is indeed a positive solution of class $C^2(0, +\infty)$ of (3.10). This is a contradiction with Theorem 3.3.

2. **Suppose** $l = 1$:

Integrating from 0 to $s_k \in]0, M]$ as in (3.25), we obtain:

$$\begin{aligned}
-\frac{k}{t_k} + 1 &= \frac{1}{(1+g(k))^\gamma} \int_0^{s_k} \frac{(1+g_k(t_k w_k(s)))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w_k^p(\tau)\, d\tau\, ds \\
&= \int_0^{s_k} \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w_k^p(\tau)\, d\tau\, ds \\
&\geq \left(\frac{k}{t_k}\right)^p \int_0^{s_k} \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1}\, d\tau\, ds \\
&= \left(\frac{k}{t_k}\right)^p \frac{s_k^{\beta-\alpha+2}}{(N+\beta)(\beta-\alpha+2)}\,.
\end{aligned}$$

Note that by hypothesis $\frac{k}{t_k} \to 1$ when $k \to +\infty$, we have that

$$s_k \to 0 \ \text{ when } \ k \to +\infty\,. \tag{3.28}$$

Integrating equation (3.22) from 0 to $y \in [0, M]$ we obtain

$$-w_n'(y) = \frac{(1+g_k(t_k w_k(y))^\gamma}{(1+g(k))^\gamma} \frac{1}{y^{N+\alpha-1}} \int_0^y \tau^{N+\beta-1} w_k^p(\tau)\, d\tau \tag{3.29}$$

Now keeping in mind that function w_k is decreasing, and integrating again from s_k to $y \in [0, M]$, we obtain

$$\begin{aligned}
-w_k(y) + \frac{k}{t_k} &= \int_{s_k}^y \left(\frac{1+g_k(t_k w_k(y))}{1+g(k)}\right)^\gamma \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w_k^p(\tau)\, d\tau \\
&= \int_{s_k}^y \left(\frac{1+g(t_k w_k(y))}{1+g(k)}\right)^\gamma \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w_k^p(\tau)\, d\tau.
\end{aligned}$$

By (3.23), hipothesis (H_2) and Lebesgue's dominated convergence theorem, and by letting $k \to +\infty$ in the last equality, we obtain

$$-w(y) + 1 = \int_0^y \frac{(w(s))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w^p(\tau)\, d\tau\,.$$

for all $y \in [0, M]$.

Thus w is a positive solution in $[0, M]$ to the initial value problem

$$
\begin{gathered}
-\left(\frac{y^{N+\alpha-1}w'}{w^{\gamma}(y)}\right)' = y^{N+\beta-1}w^{p} \\
w(0)=1\,, \quad w'(0)=0\,.
\end{gathered}
\tag{3.30}
$$

Using a diagonal iterative scheme, as in equation (3.10), w can be extended to all $\mathbb{R}_+$, as a nonnegative solution of (3.30). Furthermore, using [19], it can be shown that w is indeed a positive solution of class $C^2(0, +\infty)$ of (3.30).

Let us consider the change of variables,

$$
u(y) = w^{1-\gamma}(y) \;\text{ then }\; u'(y) = (1-\gamma)\frac{w'(y)}{w^{\gamma}(y)}\,,
\tag{3.31}
$$

then, we have that $u \in C^2(0, +\infty)$ is a nontrivial solution of

$$
\begin{gathered}
-(y^{N+\alpha-1}u')' = (1-\gamma)y^{N+\beta-1}u^{\frac{p}{1-\gamma}} \\
u(0)=1\,, \quad u'(0)=0\,.
\end{gathered}
\tag{3.32}
$$

This is a contradiction by Theorem 3.3.

3. **Suppose** $0 < l < 1$**:**

From (3.24), we have that $w_k(s_k) = \frac{k}{t_k} \to l := \frac{1}{d}$ when $k \to +\infty$ and $w_k(\frac{z_k R}{t_k}) = 0$. There exists $c_0 > 0$ such that $0 < s_k \leq c_0$ for all $k \in \mathbb{N}$. Then by compactness, there is a subsequence of $\{s_k\}$, which we still denoted by $\{s_k\}$, and $s_0 \in \mathbb{R}_+$ such that $s_k \to s_0$ when $k \to +\infty$.

Observe that for $y \in]0, s_k]$ we have that $g_k(t_k w_k(y)) = g(k)$, so that from (3.25) we have that

$$
-w_k(y) + 1 = \int_0^y \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w_k^p(\tau)\, d\tau\, ds\,.
\tag{3.33}
$$

From (3.23) and Lebesgue's dominated convergence theorem, and letting $k \to +\infty$ in (3.36), we have

$$
-w(y) + 1 = \int_0^y \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w^p(\tau)\, d\tau\, ds\,.
\tag{3.34}
$$

Then, by differentianting (3.34) we obtain

$$-w'(y) = \frac{1}{y^{N+\alpha-1}} \int_0^y \tau^{N+\beta-1} w^p(\tau)\, d\tau.$$

Thus w is a nonnegative nontrivial solution in $[0, s_0]$ to the initial value problem

$$\begin{aligned} -(y^{N+\alpha-1}w')' = y^{N+\beta-1}w^p\,, \;\; y \in (0, s_0) \\ w(0) = 1\,, \;\; w'(0) = 0\,. \end{aligned} \tag{3.35}$$

On the other side, for each $y \in [s_k, \infty)$ we have that $g_k(t_k w_k(y)) = g(t_k w_k(y))$, then integrating from s_k to y in (3.29), we have

$$-w_k(y) + \frac{k}{t_k} = \int_{s_k}^y \left(\frac{1 + g(t_k w_k(s))}{1 + g(k)} \right)^\gamma \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w_k^p(\tau)\, d\tau\, ds\,. \tag{3.36}$$

From (3.23) and Lebesgue's dominated convergence theorem, and letting $k \to \infty$ in (3.36), we obtain

$$-w(y) + \frac{1}{d} = \int_{s_0}^y \frac{(d\, w(s))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w^p(\tau)\, d\tau\, ds\,. \tag{3.37}$$

Then, by differentianting (3.37) we obtain

$$-w'(y) = \frac{(d\, w(s))^\gamma}{y^{N+\alpha-1}} \int_0^y \tau^{N+\beta-1} w^p(\tau)\, d\tau.$$

Thus w is a nonnegative nontrivial solution in $[s_0, +\infty[$ of the initial value problem

$$\begin{aligned} -\left(\frac{y^{N+\alpha-1} w'(y)}{d^\gamma w^\gamma(y)} \right)' = y^{N+\beta-1} w^p(y)\,, \;\; y \in (s_0, \infty) \\ w(s_0) = \tfrac{1}{d}\,. \end{aligned} \tag{3.38}$$

Then, we have that w is nontrivial solution of (3.35) and (3.38).

Lemma 3.2. *The Problem (3.35), (3.38) has no nontrivial positive solutions.*

Proof. In the equation (3.38) making the change of variable (3.31), we have

$$\begin{aligned} -(y^{N+\alpha-1}u'(y))' = (1-\gamma)d^\gamma y^{N+\beta-1} u^{\frac{p}{1-\gamma}}(y)\,, \;\; y \in (s_0, +\infty) \\ u(s_0) = \tfrac{1}{d^{1-\gamma}}\,. \end{aligned} \tag{3.39}$$

If u is solution of the Problem (3.39), then by Proposition 3.1, we have that

$$yu'(y) + \rho u(y)\,, \quad y \in (s_0, +\infty)\ . \tag{3.40}$$

where $\rho = N + \alpha - 2$, is a nonincreasing and nonnegative function, and the function

$$y^\rho u(y)\,, \quad y \in (s_0, +\infty)\ . \tag{3.41}$$

is nondecreasing.

Integrating the equation (3.39) on $[y, t] \subset]s_0, +\infty[$, we have

$$-[t^{N+\alpha-1}u'(t) - y^{N+\alpha-1}u'(y)] = (1-\gamma)d^\gamma \int_y^t s^{N+\beta-1} u^{\frac{p}{1-\gamma}}(s)\, ds. \tag{3.42}$$

Thus, using that u is decreasing from (3.42) and Proposition 3.1, we obtain

$$\begin{aligned}
t^{N+\alpha-2}u(t) &\geq t^{N+\alpha-1}|u'(t)| \\
&\geq (1-\gamma)d^\gamma \int_y^t s^{N+\beta-1} u^{\frac{p}{1-\gamma}}(s)\, ds \\
&= (1-\gamma)d^\gamma \int_y^t s^{N+\beta-1-\frac{\rho p}{1-\gamma}} (s^\rho u(s))^{\frac{p}{1-\gamma}}\, ds \\
&\geq (1-\gamma)d^\gamma (y^\rho u(y))^{\frac{p}{1-\gamma}} \int_y^t s^{N+\beta-1-\frac{\rho p}{1-\gamma}}\, ds \\
&\geq (1-\gamma)d^\gamma y^{\frac{\rho p}{1-\gamma}} u^{\frac{p}{1-\gamma}}(y) \left(\frac{t^{N+\beta-\frac{\rho p}{1-\gamma}} - y^{N+\beta-\frac{\rho p}{1-\gamma}}}{N+\beta-\frac{\rho p}{1-\gamma}} \right).
\end{aligned}$$

Taking $t = 2y$ and using Proposition 3.1 again, we obtain

$$\begin{aligned}
(2y)^{N+\alpha-2}u(y) &\geq (2y)^{N+\alpha-2}u(2y) \\
&\geq (1-\gamma)d^\gamma y^{N+\beta} u^{\frac{p}{1-\gamma}}(y) \left(\frac{2^{N+\beta-\frac{\rho p}{1-\gamma}} - 1}{N+\beta-\frac{\rho p}{1-\gamma}} \right),
\end{aligned}$$

hence we conclude

$$u^{\frac{p}{1-\gamma}-1}(y) \leq y^{-(\beta-\alpha+2)} \frac{2^{N+\alpha-2}}{(1-\gamma)d^{\gamma}} \left(\frac{2^{N+\beta-\frac{\rho p}{1-\gamma}}-1}{N+\beta-\frac{\rho p}{1-\gamma}} \right) . \tag{3.43}$$

On the other side, multiplying the equation (3.39) by $yu'(y)$ and integrating in $[s_0, t]$ we have that

$$-\int_{s_0}^{t} (y^{N+\alpha-1}u'(y))' yu'(y)\, dy = (1-\gamma)d^{\gamma} \int_{s_0}^{t} y^{N+\beta} u^{\frac{p}{1-\gamma}}(y) u'(y)\, dy. \tag{3.44}$$

But

$$\begin{aligned} -\int_{s_0}^{t} (y^{N+\alpha-1}u'(y))' yu'(y)\, dy &= -t^{N+\alpha}(u'(t))^2 + s_0^{N+\alpha}(u'(s_0))^2 + \int_{s_0}^{t} y^{N+\alpha-1}(u'(y))^2 dy \\ &\quad + \int_{s_0}^{t} y^{N+\alpha} u'(y)u''(y) dy \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} \int_{s_0}^{t} y^{N+\alpha-1}(u'(y))^2 dy &+ \int_{s_0}^{t} y^{N+\alpha} u'(y)u''(y) dy \\ &= \int_{s_0}^{t} y^{N+\alpha-1}(u'(y))^2 dy + \int_{s_0}^{t} y^{N+\alpha} \left(\frac{u'^2(y)}{2} \right)' dy \\ &= \int_{s_0}^{t} y^{N+\alpha-1}(u'(y))^2 dy + t^{N+\alpha} \frac{u'(t)^2}{2} \\ &\quad - s_0^{N+\alpha} \frac{u'(s_0)^2}{2} - \left(\frac{N+\alpha}{2} \right) \int_{s_0}^{t} y^{N+\alpha-1} u'^2(y) dy \\ &= -\left(\frac{N+\alpha-2}{2} \right) \int_{s_0}^{t} y^{N+\alpha-1} u'^2(y) dy + t^{N+\alpha} \frac{u'(t)^2}{2} - s_0^{N+\alpha} \frac{u'(s_0)^2}{2} . \end{aligned} \tag{3.46}$$

Replacing (3.46) in (3.45), we obtain

$$\begin{aligned} -\int_{s_0}^{t} (y^{N+\alpha-1}u'(y))' yu'(y)\, dy &= -\left(\frac{N+\alpha-2}{2} \right) \int_{s_0}^{t} y^{N+\alpha-1} u'^2(y) dy - t^{N+\alpha} \frac{u'(t)^2}{2} \\ &\quad + s_0^{N+\alpha} \frac{u'(s_0)^2}{2} . \end{aligned} \tag{3.47}$$

Now, multiplying the equation (3.39) by u and integrating on $[s_0, t]$, we obtain

$$-\int_{s_0}^{t} (y^{N+\alpha-1}u'(y))' u(y)\, dy = (1-\gamma)d^{\gamma} \int_{s_0}^{t} y^{N+\beta-1} u^{\frac{p}{1-\gamma}+1}(y)\, dy , \tag{3.48}$$

that is,

$$\begin{aligned} -\int_{s_0}^{t} (y^{N+\alpha-1}u'(y))' u(y)\, dy &= -t^{N+\alpha-1}u'(t)u(t) + s_0^{N+\alpha-1}u'(s_0)u(s_0) \\ &\quad + \int_{s_0}^{t} y^{N+\alpha-1} u'^2(y)\, dy . \end{aligned} \tag{3.49}$$

Then, from (3.48) and (3.49) we have that

$$\begin{aligned}\int_{s_0}^t y^{N+\alpha-1}u'^2(y)dy &= t^{N+\alpha-1}u'(t)u(t) - s_0^{N+\alpha-1}u'(s_0)u(s_0)\\ &\quad +(1-\gamma)d^\gamma \int_{s_0}^t y^{N+\beta-1}u^{\frac{p}{1-\gamma}}(y)u(y)\,dy\,,\end{aligned} \tag{3.50}$$

Replacing (3.50) in (3.47), we obtain

$$\begin{aligned}-\int_{s_0}^t (y^{N+\alpha-1}u'(y))'yu'(y)\,dy &= -\left(\tfrac{N+\alpha-2}{2}\right)\Big[t^{N+\alpha-1}u'(t)u(t) - s_0^{N+\alpha-1}u'(s_0)u(s_0)\\ &\quad +(1-\gamma)d^\gamma \int_{s_0}^t y^{N+\beta-1}u^{\frac{p}{1-\gamma}+1}(y)\,dy\Big] - t^{N+\alpha}\tfrac{u'(t)^2}{2}\\ &\quad +s_0^{N+\alpha}\tfrac{u'(s_0)^2}{2}\,.\end{aligned} \tag{3.51}$$

On the other hand,

$$\begin{aligned}(1-\gamma)d^\gamma \int_{s_0}^t y^{N+\beta}u^{\frac{p}{1-\gamma}}(y)u'(y)\,dy &= \tfrac{(1-\gamma)d^\gamma}{\frac{p}{1-\gamma}+1}\int_{s_0}^t y^{N+\beta}\left(u^{\frac{p}{1-\gamma}+1}(y)\right)'\,dy\\ &= \tfrac{(1-\gamma)^2d^\gamma}{p+1-\gamma}\Big[t^{N+\beta}u^{\frac{p}{1-\gamma}+1}(t) - s_0^{N+\beta}u^{\frac{p}{1-\gamma}+1}(s_0)\\ &\quad -(N+\beta)\int_{s_0}^t y^{N+\beta-1}u^{\frac{p}{1-\gamma}+1}(y)dy\Big]\,.\end{aligned} \tag{3.52}$$

Replacing (3.51) and (3.52) in (3.44), we obtain

$$\begin{aligned}(1-\gamma)d^\gamma &\left(\frac{(N+\beta)(1-\gamma)}{p+1-\gamma} - \frac{N+\alpha-2}{2}\right)\int_{s_0}^t y^{N+\beta-1}u^{\frac{p}{1-\gamma}+1}(y)\,dy =\\ &u'(t)\frac{t^{N+\alpha-1}}{2}\left[tu'(t)+(N+\alpha-2)u(t)\right] + \frac{(1-\gamma)^2d^\gamma}{p+1-\gamma}t^{N+\beta}u^{\frac{p}{1-\gamma}+1}(t)\\ &- u'(s_0)\frac{s_0^{N+\alpha-1}}{2}\left[s_0u'(s_0)+(N+\alpha-2)u(s_0)\right] - \frac{(1-\gamma)^2d^\gamma}{p+1-\gamma}s_0^{N+\beta}u^{\frac{p}{1-\gamma}+1}(s_0)\,.\end{aligned} \tag{3.53}$$

From hypotheses (H_0) and (H_1), we have that

$$(1-\gamma)d^\gamma\left(\frac{(N+\beta)(1-\gamma)}{p+1-\gamma} - \frac{N+\alpha-2}{2}\right) > 0\,, \tag{3.54}$$

then

$$(1-\gamma)d^\gamma\left(\frac{(N+\beta)(1-\gamma)}{p+1-\gamma} - \frac{N+\alpha-2}{2}\right)\int_{s_0}^t y^{N+\beta-1}u^{\frac{p}{1-\gamma}+1}(y)\,dy > 0\,, \tag{3.55}$$

for all $t \in]s_0, +\infty[$.

Since u is a decreasing function, using Proposition 3.1, we have that

$$u'(t)\frac{t^{N+\alpha-1}}{2}\left[tu'(t)+(N+\alpha-2)u(t)\right]\leq 0 \tag{3.56}$$

for all $t \in]s_0, +\infty[$.

Using inequality (3.43), we obtain

$$0 < t^{N+\beta}u^{\frac{p}{1-\gamma}+1}(t) \leq \left[\frac{2^{N+\alpha-2}}{(1-\gamma)d^{\gamma}}\left(\frac{2^{N+\beta-\frac{p}{1-\gamma}}-1}{N+\beta-\frac{p}{1-\gamma}}\right)\right]^{\frac{1-\gamma}{p+1-\gamma}} t^{N+\beta-(\beta-\alpha+2)\frac{p+1-\gamma}{p-(1-\gamma)}}. \tag{3.57}$$

From hipothesis (H_1), we have that

$$N+\beta-(\beta-\alpha+2)\frac{p+1-\gamma}{p-(1-\gamma)} = -\frac{(1-\gamma)(N+2\beta-\alpha+2)-p(N+\alpha-2)}{p-1+\gamma} < 0\,. \tag{3.58}$$

Then by (3.57) and (3.58), we see that

$$t^{N+\beta}u^{\frac{p}{1-\gamma}+1}(t)\to 0 \ \text{ when } \ t\to+\infty\,. \tag{3.59}$$

Now we will study the sign of

$$u'(s_0)\frac{s_0^{N+\alpha-1}}{2}\left[s_0u'(s_0)+(N+\alpha-2)u(s_0)\right]+\frac{(1-\gamma)^2d^{\gamma}}{p+1-\gamma}s_0^{N+\beta}u^{\frac{p}{1-\gamma}+1}(s_0)\,. \tag{3.60}$$

Making the change of variables (3.31), the equation (3.60) becomes

$$\frac{1-\gamma}{w^{2\gamma}(s_0)}\left[\frac{w'(s_0)s_0^{N+\alpha-1}}{2}\left((1-\gamma)s_0w'(s_0)+(N+\alpha-2)w(s_0)\right)+\frac{(1-\gamma)d^{\gamma}}{p+1-\gamma}s_0^{N+\beta}w^{p+1+\gamma}(s_0)\right]. \tag{3.61}$$

Multiplying equation (3.35) by w and integrating from 0 to s_0 (as in (3.14)), we obtain

$$\begin{aligned}\left(\tfrac{N+\beta}{p+1}-\tfrac{N+\alpha-2}{2}\right)\int_0^{s_0} y^{N+\beta-1}w^{p+1}(y)dy \;=\;& w^{p+1}(s_0)\tfrac{s_0^{N+\beta}}{p+1}\\ &+w'(s_0)\tfrac{s_0^{N+\alpha-1}}{2}\left[s_0w'(s_0)+(N+\alpha-2)w(s_0)\right].\end{aligned} \tag{3.62}$$

Using (3.62) and Proposition 3.1 in (3.61) , we have that

$$
\begin{aligned}
(3.61) \quad &\geq \tfrac{1-\gamma}{w^{2\gamma}(s_0)}\left[\left(\tfrac{N+\beta}{p+1}-\tfrac{N+\alpha-2}{2}\right)\int_0^{s_0} y^{N+\beta-1}w^{p+1}(y)dy - w^{p+1}(s_0)\tfrac{s_0^{N+\beta}}{p+1}\right.\\
&\quad\left.+\tfrac{(1-\gamma)d^\gamma}{p+1-\gamma}s_0^{N+\beta}w^{p+1+\gamma}(s_0)\right]\\
&\geq \tfrac{1-\gamma}{w^{2\gamma}(s_0)}\left[\left(\tfrac{N+\beta}{p+1}-\tfrac{N+\alpha-2}{2}\right)\tfrac{s_0^{N+\beta}}{N+\beta}w^{p+1}(s_0) - w^{p+1}(s_0)\tfrac{s_0^{N+\beta}}{p+1}\right.\\
&\quad\left.+\tfrac{(1-\gamma)d^\gamma}{p+1-\gamma}s_0^{N+\beta}w^{p+1+\gamma}(s_0)\right]\\
&= \tfrac{1-\gamma}{w^{2\gamma}(s_0)}s_0^{N+\beta}w^{p+1}(s_0)\left[\left(\tfrac{N+\beta}{p+1}-\tfrac{N+\alpha-2}{2}\right)\tfrac{1}{N+\beta}-\tfrac{1}{p+1}+\tfrac{(1-\gamma)d^\gamma}{p+1-\gamma}w^\gamma(s_0)\right].
\end{aligned}
\tag{3.63}
$$

Recalling that $w(s_0)=\frac{1}{d}$, we have that

$$
\begin{aligned}
(3.61) \quad &\geq \tfrac{1-\gamma}{d^{p+1-2\gamma}}s_0^{N+\beta}\left[\left(\tfrac{N+\beta}{p+1}-\tfrac{N+\alpha-2}{2}\right)\tfrac{1}{N+\beta}-\tfrac{1}{p+1}+\tfrac{(1-\gamma)}{p+1-\gamma}\right]\\
&= \tfrac{1-\gamma}{d^{p+1-2\gamma}}s_0^{N+\beta}\left[\tfrac{(1-\gamma)(N+2\beta-\alpha+2)-p(N+\alpha-2)}{2(N+\beta)(p+1-\gamma)}\right].
\end{aligned}
\tag{3.64}
$$

From hypothesis (H_1) we have that

$$
\frac{1-\gamma}{d^{p+1-2\gamma}}s_0^{N+\beta}\left[\frac{(1-\gamma)(N+2\beta-\alpha+2)-p(N+\alpha-2)}{2(N+\beta)(p+1-\gamma)}\right]>0\,. \tag{3.65}
$$

Then by (3.65), we have that (3.60) is positive.

Taking $t\to+\infty$ in (3.53), and using inequalities (3.54), (3.56) and (3.59), we obtain

$$
(1-\gamma)d^\gamma\left(\frac{(N+\beta)(1-\gamma)}{p+1-\gamma}-\frac{N+\alpha-2}{2}\right)\lim_{t\to\infty}\int_{s_0}^t y^{N+\beta-1}u^{\frac{p}{1-\gamma}+1}(y)\,dy<0\,, \tag{3.66}
$$

which contradicts inequality (3.55). Thus, we conclude that *limiting problem* (3.35) and (3.38), has no nonnegative solution. □

Finally, we conclude that exists $k_0\in\mathbb{N}$ such that $||v_{k_0}||_\infty\leq k_0$. This implies that v_{k_0} is solution of Problem (3.2).

3.3 Nonexistence of positive solution

Theorem 3.5. *Suppose hypothesis* (H_0). *Then the Problem (3.2) has no positive solutions for* $p\geq\dfrac{N+2\beta-\alpha+2}{N+\alpha-2}$.

Proof. Suppose $v \in C^1(\Omega) \cap C^0(\bar{\Omega})$ and $\dfrac{r^{N+\alpha-1}v'}{(1+g(v))^\gamma} \in C^1(\Omega)$ be a solution of Problem (3.2). Multiplying (3.2) by $rv'(r)$ and integrating on $[0,R]$, we find

$$-\int_0^R \left(\frac{r^{N+\alpha-1}v'(r)}{(1+g(v(r)))^\gamma}\right)' rv'(r)\,dr = \int_0^R r^{N+\beta}v^p(r)v'(r)\,dr\,, \tag{3.67}$$

We rewrite this expression (3.67) as

$$A = B\,. \tag{3.68}$$

The term on the left is

$$\begin{aligned} A &:= -\frac{R^{N+\alpha}v'^2(R)}{(1+g(0))^\gamma} + \int_0^R \frac{r^{N+\alpha-1}v'^2(r)}{(1+g(v(r)))^\gamma}\,dr + \int_0^R \frac{r^{N+\alpha}v''(r)v'(r)}{(1+g(v(r)))^\gamma}\,dr \\ &=: -\frac{R^{N+\alpha}v'^2(R)}{(1+g(0))^\gamma} + \int_0^R \frac{r^{N+\alpha-1}v'^2(r)}{(1+g(v(r)))^\gamma}\,dr + A_1\,. \end{aligned}$$

with,

$$\begin{aligned} A_1 &= \frac{1}{2}\int_0^R \frac{r^{N+\alpha}(v'^2(r))'}{(1+g(v(r)))^\gamma}\,dr \\ &= \frac{1}{2}\frac{R^{N+\alpha}v'^2(R)}{(1+g(0))^\gamma} - \frac{N+\alpha}{2}\int_0^R \frac{r^{N+\alpha-1}v'^2(r)}{(1+g(v(r)))^\gamma}\,dr + \frac{\gamma}{2}\int_0^R \frac{r^{N+\alpha}v'^3(r)g'(v(r))}{(1+g(v(r)))^{\gamma+1}}\,dr\,. \end{aligned}$$

Now

$$A = -\frac{1}{2}\frac{R^{N+\alpha}v'^2(R)}{(1+g(0))^\gamma} + \left(1 - \frac{N+\alpha}{2}\right)\int_0^R \frac{r^{N+\alpha-1}v'^2(r)}{(1+g(v(r)))^\gamma}\,dr + \frac{\gamma}{2}\int_0^R \frac{r^{N+\alpha}v'^3(r)g'(v(r))}{(1+g(v(r)))^{\gamma+1}}\,dr\,. \tag{3.69}$$

On the other hand, the right hand side of (3.67) is

$$\begin{aligned} B &= \int_0^R r^{N+\beta}v^p(r)v'(r)\,dr \\ &= \frac{1}{p+1}\int_0^R r^{N+\beta}(v^{p+1}(r))'\,dr \\ &= \frac{1}{p+1}r^{N+\beta}v^{p+1}(r)\,\big|_0^R - \frac{N+\beta}{p+1}\int_0^R r^{N+\beta-1}v^{p+1}(r)\,dr\,. \end{aligned}$$

So that,

$$B = -\frac{N+\beta}{p+1} \int_0^R r^{N+\beta-1} v^{p+1}(r)\, dr \,. \tag{3.70}$$

Similarly, if we multiply the equation (3.2) by v and integrating on $[0, R]$, getting

$$\int_0^R r^{N+\beta-1} v^{p+1}(r)\, dr = \int_0^R \frac{r^{N+\alpha-1} v'^2(r)}{(1+g(v(r)))^\gamma}\, dr \,. \tag{3.71}$$

Combine (3.70) and (3.71), we deduce

$$B = -\frac{N+\beta}{p+1} \int_0^R \frac{r^{N+\alpha-1} v'^2(r)}{(1+g(v(r)))^\gamma}\, dr \,. \tag{3.72}$$

This calculation and (3.68) yield

$$\left(1 - \frac{N+\alpha}{2} + \frac{N+\beta}{p+1}\right) \int_0^1 \frac{r^{N+\alpha-1}(v'(r))^2}{(1+g(v(r)))^\gamma}\, dr = \frac{1}{2}\frac{R^{N+\alpha} v'^2(R)}{(1+g(0))^\gamma} - \frac{\gamma}{2} \int_0^R \frac{r^{N+\alpha} v'^3(r) g'(v(r))}{(1+g(v(r)))^{\gamma+1}}\, dr \,. \tag{3.73}$$

Hence if v is non-negative and nontrivial solution, we have

$$\frac{1}{2}\frac{R^{N+\alpha} v'^2(R)}{(1+g(0))^\gamma} - \frac{\gamma}{2} \int_0^R \frac{r^{N+\alpha} v'^3(r) g'(v(r))}{(1+g(v(r)))^{\gamma+1}}\, dr > 0 \,. \tag{3.74}$$

Thus

$$\left(1 - \frac{N+\alpha}{2} + \frac{N+\beta}{p+1}\right) \int_0^1 \frac{r^{N+\alpha-1}(v'(r))^2}{(1+g(v(r)))^\gamma}\, dr > 0 \,, \tag{3.75}$$

which implies

$$1 - \frac{N+\alpha}{2} + \frac{N+\beta}{p+1} > 0 \,,$$

hence,

$$p < \frac{N+2\beta-\alpha+2}{N+\alpha-2} \,.$$

□

Theorem 3.6. *If $\beta - \alpha + 2 \leq 0$ then there is no solution $v \in C[0, R] \cap C^1(0, 1)$ of Problem (3.2).*

Proof. Suppose that $v \in C[0,R] \cap C^1(0,1)$ is a solution of problem (3.2). Thus the function v verifies the integral equation

$$v(r) = \int_r^R \frac{(1+g(v(s)))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau) d\tau ds\,.$$

Since v is nonnegative, nontrivial, and decreasing function, we have that, given $\varepsilon > 0$ small, there exists $r_0 \in (0, \frac{R}{2})$ such that

$$v(r_0) > \varepsilon\,.$$

Then, we have

$$\begin{aligned} v(r) &= \int_r^R \frac{(1+g(v)(s))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau) d\tau ds \\ &\geq \varepsilon^p \int_r^{r_0} \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} d\tau ds \\ &= \frac{\varepsilon^p}{N+\beta} \int_r^{r_0} \frac{1}{s^{-(\beta-\alpha+1)}} ds. \end{aligned}$$

Since $\beta - \alpha + 2 \leq 0$, we have

$$\lim_{r \to 0} v(r) = +\infty\,.$$

This proves $v \notin C[0,R] \cap C^1(0,1)$.

□

Chapter 4

Appendix

4.1 Topological Degree of Leray-Schauder

Let X be a real Banach space, let Ω be a bounded, open subset of X and let $\phi = I - T$, where I is the inclusion map of $\bar{\Omega}$ into X and $T : \Omega \to X$ is compact.
If $b \notin \phi(\partial\Omega)$, then there exists a map of finite range $T_1 : \Omega \to X_1$ (finite range means that $dim X_1 < \infty$) such that

$$\sup_{u \in \bar{\Omega}} ||T_1 u - Tu|| < \text{dis}(b, \phi(\partial\Omega)) .$$

In addition, the integer given by the Brouwer degree $\deg((I - T_1)|_{\Omega \cap X}, \Omega, b)$ is independent on T_1 . Therefore we can define the topological degree of Leray-Schauder

$$\deg(\phi, \Omega, b) = \deg((I - T_1)|_{\Omega \cap X}, \Omega, b) .$$

It satisfies the following basic properties.

(i) **Normalization property.**

$$\deg(I, \Omega, b) = 1, \text{ if } b \in \Omega .$$

(ii) **Additivity property.**
Assume that Ω_1 and Ω_2 are open, bounded and disjoint subsets of Ω. If $b \notin \phi(\Omega \setminus (\Omega_1 \cup \Omega_2))$ then

$$\deg(\phi, \Omega, b) = \deg(\phi, \Omega_1, b) + \deg(\phi, \Omega_2, b) .$$

(iii) **Homotopy property.**
Let $S \in C([0,1] \times \Omega, X)$ be a compact map and define $H(t,u) = u - S(t,u)$. If $b : [0,1] \to X$ is continuous and $b(t) \notin H([0,1] \times \partial\Omega)$, then

$$\deg(H(t,.), \Omega, b(t)) = const \quad \text{for all} \quad t \in [0,1] \, .$$

From the above properties it is easy to prove the following ones:

(iv) $\deg(\phi, \emptyset, b) = 0$.

(v) **Existence property.**
If $\deg(\phi, \Omega, b) \neq 0$, then there exists $u \in \Omega$ such that $\phi(u) = b$.

(vi) **Excision property.**
If $K \subset \Omega$ is closed and $b \notin \phi(K)$, then

$$\deg(\phi, \Omega, b) = \deg(\phi, \Omega - K, b) \, .$$

(vii)

$$S|_{\partial\Omega} = T|_{\partial\Omega} \quad \text{then} \quad \deg((I-S), \Omega, b) = \deg((I-T), \Omega, b) \, .$$

(viii) **General homotopy property.**
Let Θ be a bounded, open subset of $\mathbb{R} \times X$ and let $H : \Theta \to X$ be a compact map. For every $\lambda \in R$ we consider the λ-slice

$$\Theta_\lambda = \{u \in X : (\lambda, u) \in \Theta\} \, ,$$

and the map $H_\lambda : \Theta_\lambda \to X$ given by

$$H_\lambda(u) = H(\lambda, u) \, .$$

If $u - H_\lambda(u) = b$, for all $u \in \partial\Theta_\lambda$, for all $\lambda \in [a,b]$, then the topological degree $\deg(I - H_\lambda, \Theta_\lambda, b)$ is well-defined and independent of λ.

(ix) **Continuity.**

(a) Continuity with respect to b.
The degree is constant in each connected component of $X - \phi(\partial\Omega)$.

(b) Continuity with respect to T .
There exists a neighborhood V of T in the space $Q(\Omega, X)$ of the compact operators from $\bar{\Omega}$ in X such that

$$\deg(I - S, \Omega, b) = const \quad \text{for all} \quad S \in V \, .$$

4.1.1 A theorem of Leray and Schauder

Let X be a real Banach space, let Ω be a bounded and open subset of X, let $a < b$ in $\mathbb{R}$, and let $T : [a,b] \times \Omega \to X$ be a compact map. For $\lambda \in [a,b]$, consider the equation

$$\phi(\lambda, u) = u - T(\lambda, u) = 0, \quad u \in X\,. \tag{4.1}$$

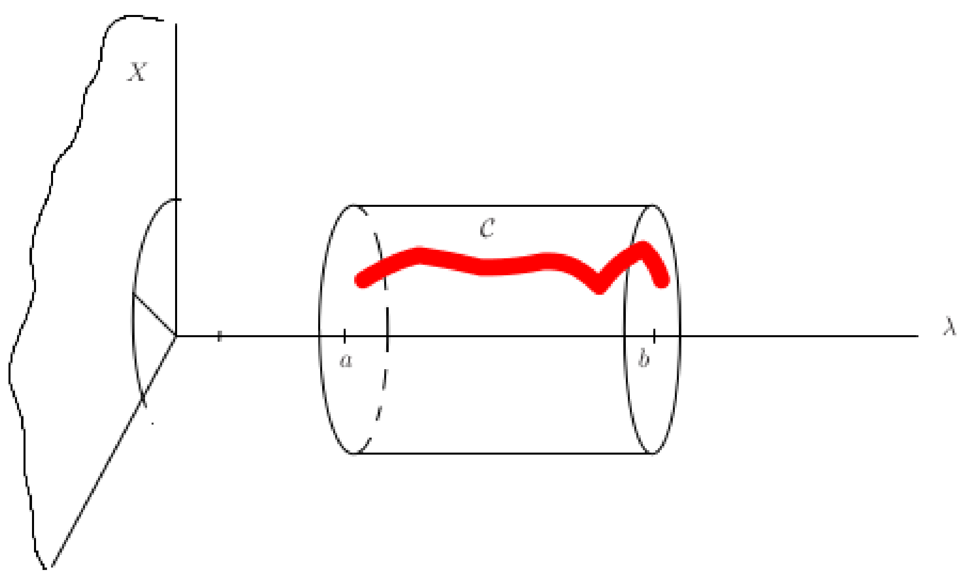
Sometimes, to put in evidence the dependence of (4.1) on λ, we refer it as $(4.1)_\lambda$. Observe that T can be seen as a family of compact operators

$$T_\lambda(u) := T(\lambda, u), \quad u \in X\,.$$

Similarly, we denote $\phi_\lambda = I - T_\lambda$. Define

$$\Sigma = \{(\lambda, u) \in [a,b] \times \Omega : \phi(\lambda, u) = 0\}\,.$$

We use the notation Σ_λ for the λ-slice, i.e. $\Sigma_\lambda = \{u \in \Omega : (\lambda, u) \in \Sigma\}$.



Theorem 4.1 (Leray-Schauder, 1934)**.** *Assume that X is a real Banach space, Ω is a bounded, open subset of X and $\phi : [a,b] \times \Omega \to X$ is given by $\phi(\lambda, u) = u - T(\lambda, u)$ with T a compact map. Suppose also that*

$$\phi(\lambda, u) = u - T(\lambda, u) = 0, \quad \textit{for all } (\lambda, u) \in [a,b] \times \partial\Omega\,.$$

If

$$deg(\phi_\lambda, \Omega, 0) \neq 0\,,$$

then

(1) $(4.1)_\lambda$ *has a solution in* Ω *for every* $a \leq \lambda \leq b$.

(2) Furthermore, there exists a compact connected set $O \subset \Theta$ *such that*

$$O \cap \Theta_a = \emptyset \;\; and \;\; O \cap \Theta_b \neq \emptyset\,.$$

4.1.2 Index of an isolated zero

Let $\phi = I - T$ be with $T : \bar{\Omega} \to X$ a compact operator. If $u_0 \in \Omega$ is an isolated solution of the equation $\phi(u) = 0$, i.e. a unique solution of this equation in a neighborhood of u_0, then, for $r_0 > 0$ sufficiently small, we deduce from the excision property that

$$\deg(\phi, Br_0(u_0), 0) = \deg(\phi, B_r(u_0), 0), \;\; \text{for all} \;\; r \in (0, r_0)\,,$$

where $B_r(u_0) = \{u \in \Omega : |u - u_0| < r\}$. Then we know that $\deg(\phi, B_r(u_0), 0)$ is the same integer for all $r \in (0, r_0]$. This number is called the index of u_0 and is denoted by $i(\phi, u_0)$.

4.1.3 Fixed–point result due to Krasnosel'skii

Definition 4.1. *A subset* C *of a Banach space* X *is said to be a cone if* $\lambda C \subset C$ *for all* $\lambda \geq 0$ *and* $C \cap (-C) = \{0\}$.

The results of this work are based on the following well Known fixed - point result due to Krasnosel'skii.

Theorem 4.2 (Krasnosel'skii)**.** *Let* C *be a cone in a Banach space, and let* $F : C \to C$ *be a compact operator such that* $F(0) = 0$. *Suppose there exists* $\delta > 0$ *verifying*

(*a*) $u \neq t\,F(u)\,,$ *for all* $||u|| = \delta$ *and* $t \in [0, 1]\,.$

Suppose further that there exist a compact homotopy $H : [0, 1] \times C \longrightarrow C$ *and* $\eta > \delta$ *such that:*

(*b*) $F(u) = H(0, u)\,,$ *for all* $u \in C$.

(*c*) $H(t, u) \neq u\,,$ *for all* $||u|| = \eta$ *and* $t \in [0, 1]$.

(*d*) $H(1,u) \neq u$, *for all* $||u|| \leq \eta$.

Then F *has a fixed point* u_0 *verifying* $\delta < ||u_0|| < \eta$.

The proof can be seen in (cf. [37])

Bibliography

[1] A. Alvino, L. Boccardo, V. Ferone, L. Orsina and G. Trombetti, *Existence results for nonlinear elliptic equations with degenerate coercivity*, Annali di Matematica. **182** (2000), 53-79.

[2] V. Anuradha, D. D. Hai, and R. Shivaji, *Existence Results for Superlinear Semipositone BVP'S*, Proc. Amer. Math. Soc. **124:3** (1996), 757–763.

[3] D. Arcoya, L. Boccardo, T. Leonori, Alessio Porretta, *Some elliptic problems with singular natural growth lower order terms*, J. Differential Equations, **249** (2010), 2771-2795.

[4] F. V. Atkinson and L. A. Peletier, *Elliptic equations with nearly critical growth*, J. Differential Equations, **70** (1987), 349-365.

[5] Z. Bai, H. Wang, *On positive solutions of some nonlinear fourth–order beam equations*, J. Math. Anal. Appl. **270** (2002), 357–368.

[6] J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Applied Mathematical Sciences, **83** Springer–Verlag, New York, 1989.

[7] J. Bebernes and D. Eberly, *Mathematical problems from combustion theory*, Applied Mathematical Sciences, **83**. Springer–Verlag, New York, 1989.

[8] A. Benkirane, A. Youssfi, D. Meskine, *Bounded solutions for nonlinear elliptic equations with degenerate coercivity and data in an L log L*, Bull. Belg Math. Soc. Simon Stevin, **15** (2008), 369-375.

[9] L. Boccardo, *Some Elliptic Problems Whit Degenerate Coercivity*, Avanced Nonlinear Studies, **6** (2006), 1-12.

[10] L. Boccardo, H. Brezis, *Some Remarks on a Class of Elliptic Equations with Degenerate Coercivity* , Bollettino U. M. I. **8** 6-B (2003), 521-530.

[11] L. Boccardo, G. Croce and L. Orsina, *Nonlinear degenerate elliptic problems with $W_0^{1,1}(\Omega)$ solutions*, Manuscripta math.**137** (2012), 419-439.

[12] L. Boccardo, A. Dall'aglio, L. Orsina, *Existence and regularity results for some elliptic equations with degenerate coercivity*, Atti Sem. Mat. Fis. Univ. Modena. **46** (1998), suppl., 51-81.

[13] L. Boccardo, L. Orsina *Semilinear elliptic equations with singular nonlinearities* , Calc. Var.**37** (2009), 363-380.

[14] L. Boccardo, S. Segura de León and C. Trombetti, *Bounded and Unbounded Solutions for a class of Quasi-Linear Elliptic Problems Whit a Quadratic Gradient Term*, J. Math. Pures Appl. **9** (2001), 919-940.

[15] A. Brezis and L. Niremberg, *Positive solutions of nonlinear elliptic equations involving critical exponents*, Comm. Pure Math. Appl, **36** (1983), 437-477.

[16] A. Cabada and R. L. Pouso *Existence theory for functional p-Laplacian equations whith variable exponents*, Nonlinear Analysis **52** (2003), 557-572.

[17] M. Conti, L. Merizzi and S. Terracini, *On the existence of many solutions for a class of superlinear elliptic systems. J. Diff. Eq.* **167** (2000), 357–387.

[18] P. Clement, D. de Figueiredo and E. Mitidieri *Quasilinear elliptic equation with critical exponents*, Topol. Methods Nonlinear Anal. **7** (1996), no. 1, 133â170.

[19] P. Clement, R. Manásevich and E. Mitidieri *Positive solutionsfor a quasilinear system via blow up*, Comm. in P.D.E. **18** (1993), 2071-2106.

[20] E. N. Dancer, *Fixed point index calculations and applications*, Topological Nonlinear Analysis; degree, singularity and variations, M. Maetzu and A. Vignoli editors, Birkhauser, Boston, 1995, pp. 303–340.

[21] E.N. Dancer, *Fixed point index calculations and applications*, Topological Nonlinear Analysis; degree, singularity and variations, M. Maetzu and A. Vignoli editors (303–340). Birkhauser, Boston 1995.

[22] D. G. de Figueiredo, *Positive solutions of semilinear elliptic equations*, Lecture Notes in Mathematics **957**, Springer–Verlag, Berlin–Heidelberg–New York, 1982, pp. 34–87.

[23] D. G. de Figueiredo and P. Ubilla, *Superlinear Systems of Second–Order ODE'S*, *Nonlinear Anal.* **68** (2008), 1765–1773.

[24] D. G. de Figueiredo, P. L. Lions, and R. D. Nussbaum, *A priori bounds and existence of positive solutions of semilinear elliptic equations*, J. Math. Pura Appl. **61** (1982), 41–63.

[25] D. G. de Figueiredo and Yang J., *A priori bounds for positive solutions of a non-variational elliptic system* , *Comm. PDE* **26** (2001), 2305–2321.

[26] K. Deimling, *Nonlinear Functional Analysis*, Springer–Verlag, New York, 1985.

[27] J. do Ó, S. Lorca and P. Ubilla, *Local superlinearity for elliptic systems involving parameters. J. Diff. Eq.* **211** (2005), 1–19.

[28] D. R. Dunninger and J. C. Kurtz, *Existence of Solutions for Some Nonlinear Singular Boundary Value Problems*, J. Math. Anal. Appl. **115** (1986), 396-405.

[29] D. R. Dunninger and Haiyan Wang, *Existence and multiplicity of positive solutions for elliptic systems*, Nonlinear Anal. **29** (1997), 1051–1060.

[30] D. R. Dunninger, and Haiyan Wang, *Multiplicity of positive radial solutions for an elliptic system on an annulus*, Nonlinear Anal. **42** (2000), 803–811.

[31] L. Evans, *Partial Differential Equations*, American Mathematical Soc. 01 June 1998.

[32] J. B. Garner and R. Shivaji, *Existence and uniqueness of solutions to a class of singular diffusion problems*, App. Anal. **30** (1988), 101-114.

[33] D. Guo and V. Lakshmikantham, *Nonlinear Problem in Abstract Cones*, Academinc Press, Orlando, FL, 1988.

[34] P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, Inc. New York, 1964.

[35] D. D. Joseph and E. M. Sparrow, *Nonlinear diffusion induced by nonlinear sources*, Quart. Appl. Math. **28** (1970), 327–342.

[36] H. B. Keller and D. S. Cohen, *Some positone problems suggested by nonlinear heat generation*, J. Math. Mech. **16** (1967), 1361–1376.

[37] M. A. Krasnosel'skii, *Positive solutions of operators equations*, Noordhoff, Groningen, 1964.

[38] Yong-Hoon Lee, *Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus*, *J. Diff. Eq.* **174** (2001), 420–441.

[39] Yongxiang Li, *On the existence and nonexistence of positive solutions for nonlinear Sturm-Liouville boundary value problems*, J. Math. Appl. **304** (2005), 74-86.

[40] Y. Li, *Positive solutions of fourth–order boundary value problems with two parameters*, J. Math. Anal. Appl. **281** (2003), 477–484.

[41] Yansheng Liu, *Multiple Positive solutions of nonlinear singular boundary value problem for fourth–order equations*, Appl. Math. Lett. **17** (2004), 747–757.

[42] Yao Qing–Liu, *Existence of solutions and positive solutions to a fourth–order two–point BVP with second derivative*, J. Zhejiang Univ. Sci. Ed. **5** (2004), 353–357.

[43] Yao–Qing Liu, Z. Bai, *Existence of positive solutions of the boundary problems for* $u^{(4)}(t) - \lambda h(t) f(u(t)) = 0$, Chinese Ann. Math. Ser. A **20** (1999), 575–578 (in Chinese).

[44] R. Ma, C. Tisdell, *Positive solutions of singular sublinear fourth–order boundary value problems*, Appl. Anal. **84** (2005), 1199–1220.

[45] R. Ma, *Multiple positive solutions for a semipositone fourth–order boundary value problem*, Hiroshima Math. J. **33** (2003), 217–227.

[46] B.P. Rynne, *Infinitely many solutions of superlinear fourth order boundary value problems*, Topol. Methods Nonlinear Anal. **19** (2002), 303–312.

[47] B. P. Rynne, *Second order, Sturm-Liouville probems whith asymmetric, Superlinear nonlinearities*, Nonlinear Analysis **54** (2003), 939-947.

[48] B. P. Rynne, *Second order, Sturm-Liouville probems whith asymmetric, Superlinear nonlinearities II*, Nonlinear Analysis **57** (2004), 905-916.

[49] Haiyan Wang, *Solvability of some singular boundary value problems for the equation* $(p(t)r(t,u)u')'/q(t) = f(t,u,pru')$, Nonlinear Analysis, Theory, Methods & Applications **18:7** (1992), 649-655.

[50] G. Weigao and R. Jingli, *New Existence theorems of positive solutions for Sturm-Liouville boundary value problems*, Applied Mathematics and Computation **148** (2004), 631-644.

[51] L. Wenbin and S. Xiaoqiu, *Nonlinear singular boundary value problems in the study of symmetric plasma*, J. Math. Anal. Appl. **248:1** (2000), 184-194.