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Higher Order Iterative Methods on Riemannian Manifolds

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HIGHER ORDER ITERATIVE METHODS ON RIEMANNIAN MANIFOLDS

ESTÉ TRABAJO DE GRADUACIÓN FUE ELABORADO BAJO LA SUPERVISIÓN DEL PROFESOR SERGIO EUGENIO PLAZA SALINAS, DEL DEPARTAMENTO DE MATEMÁTICAS Y CIENCIA DE LA COMPUTACIÓN DE LA UNIVERSIDAD DE SANTIAGO DE CHILE Y LA DRA. SONIA BUSQUIER SÁEZ, DEL DEPARTAMENTO DE MATEMÁTICA APLICADA Y ESTADÍSTICA DE LA UNIVERSIDAD POLITÉCNICA DE CARTAGENA, EL CUAL HA SIDO APROBADO POR LOS MIEMBROS DE LA COMISION CALIFICADORA.

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A la memoria del excelente profesor y amigo,
Sergio Plaza Salinas

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Informe de la Tesis del alumno Sr. Rodrigo Castro Marín titulada “**Higher Order Iterative Methods on Riemannian Manifolds** “.

La resolución de ecuaciones no lineales es un problema clásico en matemáticas. Para ello se suelen utilizar métodos iterativos. Se parte de una aproximación inicial y mediante una función de iteración se intenta mejorar las aproximaciones generando una sucesión que bajo ciertas hipótesis convergerá a la solución de la ecuación original.

Son muchos grupos los que actualmente trabajan en la creación y análisis de métodos iterativos. Los estudios contemplan tanto el caso escalar como el caso de sistemas de ecuaciones e incluso ecuaciones entre espacios de Banach.

Uno de los métodos más usado y estudiado es el método de Newton que se basa en una linealización de primer orden de la ecuación. Son muchas las variantes de este método que podemos encontrar en la literatura especializada. Recientemente, se ha hecho una extensión del método para problemas entre variedades Riemannianas. Son este tipo de problemas los abordados en la presente Tesis.

Para aproximar y entender este tipo de problemas es necesario usar herramientas de diferentes ramas de la Matemáticas. En particular, Geometría Diferencial, Análisis Funcional, Análisis Numérico y Álgebra Lineal. Esto hace que los problemas tengan una motivación extra para el investigador.

La presente Tesis está estructurada en diferentes temas bien definidos donde se estudian varias extensiones del método de Newton. En todos ellos se utiliza un paralelismo con la teoría clásica en espacios de Banach. Se estudia la convergencia de los distintos métodos y se dan resultados de existencia y unicidad de solución del problema original.

La Tesis es original, interesante, no trivial y a su vez abre caminos de futuros estudios, es por ello que propondría la máxima clasificación de 7 puntos para la misma.

Informe realizado por: Sergio Amat Plata

Informe de la Tesis del alumno Sr. Rodrigo Castro Marín titulada "**High Order Iterative Method in Riemannian Manifolds**".

En matemáticas, uno de los problemas más habituales al que nos enfrentamos es la resolución de ecuaciones. Cuando nos encontramos con la expresión $F(x) = 0$, cabe pensar en diferentes situaciones, resolución de un sistema de ecuaciones, encontrar la solución de una ecuación diferencial o hallar las raíces de un polinomio. Cuando la obtención de la solución no es posible (hecho que ocurre en numerosas ocasiones), nos debemos conformar con aproximaciones de las mismas. Este hecho da pie a los procesos numéricos, dando vida a los métodos iterativos. El ejemplo más estudiado es el método de Newton que tiene orden de convergencia dos.

El adelanto de los medios técnicos ha permitido el desarrollo de algunos métodos iterativos, para resolver ecuaciones no lineales en espacios de Banach. Por ejemplo, la simplicidad para evaluar inversas y segundas derivadas de Fréchet, en algunos casos, ha aumentado el uso de métodos de tercer orden, como, los de Halley y Chebyshev.

Una vez testada la eficiencia de un método iterativo, el aspecto más importante a considerar es la convergencia. Existen resultados ``tipo Kantarovich'' que establecen condiciones suficientes en el operador y en la primera aproximación a la solución para asegurar que la sucesión de las iteraciones del pivote converja a una solución de la ecuación, dando lugar a los llamados teoremas semilocales de convergencia.

En esta Tesis se intenta generalizar varios métodos, aparecidos en la literatura para problemas entre espacios de Banach, al caso de problemas entre variedades Riemannianas. El estudio es riguroso, donde teoremas de convergencia para todos los métodos propuestos. Las familias de métodos estudiados suelen ser de orden mayor a tres y pueden ser considerados como extensiones del método de Newton. Cabría notar que en la literatura sólo aparece la extensión de Newton para problemas entre variedades. En particular, a lo largo de la Tesis se han tenido que desarrollar maquinaría matemática para la extensión y estudios de métodos de orden mayor a dos. Esta parte es sin duda la más compleja de la memoria. Además destacar los teoremas de existencia y unicidad derivados de la teoría.

Del trabajo de la Tesis se podrán extraer un mínimo de cinco publicaciones, así pues se trata de una memoria no muy larga pero cargada de aportes nuevos. No obstante se complementa con un capítulo introductoria donde se plasman todas las herramientas que después serán usadas.

Así mi opinión es que la Tesis es merecedora de la máxima calificación, es decir, 7 puntos.

Informe realizado por: Sonia Busquier Sáez

Informe de Tesis para optar al grado de Doctor en Ciencia con Mención en Matemática

Título: "Higher order iterative methods on Riemannian Manifolds"
Alumno: Rodrigo Alberto Castro Marín.

Este trabajo de tesis consiste en la implementación y generalización de algunos métodos numéricos al contexto de variedades Riemannianas. Los métodos numéricos clásicos tales como el método de Newton, fueron desarrollados para encontrar soluciones aproximadas a ecuaciones de la forma $f(x)=0$, donde f es una función real. Naturalmente uno se pregunta si estos métodos pueden extenderse a otro tipo de ecuaciones. Una generalización importante ha sido la extensión del método de Newton a espacios de Banach por L. V. Kantorovich. Esta generalización ha resultado muy fructífera, y se usa, por ejemplo, para analizar la existencia de soluciones a ecuaciones integrales nolineales. Otra generalización natural es a variedades Riemannianas. Esta es la problemática considerada en esta tesis. Más precisamente, el problema es probar que es posible resolver, usando aproximaciones, ecuaciones del tipo $X(p)=0$, donde X es un campo vectorial sobre una variedad Riemanniana dada.

Rodrigo Castro considera varios métodos numéricos clásicos, especialmente los métodos de Kantorovich y de Chebyshev-Halley, y los implementa sobre una variedad Riemanniana. El trabajo es muy sofisticado y técnico, no se trata de una generalización inmediata. La ausencia de estructuras lineales en variedades Riemannianas, y la necesidad de trabajar usando sólo herramientas definidas geométricamente, hace que los teoremas sobre convergencia y unicidad de soluciones que aparecen en esta tesis sean altamente no-triviales.

En mi opinión los resultados presentados en este trabajo son dignos de publicación en una revista internacional y la redacción es adecuada, por lo que considero el trabajo de tesis aprobado. Por otra parte, me parece que la tesis se habría visto beneficiada si se hubiesen incluido aplicaciones, y si hubiese habido una discusión más acabada de métodos numéricos en espacios de Banach y de trabajos previos en el contexto de variedades Riemannianas.
Califico este trabajo con nota **6.8.**

Enrique Reyes García

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The main theme of the thesis *Higher order iterative methods on Riemannian manifolds* is that of introducing and studying iterative methods to find singularities of vector fields defined on a (Riemmannian) manifold. This theme grows as variations on classical root finding algorithms such as Newton's method. These methods were generalized by Kantorovich to the context of finding zeros of Banach spaced valued functions defined on Banach spaces. It is worth mentioning that Kantorovich's method and its variations refer to a semilocal study. That is, given an approximate zero of a function, design a sequence that converges to a zero of the function which hopefully is the unique one nearby. Moreover, one is interested on constructing such sequence at a minimum computational cost and, it is of theoretical and practical importance to determine how large is the region where the limiting zero is the unique one.

Castro's work is inspired on recent results (e.g. [4] and [17]) that started studying Newton-Kantorovich type methods to find singularities of vector fields on manifolds (from now on all manifolds are endowed with a Riemannian structure). This thesis brings the state of the art for these type methods on manifolds to a substantially higher level. He starts by proving, in this context, a version of Kantorovich's simplified method which is computationally cheaper but its convergence is slower when compared to the classical Kantorovich method. The proof is technical and requires appropriate and careful analysis of the error bounds, which in the context of a "non-linear" ambient space (i.e. a manifold) is much more subtle than in the context of Banach spaces. Then the author continues to analyse variations of Chebychev-Halley methods and other "higher order" methods. Here it is worth noting that it is not inmediate what are the appropriate adaptations of these methods to the context of manifold. The author proves convergence and uniqueness of the methods and/or family of methods introduced. Whether the sequences given by these "higher order" methods effectively converge faster (i.e. deserve to be called higher order methods) is left open and looks as a natural direction to proceed.

In my opinion, the work presented in this thesis is original, interesting and of importance. As I pointed out before it represents a significant contribution in the study of iterative singularity finding methods for vector fields on manifolds. One of the aspects that I particularly would like to emphasize is the fact that this thesis represents a systematic study of one theme with many new results and that opens the path for new questions. My only regret is that the writing is still rough in many parts (with plenty of misprints).

For this reason I evaluate the thesis with a 6.9.

Jan Kiwi.
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Informe de tesis:
Higher order iterative methods on Riemannian manifolds
presentada por
Rodrigo Castro Marín
para optar al gardo de Doctor en Ciencia con mención en Matemática de la Universidad de Santiago de Chile

En el trabajo arriba mencionado, el alumno Rorigo Castro se interesa en extender diversos métodos numéricos para la extracción de raíces de ecuaciones al ámbito de variedades Riemannianas. Si bien éste es un tema ya tratado para el método más célebre de todos, a saber, el método de Newton, diversos otros métodos (en general, de orden superior) no habían sido abordados en este contexto. Rodrigo demuestra un buen dominio del tema y utilizando elementos concisos de los de geometría Riemanniana, logra tratar (entre otros) la extensión de métodos del tipo Chebyshev-Halley al ámbito de las variedades.

El escrito de Rodrigo dista aún mucho de estar convenientemente escrito: además de problemas de inglés, la Introducción carece de un párrafo en el que se den motivaciones precisas del tema a tratar, y la discusión/contraposición de/entre los resultados existentes y los nuevos es claramente mejorable. Pese a esto, me parece que los resultados obtenidos darán prontamente lugar a publicaciones de interés, y que por lo tanto justifican ampliamente el otorgamiento del grado de Doctor al que se postula. Por esta razón, avalo la defensa de tesis respectiva y califico este trabajo con nota 6.3 (seis coma tres).

Andrés Navas Flores

Abstract

The iterative methods of higher order for finding zeroes of a vector function have been extensively studied for over a century. Maybe the most important ones are: Newton, Shamanskii, Halley, Super-Halley, Chevyshev, Chevyshev-Halley and the Two steps method.
These methods have been extended to Banach spaces starting with the work of Kantorovich in the 1960's. There have been several proofs of convergence and uniqueness of the sequences associated to these methods, under different types of hypotheses regarding the vector functions considered and the starting points.
In this thesis we extend the methods mentioned above to the context of complete Riemannian Manifolds. These extensions allow us to find zeroes (singularities) of vector fields on manifolds. We prove the convergence of the sequences appearing in each method, and uniqueness of the singularities determined by them. For this we use non-trivial adaptations to the context of manifolds of some of the techniques studied in Banach spaces such as the majorizing sequence.

Introduction

Recently; [1], [4], [17], [18], there has been a growing interest in studying numerical algorithms on manifolds. There are many numerical problems posed on manifolds that arise naturally in many contexts. For example, finding roots of vector fields defined on surfaces or in spaces of matrices (see [6]).

For us "a method" as the construction of a sequence of points, so we say that a method converges if the associated sequence converges. In vector spaces, the most famous method to approximatively solve a nonlinear differentiable equation $F(x) = 0$ is Newton's method, where F is a differentiable mapping from a vector space X into a vector space Y. This method, can be extended to Banach spaces, using Fréchet-derivatives, as proven by Kantorovich [12]. In addition, he introduces the called "simplified method of Kantorovich", in which the derivative is computed only at a single point. In 2001, using Kantorovich's ideas [17], Newton's method was extended to Riemannian manifolds. In this work, we will extend the Kantorovich simplified method to Riemannian manifolds. Moreover, we will prove, in this context, that the orders of convergence of the Kantorovich method and the simplified Kantorovich method are two and one respectively. We will also extend the higher order method called the Shamanskii method [2], [10], [27], to this new context. This method combines the Kantorovich and the simplified Kantorovich methods.

As it is well known, there are several kinds of cubic order generalizations for Newton's method. The most important ones are the Chebyshev method and the Halley method [21], [23], [24]. Another more general family of cubic extensions is the family of Euler-Halley type methods on Banach spaces.

The Chebyshev-Halley method are probably the best known cubically convergent iterative procedures for solving nonlinear equations, in the same way as the Kantorovich method in Banach spaces; it requires an evaluation of the second Fréchet-derivative at each step and was proven for Ioannis K. Argyros in 1997 [5]. In this work we will extend this method to Riemannian manifolds.

Within the methods of the Euler-Halley family, perhaps, the best known are Chebyshev, Halley, Two-step, α-Methods and some of their approximations using divided differences or

similar techniques, in Banach spaces, some generalizations of them were made in [21], [24], [22]. They are often used to find solutions of differential equations and integro-differential equations in the space of continuous functions defined on an interval [5], [20], [21], they have also been used to calculate basins of attraction of functions defined in the complex plane, that give rise to beautiful fractals [22]. We will prove convergence and uniqueness on Riemannian manifolds, with some changes in the hypotheses, for the methods described in [22] and [24].
There exist other methods of cubic convergence without evaluating any second derivative [23], even without evaluating any first derivative or any bilinear operator [20]. these methods use divided differences, techniques that we will generalize to the context of Riemannian manifolds.
We organize this work as follows:
In the first Chapter we introduce some basic concepts of Riemanian Geometry, which we will use in chapters 2 and 3, such as the covariant derivative, geodesic, paralell transport, etc. In the second Chapter we will prove a theorem on existence and uniqueness for the Kantorovich method but fixing the covariant derivative in the point P_0 in all iterations, this method will be called simplified Kantorovich method on Riemanian manifolds. Next combining the Kantorovich method with the simplified Kantorovich method, we will present our first method of higher order on manifolds.
In the third Chapter, we study diferent kinds of third order methods, which generalize to manifolds the higher order methods mentioned before. The first three of them involve the second covariant derivate, in contrast with the last, which just involve one covariant derivative by using divided differences, which is pleasant from the computational point of view.

Contents

Chapter 1

Basic results

1.1 Basic definitions and preliminary results

In this section, we introduce some fundamental properties and notation of Riemannian manifolds.

Definition 1 *A differentiable manifold of dimension m is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^m \longrightarrow M$ of open sets U_α of $\mathbb{R}^m$ into M such that:*

(i) $\cup_\alpha x_\alpha (U_\alpha) = M$.

(ii) for any pair α, β with $x_\alpha (U_\alpha) \cap x_\beta (U_\beta) = W \neq \phi$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^m$ and the mappings $x_\beta^{-1} \circ x_\alpha$ are differentiable.

(iii) The family $\{(U_\alpha, x_\alpha)\}$ is maximal relative to the conditions (i) and (ii).

The pair (U_α, x_α) (or the mapping x_α) with $p \in x_\alpha (U_\alpha)$ is called a parametrization (or system of coordinates) of M at p; $x_\alpha (U_\alpha)$ is then called neighborhood at p and $\left(x_\alpha (U_\alpha), x_\alpha^{-1}\right)$ is called a coordinate chart. A family $\{(U_\alpha, x_\alpha)\}$ satisfying (i) and (ii) is called a differentiable structure on M.

Let M denote a differentiable manifold, given $p \in M$ and T_pM denotes the *tangent space* at p to M, let $x : U \subset \mathbb{R}^m \longrightarrow M$ be a system of coordinates around p whit $x(x_1, x_2, ...x_m) = p$. Let basis $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p, ..., \frac{\partial}{\partial x_m}\Big|_p \right\}$ be the asociated basis of T_pM. The *tangent bundle* TM is defined as

$$TM = \{(p, v)\,;\; p \in M \text{ and } v \in T_pM\} = \bigcup_{p\in M} T_pM$$

the set TM admits a differentiable structure of dimension $2m$ and the functions $X \in C^k\left(M, T_{(.)}M\right)$ are called *vector fields of class* C^k (see [3]). Next we define the concept of Riemannian metric

Definition 2 *A Riemannian metric on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $\langle ., . \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM, which varies differentiably in the following sense: $x : U \subset \mathbb{R}^m \longrightarrow M$ is a system of coordinates around p whit $x(x_1, x_2, ...x_m) = p$, then*

$$\begin{aligned} g_{ij}(x_1, x_2, ...x_m) &:= \left\langle \left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right\rangle_p \\ &= \left\langle dx^{-1}\left(\left.\frac{\partial}{\partial x_i}\right|_p\right), dx^{-1}\left(\left.\frac{\partial}{\partial x_j}\right|_p\right)\right\rangle, \end{aligned}$$

in which dx^{-1} is the tangent map of x^{-1}, is a differentiable function on U for each $i, j = 1, 2, .., n$. The functions g_{ij} are called the local representatives of the Riemannian metric.

This definition does not depend on the choice of a coordinate system, (see [3]).

Hereafter we will always assume that M is equipped with a Riemmannian metric g.The inner product $\langle ., . \rangle_p$ induces in a natural way the norm $||.||_p$. The subscript p is usually deleted if there is not possibility of confusion.

Let $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ be a piecewise smooth curve. If we choose a parametrizacion $x : U \subset \mathbb{R}^m \longrightarrow M$, we can express the curve γ in this parametrization by

$$x^{-1} \circ \gamma(t) = (x_1(t), x_2(t), ..., x_n(t)).$$

Thus, the vector $\gamma'(t)$ can be expressed in the parametrization x by

$$\gamma'(t) = \sum_{i=1}^{n} x_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{x^{-1}\circ\gamma(t)}.$$

If p and q are two points of a manifold M, let $\gamma : [0,1] \longrightarrow M$ be a piecewise smooth curve connecting p and q. The *arc length of* γ is defined by

$$\begin{aligned} l(\gamma) &= \int_0^1 \left|\left|\gamma'(t)\right|\right| dt \\ &= \int_0^1 \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle^{1/2} dt, \end{aligned} \tag{1.1}$$

and the *Riemannian distance from* p *to* q by, (see [3], [15])

$$d(p,q) = \inf_{\gamma} l(\gamma). \tag{1.2}$$

Definition 3 *Let* $\chi(M)$ *be the set of all vector fields of class* C^∞ *on* M *and* $\mathcal{D}(M)$ *the ring of real-valued functions of class* C^∞ *defined on* M*, that is:*

$$\begin{aligned} &\chi(M) = \left\{X \in TM : X \in C^\infty\left(M, T_{(.)}M\right)\right\}, \\ &\mathcal{D}(M) = C^\infty(M, \mathbb{R}). \end{aligned}$$

An affine connection ∇ *on* M *is a mapping*

$$\begin{aligned} \nabla : \quad \chi(M) \times \chi(M) &\longrightarrow \chi(M) \\ (X,Y) &\longmapsto \nabla_X Y \end{aligned} \tag{1.3}$$

that satisfies the following properties:

$$\begin{aligned} &i) \quad \nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z. \\ &ii) \quad \nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z. \\ &iii) \quad \nabla_X (fY) = f\nabla_X Y + X(f)Y, \end{aligned}$$

where X, Y, $Z \in \chi(M)$ *and* $f, g \in \mathcal{D}(M)$.

Definition 4 *If* X *is a* C^1 *vector field on* M, *the covariant derivative of* X *determined by the connection* ∇ *defines on each* $p \in M$ *a linear application of* T_pM *in itself*

$$\begin{aligned} \mathcal{D}X(p) : \quad T_pM &\longrightarrow T_pM \\ v &\longmapsto \mathcal{D}X(p)(v) = \nabla_Y X(p) \end{aligned} \tag{1.4}$$

where Y is a vector field satisfying $Y(p) = v$. The value $\mathcal{D}X(p)(v)$ depends only on the tangent vector $v = Y(p)$ since ∇ is linear in Y. In this way we can write

$$\mathcal{D}X(p)(v) = \nabla_v X(p).$$

Let us consider the curve $\gamma : [a,b] \longrightarrow M$ and vector field X along γ i.e $X(p) \in T_{\gamma(t)}M$, where $\gamma(t) = p$ for all t. We say that a vector field X is parallel along of γ (with respect to ∇) if $\mathcal{D}X(p)(\gamma'(t)) = 0$. The affine connection is *compatible* with the metric $\langle .,. \rangle$, when for any smooth curve γ and any pair of parallel vector fields P and P' along γ, we have that $\langle P, P' \rangle$ is constant; equivalently,

$$\frac{d}{dt}\langle X, Y\rangle = \left\langle \nabla_{\gamma'(t)}X, Y \right\rangle + \left\langle X, \nabla_{\gamma'(t)}Y \right\rangle,$$

where X and Y are vector fields along the differentiable curve $\gamma : I \longrightarrow M$ (see [3], [15]). We say that ∇ is *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y] \text{ for all } X, Y \in \chi(M).$$

The theorem of Levi-Civita establishes that there exists an unique affine connection ∇ on M compatible with the metric and symmetric (see [15]). This connection is called *the Levi-Civita connection.*

Definition 5 *A parametrized curve $\gamma : I \longrightarrow M$ is a geodesic at $t_0 \in I$ if $\nabla_{\gamma'(t)}\gamma'(t) = 0$ in the point t_0. If γ is a geodesic at t, for all $t \in I$, we say that γ is a geodesic. If $[a,b] \subseteq I$,the restriction of γ to $[a,b]$ is called a geodesic segment joining $\gamma(a)$ to $\gamma(b)$.*

In some cases, by abuse of language, we refer to the image $\gamma(I)$, of a geodesic γ, as a geodesic. A basic property of a geodesic is that $\gamma'(t)$ is parallel along of $\gamma(t)$; this implies that $||\gamma'(t)||$ is constant.

Let $B(p,r)$ and $B[p,r]$ denote respectively the *open geodesic and the closed geodesic balls* with center p and radius r, that is:

$$B(p,r) = \{q \in M : d(p,q) < r\}$$
$$B[p,r] = \{q \in M : d(p,q) \leq r\}.$$

The Hopf and Rinnov's theorem [3], establishes that if M is a complete metric space the for every $p, q \in M$ there exists a geodesic γ, called *minimizing geodesic,* joining p to q with

$$l(\gamma) = d(p, q). \tag{1.5}$$

Moreover, if $v \in T_pM$, there exists an unique locally minimizing geodesic γ such that $\gamma(0) = p$ and $\gamma'(0) = v$. The point $\gamma(1)$ is called the image of v by the *exponential map at* p, that is, there exists a well-defined map

$$\exp_p : \quad T_pM \longrightarrow M \tag{1.6}$$

such that

$$\exp_p(v) = \gamma(1),$$

and for any $t \in [0, 1]$,

$$\gamma(t) = \exp_p(tv).$$

It can be shown that $\exp_p$ defines a diffeomorphism of a neighborhood $\widehat{U}$ of the origin $0_p \in T_pM$ onto a neighborhood U of $p \in M$, called *normal neighborhood* of p, (see [14]).

Let $p \in M$ and U a normal neighborhood of p. Let us consider an orthonormal basis $\{e_i\}_{i=1}^m$ of T_pM. This basis gives the isomorphism $f : \mathbb{R}^m \longrightarrow T_pM$ defined by $f(u_1, ..., u_n) = \sum_{i=1}^m u_i e_i$. If $q = \exp_p\left(\sum_{i=1}^m u_i e_i\right)$, we say that $(u_1, ..., u_n)$ are *normal coordinates* of q in the normal neighborhood U of p and the coordinate chart is the composition

$$\varphi := \exp_p \circ f : \mathbb{R}^m \longrightarrow U.$$

One of the most important properties of the normal coordinates is that the geodesies passing through p are given by linear equations, see [15].

The exponential map has many important properties [3], [14]. When the exponential map is defined for each value of the parameter $t \in \mathbb{R}$, we will say that the Riemannian manifold M is geodesically complete or, simply, complete. The Hopf and Rinnov's theorem, also establishes that the property of the Riemannian manifold of being geodesically complete is equivalent to being complete as a metric space.

Definition 6 *Let γ be a piecewise smooth curve. For any $a, b \in \mathbb{R}$, we define the parallel* *transport along γ which is denoted by P_γ as*

$$P_{\gamma,a,b}: \begin{array}{ccc} T_{\gamma(a)}M & \longrightarrow & T_{\gamma(b)}M \\ v & \longmapsto & V\left(\gamma\left(b\right)\right), \end{array} \tag{1.7}$$

where V is the unique vector field along γ such that $\nabla_{\gamma'(t)}V = 0$ and $V\left(\gamma\left(a\right)\right) = v$.

It is easy to show that $P_{\gamma,a,b}$ is linear and one-one, so that $P_{\gamma,b,a}$ is an isomorphism between the tangent spaces $T_{\gamma(a)}M$ and $T_{\gamma(b)}M$. Its inverse is the parallel transport along the reversed portion of γ from $V\left(\gamma\left(b\right)\right)$ to $V\left(\gamma\left(a\right)\right)$. Thus $P_{\gamma,a,b}$ is a isometry between $T_{\gamma(a)}M$ and $T_{\gamma(b)}M$.

Moreover, for a positive integer i and for all $(v_1, v_2, ..., v_i) \in T_{\gamma(b)}M \times T_{\gamma(b)}M \times ... \times T_{\gamma(b)}M$, we define P^i_γ as

$$P^i_{\gamma,a,b}: \underbrace{T_{\gamma(a)}M \times ... \times T_{\gamma(a)}M}_{i\text{-times}} \longrightarrow \underbrace{T_{\gamma(b)}M \times ... \times T_{\gamma(b)}M}_{i\text{-times}},$$

where

$$P^i_{\gamma,a,b}\left(v_1, v_2, ..., v_i\right) = \left(P_{\gamma,a,b}\left(v_1\right), P_{\gamma,a,b}\left(v_2\right), ..., P_{\gamma,a,b}\left(v_i\right)\right).$$

The parallel transport has the important properties:

$$\begin{array}{l} P_{\gamma,a,b} \circ P_{\gamma,b,d} = P_{\gamma,a,d}, \\ P^{-1}_{\gamma,b,a} = P_{\gamma,a,b}. \end{array} \tag{1.8}$$

Next we extend the concept of covariant derivative to higher order

$$\mathcal{D}X: \begin{array}{ccc} C^k\left(TM\right) & \longrightarrow & C^{k-1}\left(TM\right) \\ (v, .) & \longmapsto & \mathcal{D}X\left(Y\right) = \nabla_Y X, \end{array} \tag{1.9}$$

where TM is the tangent bundle. Similar of the higher order Fréchet-derivative (see [28]). We define the *higher order covariant derivatives,* (see [16], [25]), as the multilinear map or j-tensor:

$$\mathcal{D}^j X: \underbrace{C^k\left(TM\right) \times C^k\left(TM\right) \times ... \times C^k\left(TM\right)}_{j\text{-times}} \longrightarrow C^{k-j}\left(TM\right)$$

given by

$$\mathcal{D}^j X\left(Y_1, Y_2, ...Y_{j-1}, Y\right) = \nabla_Y \mathcal{D}^{j-1}\left(X\left(Y_1, Y_2, ...Y_{j-1}\right)\right) - \sum_{i=1}^{j-1} \mathcal{D}^{j-1} X\left(Y_1, Y_2, ..., \nabla_Y Y_i, ...Y_{j-1},\right) \tag{1.10}$$

for each $Y_1, Y_2, ...Y_{j-1} \in C^k(TM)$. In the case $j = 2$, we have

$$\mathcal{D}^2 X: \quad C^k(TM) \times C^k(TM) \quad \longrightarrow \quad C^{k-2}(TM)$$

and

$$\begin{aligned} \mathcal{D}^2 X\left(Y_1, Y\right) &= \nabla_Y \mathcal{D}X\left(Y_1\right) - \mathcal{D}X\left(\nabla_Y Y_1\right) \\ &= \nabla_Y\left(\nabla_{Y_1} X\right) - \nabla_{\nabla_Y Y_1} X. \end{aligned} \tag{1.11}$$

The multilinearity refers to the structure of $C^k(M)$-module, in that, the value of $\mathcal{D}^j X\left(Y_1, Y_2, ...Y_{j-1}, Y\right)$ at $p \in M$ only depends on the j-tuple of tangent vectors

$$\left(v_1, v_2, ...v_j\right) = \left(Y_1(p), Y_2(p), ...Y_{j-1}(p), Y(p)\right) \in \left(T_p M\right)^j.$$

Therefore, for any $p \in M$, we define the map

$$\mathcal{D}^j X(p): \quad \left(T_p M\right)^j \quad \longrightarrow \quad T_p M$$

by

$$\mathcal{D}^j X(p)\left(v_1, v_2, ...v_j\right) = \mathcal{D}^j X\left(Y_1, Y_2, ...Y_{j-1}, Y\right)(p). \tag{1.12}$$

Definition 7 *Let M be a Riemannian manifold, $\Omega \subseteq M$ an open convex set and $X \in \chi(M)$. The covariant derivative $\mathcal{D}X = \nabla_{(.)} X$ is Lipschitz with constant $L > 0$, if for any geodesic γ and $a, b \in \mathbb{R}$ so that $\gamma[a, b] \subseteq \Omega$, it holds that:*

$$\left\| P_{\gamma,b,a} \mathcal{D}X\left(\gamma(b)\right) P_{\gamma,a,b} - \mathcal{D}X\left(\gamma(a)\right) \right\| \leq L \int_a^b \left\|\gamma'(t)\right\| dt. \tag{1.13}$$

We will write $\mathcal{D}X \in Lip_L(\Omega)$, (see [4],[17]).

Note that $P_{\gamma,b,a}\mathcal{D}X\left(\gamma\left(b\right)\right)P_{\gamma,b,a}$ and $\mathcal{D}X\left(\gamma\left(a\right)\right)$ are both operators defined in the same tangent plane $T_{\gamma(a)}M$. If M is the euclidean space, the above definition coincides with the usual Lipschitz definition for the operator $DF : M \longrightarrow M$.

Proposition 8 *Let γ be a curve in M and X a C^1 vector field on M, then the covariant derivative of X in the direction of $\gamma'\left(s\right)$ is (see Fig. 1).*

$$\begin{aligned} \mathcal{D}X\left(\gamma\left(s\right)\right)\gamma'\left(s\right) &= \nabla_{\gamma'(s)}X_{\gamma(s)} \\ &= \lim_{h\to 0}\frac{1}{h}\left(P_{\gamma,s+h,s}X\left(\gamma\left(s+h\right)\right)-X\left(\gamma\left(s\right)\right)\right) \end{aligned} \tag{1.14}$$

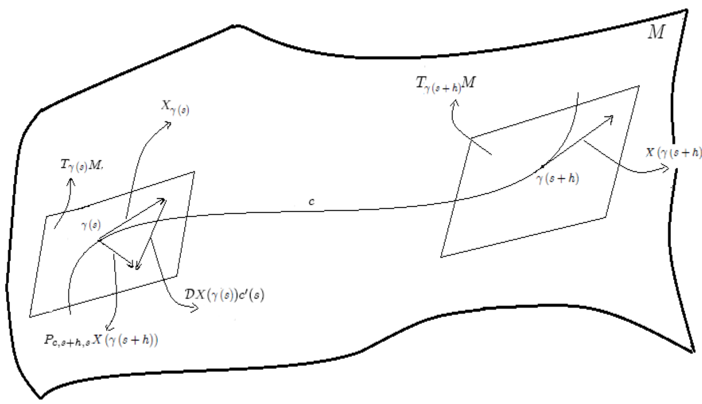

Fig. 1.

Note that if $M = \mathbb{R}^n$, the previous proposition agrees with the definition of classic directional derivative in $\mathbb{R}^n$; for the proof see [15].

Let us recall that if $A : T_pM \longrightarrow T_pM$ is linear, then $\|A\| = \sup\left\{\|Av\| : v \in T_pM,\ \ \|v\| = 1\right\}.$

The following is an important lemma, that allows to know when an operator is invertible and also allows to give on estimate for its inversse.

Lemma 9 *(Banach's lemma) Let A be an invertible bounded linear operator in an Banach space E and B a bounded linear operator B in E. If*

$$\left\|A^{-1}B - I\right\| < 1,$$

then B^{-1} *exists and*

$$\left\|B^{-1}\right\| \leq \frac{\left\|A^{-1}\right\|}{1-\left\|A^{-1}B-I\right\|} \leq \frac{\left\|A^{-1}\right\|}{1-\left\|A^{-1}\right\|\left\|B-A\right\|}.$$

Moreover,

$$\left\|B^{-1}A\right\| \leq \frac{1}{1-\left\|A^{-1}B-I\right\|} \leq \frac{1}{1-\left\|A^{-1}\right\|\left\|B-A\right\|}.$$

1.2 Taylor-like approximations

Next we will show some of the Taylor-type expansions on Riemannian manifolds, which will be used in the subsequent chapters, in the prove the convergence of the sequences appearing in each method, and uniqueness of the singularities determined by them.

It is also possible to obtain a version of the Fundamental Theorem of Calculus for manifolds as it establishe in the following theorem.

Theorem 10 *Let* γ *be a geodesic in* M *and let be* X *a* C^1 *vector field on* M*. Then*

$$P_{\gamma,t,o}X\left(\gamma\left(t\right)\right) = X\left(\gamma\left(0\right)\right) + \int_0^t P_{\gamma,s,o}\left(\mathcal{D}X\left(\gamma\left(s\right)\right)\gamma'\left(s\right)\right)ds. \tag{1.15}$$

Proof. Let us consider the curve

$$f\left(s\right) = P_{\gamma,s,o}X\left(\gamma\left(s\right)\right)$$

in $T_{\gamma(0)}M$. Then

$$\begin{aligned} f'(s) &= \lim_{h\to 0} \frac{f(s+h)-f(s)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(P_{\gamma,s+h,o}X(\gamma(s+h)) - P_{\gamma,s,o}X(\gamma(s))\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(P_{\gamma,s,o}\circ P_{\gamma,s+h,s}X(\gamma(s+h)) - P_{\gamma,s,o}X(\gamma(s))\right). \end{aligned}$$

Since $P_{\gamma,s,o}$ is linear and continuous, we have

$$f'(s) = P_{\gamma,s,o}\left(\lim_{h\to 0}\frac{1}{h}\left(P_{\gamma,s+h,s}X(\gamma(s+h)) - X(\gamma(s))\right)\right),$$

Using (1.14)

$$\begin{aligned} f'(s) &= P_{\gamma,s,o}\left(\nabla_{\gamma'(s)}X(\gamma(s))\right) \\ &= P_{\gamma,s,o}\left(\mathcal{D}X(\gamma(s))\gamma'(s)\right), \end{aligned} \tag{1.16}$$

and from

$$\int_0^t f'(s)\,ds = f(t) - f(0)$$

we obtain

$$P_{\gamma,t,o}X(\gamma(t)) = X(\gamma(0)) + \int_0^t P_{\gamma,s,o}\left(\mathcal{D}X(\gamma(s))\gamma'(s)\right)ds.$$

■

It is not hard to prove (using induction) that

$$f^{(n)}(s) = P_{\gamma,s,o}\mathcal{D}^{(n)}X(\gamma(s))\underbrace{\left(\gamma'(s),\gamma'(s),...,\gamma'(s)\right)}_{n-times} \tag{1.17}$$

Theorem 11 *Let γ be a geodesic in M and let be X a γ^2 vector field on M. Then*

$$P_{\gamma,t,o}\mathcal{D}X(\gamma(t))\gamma'(t) = \mathcal{D}X(\gamma(0))\gamma'(0) + \int_0^t P_{\gamma,s,o}\left(\mathcal{D}^2X(\gamma(s))\left(\gamma'(s),\gamma'(s)\right)\right)ds. \tag{1.18}$$

Proof. *Let us consider the vector field along of the geodesic* $\gamma(s)$:

$$Y(\gamma(s)) = \mathcal{D}X(\gamma(s))\gamma'(s).$$

By the previous theorem,

$$P_{\gamma,t,o}Y(\gamma(t)) = Y(\gamma(0)) + \int_0^t P_{\gamma,s,o}\left(\mathcal{D}Y(\gamma(s))\gamma'(s)\right)ds,$$

hence

$$P_{\gamma,t,o}\mathcal{D}X(\gamma(t))\gamma'(t) = \mathcal{D}X(\gamma(0))\gamma'(0) + \int_0^t P_{\gamma,s,o}\left(\mathcal{D}\left(\mathcal{D}X(\gamma(s))\gamma'(s)\right)\gamma'(s)\right)ds$$

by (1.11),

$$\begin{aligned}\mathcal{D}^2X(\gamma(s))\left(\gamma'(s),\gamma'(s)\right) &= \nabla_{\gamma'(s)}\mathcal{D}\left(X(\gamma(s))\left(\gamma'(s)\right)\right) - \mathcal{D}X(\gamma(s))\left(\nabla_{\gamma'(s)}\gamma'(s)\right)\\ &\mathcal{D}\left(\mathcal{D}X(\gamma(s))\gamma'(s)\right)\gamma'(s) - \mathcal{D}X(\gamma(s))\left(\nabla_{\gamma'(s)}\gamma'(s)\right)\end{aligned}$$

since $\gamma(s)$ *is geodesic, we have* $\nabla_{\gamma'(s)}\gamma'(s) = 0$, *hence*

$$\mathcal{D}^2X(\gamma(s))\left(\gamma'(s),\gamma'(s)\right) = \mathcal{D}\left(\mathcal{D}X(\gamma(s))\gamma'(s)\right)\gamma'(s),$$

Therefore,

$$P_{\gamma,t,o}\mathcal{D}X(\gamma(t))\gamma'(t) = \mathcal{D}X(\gamma(0))\gamma'(0) + \int_0^t P_{\gamma,s,o}\mathcal{D}^2X(\gamma(s))\left(\gamma'(s),\gamma'(s)\right)ds,$$

■

Note that (1.18) is equivalent to

$$P_{\gamma,t,0}\mathcal{D}X(\gamma(t))P_{\gamma,0,t} - \mathcal{D}X(\gamma(0)) = \int_0^s P_{\gamma,s,0}\mathcal{D}^2X(\gamma(s))P_{\gamma,0,s}\left(\gamma'(0),.\right)ds.$$

In a similar way, using induction, we can prove that

$$P_{\gamma,t,0}\mathcal{D}^nX(\gamma(t))P^n_{\gamma,0,t} - \mathcal{D}^nX(\gamma(0)) = \int_0^s P_{\gamma,s,0}\left(\mathcal{D}^nX(\gamma(s))P^n_{\gamma,0,s}\left(\gamma'(0),...,.\right)\right)ds. \quad (1.19)$$

Theorem 12 *Let M be a Riemannian manifold, $\Omega \subseteq M$ an open convex set, $X \in \chi(M)$ and $\mathcal{D}X \in Lip_L(\Omega)$. Take $p \in B(p_0, r) \subseteq \Omega$, $v \in T_pM$, $\sigma : [0,1] \longrightarrow M$ be a minimizing geodesic connecting p_0, p and*

$$\gamma(t) = \exp_p(tv).$$

Then

$$P_{\gamma,t,0}X(\gamma(t)) = X(p) + P_{\sigma,0,1}t\mathcal{D}X(p_0)P_{\sigma,1,0}v + R(t),$$

with

$$\|R(t)\| \leq L\left(\frac{t}{2}\|v\| + d(p_0,p)\right)t\|v\|.$$

Proof. From Theorem 10, it follows that

$$P_{\gamma,t,0}X(\gamma(t)) - X(\gamma(0)) = \int_0^t P_{\gamma,s,o}\left(\mathcal{D}X(\gamma(s))\gamma'(s)\right)ds,$$

since γ is a minimizing geodesic, then $\gamma'(t)$ is parallel and $\gamma'(s) = P_{\gamma,o,s}\gamma'(0)$. Moreover $\gamma'(0) = v$ then

$$P_{\gamma,t,0}X(\gamma(t)) - X(p) = \int_0^t P_{\gamma,s,o}(\mathcal{D}X(\gamma(s))P_{\gamma,o,s}v)\,ds.$$

Thus

$$\begin{aligned}
&P_{\gamma,t,0}X(\gamma(t)) - X(p) - P_{\sigma,0,1}t\mathcal{D}X(p_0)P_{\sigma,1,0}v \\
&= \int_0^t P_{\gamma,s,o}(\mathcal{D}X(\gamma(s))P_{\gamma,o,s}v)\,ds - P_{\sigma,0,1}\mathcal{D}X(p_0)P_{\sigma,1,0}v \\
&= \int_0^t (P_{\gamma,s,o}\mathcal{D}X(\gamma(s))P_{\gamma,o,s}v - P_{\sigma,0,1}\mathcal{D}X(p_0)P_{\sigma,1,0}v)\,ds,
\end{aligned}$$

letting

$$R(t) = \int_0^t (P_{\gamma,s,o}\mathcal{D}X(\gamma(s))P_{\gamma,o,s}v - P_{\sigma,0,1}\mathcal{D}X(p_0)P_{\sigma,1,0}v)\,ds,$$

and using the, by hypothesis, $\mathcal{D}X \in Lip_L(\Omega)$, we obtain

$$
\begin{aligned}
\|R(t)\| &\le \int_0^t \|(P_{\gamma,s,o}\mathcal{D}X(\gamma(s))P_{\gamma,o,s} - \mathcal{D}X(p) + \mathcal{D}X(p) - P_{\sigma,0,1}\mathcal{D}X(p_0)P_{\sigma,1,0})\| \|v\| ds \\
&\le \int_0^t (\|P_{\gamma,s,o}\mathcal{D}X(\gamma(s))P_{\gamma,o,s} - \mathcal{D}X(p)\| + \|\mathcal{D}X(p) - P_{\sigma,0,1}\mathcal{D}X(p_0)P_{\sigma,1,0}\|) \|v\| ds \\
&= \int_0^1 (\|P_{\gamma,s,o}\mathcal{D}X(\gamma(s))P_{\gamma,o,s} - \mathcal{D}X(\gamma(0))\| + \|\mathcal{D}X(\sigma(1)) - P_{\sigma,0,1}\mathcal{D}X(\sigma(0))P_{\sigma,1,0}\|) \|v\| ds \\
&\le L\int_0^t \left(\int_0^s \left\|\gamma'(\tau)\right\| d\tau + d(p_0,p)\right) \|v\| ds \\
&= L\int_0^t \left(\int_0^s \left\|\gamma'(0)\right\| d\tau + d(p_0,p)\right) \|v\| ds \\
&= L\int_0^t \left(s\left\|\gamma'(0)\right\| + d(p_0,p)\right) \|v\| ds \\
&= L\left(\frac{t^2}{2}\|v\| + td(p_0,p)\right)\|v\|.
\end{aligned}
$$

Therefore,

$$
\|R(t)\| \le L\left(\frac{t}{2}\|v\| + d(p_0,p)\right) t\|v\|.
$$

■

Theorem 13 *Let M be a Riemannian manifold, $\Omega \subseteq M$ an open set, $X \in \chi(M)$ and $\mathcal{D}X \in Lip_L(\Omega)$. Let us take $p \in \Omega$, $v \in T_pM$ and let*

$$
\gamma(t) = \exp_p(tv).
$$

If $\gamma[0,t) \subseteq \Omega$, then

$$
P_{\gamma,t,o}X(\gamma(t)) = X(p) + t\mathcal{D}X(p)v + R(t),
$$

with

$$
\|R(t)\| \le \frac{L}{2}t^2\|v\|^2.
$$

Proof. From Theorem 10, it follows that

$$
P_{\gamma,t,o}X(\gamma(t)) - X(\gamma(0)) = \int_0^t P_{\gamma,s,o}\left(\mathcal{D}X(\gamma(s))\gamma'(s)\right) ds.
$$

Given that γ is a geodesic, we have that $\gamma'(t)$ is parallel and $\gamma'(s) = P_{\gamma,o,s}\gamma'(0)$. Moreover, since $\gamma'(0) = v$ then

$$P_{\gamma,t,o}X(\gamma(t)) - X(p) = \int_0^t P_{\gamma,s,o}(\mathcal{D}X(\gamma(s))\,P_{\gamma,o,s}v)\,ds.$$

Therefore

$$\begin{aligned}
&P_{\gamma,t,o}X(\gamma(t)) - X(p) - t\mathcal{D}X(p)\,v \\
&= \int_0^t P_{\gamma,s,o}(\mathcal{D}X(\gamma(s))\,P_{\gamma,o,s}v)\,ds - t\mathcal{D}X(p)\,v \\
&= \int_0^t (P_{\gamma,s,o}(\mathcal{D}X(\gamma(s))\,P_{\gamma,o,s}v) - \mathcal{D}X(p)\,v)\,ds,
\end{aligned}$$

let

$$R(t) = \int_0^t (P_{\gamma,s,o}\mathcal{D}X(\gamma(s))\,P_{\gamma,o,s}v - \mathcal{D}X(p)\,v)\,ds.$$

By hypothesis, $\mathcal{D}X \in Lip_L(\Omega)$, hence

$$\begin{aligned}
\|R(t)\| &\leq \int_0^t \|(P_{\gamma,s,o}\mathcal{D}X(\gamma(s))\,P_{\gamma,o,s}v - \mathcal{D}X(p)\,v)\|\,ds \\
&\leq \int_0^t \|(P_{\gamma,s,o}\mathcal{D}X(\gamma(s))\,P_{\gamma,o,s} - \mathcal{D}X(p))\|\,\|v\|\,ds \\
&\leq \int_0^t \left(L\int_0^s \left\|\gamma'(\tau)\right\|d\tau\right)\|v\|\,ds.
\end{aligned}$$

Since γ is a geodesic, $\|\gamma'(\tau)\|$ is constant. Therefore,

$$\left\|\gamma'(\tau)\right\| = \left\|\gamma'(0)\right\| = \|v\|\,,$$

thus

$$\begin{aligned}
\|R(t)\| &\leq \int_0^t \left(L\int_0^s \|v\|\,d\tau\right)\|v\|\,ds \\
&= \int_0^t L\,\|v\|\,s\,\|v\|\,ds \\
&= \frac{L}{2}t^2\,\|v\|^2\,.
\end{aligned}$$

■

Theorem 14 *Let γ be a geodesic in M such that $[0,1] \subseteq Dom(\gamma)$. Let X be a C^2 vector field on M. Then*

$$P_{\gamma,1,0}X(\gamma(1)) = X(\gamma(0)) + \mathcal{D}X(\gamma(0)).\gamma'(0) + \int_0^1 (1-t) P_{\gamma,t,0}\mathcal{D}^2X(\gamma(t))\big(\gamma'(t), \gamma'(t)\big)\, dt. \tag{1.20}$$

Proof. Consider the curve

$$f(t) = P_{\gamma,t,0}X(\gamma(t)),$$

in $T_{\gamma(0)}M$. By (1.17),

$$f''(t) = P_{\gamma,t,0}\mathcal{D}^2X(\gamma(t))\big(\gamma'(t), \gamma'(t)\big),$$

and from Taylor's theorem

$$f(1) = f(0) + f'(0)(1-0) + \int_0^1 (1-t) f''(t)\, dt.$$

Therefore

$$P_{\gamma,1,0}X(\gamma(1)) = X(\gamma(0)) + \mathcal{D}X(\gamma(0))\gamma'(0) + \int_0^1 (1-t) P_{\gamma,t,0}\mathcal{D}^2X(\gamma(t))\big(\gamma'(t), \gamma'(t)\big)\, dt.$$

■

Theorem 15 *Let γ be a geodesic in M, $[0,1] \subseteq Dom(\gamma)$ and let be X a C^2 vector field on M. Then*

$$\begin{aligned} P_{\gamma,s,0}X(\gamma(s)) &= X(\gamma(0)) + s\mathcal{D}X(\gamma(0))\gamma'(0) + \frac{1}{2}s^2\mathcal{D}^2X(\gamma(0))\big(\gamma'(0), \gamma'(0)\big) \\ &+ \frac{1}{2}\int_0^s (s-t)^2 P_{\gamma,t,0}\mathcal{D}^3X(\gamma(t))\big(\gamma'(t), \gamma'(t), \gamma'(t)\big)\, dt. \end{aligned}$$

Proof. Consider the curve

$$f(s) = P_{\gamma,s,0}X(\gamma(s))$$

in $T_{\gamma(0)}M$. By (1.17)

$$f'''(s) = P_{\gamma,s,0}\mathcal{D}^3X(\gamma(s))\big(\gamma'(s), \gamma'(s), \gamma'(s)\big),$$

and by Taylor's theorem,

$$f(s)=f(0)+f'(0)(s-0)+\frac{1}{2}f''(0)(s-0)^2+\frac{1}{2}\int_0^s(s-t)^2 f'''(t)\,dt.$$

We thus conclude that

$$\begin{aligned}P_{\gamma,s,0}X(\gamma(s)) &= X(\gamma(0))+s\mathcal{D}X(\gamma(0))\gamma'(0)+\frac{1}{2}s^2\mathcal{D}^2X(\gamma(0))\left(\gamma'(0),\gamma'(0)\right)\\ &+\frac{1}{2}\int_0^s(s-t)^2 P_{\gamma,t,0}\mathcal{D}^3X(\gamma(t))\left(\gamma'(t),\gamma'(t),\gamma'(t)\right)dt.\end{aligned}$$

■

Theorem 16 *Let M be a Riemannian manifold, $\Omega\subseteq M$ an open convex set, $X\in C^3\left(M,T_{(.)}M\right)$. Suppose that there exist a real $c>0$ such that for any geodesic γ and $\tau_1,\tau_2\in\mathbb{R}$ with $\gamma[\tau_1,\tau_2]\subseteq\Omega$ it holds that*

$$\left\|P_{\gamma,\tau_2,\tau_1}\mathcal{D}^2(\gamma(\tau_2))P^2_{\gamma,\tau_1,\tau_2}-\mathcal{D}^2(\gamma(\tau_1))\right\|\le c\int_{\tau_1}^{\tau_2}\left\|\gamma'(t)\right\|dt,$$

provided that $[0,1]\subseteq[\tau_1,\tau_2]$. Then

$$\begin{aligned}&\|(P_{\gamma,s,0}X(\gamma(s))-X(\gamma(0))-s\mathcal{D}X(\gamma(0))\gamma'(0)\\ &-\tfrac{1}{2}s^2\mathcal{D}^2X(\gamma(0))(\gamma'(0),\gamma'(0)))\|\le\tfrac{1}{6}s^3\|\gamma'(0)\|^3\end{aligned}\tag{1.21}$$

and

$$\|(P_{\gamma,s,0}\mathcal{D}X(\gamma(s))P_{\gamma,0,s}-\mathcal{D}X(\gamma(0)))\|\le\left(\left\|\mathcal{D}^2X(\gamma(0))\right\|+\frac{1}{2}sc\left\|\gamma'(0)\right\|\right)s\left\|\gamma'(0)\right\|.\tag{1.22}$$

Proof. We start with the following Taylor's expansion (Theorem 15):

$$\begin{aligned}P_{\gamma,s,0}X(\gamma(s)) &= X(\gamma(0))+s\mathcal{D}X(\gamma(0))\gamma'(0)+\frac{1}{2}s^2\mathcal{D}^2X(\gamma(0))\left(\gamma'(0),\gamma'(0)\right)\\ &+\frac{1}{2}\int_0^s(s-t)^2 P_{\gamma,t,0}\mathcal{D}^3X(\gamma(t))\left(\gamma'(t),\gamma'(t),\gamma'(t)\right)dt,\end{aligned}$$

we have

$$\begin{aligned}
&\left\| P_{\gamma,s,0} X\left(\gamma\left(s\right)\right) - X\left(\gamma\left(0\right)\right) - s\mathcal{D}X\left(\gamma\left(0\right)\right)\gamma'\left(0\right) - \frac{1}{2}s^{2}\mathcal{D}^{2}X\left(\gamma\left(0\right)\right)\left(\gamma'\left(0\right),\gamma'\left(0\right)\right)\right\| \\
&\leq \frac{1}{2}\int_{0}^{s}\left(s-t\right)^{2}\left\|\mathcal{D}^{3}X\left(\gamma\left(t\right)\right)\right\|\left\|\gamma'\left(0\right)\right\|^{3}dt \\
&\leq \frac{1}{6}s^{3}\left\|\gamma'\left(0\right)\right\|^{3}.
\end{aligned}$$

Using (1.18) and (1.19), we obtain

$$P_{\gamma,t,0}\mathcal{D}^{2}X\left(\gamma\left(s\right)\right)P_{\gamma,0,t}^{2} - \mathcal{D}^{2}X\left(\gamma\left(0\right)\right) = \int_{0}^{s}P_{\gamma,t,o}\left(\mathcal{D}^{3}X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}^{3}\left(\gamma'\left(0\right),.,.\right)\right)dt$$

and

$$P_{\gamma,s,0}\mathcal{D}X\left(\gamma\left(s\right)\right)P_{\gamma,0,s} - \mathcal{D}X\left(\gamma\left(0\right)\right) = \int_{0}^{s}P_{\gamma,t,0}\left(\mathcal{D}^{2}X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}^{2}\left(\gamma'\left(0\right),.\right)\right)dt.$$

Therefore,

$$\begin{aligned}
&\left\|P_{\gamma,s,0}\mathcal{D}X\left(\gamma\left(s\right)\right)P_{\gamma,0,s} - \mathcal{D}X\left(\gamma\left(0\right)\right)\right\| \\
&= \left\|\int_{0}^{s}P_{\gamma,t,0}\left(\mathcal{D}^{2}X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}^{2}\left(\gamma'\left(0\right),.\right)\right)dt\right\| \\
&= \left\|\int_{0}^{s}\left(\mathcal{D}^{2}X\left(\gamma\left(0\right)\right)\left(\gamma'\left(0\right)\right) + \int_{0}^{t}P_{\gamma,\tau,o}\left(\mathcal{D}^{3}X\left(\gamma\left(\tau\right)\right)P_{\gamma,0,\tau}^{3}\left(\gamma'\left(0\right),\gamma'\left(0\right),.\right)\right)d\tau\right)dt\right\| \\
&\leq \left\|\int_{0}^{s}\mathcal{D}^{2}X\left(\gamma\left(0\right)\right)\left(\gamma'\left(0\right)\right)dt\right\| + \left\|\int_{0}^{s}\int_{0}^{t}P_{\gamma,\tau,o}\left(\mathcal{D}^{3}X\left(\gamma\left(\tau\right)\right)P_{\gamma,0,\tau}^{3}\left(\gamma'\left(0\right),\gamma'\left(0\right),.\right)\right)d\tau dt\right\| \\
&\leq s\left\|\mathcal{D}^{2}X\left(\gamma\left(0\right)\right)\right\|\left\|\gamma'\left(0\right)\right\| + \int_{0}^{s}\int_{0}^{t}\left\|\mathcal{D}^{3}X\left(\gamma\left(\tau\right)\right)\right\|\left\|\gamma'\left(0\right)\right\|^{2}d\tau dt \\
&\leq s\left\|\mathcal{D}^{2}X\left(\gamma\left(0\right)\right)\right\|\left\|\gamma'\left(0\right)\right\| + \int_{0}^{s}\int_{0}^{t}c\left\|\gamma'\left(0\right)\right\|^{2}d\tau dt \\
&\leq s\left\|\mathcal{D}^{2}X\left(\gamma\left(0\right)\right)\right\|\left\|\gamma'\left(0\right)\right\| + \tfrac{1}{2}s^{2}c\left\|\gamma'\left(0\right)\right\|^{2} \\
&= \left(\left\|\mathcal{D}^{2}X\left(\gamma\left(0\right)\right)\right\| + \tfrac{1}{2}sc\left\|\gamma'\left(0\right)\right\|\right)s\left\|\gamma'\left(0\right)\right\|.
\end{aligned}$$

■

Chapter 2

Kantorovich method on Riemannian manifold

As a preamble for the Chebyshev-Halley method, in this section we will study the Newton's method for Riemannian manifolds. We will begin by establishing a parallel between Newton's method for Banach spaces and Newton's method for Riemannian manifolds.

Let us recall Kantorovich's Theorem or Newton's method in Banach spaces, (see [12]).

Theorem 17 *(Kantorovich) Let E be a Banach space, $\Omega \subseteq E$ be an open convex set, $F : \Omega \longrightarrow \Omega$ be a continuous function, $F \in C^1$ and DF Lipschitz in Ω , so that, there existe $L > 0$ such that:*

$$\|DF(x) - DF(y)\| \leq L \|x - y\|, \text{ for all } x, y \in \Omega.$$

Suppose that for some $x_0 \in \Omega$, $DF(x_0)$ is invertible and that for some $a > 0$ and $b \geq 0$:

$$\begin{array}{ll} (1) & \left\| DF(x_0)^{-1} \right\| \leq a \\ (2) & \left\| DF(x_0)^{-1} F(x_0) \right\| \leq b \\ (3) & c = abL \leq \frac{1}{2} \\ (4) & B(x_0, t_*) \subseteq \Omega \text{ where } t_* = \frac{1}{aL}\left(1 - \sqrt{1 - 2c}\right). \end{array}$$

If

$$\begin{aligned} v_k &= -DF(x_k)^{-1} F(x_k), \\ x_{k+1} &= x_k + v_k, \end{aligned}$$

then $\{x_k\}_{k\in\mathbb{N}} \subseteq B(x_0, t_*)$ *and* $x_k \longrightarrow p_*$*, which is the unique root of* F *in* $B[x_0, t_*]$. *Furthermore, if* $c < \frac{1}{2}$ *and* $B(x_0, r) \subseteq \Omega$*, with*

$$t_* < r \leq t_{**} = \frac{1}{aL}\left(1 + \sqrt{1-2c}\right),$$

then p_* *is also the unique root of* F *in* $B(x_0, r)$. *Also, the error bound is:*

$$\|x_k - x_*\| \leq (2c)^{2^k} \tfrac{b}{c}, \quad k = 1, 2, ...$$

Although the concepts will be defined later on, to extended the methodto riemannian manifolds, preliminarily we will say that the derivative of F at x_n is replaced by the covariant derivative of X at p_n :

$$\begin{aligned} \nabla_{(.)}X(p_n): \quad & T_{p_n}M \longrightarrow T_{p_n}M \\ & v \longrightarrow \nabla_Y X, \end{aligned}$$

where Y is a vector field satisfying $Y(p) = v$. We adopt the notation $\mathcal{D}X(p)\, v = \nabla_Y X(p)$; hence $\mathcal{D}X(p)$ is a linear mapping of T_pM into T_pM. So, in this new context,

$$-F'(x_n)^{-1} F(x_n)$$

is written as

$$-\mathcal{D}X(p_n)^{-1} X(p_n),$$

or

$$\left(\nabla_{(.)}X(p_n)\right)^{-1} X(p_n).$$

Now we can write Kantorovich's theorem in the new context. We will say that a singularity of a vector field X, is a point $p \in M$ for which $X(p) = 0$.

Theorem 18 *(Kantorovich theorem on Riemannian manifolds) Let* M *be a Riemannian manifold,* $\Omega \subseteq M$ *an open convex set,* $X \in \chi(M)$ *and* $\mathcal{D}X \in Lip_L(\Omega)$*. Suppose that for* *some* $p_0 \in \Omega$, $\mathcal{D}X(p_0)$ *is invertible and that for some* $a > 0$ *and* $b \geq 0$:

$$\begin{array}{lll}
(1) & \left\| \mathcal{D}X(p_0)^{-1} \right\| \leq a & \left(\left\| \left(\nabla_{(.)} X(p_0) \right)^{-1} \right\| \leq a \right) \\
(2) & \left\| \mathcal{D}X(p_0)^{-1} X(p_0) \right\| \leq b & \left(\left\| \left(\nabla_{(.)} X(p_0) \right)^{-1} X(p_0) \right\| \leq b \right) \\
(3) & c = abL \leq \frac{1}{2} & \\
(4) & B(p_0, t_*) \subseteq \Omega \text{ where } t_* = \frac{1}{aL}\left(1 - \sqrt{1-2c}\right). &
\end{array}$$

If (see Fig.2),

$$v_k = -\mathcal{D}X(p_k)^{-1} X(p_k),$$
$$p_{k+1} = \exp_{p_k}(v_k),$$

then $\{p_k\}_{k\in\mathbb{N}} \subseteq B(p_0, t_*)$ *and* $p_k \longrightarrow p_*$ *which is the unique singularity of* X *in* $B[p_0, t_*]$. *Furthermore, if* $c < \frac{1}{2}$ *and* $B(p_0, r) \subseteq \Omega$ *with*

$$t_* < r \leq t_{**} = \frac{1}{aL}\left(1 + \sqrt{1-2c}\right),$$

then p_* *is also the unique singularity of* X *in* $B(p_0, r)$. *The error bound is:*

$$d(p_k, p_*) \leq \frac{b}{c}(2c)^{2^k}; \quad k = 1, 2, ... \tag{2.1}$$

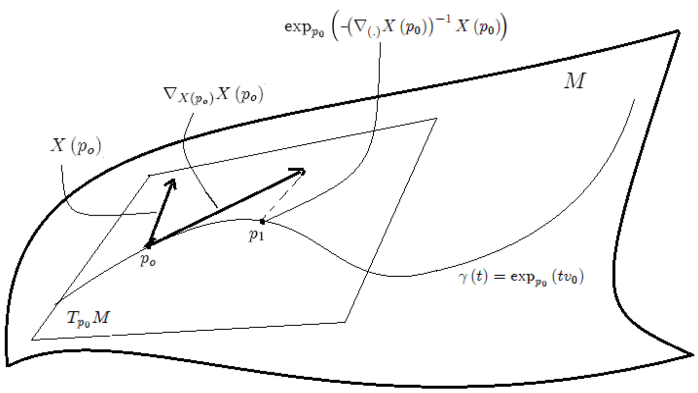

Fig. 2.

Proof. See [4], [17] ■

2.1 Simplified Kantorovich method on Riemannian manifolds

Next we will prove the method of Kantorovich on Riemannian manifold fixing $\mathcal{D}X(p_0)^{-1}$ in each iteration. This method we be called the Kantorovich's simplified method on Riemannian manifolds. In Banach spaces, this method was developed by Kantorovich (for the proof, see [12]).

Theorem 19 *(Kantorovich simplified method on Riemannian manifold) Let M be a Riemannian manifolds, $\Omega \subseteq M$ an open convex set, X a C^1 vector field, $\mathcal{D}X \in Lip_L(\Omega)$. Suppose that for some $p_0 \in \Omega$, $\mathcal{D}X(p_0)$ is invertible and that for some $a > 0$ and $b \geq 0$:*

$$
\begin{array}{ll}
(1) & \left\|\mathcal{D}X(p_0)^{-1}\right\| \leq a, \\
(2) & \left\|\mathcal{D}X(p_0)^{-1} X(p_0)\right\| \leq b, \\
(3) & c = abL \leq \frac{1}{2}, \\
(4) & B(p_0, t_*) \subseteq \Omega, \text{ where } t_* = \frac{1}{aL}\left(1 - \sqrt{1-2c}\right).
\end{array}
$$

Let

$$
\begin{array}{l}
v_k = -P_{\sigma_k,0,1}\mathcal{D}X(p_0)^{-1} P_{\sigma_k,1,0}X(p_k), \\
p_{k+1} = \exp_{p_k}(v_k),
\end{array}
\tag{2.2}
$$

where $\{\sigma_k : [0,1] \longrightarrow M\}_{k\in\mathbb{N}}$ is the minimizing geodesic family connecting p_0, p_k. Then $\{p_k\}_{k\in\mathbb{N}} \subseteq B(p_0, t_*)$ and $p_k \longrightarrow p_*$ which is the only one singularity of X in $B[p_0, t_*]$. Furthermore , if $c < \frac{1}{2}$ and $B(p_0, r) \subseteq \Omega$ with

$$
t_* < r \leq t_{**} = \frac{1}{al}\left(1 + \sqrt{1-2c}\right),
$$

then p_* is also the only singularity of X in $B(p_0, r)$. The error bound is:

$$
d(p_k, p_*) \leq \tfrac{b}{c}\left(1 - \sqrt{1-2c}\right)^{k+1}, \quad k = 1, 2, ... \tag{2.3}
$$

Before proceding to the proof of the theorem, we will establish some results that are of primary relevance in this proof.

Lemma 20 *Let M be a Riemannian manifold, $\Omega \subseteq M$ an open convex set, $X \in \chi(M)$, $\mathcal{D}X \in Lip_L(\Omega)$ and $\sigma : [0,1] \longrightarrow M$ be a minimizing geodesic connecting p_0, p . Take $p \in \Omega$,*

$v \in T_pM$ and let

$$\gamma(t) = \exp_p(tv).$$

If $\gamma[0,t) \subseteq \Omega$ and $P_{\sigma,0,1}\mathcal{D}X(p_0)\,P_{\sigma,1,0}v = -X(p)$, then

$$\|P_{\gamma,1,o}X(\gamma(1))\| \leq L\left(\frac{1}{2}\|v\| + d(p_0,p)\right)\|v\|. \tag{2.4}$$

Proof. It is immediate, by Theorem 12. ■

Now we can prove the simplified Kantorovich theorem on Riemannian manifolds (Theorem 14). The proof of this theorem will be divided in two parts. First, we will prove that simplified Kantorovich method is well defined, i.e. $\{p_k\}_{k\in\mathbb{N}} \subseteq B(p_0,t_*)$; we will also prove the convergence of the method. In the second part, we will establish uniqueness.

We will consider the auxiliary real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(t) = \frac{L}{2}t^2 - \frac{1}{a}t + \frac{b}{a}. \tag{2.5}$$

Its discriminant is

$$\triangle = \frac{1}{a^2}(1 - 2Lba),$$

which is positive, because $abL \leq \frac{1}{2}$. Thus, f has a least one real root (unique when $c = \frac{1}{2}$). If t_* is the smallest root, a direct calculations show that $f'(t) < 0$ for $0 \leq t < t_*$, so f is strictly decreasing in $[0,t_*]$. Therefore (see [5]), Newton's method can be applied to f, in other words:

If $t_0 \in [0,t_*)$, for $k = 0,1,2,..$ define

$$t_{k+1} = t_k - \frac{f(t_k)}{f'(t_0)}.$$

Then $\{t_k\}_{k\in\mathbb{N}}$ is well defined, it is strictly increasing, and it converges to t_*. Furthermore, if $c = abL < \frac{1}{2}$, then

$$t_* - t_k \leq \frac{b}{c}\left(1 - \sqrt{1-2c}\right)^{k+1}, \quad k = 1,2,... \tag{2.6}$$

Let us take as starting point $t_0 = 0$. We want to show that Newton's iteration is well-defined for any $q \in B(p_0,t_*) \subseteq \Omega$.

Now, define

$$k(t)=\left\{q\in B[p_0,t]:\left\|P_{\sigma,0,1}\mathcal{D}X(p_0)^{-1}P_{\sigma,1,0}X(q)\right\|\le\frac{f(t)}{|f'(0)|}=af(t),\ 0\le t<t_*\right\},\quad(2.7)$$

where $\sigma[0,1]\longrightarrow M$ be the minimizing geodesic connecting p_0 and q. Note that $k(t)\neq\phi$ since $p_0\in k(t)$.

Now we can prove te following proposition, which will be used in several places

Proposition 21 *Under the hypotheses of either the Kantorovich or the simplified Kantorovich method, if $q\in B(p_0,t_*)$, then $\mathcal{D}X(q)$ is nonsingular and*

$$\left\|\mathcal{D}X(q)^{-1}\right\|\le\frac{1}{|f'(\lambda)|},\quad \textit{where } \lambda=d(p_0,q)<t_*.$$

Proof. Let $\lambda=d(p_0,q)$ and $\alpha:[0,1]\longrightarrow M$ be a geodesic with $\alpha(0)=p_0$, $\alpha(1)=q$ and $\|\alpha'(0)\|=\lambda$. Define $\phi:T_qM\longrightarrow T_qM$ by letting

$$\phi=P_{\alpha,1,0}\mathcal{D}X(p_0)P_{\alpha,0,1}.\tag{2.8}$$

Since $P_{\alpha,1,0}$ and $P_{\alpha,0,1}$ are linear, isometric and $\mathcal{D}X(p_0)$ is nonsingular, we have that ϕ is linear, nonsingular and

$$\left\|\phi^{-1}\right\|=\left\|\mathcal{D}X(p_0)^{-1}\right\|\le a=\frac{1}{|f'(0)|},$$

with $\alpha([0,1])\subseteq B(p_0,t_*)$. Since $d(p_0,q)<t_*$, $\mathcal{D}X\in Lip_L(\Omega)$ and $\|\alpha'(0)\|=\lambda$. Therefore

$$\|\mathcal{D}X(q)-\phi\|\le L\lambda.\tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned}\left\|\phi^{-1}\right\|\|\mathcal{D}X(q)-\phi\|&\le aL\lambda\\&\le aLt_*\\&=aL\frac{1}{aL}\left(1-\sqrt{1-2abL}\right)\\&\le 1.\end{aligned}$$

Using Banach's lemma, we conclude that $\mathcal{D}X(q)$ is nonsingular, and

$$\begin{aligned}\left\|\mathcal{D}X(q)^{-1}\right\| &\leq \frac{\left\|\phi^{-1}\right\|}{1-\left\|\phi^{-1}\right\|\left\|\mathcal{D}X(q)-\phi\right\|} \\ &\leq \frac{a}{1-aL\lambda} \\ &\leq \frac{1}{\left|f'(\lambda)\right|}.\end{aligned}$$

■

Therefore, for any $q \in B(p_0, t_*)$, we can apply the Kantorovich methods.

Lemma 22 *Let $q \in k(t)$. Define*

$$\begin{aligned}t_+ &= t - \tfrac{f(t)}{f'(0)} \\ q_+ &= \exp_q\left(-P_{\sigma,0,1}\mathcal{D}X(p_0)^{-1} P_{\sigma,1,0}X(q)\right).\end{aligned}$$

Then $t < t_+ < t_*$ and $q_+ \in k(t_+)$.

Proof. Consider the geodesic $\gamma : [0,1] \longrightarrow M$ defined by

$$\gamma(\theta) = \exp_q\left(-\theta P_{\sigma,0,1}\mathcal{D}X(p_0)^{-1} P_{\sigma,1,0}X(q)\right),$$

we have

$$\begin{aligned}d(p_0, \gamma(\theta)) &\leq d(p_0, q) + d(q, \gamma(\theta)) \\ &\leq t + \left\|\theta P_{\sigma,0,1}\mathcal{D}X(p_0)^{-1} P_{\sigma,1,0}X(q)\right\| \\ &\leq t + \theta\frac{f(t)}{\left|f'(0)\right|}.\end{aligned}$$

Since

$$\gamma(1) = \exp_q\left(-P_{\sigma,0,1}\mathcal{D}X(p_0)^{-1} P_{\sigma,1,0}X(q)\right) = q_+,$$

this implies that

$$d(p_0, q_+) = d(p_0, \gamma(1)) \leq t + \frac{f(t)}{\left|f'(0)\right|} = t_+,$$

therefore

$$q_+ \in B(p_0, t_+) \subset B(p_0, t_*) .$$

Moreover, if $\sigma_+ [0,1] \longrightarrow M$ is the minimizing geodesic connecting p_0 and q_+, then

$$\left\| -P_{\sigma_+,0,1} \mathcal{D}X(p_0)^{-1} P_{\sigma_+,1,0} X(q_+) \right\| \leq \left\| \mathcal{D}X(p_0)^{-1} \right\| \|X(q_+)\| .$$

furthermore, if $v = -P_{\sigma,0,1} \mathcal{D}X(p_0)^{-1} P_{\sigma,1,0} X(q)$, then

$$\begin{aligned} P_{\sigma,0,1} \mathcal{D}X(p_0) P_{\sigma,1,0} v &= P_{\sigma,0,1} \mathcal{D}X(p_0) P_{\sigma,1,0} \left(-P_{\sigma,0,1} \mathcal{D}X(p_0)^{-1} P_{\sigma,1,0} X(q) \right) \\ &= -P_{\sigma,0,1} \mathcal{D}X(p_0) \mathcal{D}X(p_0)^{-1} P_{\sigma,1,0} X(q) \\ &= -X(q) . \end{aligned}$$

By Lemma 20,

$$\begin{aligned} \|X(q_+)\| &= \|X(\gamma(1))\| \\ &\leq L \left(\frac{1}{2} \|v\| + d(p_0, p) \right) \|v\| \\ &\leq L \left(\frac{1}{2} \left\| -P_{\sigma_+,0,1} \mathcal{D}X(p_0)^{-1} P_{\sigma_+,1,0} X(q) \right\| + t \right) \left\| -P_{\sigma_+,0,1} \mathcal{D}X(p_0)^{-1} P_{\sigma_+,1,0} X(q) \right\| \\ &\leq L \left(\frac{1}{2} \left(\frac{f(t)}{|f'(0)|} \right) + t \right) \left(\frac{f(t)}{|f'(0)|} \right), \end{aligned}$$

thus, by (2.7), after some calculations,

$$\begin{aligned} \left\| -P_{\sigma_+,0,1} \mathcal{D}X(p_0)^{-1} P_{\sigma_+,1,0} \right\| \|X(q_+)\| &\leq \left(\frac{1}{|f'(0)|} \right) L \left(\frac{1}{2} \left(\frac{f(t)}{|f'(0)|} \right) + t \right) \left(\frac{f(t)}{|f'(0)|} \right) \\ &= \frac{1}{8} L \left(2b - 2t + aLt^2 \right) \left(2b + 2t + aLt^2 \right) \\ &= \frac{f(t_+)}{|f'(0)|}, \end{aligned}$$

we thus conclude

$$\|X(q_+)\| \leq \frac{f(t_+)}{|f'(0)|},$$

and therefore

$$q_+ \in k(t_+).$$

■

Now we are going to prove that starting with any point of $k(t)$, the Newton method converges.

Lemma 23 *Take $0 \leq t < t_*$ and $q \in k(t)$, and define*

$$\begin{aligned} &\tau_0 = t \\ &\tau_{k+1} = \tau_k - \tfrac{f(\tau_k)}{|f'(0)|}, \ k = 0, 1, ... \end{aligned}$$

Then the sequence generated by Newton's method starting with the point $q_0 = q$ is well defined for any k and

$$q_k \in k(\tau_k). \tag{2.10}$$

Moreover $\{q_k\}_{k\in\mathbb{N}}$ converges to some $q_* \in B(p_0, t_*)$, $X(q_*) = 0$ and

$$d(q_k, q_*) \leq t_* - \tau_k, \text{ for all } k.$$

Proof. It is clear that the sequence $\{\tau_k\}_{k\in\mathbb{N}}$ is the sequence generated by Newton's method for solving $f(t) = 0$. Therefore, $\{\tau_k\}_{k\in\mathbb{N}}$ is well defined, strictly increasing and it converges to the root t_* (see the definition of f). By hypothesis, $q_0 \in k(\tau_0)$; suppose that the points $q_0, q_1, .., q_k$ are well defined. Then, using Lemma 23, we conclude that q_{k+1} is well defined. Furthermore,

$$d(q_{k+1}, q_k) \leq \left\| -P_{\sigma_k,0,1}\mathcal{D}X(p_0)^{-1} P_{\sigma_k,1,0}X(q_k) \right\|.$$

Since

$$q_{k+1} = \exp_{q_k}\left(-P_{\sigma_k,0,1}\mathcal{D}X(p_0)^{-1} P_{\sigma_k,1,0}X(q_k) \right)$$

and $\sigma_k : [0,1] \longrightarrow M$ is the minimizing geodesic connecting p_0, q_k, from Lemma 23 and using (2.10) we obtain

$$d(q_{k+1}, q_k) \leq \frac{f(\tau_k)}{|f'(0)|} = \tau_{k+1} - \tau_k. \tag{2.11}$$

Hence, for $k \geq s$, $s \in \mathbb{N}$,

$$d(q_k, q_s) \leq \tau_s - \tau_k. \tag{2.12}$$

It follows that $\{q_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Since M is complete, it converges to the some $q_* \in M$. Moreover $q_k \in k(\tau_k) \subseteq B[p_0, t_*]$, therefore $q_* \in B[p_0, t_*]$.

Next, we prove that $X(q_*) = 0$. We have next

$$\begin{aligned}
\|X(q_k)\| &= \left\|P_{\sigma_k,0,1}\mathcal{D}X(p_0)\, P_{\sigma_k,1,0}P_{\sigma_k,0,1}\mathcal{D}X(p_0)^{-1}\, P_{\sigma_k,1,0}X(q_k)\right\| \\
&\leq \|\mathcal{D}X(p_0)\| \left\|\mathcal{D}X(p_0)^{-1}\, X(q_k)\right\| \\
&\leq (\|\mathcal{D}X(p_0)\|)\, \frac{f(\tau_k)}{|f'(0)|} \\
&= (\|\mathcal{D}X(p_0)\|)\,(\tau_{k+1} - \tau_k).
\end{aligned}$$

Passing to the limit in k, we conclude $X(q_*) = 0$. Finally, letting $s \longrightarrow \infty$ in (2.12), we get

$$d(q_*, q_k) \leq t_* - \tau_k.$$

■

These lemmas allows as to proved the convergence part of Theorem 17, by (2.6)

$$d(q_*, q_k) \leq \frac{b}{c}\left(1 - \sqrt{1-2c}\right)^{k+1}, \quad k = 1, 2, ...$$

By hypothesis, $p_0 \in k(0)$, thus by the lemma 23, the sequence $\{p_k\}_{k\in\mathbb{N}}$ generated by (2.2) is well defined, contained in $B(p_0, t_*)$ and converges to some p_*, which is a singular point of X in $B[p_0, t_*]$. Moreover, if $c < 1/2$, then

$$d(p_k, p_*) \leq \frac{b}{c}\left(1 - \sqrt{1-2c}\right)^{k+1}.$$

Next we will show the uniqueness. This proof will be made in an indirect way, by contradiction. Before we are going to establish some preliminary results. The first step is to prove a stronger version of Lemma 22.

Lemma 24 *Take* $0 \leq t < t_*$ *and* $q \in k(t)$, *and let*

$$\begin{aligned}
&A^{-1} = -P_{\sigma,0,1}\mathcal{D}X(p_0)^{-1}\, P_{\sigma,1,0} \\
&v = A^{-1}X(q),
\end{aligned}$$

where $\sigma : [0,1] \longrightarrow M$ *is the minimizing geodesic connecting* p_0, p_k*. Define for* $\theta \in \mathbb{R}$,

$$\tau(\theta) = t + \theta a f(t),$$
$$\gamma(\theta) = \exp_q(\theta v).$$

Then, for $\theta \in [0,1]$,

$$t < \tau(\theta) < t_* \quad \text{and} \quad \gamma(\theta) \in k(\tau(\theta)).$$

Proof. Because γ is a minimizing geodesic, for all $\theta \in [0,1]$ we have

$$\begin{aligned} d(p_{0,}\gamma(\theta)) &\leq d(p_{0,}q) + d(q, \gamma(\theta)) \\ &\leq t + \theta \|v\| \\ &\leq t + \theta a f(t) \\ &= \tau(\theta). \end{aligned}$$

This implies that

$$t \leq \tau(\theta) \leq \tau(1) \leq t_* \text{ and } \gamma([0,\theta]) \subset B(p_0, t_*). \tag{2.13}$$

Using the Theorem 12, we obtain

$$X(\gamma(\theta)) = P_{\gamma,0,\theta}\left(X(p) + P_{\sigma,0,1}\theta \mathcal{D}X(p_0) P_{\sigma,1,0}v + R(\theta)\right),$$

with

$$R(\theta) = \int_0^\theta \left(P_{\gamma,s,0}\mathcal{D}X(\gamma(s)) P_{\gamma,0,s}v - P_{\sigma,0,1}\mathcal{D}X(p_0) P_{\sigma,1,0}v\right) ds,$$

and

$$\|R(\theta)\| \leq L\left(\frac{\theta}{2}\|v\| + d(p_0, q)\right)\theta \|v\|.$$

After some calculations, this yields

$$\begin{aligned}
\left\|A^{-1}X\left(\gamma\left(\theta\right)\right)\right\| &= \left\|A^{-1}P_{\gamma,0,\theta}\left(X\left(q\right)-\int_0^\theta P_{\gamma,s,0}\mathcal{D}X\left(\gamma\left(s\right)\right)P_{\gamma,0,s}v\right)ds\right\| \\
&= \left\|A^{-1}P_{\gamma,0,\theta}\left(\left(1-\theta\right)X\left(q\right)-\int_0^\theta\left(P_{\gamma,s,0}\mathcal{D}X\left(\gamma\left(s\right)\right)P_{\gamma,0,s}-\mathcal{D}X\left(q\right)\right)v\right)ds\right\| \\
&\leq \left\|A^{-1}P_{\gamma,0,\theta}\left(1-\theta\right)X\left(q\right)\right\| + \left\|A^{-1}P_{\gamma,0,\theta}\int_0^\theta\left(P_{\gamma,s,0}\mathcal{D}X\left(\gamma\left(s\right)\right)P_{\gamma,0,s}-\mathcal{D}X\left(q\right)\right)vds\right\| \\
&\leq \left(1-\theta\right)af\left(t\right)+a\left\|R\left(\theta\right)\right\| \\
&\leq \left(1-\theta\right)af\left(t\right)+aL\left(\frac{\theta}{2}\left\|v\right\|+d\left(p_0,q\right)\right)\theta\left\|v\right\| \\
&\leq \left(1-\theta\right)af\left(t\right)+aL\left(\frac{\theta}{2}af\left(t\right)+t\right)\theta af\left(t\right) \\
&= \frac{1}{8}\left(2b-2t+aLt^2\right)\left(-4\theta+a^2L^2\theta^2t^2+4aL\theta t+2abL\theta^2-2aL\theta^2t+4\right) \\
&= af\left(\tau\left(\theta\right)\right).
\end{aligned}$$

Therefore

$$\gamma\left(\theta\right)\in k\left(\tau\left(\theta\right)\right),$$

and the Lemma is proved. ■

Lemma 25 *Let* $0\leq t<t_*$ *and* $q\in k\left(t\right)$. *Suppose that* $q_*\in B\left[p_0,t_*\right]$ *is a singularity of the vector field* X *and*

$$t+d\left(q,q_*\right)=t_*.$$

Then

$$d\left(p_0,q\right)=t.$$

Moreover, letting

$$\begin{aligned}
t_+ &= t+af\left(t\right), \\
q_+ &= \exp_q\left(A^{-1}X\left(q\right)\right),
\end{aligned}$$

then $t<t_+<t_*$, $q_+\in k\left(t_+\right)$ *and*

$$t_++d\left(q_+,q_*\right)=t_*.$$

Proof. Consider the minimizing geodesic $\alpha : [0,1] \longrightarrow M$ joining q to q_*. Since $q \in k(t)$, we have

$$\begin{aligned} d(p_0, \alpha(\theta)) &\leq d(p_0, q) + d(q, \alpha(\theta)) \\ &\leq t + \theta d(q, q_*) \\ &\leq t + d(q, q_*) \\ &= t_*. \end{aligned}$$

It follows that $\alpha([0,1]) \subset B(p_0, t_*)$. Taking $u = \alpha'(0)$, by Theorem 12 we have

$$P_{\alpha,1,0} X(\alpha(1)) = X(q) + P_{\sigma,0,1} \mathcal{D}X(p_0) P_{\sigma,1,0} u + R(1),$$

with

$$\|R(1)\| \leq L\left(\frac{1}{2}\|u\| + d(p_0, q)\right)\|u\|.$$

Therefore

$$\begin{aligned} \|R(1)\| &\leq L\left(\frac{1}{2} d(q, q_*) + d(p_0, q)\right) d(q, q_*) && (2.14) \\ &= L\left(\frac{1}{2}(t_* - t) + d(p_0, q)\right)(t_* - t) \\ &\leq L\left(\frac{1}{2}(t_* - t) + t\right)(t_* - t) && (2.15) \\ &= L\frac{1}{2}(t_* + t)(t_* - t). \end{aligned}$$

On the other hand, since $|f(t)|$ is strictly decreasing in $[0,t_*]$ and $0\le d(p_0,q)\le t<t_*$,

$$\begin{aligned}\|R(1)\| &= \|X(q)+Au\| \\ &\ge \frac{1}{\left\|\mathcal{D}X(p_0)^{-1}\right\|}\left\|A^{-1}X(q)+u\right\| \\ &\ge \left|f'(0)\right|\left\|A^{-1}X(q)+u\right\| \\ &\ge \left|f'(0)\right|\left(\|u\|-\left\|A^{-1}X(q)\right\|\right) \\ &\ge \left|f'(0)\right|\left(\|u\|-af(t)\right) \\ &= -f'(0)(t_*-t)-f(t)>0.\end{aligned} \tag{2.16}$$

Because

$$\begin{aligned}&f''(t)=L,\\ &0=f'(t_*)=f(t)+f'(t)(t_*-t)+\tfrac{1}{2}f''(t)(t_*-t)^2,\end{aligned}$$

and

$$f'(t)=f'(0)+\int_0^t f''(t)\,dt,$$

therefore

$$0=f(t)+\left(f'(0)+tL\right)(t_*-t)+\frac{1}{2}L(t_*-t)^2,$$

hence

$$\frac{1}{2}L(t_*+t)(t_*-t)=-f'(0)(t_*-t)-f(t).$$

Thus, the last term in (2.14) is equal to the last term in the inequality (2.16), we conclude that all these inequalities in (2.16) are equalities, in particular

$$\begin{aligned}&\tfrac{1}{\left\|\mathcal{D}X(p_0)^{-1}\right\|}=|f'(0)|=a,\\ &\|u\|-\left\|A^{-1}X(q)\right\|=\left\|A^{-1}X(q)+u\right\|>0,\\ &\left\|A^{-1}X(q)\right\|=af(t),\\ &L\left(\tfrac{1}{2}(t_*-t)+d(p_0,q)\right)(t_*-t)=L\left(\tfrac{1}{2}(t_*-t)+t\right)(t_*-t).\end{aligned} \tag{2.17}$$

From the last equation in (2.17), we obtain

$$d(p_0,q)=t,$$

the second equation in (2.17) implies that u and $A^{-1}X(q)$ are linearly dependent vectors in T_qM, so that there exists $r \in \mathbb{R}$ such that

$$A^{-1}X(q) = -ru.$$

Thus, the second equation implies

$$1 - |r| = |1 - r|,$$

and because $r \neq 0$ and $r \neq 1$, we have $0 < r < 1$, thus

$$q_+ = \exp_q(ru) = \alpha(r).$$

Moreover, given that α is a minimizing geodesic joining q to q_*, we have that q, $\alpha(r)$ and q_* are in the same geodesic line, thus

$$d(q, \alpha(r)) + d(\alpha(r), q_*) = d(q, q_*),$$

therefore,

$$d(q, q_+) + d(q_+, q_*) = d(q, q_*).$$

Moreover,

$$\begin{aligned} d(q, q_+) &= \|ru\| \\ &= \left\|A^{-1}X(q)\right\| \\ &= af(t) \\ &= t_+ - t, \end{aligned}$$

hence

$$\begin{aligned} d(q_+, q_*) &= d(q, q_*) - d(q, q_+) \\ &= (t_* - t) - (t_+ - t) \\ &= t_* - t_+, \end{aligned}$$

that is

$$d\left(q_{+}, q_{*}\right) + t_{+} = t_{*}.$$

■

Lemma 26 *Suppose that $q_* \in B\left[p_0, t_*\right]$ is a singularity of the vector field X. If there exist $\tilde{t}$ and $\tilde{q}$ such that*

$$0 \leq \tilde{t} < t_*, \quad \tilde{q} \in k\left(\tilde{t}\right) \quad \text{and} \quad \tilde{t} + d\left(\tilde{q}, q_*\right) = t_*,$$

then

$$d\left(p_0, q_*\right) = t_*.$$

Proof. Changing τ_0 by $\tilde{t}$ and q_0 by $\tilde{q}$ in Lemma 23, we obtain that

$$q_k \in k\left(\tau_k\right), \text{ for all } k \in \mathbb{N},$$

$\left\{\tau_k\right\}_{k\in\mathbb{N}}$ converges to t_*, $\left\{q_k\right\}_{k\in\mathbb{N}}$ converges to some $\tilde{q}_* \in B\left(p_0, t_*\right)$, and $X\left(q_*\right) = 0$. Moreover, by Lemma 25 and applying induction, it is easy to show that for all k,

$$d\left(p_0, q_k\right) = \tau_k \quad \text{and} \quad d\left(q_k, q_*\right) = t_* - \tau_k.$$

Passing to the limit, we obtain

$$d\left(p_0, \tilde{q}_*\right) = t_* \quad \text{and} \quad d\left(\tilde{q}_*, q_*\right) = 0.$$

Therefore $\tilde{q}_* = q_*$ and

$$d\left(p_0, q_*\right) = t_*.$$

■

The two following lemmas complete the proof of the uniqueness.

Lemma 27 *The limit p_* of the sequence $\left\{p_k\right\}_{k\in\mathbb{N}}$ is the unique singularity of X in $B\left[p_0, t_*\right]$.*

Proof. Let $q_* \in B\left[p_0, t_*\right]$ a singularity of the vector field X. Using induction, we neet show that

$$d\left(p_k, q_*\right) + t_k \leq t_*.$$

We need to consider two cases:

Case 1. $(d(p_0, q_*) < t_*)$. First we show by induction that for all $k \in \mathbb{N}$,

$$d(p_k, q_*) + t_k < t_*. \tag{2.18}$$

Indeed, for $k = 0$ (2.18) is immediately true, because $t_0 = 0$. Now, suppose the property is true for some k. Let us take the geodesic

$$\gamma_k(\theta) = \exp_{p_k}(-\theta v_k),$$

where v_k is defined in (2.2). From Lemma 24, for all $\theta \in [0,1]$,

$$\gamma_k(\theta) \in k(t_k + \theta(t_{k+1} - t_k)). \tag{2.19}$$

Define $\phi : [0,1] \longrightarrow M$ by

$$\phi(\theta) = d(\gamma_k(\theta), q_*) + t_k + \theta(t_{k+1} - t_k). \tag{2.20}$$

We know that

$$\phi(0) = d(p_k, q_*) + t_k < t_*.$$

We next show, by contradiction, that $\phi(\theta) \neq t_*$ for all $\theta \in [0,1]$.

Suppose that there exists a $\tilde{\theta} \in [0,1]$ such that $\phi\left(\tilde{\theta}\right) = t_*$, and let $\tilde{q} = \gamma_k\left(\tilde{\theta}\right)$ and $\tilde{t} = t_k + \tilde{\theta}(t_{k+1} - t_k)$. By (2.19) and (2.20),

$$\tilde{q} \in k\left(\tilde{t}\right) \text{ and } d(\tilde{q}, q_*) + \tilde{t} = t_*.$$

Applying Lemma 26, we conclude that

$$d(p_0, q_*) = t_*,$$

which contradicts our assumption. Thus $\phi(\theta) \neq t_*$ for all $\theta \in [0,1]$, Since $\phi(0) < t_*$ and ϕ is continuous, we have that $\phi(\theta) < t_*$ for all $\theta \in [0,1]$. In particular, by (2.20),

$$d(\gamma_k(1), q_*) + t_{k+1} = \phi(1) < t_*.$$

Thus,

$$d(p_{k+1}, q_*) + t_{k+1} < t_*,$$

in this way (2.18) is true for all $k \in \mathbb{N}$.

Case 2. $(d(p_0, q_*) = t_*)$. Using induction, let us prove that for all $k \in \mathbb{N}$,

$$d(p_k, q_*) + t_k = t_*. \tag{2.21}$$

Indeed, for $k = 0$, this is immediately true, because $t_0 = 0$. Now, suppose that (2.21) is true for some k. Since $p_k \in k(t_k)$, by Lemma 25 we conclude that

$$d(p_{k+1}, q_*) + t_{k+1} = t_*.$$

Finally, by (2.18) and (2.21), we conclude that for all $k \in \mathbb{N}$,

$$d(p_k, q_*) + t_k \leq t_*,$$

and passing to the limit $k \longrightarrow \infty$, we obtain $d(p_*, q_*) = 0$, and therefore

$$p_* = q_*.$$

■

Lemma 28 *If $c = abL < \frac{1}{2}$ and $B(p_0, r) \subseteq \Omega$, with*

$$t_* < r \leq t_{**} = \frac{1}{aL}\left(1 + \sqrt{1 - 2c}\right),$$

then the limit p_* of the sequence $\{p_k\}_{k \in \mathbb{N}}$ is the unique singularity of the vector field X in $B(p_0, r)$.

Proof. Let $q_* \in B(p_0, r)$ be a singularity of the vector field X in $B(p_0, r)$. Let us consider the minimizing geodesic $\alpha : [0, 1] \longrightarrow M$ joining p_0 to q_*. By Lemma 20,

$$P_{\alpha,1,0} X(\alpha(1)) = X(p_0) + P_{\sigma,0,1} \mathcal{D}X(p_0) P_{\sigma,1,0} u + R(1),$$

where

$$\|R(1)\| \leq L\left(\frac{1}{2}\|u\| + d(p_0, p_0)\right)\|u\| = \frac{L}{2} d(p_0, q_*)^2 \text{ and } \|u\| = d(p_0, q_*). \tag{2.22}$$

In a similar way to the inequality (2.16), is easy to prove that

$$\begin{aligned}\|R(1)\| &\geq \frac{1}{a}\left(\|u\| - \left\|\mathcal{D}X(p_0)^{-1} X(p_0)\right\|\right) \\ &\geq \frac{1}{a} d(p_0, q_*) - \frac{b}{a}.\end{aligned}$$

Therefore

$$\frac{L}{2} d(p_0, q_*)^2 \geq \frac{1}{a} d(p_0, q_*) - \frac{b}{a},$$

hence

$$f(d(p_0, q_*)) \geq 0,$$

since $d(p_0, q_*) \leq r \leq t_{**}$, then

$$d(p_0, q_*) \leq t_*.$$

Finally, from Lemma 27,

$$p_* = q_*.$$

■

2.1.1 Order of convergence of Kantorovich methods

It is well-known that in Banach spaces the Kantorovich and simplified Kantorovich methods have order of convergence one (linear) and two (quadratic), respectively (see [27]). Since the analysis of the order of convergence is made in a local way, that is, in a neighborhood of the singularity of the vector field which we are considering. We can define the order of convergence on Riemannian manifolds in the following way:

Definition 29 *Let M be a complete Riemannian manifold and let $\{p_k\}_{k\in\mathbb{N}}$ be a sequence on M converging to p_*. If there exist a system of coordinates (U, x) of M whith $p_* \in U_\alpha$, constants $p > 0, c \geq 0$, and $K \geq 0$ such that, for all $k \geq K$, $\{p_k\}_{k=K}^{\infty}$ is contained in U_α and* *the following inequality holds:*

$$\left\| x^{-1}(p_{k+1}) - x^{-1}(p_*) \right\| \leq c \left\| x^{-1}(p_k) - x^{-1}(p_*) \right\|^p, \tag{2.23}$$

then it is said that $\{p_k\}_{k \in \mathbb{N}}$ *converges to* p_* *with order at least* p.

It can be shown that the definition above do not depend on the choice of the coordinates system and the multiplictive constant c depends on the chart, but for any chart, there exists such a constant, (see [18]).

Notice that in normal coordinates of 0_{p_k},

$$\left\| \exp_{p_k}^{-1}(p) - \exp_{p_k}^{-1}(q) \right\| = d(p,q),$$

thus, in normal coordinates, (2.23) is transformed into

$$d(p_{k+1}, p_*) \leq c d(p_k, p_*)^p.$$

Lemma 30 *i) The order of convergence of the Kantorovich method on Riemannian manifolds is two (quadratic convergence).*

ii) The order of convergence of the simplified Kantorovich method on Riemannian manifolds is one (linear convergence).

Proof. Let k be sufficiently large in such a way that p_*, p_k, $p_{k+1}, \ldots$belong to a normal neighborhood U of p_k. Let us consider the geodesic γ_k joining p_k to p_* defined by

$$\gamma_k(t) = \exp_{p_k}(t u_k),$$

where $u_k \in T_{p_k}M$ and $d(p_k, p_*) = \|u_k\|$.

We know that if p, q be in one normal neighborhood U of p_k, then

$$\left\| \exp_{p_k}^{-1}(p) - \exp_{p_k}^{-1}(q) \right\| = d(p,q).$$

i) By Theorem 13,

$$P_{\gamma,t,o} X(p_*) = X(p_k) + \mathcal{D}X(p_k) u_k + R(1),$$

with

$$\|R(1)\| \leq \frac{L}{2}\|u_k\|^2 \text{ and } \|u_k\| = d(p_k, p_*).$$

Hence,

$$0 = \mathcal{D}X(p_k)^{-1} X(p_k) + u_k + \mathcal{D}X(p_k)^{-1} R(1).$$

Since

$$-\mathcal{D}X(p_k)^{-1} X(p_k) = \exp_{p_k}^{-1}(p_{k+1}) \quad \text{and} \quad u_k = \exp_{p_k}^{-1}(p_*),$$

we have

$$\exp_{p_k}^{-1}(p_{k+1}) - \exp_{p_k}^{-1}(p_*) = \mathcal{D}X(p_k)^{-1} R(1),$$

thus

$$d(p_{k+1}, p_*) \leq \left\|\mathcal{D}X(p_k)^{-1}\right\| \frac{L}{2}\|u_k\|^2.$$

Moreover, by Proposition 21,

$$\begin{aligned}
\left\|\mathcal{D}X(p_k)^{-1}\right\| &\leq \frac{a}{1 - aLd(p_k, p_0)} \\
&\leq \frac{a}{1 - aL\tau_k} \\
&\leq \frac{a}{1 - aLt_*} \\
&= \frac{a}{\sqrt{1 - 2abL}}.
\end{aligned}$$

Therefore

$$d(p_{k+1}, p_*) \leq Cd(p_k, p_*)^2,$$

with

$$C = \frac{La}{2\sqrt{1 - 2abL}}.$$

ii) Let p_0 be sufficiently near to p_* in such a way that p_0 is in the normal neighborhood U of 0_{p_k}, By Theorem 12, if $\sigma_k : [0,1] \longrightarrow M$ is the minimizing geodesic connecting p_0, p_k, then

$$P_{\gamma,1,0} X(p_*) = X(p_k) + P_{\sigma_k,0,1} \mathcal{D}X(p_0) P_{\sigma_k,1,0} u_k + R(1),$$

with

$$\left\|R\left(1\right)\right\| \leq L\left(\frac{1}{2}\left\|u_k\right\| + d\left(p_0, p_k\right)\right)\left\|u_k\right\|.$$

Therefore

$$0 = P_{\sigma_k,0,1}\mathcal{D}X\left(p_0\right)^{-1}P_{\sigma_k,1,0}X\left(p_k\right) + u_k + P_{\sigma_k,0,1}\mathcal{D}X\left(p_0\right)^{-1}P_{\sigma_k,1,0}R\left(1\right).$$

Since

$$-P_{\sigma_k,0,1}\mathcal{D}X\left(p_0\right)^{-1}P_{\sigma_k,1,0}X\left(p_k\right) = \exp_{p_k}^{-1}\left(p_{k+1}\right) \quad \text{and} \quad u_k = \exp_{p_k}^{-1}\left(p_*\right),$$

we have

$$\exp_{p_k}^{-1}\left(p_{k+1}\right) - \exp_{p_k}^{-1}\left(p_*\right) = P_{\sigma_k,0,1}\mathcal{D}X\left(p_0\right)^{-1}P_{\sigma_k,1,0}R\left(1\right).$$

We thus conclude that

$$\begin{aligned}
d\left(p_{k+1}, p_*\right) &= \left\|\exp_{p_k}^{-1}\left(p_{k+1}\right) - \exp_{p_k}^{-1}\left(p_*\right)\right\| \\
&= \left\|P_{\sigma_k,0,1}\mathcal{D}X\left(p_0\right)^{-1}P_{\sigma_k,1,0}R\left(1\right)\right\| \\
&\leq \left\|\mathcal{D}X\left(p_0\right)^{-1}\right\|\left\|R\left(1\right)\right\| \\
&\leq aL\left(\frac{1}{2}\left\|u_k\right\| + d\left(p_0, p_k\right)\right)\left\|u_k\right\| \\
&= aL\left(\frac{1}{2}d\left(p_k, p_*\right) + d\left(p_0, p_k\right)\right)d\left(p_k, p_*\right) \\
&= aL\left(\frac{1}{2}\frac{d\left(p_k, p_*\right)}{d\left(p_0, p_k\right)} + 1\right)d\left(p_0, p_k\right)d\left(p_k, p_*\right).
\end{aligned}$$

If k is sufficiently large, then $d\left(p_k, p_*\right) \leq d\left(p_0, p_k\right)$, and therefore

$$\left(\frac{1}{2}\frac{d\left(p_k, p_*\right)}{d\left(p_0, p_k\right)} + 1\right) \leq \frac{3}{2},$$

and therefore, for p_0 sufficiently close to p_*,

$$d\left(p_{k+1}, p_*\right) \leq K_0 d\left(p_0, p_k\right)d\left(p_k, p_*\right),$$

with $K_0 \leq \frac{3aL}{2}$. ■

Remark 31 *Note that if instead of putting in the Kantorovich method the point* p_0*, we fix* p_j *sufficiently close to* p_**, we will obtain a new convergent method. Indeed the calculations made in the previous lemma become in*

$$d(p_{k+1}, p_*) \leq K_j d(p_j, p_*) d(p_k, p_*),$$

with $K_j \leq \frac{3aL}{2}$*. Thus,*

$$d(p_{k+1}, p_*) \leq K d(p_j, p_*) d(p_k, p_*), \tag{2.24}$$

with $K \leq \frac{3aL}{2}$*.*

2.2 A family of higher order Newton type methods

There is a method in Banach spaces, the Shamanskii method, which combines the Newton and simplified Newton methods; this method was introduced by V. E. Shamanskii [27]. Next, we describe the method.

Given an integer m and an initial point x_0 in a Banach space, we move from x_n to x_{n+1} through an intermediate sequence $\left\{y_n^i\right\}_{i=1}^{\infty}$ which is a combination of the Kantorovich and the simplified Kantorovich methods

$$M_{p,f} := \begin{cases} y_n^1 = x_n - \mathcal{D}F(x_n)^{-1} F(x_n) \\ y_n^2 = y_n^1 - \mathcal{D}F(x_n)^{-1} F\left(y_n^1\right) \\ \vdots \\ y_n^m = x_{n+1} = y_n^{m-1} - \mathcal{D}F(x_n)^{-1} F\left(y_n^{(m-1)}\right). \end{cases}$$

Under appropriate conditions, this method or family of methods is convergent to a root x_* of the equation $F(x) = 0$. Moreover, if x_0 is sufficiently close to x_*, the convergence of the Shamanskii method has order at least $m+1$ (see [2], [10], [27]). We are going extend this result to the context of Riemannian manifolds.

Under the hypotheses of the Kantorovich theorems, let us consider the family of methods

$$M_{p,f} := \begin{cases} q_n^1 = \exp_{p_n}\left(-\mathcal{D}X\left(p_n\right)^{-1} X\left(p_n\right)\right) \\ q_n^2 = \exp_{q_n^1}\left(-P_{\sigma_1,0,1}\mathcal{D}X\left(p_n\right)^{-1} P_{\sigma_1,1,0}X\left(q_n^1\right)\right) \\ \vdots \\ q_n^m = p_{n+1} = \exp_{q_n^{m-1}}\left(-P_{\sigma_{m-1},0,1}\mathcal{D}X\left(p_n\right)^{-1} P_{\sigma_{m-1},1,0}X\left(q_n^{(m-1)}\right)\right), \end{cases} \tag{2.25}$$

where $\sigma_k : [0,1] \longrightarrow M$ is the minimizing geodesic joining the points p_n and q_n^k, whit $k = 1, 2, \ldots (m-1)$. Thus,

$$\sigma_k(0) = p_n \quad \text{and} \quad \sigma_k(1) = q_n^k.$$

Note that for $p = 1$ this yields the Kantorovich's method. Moreover, at each step we use the Kantorovich method fixing the initial point or the simplified Kantorovich method.

Theorem 32 *Under the hypotheses of the Kantorovich theorem, the method described in (2.25) converges with order of convergence $m+1$.*

Proof. Let us observe that

$$d\left(p_{n+1}, p_n\right) \le d\left(p_{n+1}, q_n^{(m-1)}\right) + d\left(q_n^{(m-1)}, q_n^{(m-2)}\right) + \ldots + d\left(q_n^2, q_n^1\right) + d\left(q_n^1, p_n\right).$$

Now, if we define $p_{n+1} = q_n^m$, $p_n = q_n^1$, looking at each step as a different method according to (2.25) , then by Kantorovich theorem in the first step and by the simplified Kantorovich theorem for the following steps, each one of the sequences $\{q_n^m\}_{m\in\mathbb{N}}$ for fixed n, is convergent to the same point $p_* \in M$. Therefore, $\{p_n\}_{p\in\mathbb{N}}$ is convergent to p_*. Moreover, for Lemma 30 i) and (2.24),

$$\begin{aligned} d\left(p_{n+1}, p_*\right) &\le Kd\left(p_n, p_*\right) d\left(q_n^{(m-1)}, p_*\right) \le Kd\left(p_n, p_*\right) Kd\left(p_n, p_*\right) d\left(q_n^{(m-2)}, p_*\right) \\ &\le \ldots \le K^{m-1} d\left(p_n, p_*\right)^{m-1} d\left(q_n^1, p_*\right) \\ &\le K^{m-1} d\left(p_n, p_*\right)^{m-1} Cd\left(p_n, p_*\right)^2. \end{aligned}$$

Therefore,

$$d\left(p_{n+1}, p_*\right) \le CK^{m-1} d\left(p_n, p_*\right)^{m+1}.$$

■

Chapter 3

Third-order iterative methods on Riemannian manifolds

A review to the amount of literature on high order iterative methods in Banach spaces, in the two last decades, shows the importance of higher order schemes, see [21] and references therein. In general the methods of third order ([5], [7-9], [11], [13], [20-24]), present the difficulty of evaluating the second order Fréchet derivative. For a nonlinear system of m equations and m unknows, the first Fréchet derivative is a matrix with m^2 entries, while the second Fréchet derivative has m^3 entries. This implies a huge amount of operations, but a high convergence order.

Let us suppose that F is an operator defined on an open convex subset Ω of a Banach space E of class C^2,

$$L_F(x_n) = DF(x_n)^{-1} D^2F(x_n) DF(x_n)^{-1} F(x_n).$$

Some of the most famous methods of third order to find a root of the equation $F(x) = 0$ are:

- *Halley:*

$$x_{n+1} = x_n - \left(I + \frac{1}{2} L_F(x_n)\right)^{-1} DF(x_n)^{-1} F(x_n).$$

- *Super-Halley:*

$$x_{n+1} = x_n - \left(I + \frac{1}{2}\left(I - L_F\left(x_n\right)\right)^{-1} L_F\left(x_n\right)\right) DF\left(x_n\right)^{-1} F\left(x_n\right).$$

- *Chebyshev:*

$$x_{n+1} = x_n - \left(I + \frac{1}{2} L_F\left(x_n\right)\right) DF\left(x_n\right)^{-1} F\left(x_n\right).$$

- *Chebyshev like methods:* for $0 \leq \lambda \leq 2$,

$$x_{n+1} = x_n - \left(I + \frac{1}{2} L_{F(x_n)} + \lambda L_F^2\left(x_n\right)\right) DF\left(x_n\right)^{-1} F\left(x_n\right).$$

- *Chebyshev-Halley method:*

$$x_{n+1} = y_n - \frac{1}{2}\left[I - \frac{\lambda}{2} L_F\left(x_n\right)\right]^{-1} L_F\left(x_n\right) DF\left(x_n\right)^{-1} F\left(x_n\right).$$

- *Two-step:*

$$\begin{cases} y_n = x_n - DF\left(x_n\right)^{-1} F\left(x_n\right) \\ x_{n+1} = y_n - DF\left(x_n\right)^{-1} F\left(y_n\right), \end{cases}$$

  where I is the identity operator in E.

In this chapter, we will prove some methods on Riemannian manifolds that generalize the methods described previously.

3.1 Chebyshev-Halley method on Riemannian manifolds

Let us first recall first the Chebyshev-Halley method to approach a local unique solution x^* of the equation

$$F\left(x\right) = 0, \tag{3.1}$$

where F is a nonlinear operator defined in an open convex subset Ω of a Banach space E into itself. Take $x_0 \in E$. For $n = 0, 1, ...,$ define:

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1} F(x_n), \\
G_n &= F'(x_n)^{-1} F''(x_n)(y_n - x_n), \\
x_{n+1} &= y_n - \tfrac{1}{2} F'(x_n)^{-1} \left[I + \tfrac{\lambda}{2} G_n\right]^{-1} F''(x_n)(y_n - x_n)^2 .
\end{aligned}
\tag{3.2}
$$

Equivaletly

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1} F(x_n), \\
G_n &= F'(x_n)^{-1} F''(x_n)(y_n - x_n), \\
x_{n+1} &= y_n - \tfrac{1}{2} \left[I + \tfrac{\lambda}{2} G_n\right]^{-1} G_n (y_n - x_n) .
\end{aligned}
\tag{3.3}
$$

Under certain conditions, the convergence of these methods was proven by Argyros [5]; these methods are actually cubically convergent. Here, $F'(x_n)$ and $F''(x_n)$ denote the first and second Fréchet-derivative of F evaluated at x_n and λ is a nonnegative parameter. We obtain the Halley Methods for $\lambda = 1$ and the Euler-Chebyshev method for $\lambda = 0$. In general (3.2) or (3.3) are called The Chebyshev-Halley family.

We study an extension of this method to the problem of finding singularities of a vector field X defined on a real, complete and connected m-dimensional Riemannian manifold M, that is, we wish to solve

$$X(p) = 0, \quad p \in M.$$

Recall that in the first line of (3.2),

$$y_n - x_n = -F'(x_n)^{-1},$$

on a manifold M this equation is transformed in

$$u_n = -\mathcal{D}X(p_n)^{-1} X(p_n),$$

where $y_n - x_n$ is replaced by u_n, $F(x_n)$ by $X(p_n)$ and $F'(x_n)$ by $\mathcal{D}X(p_n)$. We can hence write (3.3) as

$$
\begin{array}{l}
u_n = -\mathcal{D}X\left(p_n\right)^{-1} X\left(p_n\right), \\
G_n = -\mathcal{D}X\left(p_n\right)^{-1} \mathcal{D}^2 X\left(p_n\right) u_n, \\
q_n = \exp_{p_n}\left(u_n\right), \\
\sigma\left(t\right) = \exp_{p_n}\left(t u_n\right), \\
H_n = \left[I_{T_{p_n}M} + \frac{\lambda}{2} G_n\right], \\
v_n = -\frac{1}{2} P_{\gamma,0,1} \mathcal{D}X\left(p_n\right)^{-1} H_n^{-1} \mathcal{D}^2 X\left(p_n\right)\left(u_n, u_n\right), \\
p_{n+1} = \exp_{q_n}\left(v_n\right),
\end{array}
\tag{3.4}
$$

where

$$
\begin{aligned}
\mathcal{D}X\left(p_n\right) &= \nabla_{(.)} X\left(p_n\right), \\
\mathcal{D}^2 X\left(p_n\right) &= \nabla_{(.)}\left(\nabla_{(.)} X\left(p_n\right)\right) - \nabla_{\nabla_{(.)}(.)} X\left(p_n\right),
\end{aligned}
$$

and $I_{T_{p_n}M}$ is the identity on $T_{p_n}M$. We next proceed to the extension of this method to the problem of finding a solution of $X\left(p\right) = 0$ for a vector field $X \in TM$ defined on a real, complete and connected m-dimensional Riemannian manifold M. In fact, we will consider a much more general method than the one described in (3.4). This new method is:

$$
\begin{array}{l}
u_n = -\mathcal{D}X\left(p_n\right)^{-1} X\left(p_n\right), \\
q_n = \exp_{p_n}\left(u_n\right), \\
\sigma\left(t\right) = \exp_{p_n}\left(t u_n\right), \\
G_n = \mathcal{D}X\left(p_n\right)^{-1} B\left(p_n\right)\left(u_n, .\right), \\
H_n = \left[I_{T_{p_n}M} + \frac{\lambda}{2} G_n\right], \\
v_n = -\frac{1}{2} P_{\gamma,0,1} \mathcal{D}X\left(p_n\right)^{-1} H_n^{-1} B\left(p_n\right)\left(u_n, u_n\right), \\
p_{n+1} = \exp_{q_n}\left(v_n\right),
\end{array}
\tag{3.5}
$$

where $B\left(p_n\right) : T_{p_n}M \times T_{p_n}M \longrightarrow T_{p_n}M$ is a bilinear operator. Note that for $B = 0$ and $B = \mathcal{D}^2 X$, we obtain Newton's and Chebyshev-Halley methods ([16]) on Riemannian manifolds, respectively.

Before proving the theorem, introduce some notation. Let $p_0 \in \Omega \subseteq M$; $a, b, \eta, N, c, d, \lambda \geq 0, \beta > 0$ and $\left\|\mathcal{D}X\left(p_0\right)^{-1} X\left(p_0\right)\right\| \leq g_0 < \eta$ we define the sequences for all $n \geq 0$:

$$
\begin{aligned}
&\overline{\alpha}_{n+1} = \tfrac{N}{2}\left\|v_n\right\|^2 + \left[\tfrac{a}{2}\left\|u_n\right\|^2 + \left(1+\tfrac{\lambda}{2}\right)\tfrac{bN}{2}\left(\tfrac{1}{\beta} - Nd\left(p_n,p_0\right)\right)^{-1}\left\|u_n\right\|^3\right]\overline{h}_n,\\
&\overline{h}_n = \left[1 - \tfrac{\lambda b}{2}\left\|u_n\right\|\left(\tfrac{1}{\beta} - Nd\left(p_n,p_0\right)\right)^{-1}\right]^{-1},\\
&\overline{\beta}_{n+1} = \left(\tfrac{1}{\beta} - Nd\left(p_{n+1},p_0\right)\right)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
f_0 &= 0,\\
g_{n+1} &= f_{n+1} + \beta_{n+1}\alpha_{n+1},\\
f_{n+1} &= g_n + \tfrac{1}{2}\beta_n h_n b\left(g_n - f_n\right)^2,
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
&\alpha_{n+1} = \tfrac{N}{2}\left(f_{n+1} - g_n\right)^2 + \left[\tfrac{a}{2}\left(g_n - f_n\right)^2 + \tfrac{bN}{2}\beta_n\left(1+\tfrac{\lambda}{2}\right)\left(g_n - f_n\right)^3\right]h_n,\\
&h_n = \left[1 - \tfrac{\lambda b}{2}\beta_n\left(g_n - f_n\right)\right]^{-1},\\
&\beta_n = \left(\tfrac{1}{\beta} - Nf_n\right)^{-1}.
\end{aligned}
$$

We also consider the auxiliary function

$$
T\left(r\right) = \eta + \tfrac{\beta b r^2}{2-2\beta N r - \lambda b \beta r} + \tfrac{\beta}{1-\beta N r}\left[\tfrac{N}{2}r^2 + \tfrac{\alpha\left(1-\beta N r\right)r^2 + \left(1+\frac{\lambda}{2}\right)\beta b N r^3}{2-2\beta N r - \lambda b \beta r}\right].
$$

Theorem 33 *(Chebyshev-Halley's method on Riemannian manifolds) Let M be a complete Riemannian manifold, $\Omega \subseteq M$ an open convex set, $X \in \chi\left(M\right)$. Suppose that the bilinear operator $\mathcal{D}^2X$ exists on Ω.*

Assume That:

1. *The continuous linear operator $\mathcal{D}X\left(p_0\right)^{-1}$ exists on $B\left(p_0,R\right)$ and satisfies*

$$
\left\|\mathcal{D}X\left(p_0\right)^{-1}\right\| \leq \beta \text{ for } p_0 \in \Omega,\ \beta > 0,\ R > 0;
$$

2. $$
\left\|\mathcal{D}^2X\left(p\right)\right\| \leq N \text{ for all } p \in B\left(p_0,R\right),\ N > 0;
$$

3. *If $B\left(p\right) : T_pM \times T_pM \longrightarrow T_pM$ is a bilinear operator such that $\left\|B\left(p\right)\right\| \leq b$, $b \geq 0$, then*

$$
\left\|P_{\alpha,t,0}\mathcal{D}^2X\left(\alpha\left(t\right)\right)P^2_{\alpha,0,t} - B\left(p\right)\right\| \leq a \text{ for all } p \in B\left(p_0,R\right) \quad a > 0,
$$

for all geodesic $\alpha : [0,1] \longrightarrow M$*, such that* $\alpha(0) = p$;

4.
$$\left\| \mathcal{D}X(p_0)^{-1} X(p_0) \right\| \leq \eta;$$

5. *There exists a minimum nonnegative number* r^* *satisfying*

$$T(r^*) \leq r^*;$$

6. *The numbers* r^*, R, *also satisfy*

$$r^* \leq R, \quad \beta(2N + \lambda b) r^* < 2, \quad \beta N (3r^* + R) < 2, \quad \text{and } B(p_0, R) \subseteq \Omega. \tag{3.7}$$

Then

(a) The scalar sequences $\{g_n\}$, $\{f_n\}$ $(n \in N)$ *generated by (3.6) are monotonically increasing and bounded from above by*

$$r^* = \lim_{n \longrightarrow \infty} g_n = \lim_{n \longrightarrow \infty} f.$$

(b) The Chebyshev-Halley like iteration $\{p_k\}_{k \in \mathbb{N}}$ *generated by (3.5) is well defined in* $B(p_0, r^*)$, *that is* $\{p_k\}_{k \in \mathbb{N}} \subset B(p_0, r^*)$, *and it converges to the unique root* p_* *of the equation* $X(p) = 0$ *in* $B(p_0, R)$.

Moreover, the following error bounds hold for all $n \geq 0$:

$$\begin{aligned} d(q_n, p_n) &\leq g_n - f_n, \quad d(p_{n+1}, q_n) \leq f_{n+1} - g_n, \\ d(q_n, p_*) &\leq r^* - g_n, \quad d(p_n, p_*) \leq r^* - f_n. \end{aligned}$$

Before proving the Chebyshev-Halley method on Riemannian manifolds, we establish some preliminary results.

Lemma 34 *Assume that the method (3.5) is well defined for all* $n \geq 0$, *let* $\sigma(t) = \exp_{p_n}(tu_n)$, *then*

$$P_{\sigma,1,0} X(q_n) = \int_0^1 (1-t) P_{\sigma,t,0} \mathcal{D}^2 X(\sigma(t)) (P_{\sigma,0,t} u_n, P_{\sigma,0,t} u_n) \, dt. \tag{3.8}$$

Proof. From the theorem 14,

$$\begin{aligned}
P_{\sigma,1,0}X(q_n) &= X(p_n) + \mathcal{D}X(p_n)\,u_n + \int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\left(\sigma'(t),\sigma'(t)\right)dt \\
&= X(p_n) + \mathcal{D}X(p_n)\left(-\mathcal{D}X(p_n)^{-1}X(p_n)\right) + \int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\left(\sigma'(t),\sigma'(t)\right)dt \\
&= \int_0^1 (1-t)\,P_{\sigma,s,0}\mathcal{D}^2X(\sigma(t))\left(\sigma'(t),\sigma'(t)\right)dt,
\end{aligned}$$

Because σ is a geodesic, then $\sigma'(t)$ is parallel and $\sigma'(t) = P_{\sigma,o,t}\sigma'(0)$, and because $\sigma'(0) = u_n$, then

$$P_{\sigma,1,0}X(q_n) = \int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\left(P_{\sigma,0,t}u_n, P_{\sigma,0,t}u_n\right)dt.$$

■

Lemma 35 *Let M be a complete Riemannian manifold, $\Omega \subseteq M$ an open convex set, $X \in C^2$. Suppose that the bilinear operator $\mathcal{D}^2X$ exists on Ω. If*

$$\gamma(t) = \exp_{q_n}(tv_n),$$

assume that the method (3.5) is well defined for all $n \geq 0$. Then the following representation of X is true for all $n \geq 0$:

$$\begin{aligned}
&P_{\gamma,1,0}X(p_{n+1}) = \int_0^1 (1-t)\,P_{\gamma,t,0}\mathcal{D}^2X(\gamma(t))\,P^2_{\gamma,0,t}(v_n,v_n)\,dt \\
&+P_{\sigma,0,1}H_n^{-1}\left(\int_0^1 (1-t)\left[P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P^2_{\sigma,0,t} - B(p_n)\right](u_n,u_n)\,dt\right. \\
&+\tfrac{\lambda}{2}P_{\sigma,0,1}\mathcal{D}X(p_n)^{-1}B(p_n)\left.\left(u_n, \int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P^2_{\sigma,0,t}(u_n,u_n)\,dt\right)\right) \\
&-\tfrac{1}{2}P_{\sigma,0,1}\int_0^1 P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P_{\sigma,0,t}\left(u_n, \mathcal{D}X(p_n)^{-1}H_n^{-1}B(p_n)(u_n,u_n)\right)dt.
\end{aligned}$$

Proof. Note that

$$\sigma(0) = p_n, \quad \sigma(1) = q_n,$$

and

$$\gamma(0) = q_n, \quad \gamma(1) = p_{n+1}.$$

It is clear that

$$P_{\gamma,1,0}X\left(p_{n+1}\right)=P_{\gamma,1,0}X\left(p_{n+1}\right)-X\left(q_{n}\right)-\mathcal{D}X\left(q_{n}\right)\left(v_{n}\right)+X\left(q_{n}\right)+\mathcal{D}X\left(q_{n}\right)\left(v_{n}\right), \qquad (3.9)$$

Now, since

$$\begin{aligned}X\left(q_{n}\right)+\mathcal{D}X\left(q_{n}\right)\left(v_{n}\right) &= X\left(q_{n}\right)+\mathcal{D}X\left(q_{n}\right)\left(-\frac{1}{2}P_{\sigma,0,1}\mathcal{D}X\left(p_{n}\right)^{-1}H_{n}^{-1}B\left(p_{n}\right)\left(u_{n},u_{n}\right)\right)\\
&= X\left(q_{n}\right)-\frac{1}{2}\mathcal{D}X\left(q_{n}\right)\left(P_{\sigma,0,1}\mathcal{D}X\left(p_{n}\right)^{-1}H_{n}^{-1}B\left(p_{n}\right)\left(u_{n},u_{n}\right)\right)\\
&+\frac{1}{2}P_{\sigma,0,1}\mathcal{D}X\left(p_{n}\right)\left(\mathcal{D}X\left(p_{n}\right)^{-1}H_{n}^{-1}B\left(p_{n}\right)\left(u_{n},u_{n}\right)\right)-\frac{1}{2}P_{\sigma,0,1}H_{n}^{-1}B\left(p_{n}\right)\left(u_{n},u_{n}\right)\\
&= X\left(q_{n}\right)-\frac{1}{2}\left[\mathcal{D}X\left(q_{n}\right)P_{\sigma,0,1}-P_{\sigma,0,1}\mathcal{D}X\left(p_{n}\right)\right]\left(\mathcal{D}X\left(p_{n}\right)^{-1}H_{n}^{-1}B\left(p_{n}\right)\left(u_{n},u_{n}\right)\right)\\
&-\frac{1}{2}P_{\sigma,0,1}H_{n}^{-1}B\left(p_{n}\right)\left(u_{n},u_{n}\right).\end{aligned}$$

By (1.18), we obtain

$$P_{\sigma,1,0}\mathcal{D}X\left(\sigma\left(1\right)\right)\sigma'\left(1\right)=\mathcal{D}X\left(\sigma\left(0\right)\right)\sigma'\left(0\right)+\int_{0}^{1}P_{\sigma,t,0}\mathcal{D}^{2}X\left(\sigma\left(t\right)\right)\left(\sigma'\left(t\right),\sigma'\left(t\right)\right)dt,$$

since

$$\sigma'\left(0\right)=u_{n} \text{ and } \sigma'\left(t\right)=P_{\sigma,0,t}u_{n},$$

this yields

$$P_{\sigma,1,o}\mathcal{D}X\left(q_{n}\right)P_{\sigma,0,1}u_{n}=\mathcal{D}X\left(p_{n}\right)u_{n}+\int_{0}^{1}P_{\sigma,t,0}\mathcal{D}^{2}X\left(\sigma\left(t\right)\right)\left(P_{\sigma,0,t}u_{n},P_{\sigma,0,t}u_{n}\right)dt.$$

Therefore

$$\mathcal{D}X\left(q_{n}\right)P_{\sigma,0,1}-P_{\sigma,0,1}\mathcal{D}X\left(p_{n}\right)=P_{\sigma,0,1}\int_{0}^{1}P_{\sigma,t,0}\mathcal{D}^{2}X\left(\sigma\left(t\right)\right)\left(P_{\sigma,0,t}u_{n},P_{\sigma,0,t}\left(.\right)\right)dt,$$

hence

$$
\begin{aligned}
&X(q_n) + \mathcal{D}X(q_n)(v_n) \\
&= X(q_n) - \tfrac{1}{2}\left[P_{\sigma,0,1}\int_0^1 P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P^2_{\sigma,0,t}(u_n,.)\,dt\right]\left(\mathcal{D}X(p_n)^{-1}H_n^{-1}B(p_n)(u_n,u_n)\right) \\
&-\tfrac{1}{2}P_{\sigma,0,1}H_n^{-1}B(p_n)(u_n,u_n) \\
&= X(q_n) - \tfrac{1}{2}P_{\sigma,0,1}H_n^{-1}B(p_n)(u_n,u_n) \\
&-\tfrac{1}{2}P_{\sigma,0,1}\int_0^1 P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P^2_{\sigma,0,t}\left(u_n,\mathcal{D}X(p_n)^{-1}H_n^{-1}B(p_n)(u_n,u_n)\right)dt,
\end{aligned}
\tag{3.10}
$$

in the other side

$$
\begin{aligned}
&X(q_n) - \tfrac{1}{2}P_{\sigma,0,1}H_n^{-1}B(p_n)(u_n,u_n) \\
&= P_{\sigma,0,1}H_n^{-1}\left(H_nP_{\sigma,1,0}X(q_n) - \tfrac{1}{2}B(p_n)(u_n,u_n)\right) \\
&= P_{\sigma,0,1}H_n^{-1}\left(\left[I_{T_{p_n}M} + \tfrac{\lambda}{2}G_n\right]P_{\sigma,1,0}X(q_n) - \tfrac{1}{2}B(p_n)(u_n,u_n)\right) \\
&= P_{\sigma,0,1}H_n^{-1}\left(\left[I_{T_{p_n}M}(P_{\sigma,1,0}X(q_n)) + \tfrac{\lambda}{2}\mathcal{D}X(p_n)^{-1}B(p_n)(u_n,.)\right]P_{\sigma,1,0}X(q_n) - \tfrac{1}{2}B(p_n)(u_n,u_n)\right) \\
&= P_{\sigma,0,1}H_n^{-1}\left(\left[P_{\sigma,1,0}X(q_n) + \tfrac{\lambda}{2}\mathcal{D}X(p_n)^{-1}B(p_n)(u_n,P_{\sigma,1,0}X(q_n))\right] - \tfrac{1}{2}B(p_n)(u_n,u_n)\right).
\end{aligned}
$$

By (3.8), this yields

$$
\begin{aligned}
&X(q_n) - \tfrac{1}{2}P_{\sigma,0,1}H_n^{-1}B(p_n)(u_n,u_n) \\
&= P_{\sigma,0,1}H_n^{-1}\left(P_{\sigma,1,0}X(q_n) + \tfrac{\lambda}{2}\mathcal{D}X(p_n)^{-1}B(p_n)(u_n,P_{\sigma,1,0}X(q_n)) - \tfrac{1}{2}B(p_n)(u_n,u_n)\right) \\
&= P_{\sigma,0,1}H_n^{-1}\left(\int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))(P_{\sigma,0,t}u_n,P_{\sigma,0,t}u_n)\,dt - \tfrac{1}{2}B(p_n)(u_n,u_n)\right. \\
&\left.+\tfrac{\lambda}{2}\mathcal{D}X(p_n)^{-1}B(p_n)\left(u_n,\int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))(P_{\sigma,0,t}u_n,P_{\sigma,0,t}u_n)\,dt\right)\right) \\
&= P_{\sigma,0,1}H_n^{-1}\left(\int_0^1 \left((1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P^2_{\sigma,0,t}(u_n,u_n) - \tfrac{1}{2}B(p_n)(u_n,u_n)\right)dt\right. \\
&\left.+\tfrac{\lambda}{2}P_{\sigma,0,1}\mathcal{D}X(p_n)^{-1}B(p_n)\left(u_n,\int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P^2_{\sigma,0,t}(u_n,u_n)\,dt\right)\right) \\
&= P_{\sigma,0,1}H_n^{-1}\left(\int_0^1 \left((1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P_{\sigma,0,t} - \tfrac{1}{2}B(p_n)\right)(u_n,u_n)\,dt\right. \\
&\left.+\tfrac{\lambda}{2}P_{\sigma,0,1}\mathcal{D}X(p_n)^{-1}B(p_n)\left(u_n,\int_0^1 (1-t)\,P_{\sigma,t,0}\mathcal{D}^2X(\sigma(t))\,P^2_{\sigma,0,t}(u_n,u_n)\,dt\right)\right),
\end{aligned}
$$

Replacing in (3.10), we obtain

$$
\begin{aligned}
&X(q_n) + \mathcal{D}X(q_n)(v_n) \\
&= P_{\sigma,0,1} H_n^{-1} \left(\int_0^1 \left((1-t) P_{\sigma,t,0} \mathcal{D}^2 X(\sigma(t)) P_{\sigma,0,t} - \tfrac{1}{2} B(p_n) \right)(u_n, u_n)\, dt \right. \\
&\left. + \tfrac{\lambda}{2} P_{\sigma,0,1} \mathcal{D}X(p_n)^{-1} B(p_n) \left(u_n, \int_0^1 (1-t) P_{\sigma,t,0} \mathcal{D}^2 X(\sigma(t)) P_{\sigma,0,t}(u_n, u_n)\, dt \right) \right) \\
&- \tfrac{1}{2} \left[P_{\sigma,0,1} \int_0^1 P_{\sigma,t,0} \mathcal{D}^2 X(\sigma(t)) P_{\sigma,0,t}(u_n, .)\, dt \right] \left(\mathcal{D}X(p_n)^{-1} H_n^{-1} B(p_n)(u_n, u_n) \right).
\end{aligned}
\tag{3.11}
$$

Since

$$
P_{\gamma,1,0} X(\gamma(1)) = X(\gamma(0)) - \mathcal{D}X(\gamma(0))\gamma'(0) + \int_0^1 (1-t) P_{\gamma,t,0} \mathcal{D}^2 X(\gamma(t)) \left(\gamma'(t), \gamma'(t) \right) dt,
$$

we have

$$
P_{\gamma,1,0} X(p_{n+1}) = X(q_n) + \mathcal{D}X(q_n) v_n + \int_0^1 (1-t) P_{\gamma,t,0} \mathcal{D}^2 X(\gamma(t)) P^2_{\gamma,0,t}(v_n, v_n)\, dt.
$$

Finally, replacing (3.10) and (3.11) in (3.9), we obtain

$$
\begin{aligned}
&P_{\gamma,1,0} X(p_{n+1}) = \int_0^1 (1-t) P_{\gamma,t,0} \mathcal{D}^2 X(\gamma(t)) P^2_{\gamma,0,t}(v_n, v_n)\, dt \\
&+ P_{\sigma,0,1} H_n^{-1} \left(\int_0^1 (1-t) \left[P_{\sigma,t,0} \mathcal{D}^2 X(\sigma(t)) P^2_{\sigma,0,t} - B(p_n) \right](u_n, u_n)\, dt \right. \\
&\left. + \tfrac{\lambda}{2} P_{\sigma,0,1} \mathcal{D}X(p_n)^{-1} B(p_n) \left(u_n, \int_0^1 (1-t) P_{\sigma,t,0} \mathcal{D}^2 X(\sigma(t)) P^2_{\sigma,0,t}(u_n, u_n)\, dt \right) \right) \\
&- \tfrac{1}{2} P_{\sigma,0,1} \int_0^1 P_{\sigma,t,0} \mathcal{D}^2 X(\sigma(t)) P_{\sigma,0,t} \left(u_n, \mathcal{D}X(p_n)^{-1} H_n^{-1} B(p_n)(u_n, u_n) \right) dt.
\end{aligned}
$$

■

Now we are ready to prove the Chebyshev-Halley method (Theorem 33):

Proof. For the proof of the part (a), (see [11]). Actually, therein it is proved that:

$$
f_k \le g_k \le f_{k+1} \le g_{k+1} ... \le r^*, \quad k \in \mathbb{N}. \tag{3.12}
$$

For the part (b), by the Principle of Mathematical Induction, it suffices to show for all $k \in \mathbb{N}$:

1. $p_k \in B(p_0, f_k)$,
2. $\left\| \mathcal{D}X\left(p_k\right)^{-1} \right\| \leq \overline{\beta}_k \leq \beta_k$,
3. $d\left(q_k, p_k\right) \leq g_k - f_k$,
4. $q_k \in B\left(p_0, g_k\right)$,
5. $d\left(p_{k+1}, q_k\right) \leq f_{k+1} - g_k$.

Let us see that these five properties are true for $k = 0$. This is clear for the properties 1 and 2, but we have to prove 3, 4 and 5. We start with 3 and 4:

$$q_k = \exp_{p_k}\left(u_k\right),$$

we have

$$d\left(q_0, p_0\right) = \|u_0\| = \left\| \mathcal{D}X\left(p_0\right)^{-1} X\left(p_0\right) \right\| \leq g_0 = g_0 - f_0.$$

which shows the properties 3 and 4. Now 5:

Since

$$p_{k+1} = \exp_{q_k}\left(v_k\right),$$

it holds

$$
\begin{aligned}
d\left(p_1, q_0\right) &= \left\|v_0\right\| \\
&= \left\|\mathcal{D}X\left(p_0\right)^{-1} H_0^{-1} B\left(p_0\right)\left(u_0, u_0\right)\right\| \\
&\leq \left\|\mathcal{D}X\left(p_0\right)^{-1}\right\| \left\|H_0^{-1}\right\| \left\|B\left(p_0\right)\right\| \left\|u_0\right\|^2 \\
&\leq \beta \left\|\left[I_{T_{p_0}M} + \frac{\lambda}{2}\mathcal{D}X\left(p_0\right)^{-1} B\left(p_0\right)\left(u_0, .\right)\right]\right\|^{-1} b g_0^2 \\
&\leq \beta \left(\left\|I_{T_{p_0}M}\right\| - \left\|\frac{\lambda}{2}\mathcal{D}X\left(p_0\right)^{-1} B\left(p_0\right)\left(u_0, .\right)\right\|\right)^{-1} g_0^2 \\
&\leq \beta \left(1 - \frac{\lambda}{2}\left\|\mathcal{D}X\left(p_0\right)^{-1}\right\| \left\|B\left(p_0\right)\right\| \left\|u_0\right\|\right)^{-1} g_0^2 \\
&\leq \beta \left(1 - \frac{\lambda}{2}\beta b g_0\right)^{-1} g_0^2 \\
&= f_1 - g_0.
\end{aligned}
$$

Therefore the five properties are true for $k = 0$.

Assume that the five properties are true for $k = 0, 1, 2, ..., n$, Let us prove that the five properties are true for $k = n + 1$.

1. We have

$$
\begin{aligned}
d\left(p_{n+1}, p_0\right) &\leq d\left(p_{n+1}, q_n\right) + d\left(q_n, p_n\right) + d\left(p_n, p_0\right) \\
&\leq \left(f_{n+1} - g_n\right) + \left(g_n - f_n\right) + f_n \\
&= f_{n+1},
\end{aligned}
$$

so that

$$p_{n+1} \in B\left(p_0, f_{n+1}\right). \tag{3.13}$$

2. Let $\sigma : [0, 1] \rightarrow M$ be a geodesic with $\sigma(0) = p_0$, $\sigma(1) = p_{n+1}$ and $\left\|\sigma'(0)\right\| = d\left(p_{n+1}, p_0\right)$.By (1.18),

$$P_{\sigma,1,o}\mathcal{D}X\left(\sigma(1)\right)\sigma'(1) = \mathcal{D}X\left(\sigma(0)\right)\sigma'(0) + \int_0^1 P_{\sigma,s,o}\mathcal{D}^2 X\left(\sigma(s)\right)\left(\sigma'(s), \sigma'(s)\right) ds.$$

Therefore,

$$P_{\sigma,1,o}\mathcal{D}X\left(p_{n+1}\right)P_{\sigma,0,1}\sigma'\left(0\right)-\mathcal{D}X\left(p_0\right)\sigma'\left(0\right)=\int_0^1 P_{\sigma,s,o}\mathcal{D}^2X\left(\sigma\left(s\right)\right)P^2_{\sigma,0,s}\left(\sigma'\left(0\right),\sigma'\left(0\right)\right)ds,$$

hence

$$\begin{aligned}\left\|P_{\sigma,1,o}\mathcal{D}X\left(p_{n+1}\right)P_{\sigma,0,1}-\mathcal{D}X\left(p_0\right)\right\| &\leq \int_0^1\left\|P_{\sigma,s,o}\mathcal{D}^2X\left(\sigma\left(s\right)\right)P^2_{\sigma,0,s}\right\|ds\left\|\sigma'\left(0\right)\right\|\\ &\leq d\left(p_{n+1},p_0\right)\int_0^1\left\|\mathcal{D}^2X\left(\sigma\left(s\right)\right)\right\|ds\\ &\leq d\left(p_{n+1},p_0\right)\int_0^1 N ds\\ &= Nd\left(p_{n+1},p_0\right).\end{aligned}$$

By (3.13),

$$\left\|P_{\sigma,1,o}\mathcal{D}X\left(p_{n+1}\right)P_{\sigma,0,1}-\mathcal{D}X\left(p_0\right)\right\|\leq Nf_{n+1}\leq Nr^*.$$

In this way, using the hypotheses

$$\beta\left(2N+\lambda b\right)r^*<2,$$

we obtain

$$\begin{aligned}\left\|\mathcal{D}X\left(p_0\right)^{-1}\right\|\left\|P_{\sigma,1,o}\mathcal{D}X\left(p_{n+1}\right)P_{\sigma,0,1}-\mathcal{D}X\left(p_0\right)\right\| &\leq \beta Nr^*\\ &\leq 1-\lambda br^*\\ &\leq 1.\end{aligned}$$

By Banach's lemma, the operator $P_{\sigma,1,o}\mathcal{D}X\left(p_{n+1}\right)P_{\sigma,0,1}$ is invertible; moreover

$$\begin{aligned}
\left\|\mathcal{D}X\left(p_{n+1}\right)\right\|^{-1} &= \left\|P_{\sigma,1,o}\mathcal{D}X\left(p_{n+1}\right)P_{\sigma,0,1}\right\|^{-1} \\
&\leq \frac{\left\|\mathcal{D}X\left(p_0\right)^{-1}\right\|}{1-\left\|\mathcal{D}X\left(p_0\right)^{-1}\right\|\left\|P_{\sigma,1,o}\mathcal{D}X\left(p_{n+1}\right)P_{\sigma,0,1}-\mathcal{D}X\left(p_0\right)\right\|} \\
&\leq \frac{\beta}{1-\beta N d\left(p_{n+1},p_0\right)} \\
&= \bar{\beta}_{n+1} \\
&\leq \frac{\beta}{1-\beta N f_{n+1}} \\
&= \left(\frac{1}{\beta}-N f_{n+1}\right)^{-1} \\
&= \beta_{n+1}.
\end{aligned}$$

thus

$$\left\|\mathcal{D}X\left(p_{n+1}\right)\right\|^{-1} \leq \bar{\beta}_{n+1} \leq \beta_{n+1}. \tag{3.14}$$

3. By Lemma 35,

$$\begin{aligned}
&\left\|X\left(p_{n+1}\right)\right\| \\
&= \left\|P_{\gamma,1,0}X\left(p_{n+1}\right)\right\| \\
&\leq \int_0^1 (1-t)\left\|P_{\gamma,t,0}\mathcal{D}^2X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}\left(v_n,v_n\right)\right\|dt \\
&+\left\|H_n^{-1}\right\|\left(\int_0^1 (1-t)\left\|\left[P_{\sigma,t,0}\mathcal{D}^2X\left(\sigma\left(t\right)\right)P_{\sigma,0,t}-B\left(p_n\right)\right]\left(u_n,u_n\right)\right\|dt\right. \\
&+\tfrac{\lambda}{2}\left\|\mathcal{D}X\left(p_n\right)^{-1}B\left(p_n\right)\right\|\left(u_n,\int_0^1 (1-t)\left\|P_{\sigma,t,0}\mathcal{D}^2X\left(\sigma\left(t\right)\right)P_{\sigma,0,t}\left(u_n,u_n\right)\right\|dt\right)\Bigg) \\
&+\tfrac{1}{2}\int_0^1 \left\|\mathcal{D}^2X\left(\sigma\left(t\right)\right)P_{\sigma,0,t}\left(u_n,\mathcal{D}X\left(p_n\right)^{-1}H_n^{-1}B\left(p_n\right)\left(u_n,u_n\right)\right)\right\|dt.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\|X(p_{n+1})\| \leq \|v_n\|^2 \int_0^1 (1-t)\left\|\mathcal{D}^2 X(\gamma(t))\right\| dt \\
&+\left\|H_n^{-1}\right\| \left(\|u_n\|^2 \int_0^1 (1-t)\left\|\left[P_{\sigma,t,0}\mathcal{D}^2 X(\sigma(t))\, P_{\sigma,0,t}^2 - B(p_n)\right]\right\| dt \right.\\
&\left. +\tfrac{\lambda}{2}\left\|\mathcal{D}X(p_n)^{-1}\right\| \|B(p_n)\| \|u_n\|^3 \int_0^1 (1-t)\left\|\mathcal{D}^2 X(\sigma(t))\right\| dt \right)\\
&+\tfrac{1}{2}\int_0^1 \left\|\mathcal{D}^2 X(\sigma(t))\right\| \|u_n\|^3 \left\|\mathcal{D}X(p_n)^{-1}\right\| \left\|H_n^{-1}\right\| \|B(p_n)\|\, dt,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\|X(p_{n+1})\| \leq \tfrac{N}{2}\|v_n\|^2 + \left\|H_n^{-1}\right\| \left(\tfrac{a}{2}\|u_n\|^2 + \tfrac{\lambda}{2}\beta_n b \|u_n\|^3 \tfrac{N}{2}\right) - \tfrac{1}{2}N\|u_n\|^3 \beta_n b \left\|H_n^{-1}\right\| \\
&= \tfrac{N}{2}\|v_n\|^2 + \left(\tfrac{a}{2}\|u_n\|^2 + \tfrac{\lambda}{2}\beta_n b\|u_n\|^3 \tfrac{N}{2} - \tfrac{1}{2}N\|u_n\|^3 \beta_n b\right)\left\|H_n^{-1}\right\| \\
&= \tfrac{N}{2}\|v_n\|^2 + \left(\tfrac{a}{2}\|u_n\|^2 + \left(1+\tfrac{\lambda}{2}\right)\tfrac{Nb}{2}\beta_n\|u_n\|^3\right)\left\|H_n^{-1}\right\|.
\end{aligned}
$$

By the induction hypotheses

$$
\|v_n\| = d(p_{n+1}, q_n) \leq f_{n+1} - g_n \quad \text{and} \quad \|u_n\| = d(q_n, p_n) = g_n - f_n,
$$

hence

$$
\begin{aligned}
\left\|H_n^{-1}\right\| &= \left\| I_{T_{p_n}M} + \frac{\lambda}{2}\mathcal{D}X(p_n)^{-1} B(p_n)(u_n, .)\right\|^{-1} \\
&\leq \left(\left\|I_{T_{p_n}M}\right\| - \frac{\lambda}{2}\left\|\mathcal{D}X(p_n)^{-1} B(p_n)(u_n, .)\right\| \right)^{-1} \\
&\leq \left(\left\|I_{T_{p_n}M}\right\| - \frac{\lambda}{2}\|u_n\|\beta_n b \right)^{-1} \\
&\leq \left(1 - \frac{\lambda b}{2}\beta_n (g_n - f_n) \right)^{-1} \\
&= h_n,
\end{aligned}
$$

and

$$\begin{aligned}\|X(p_{n+1})\| \leq& \frac{N}{2}\|v_n\|^2 + \left\|H_n^{-1}\right\| \left(\frac{a}{2}\|u_n\|^2 + \left(1+\frac{\lambda}{2}\right)\frac{Nb}{2}\beta_n \|u_n\|^3\right) \\ \leq& \frac{N}{2}(f_{n+1}-g_n)^2 + \left(\frac{a}{2}(g_n-f_n)^2 + \left(1+\frac{\lambda}{2}\right)\frac{Nb}{2}\beta_n (g_n-f_n)^3\right) h_n \\ =&\alpha_{n+1}.\end{aligned}$$

We thus conclude that

$$\begin{aligned}d(q_{n+1}, p_{n+1}) &= \|u_{n+1}\| \\ &\leq \left\|\mathcal{D}X(p_{n+1})^{-1}\right\| \|X(p_{n+1})\| \\ &\leq \beta_{n+1}\alpha_{n+1} \\ &= g_{n+1} - f_{n+1}.\end{aligned}$$

4. It is clear that

$$\begin{aligned}d(q_{n+1}, p_0) &\leq d(q_{n+1}, p_{n+1}) + d(p_{n+1}, q_n) + d(q_n, p_n) + d(p_n, p_0) \\ &\leq g_{n+1} - f_{n+1.} + f_{n+1} - g_n + g_n - f_n + f_0 \\ &= g_{n+1},\end{aligned}$$

hence

$$q_{n+1} \in B(p_0, g_{n+1}).$$

5. Note that

$$\begin{aligned}\left\|\left(I_{T_{p_{n+1}}M} - H_{n+1}\right)\right\| &= \left\|\frac{\lambda}{2}\mathcal{D}^2 X(p_{n+1}) B(p_{n+1}) u_{n+1}\right\| \\ &\leq \frac{\lambda}{2}\beta_{n+1} b \|u_{n+1}\| \\ &\leq \frac{\lambda b}{2}\beta_{n+1}(g_{n+1} - f_{n+1.}).\end{aligned} \tag{3.15}$$

Moreover, By (3.12),

$$f_{n+1} \leq r^* \quad \text{and} \quad f_{n+1} \leq g_{+1} \leq r^*.$$

Therefore,

$$\begin{aligned}\beta_{n+1} &= \left(\frac{1}{\beta} - Nf_{n+1}\right)^{-1} \\ &\leq \left(\frac{1}{\beta} - Nr^*\right)^{-1}\end{aligned}$$

and

$$\begin{aligned}0 &\leq g_{n+1} - f_{n+1} \\ &\leq r^* - f_{n+1} \\ &\leq r^* - f_0 \\ &= r^*.\end{aligned}$$

Replacing this into (3.15) we obtain

$$\begin{aligned}\left\|\left(I_{T_{p_{n+1}}M} - H_{n+1}\right)\right\| &\leq \frac{\lambda b}{2}\left(\frac{1}{\beta} - Nr^*\right)^{-1} r^* \\ &= \frac{\lambda b \beta r^*}{2\left(1 - \beta N r^*\right)} \\ &\leq 1.\end{aligned}$$

Therefore, by Banach's lemma, H_{n+1} is invertible, and

$$\begin{aligned}\left\|H_{n+1}^{-1}\right\| &\leq \frac{1}{1 - \frac{\lambda b}{2}\beta_{n+1}\left(g_{n+1} - f_{n+1.}\right)} \\ &\leq \left(1 - \frac{\lambda b}{2}\beta_{n+1}\left(g_{n+1} - f_{n+1.}\right)\right)^{-1} \\ &= h_{n+1}.\end{aligned}$$

Thus,

$$\begin{aligned}
d\left(p_{n+2}, q_{n+1}\right) &= \|v_{n+1}\| \\
&= \frac{1}{2}\left\|\mathcal{D}X\left(p_{n+1}\right)^{-1} H_{n+1}^{-1} B\left(p_{n+1}\right)\left(u_{n+1}, u_{n+1}\right)\right\| \\
&\leq \frac{1}{2}\left\|\mathcal{D}X\left(p_{n+1}\right)^{-1}\right\| \left\|H_{n+1}^{-1}\right\| \left\|B\left(p_{n+1}\right)\right\| \left\|u_{n+1}\right\|^{2} \\
&\leq \frac{1}{2}\beta_{n+1} h_{n+1} b\left(d\left(p_{n+1}, q_{n+1}\right)\right)^{2} \\
&\leq \frac{1}{2}\beta_{n+1} h_{n+1} b\left(g_{n+1} - f_{n+1}\right)^{2} \\
&= f_{n+2} - g_{n+1}.
\end{aligned}$$

The induction has been completed, thus for all $n \in \mathbb{N}$:

$$\begin{aligned}
d\left(p_{n+1}, p_{n}\right) &\leq d\left(p_{n+1}, q_{n}\right) + d\left(q_{n}, p_{n}\right) \\
&\leq \left(f_{n+1} - g_{n}\right) + \left(g_{n} - f_{n}\right) \\
&= f_{n+1} - f_{n},
\end{aligned}$$

and similarly

$$d\left(q_{n+1}, q_{n}\right) \leq g_{n+1} - g_{n}.$$

We have hence showed that the sequences $\left\{p_{n}\right\}_{n\in\mathbb{N}}$, $\left\{q_{n}\right\}_{n\in\mathbb{N}}$ are Cauchy sequences. Since M is complete, they must converge, and by construction, their limit point coincide:

$$p^{*} = \lim_{n\to\infty} p_{n} = \lim_{n\to\infty} q_{n},$$

where $p^{*} \in B\left[p_{0}, r^{*}\right]$.

Now we prove that $X\left(p^{*}\right) = 0$. By (3.8),

$$P_{\sigma,1,0} X\left(q_{n}\right) = \int_{0}^{1}\left(1-t\right) P_{\sigma,t,0} \mathcal{D}^{2} X\left(\sigma\left(t\right)\right)\left(P_{\sigma,0,t} u_{n}, P_{\sigma,0,t} u_{n}\right) dt.$$

Hence,

$$
\begin{aligned}
\|X(q_n)\| &\leq \int_0^1 (1-t)\left\|\mathcal{D}^2 X(\sigma(t))\right\| \|u_n\|^2\, dt \\
&\leq \int_0^1 (1-t)\left\|\mathcal{D}^2 X(\sigma(t))\right\| \|u_n\|^2\, dt \\
&\leq \int_0^1 (1-t) N \|u_n\|^2\, dt \\
&= \frac{1}{2} N \|u_n\|^2 \\
&= \frac{1}{2} N (d(q_n, p_n))^2 .
\end{aligned}
$$

Passing to the limit as $n \to \infty$, we obtain $X(p^*) = 0$.

To show uniqueness, let us assume that there exist a second solution $q^* \in B(p_0, R)$. Let $\delta : [0,1] \longrightarrow M$ be a minimizing geodesic joining p^* and q^*. By (1.18), this yields

$$
P_{\delta,t,o}\mathcal{D}X(\delta(t))\,\delta'(t) = \mathcal{D}X(\delta(0))\,\delta'(0) + \int_0^t P_{\delta,s,o}\left(\mathcal{D}^2 X(\delta(s))\left(\delta'(s), \delta'(s)\right)\right) ds,
$$

hence

$$
P_{\delta,t,0}\mathcal{D}X(\delta(t))\,P_{\delta,0,t} - \mathcal{D}X(p^*) = \int_0^t P_{\delta,s,0}\left(\mathcal{D}^2 X(\delta(s))\,P^2_{\delta,0,s}\left(\delta'(0), .\right)\right) ds.
$$

Therefore, by hypothesis and the triangle inequality

$$
\begin{aligned}
\|P_{\delta,t,0}\mathcal{D}X(\delta(t))\,P_{\delta,0,t} - \mathcal{D}X(p^*)\| &\leq \int_0^t \left\|\mathcal{D}^2 X(\delta(s))\right\| \left\|\delta'(0)\right\| ds \\
&\leq Nt\left\|\delta'(0)\right\| \\
&= Ntd(p^*, q^*) \\
&\leq Nt\left(d(p_0, p^*) + d(p_0, q^*)\right).
\end{aligned}
$$

As a consecuence,

$$\begin{aligned}
&\left\| \mathcal{D}X\left(p^*\right)^{-1} \right\| \left\| \int_0^1 P_{\delta,t,0} \mathcal{D}X\left(\delta\left(t\right)\right) P_{\delta,0,t} dt - \mathcal{D}X\left(p^*\right) \right\| \\
&\leq \left(\tfrac{1}{\beta} - Nr^*\right)^{-1} \int_0^1 Nt\left(d\left(p_0, p^*\right) + d\left(p_0, q^*\right)\right) dt \\
&= \left(\tfrac{1}{\beta} - Nr^*\right)^{-1} \tfrac{N}{2}\left(r^* + R\right) \\
&\leq 1.
\end{aligned}$$

By Banach's Lemma, the operator

$$\int_0^1 P_{\delta,t,0} \mathcal{D}X\left(\delta\left(t\right)\right) P_{\delta,0,t} dt,$$

is invertible. Finally

$$\begin{aligned}
0 &= X\left(q^*\right) - X\left(p^*\right) \\
&= \int_0^1 P_{\delta,t,0} \mathcal{D}X\left(\delta\left(t\right)\right) P_{\delta,0,t}\left(\delta'\left(0\right)\right) dt.
\end{aligned}$$

Therefore,

$$\delta'\left(0\right) = 0,$$

thus

$$0 = \left\|\delta'\left(0\right)\right\| = d\left(p^*, q^*\right),$$

and we conclude that

$$p^* = q^*.$$

■

3.2 Third-order iterative methods on Riemannian manifolds under Kantorovich conditions

When conditions as in the Kantorovich theorem [12], are imposed on x_0 and on F, to ensure the convergence of sequence $\{x_n\}_{n\in\mathbb{N}}$ to a solution of $F\left(x\right) = 0$, we will say that we have imposed *conditions of Kantorovich type.*

In this section, we will prove convergence and uniqueness of another method which also generalizes the Chebyshev and Halley methods to Riemannian manifolds. Moreover, this method also generalizes others third order iterative methods, ([5], [7-9], [11], [13], [20-24]).

In Banach spaces, this method is described for:

$$\begin{aligned} u_n &= F'(x_n)^{-1} F(x_n), \\ T_n &= \tfrac{1}{2} F'(x_n)^{-1} F''(x_n) u_n, \\ x_{n+1} &= x_n - \left(I + T_n + \textstyle\sum_{k\geq 2} \beta_k T_n^k \right) u_n, \end{aligned}$$

where F is a nonlinear operator defined from an open convex subset Ω of a Banach space E in itself and I is the identity operator in E. Under certain hypotheses, which will be stated later, S. Amat and S. Busquier, showed the convergence of the method to a root of the nonlineal operator F (See [21]).

Let $X \in C^3$ be a vector field on the complete and connected m-dimensional Riemannian manifold M, let $p_0 \in M$ and

$$\begin{aligned} u_n &= \mathcal{D}X(p_n)^{-1} X(p_n), \\ T_n &= \tfrac{1}{2} \mathcal{D}X(p_n)^{-1} \left(\mathcal{D}^2 X(p_n)(u_n, .) \right), \\ v_n &= -\left(I_{T_{p_n}M} + T_n + \textstyle\sum_{k\geq 2} \beta_k T_n^k \right) u_n, \\ p_{n+1} &= \exp_{p_n}(v_n), \end{aligned} \tag{3.16}$$

where $\{\beta_k\}_{k\geq 2}$ is a decreasing sequence of positive real numbers such that

$$\sum_{k\geq 2} \beta_k \theta_n^k \leq d \left| \theta_n^2 \right|, \ 0 \leq d \leq 2. \tag{3.17}$$

We will find conditions so that the method (3.16) converges to the unique solution of the equation $X(p) = 0$, but before doing so, we will establish some results which will allow us to build a majorizing sequence and so we can prove the convergence of the method.

Lemma 36 *Let a, b, c be positive real numbers with $a \neq 0$ and $c \neq 0$. We define*

$$g(t) = \frac{c}{6} t^3 + \frac{b}{2} t^2 - t + a.$$

Then, $g(t)$ has two positive roots, t_0, t_0' if and only if

$$a \leq \frac{\left(b^2+2c\right)^{\frac{3}{2}} - b\left(b^2+3c\right)}{3c^2} \tag{3.18}$$

and $0 < t_0 < t_2 < t_0'$, where

$$t_2 = \frac{1}{c}\left(-b + \sqrt{b^2+2c}\right), \text{ see Fig.3.}$$

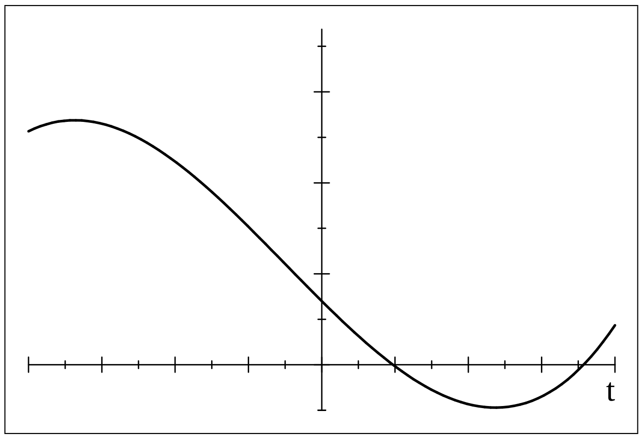

Fig. 3.

Proof. See [10]. ■

Corollary 37 *If a, b, c verify (3.18), then*

$$ab \leq \frac{1}{2}.$$

Proof. See [21]. ■

Proposition 38 *Let a, b, c be positive real numbers such that (3.18) is true. Then there exists a third degree polynomial $f(t)$ verifying:*

1. $f(0) = 0,$

2. $f(t_0) = a,$

3. $f'(t_0) = 1,$

4. $f''(t_0) = b,$

5. $f'''(t) = -c$ *for all* $t \in \mathbb{R}$.

Proof. It is sufficient to take the polynomial

$$f(t) = t\left(-\frac{c}{6}t^2 + \beta t + \gamma\right),$$

where

$$\beta = \frac{1}{2}(b + ct_0), \gamma = 1 - \frac{1}{2}ct_0^2 - bt_0,$$

and to consider Lemma 36. ■

Remark 39 *It is important to observe that for all* $t \in (0, t_0)$, *the previous polynomial verifies:* $f(t), f'(t), f''(t) > 0, f'''(t) < 0,$ *and* $f'(t)^2 > \frac{1}{2}f''(t) f(t)$

Proposition 40 *Let us consider the sequence*

$$t_{n+1} = t_n - \left(1 + \theta_n + \sum_{k\geq 2} \beta_k \theta_n^k\right) \frac{f(t_n)}{f'(t_n)}, \tag{3.19}$$

where

$$\theta_n = \frac{1}{2}\frac{f''(t_n) f(t_n)}{[f'(t_n)]^2}.$$

If (3.17) is true, then $\{t_n\}_{n\in\mathbb{N}}$ *converges monotonically, and its limit is* 0.

Proof. See [21]. ■

Lemma 41 *Under the considerations of the method (3.16), if* γ_n *is the geodesic defined by*

$$\gamma_n(t) = \exp_{p_n}(tv_n).$$

Then the following representation is true

$$P_{\gamma_n,1,0}X\left(p_{n+1}\right)=\frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(u_n,\left(\sum_{k\geq 2}\left(\beta_{k-1}-\beta_k\right)T_n^{k-1}u_n\right)\right)$$
$$+\frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(T_nH\left(T_n\right)u_n,\left(I_{T_{p_n}M}+T_nH\left(T_n\right)\right)u_n\right)$$
$$+\int_0^1\left(1-t\right)\left[P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)P^2_{\gamma_n,0,t}-\mathcal{D}^2X\left(p_n\right)\right]\left(v_n,v_n\right)dt,$$

where $H\left(T_n\right)=I_{T_{p_n}M}+\sum_{k\geq 2}\beta_kT_n^{k-1}$ *and* $\beta_1=1$.

Proof. By (1.20),

$$P_{\gamma_n,1,0}X\left(p_{n+1}\right)=P_{\gamma_n,1,0}X\left(\gamma_n\left(1\right)\right)$$
$$=X\left(\gamma_n\left(0\right)\right)+\mathcal{D}X\left(\gamma_n\left(0\right)\right)\gamma_n'\left(0\right)+\int_0^1\left(1-t\right)P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)\left(\gamma_n'\left(t\right),\gamma_n'\left(t\right)\right)dt$$
$$=X\left(\gamma_n\left(0\right)\right)+\mathcal{D}X\left(\gamma_n\left(0\right)\right)\gamma_n'\left(0\right)+\frac{1}{2}\mathcal{D}^2X\left(\gamma_n\left(0\right)\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right)$$
$$+\int_0^1\left(1-t\right)\left[P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)P^2_{\gamma_n,0,t}\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right)-\mathcal{D}^2X\left(\gamma_n\left(0\right)\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right)\right]dt$$
$$=X\left(p_n\right)+\mathcal{D}X\left(p_n\right)v_n+\frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(v_n,v_n\right)$$
$$+\int_0^1\left(1-t\right)\left[P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)P^2_{\gamma_n,0,t}\left(v_n,v_n\right)-\mathcal{D}^2X\left(p_n\right)\left(v_n,v_n\right)\right]dt.$$

Note that

$$I_{T_{p_n}M}+T_n+\sum_{k\geq 2}\beta_kT_n^k$$
$$=I_{T_{p_n}M}+T_n\left(I_{T_{p_n}M}+\sum_{k\geq 2}\beta_kT_n^{k-1}\right)$$
$$=I_{T_{p_n}M}+T_nH\left(T_n\right).$$

By the definition of the method (3.16), this implies that

$$\begin{aligned}
\mathcal{D}X(p_n)\,v_n &= \mathcal{D}X(p_n)\left(-\left(I_{T_{p_n}M}+T_nH(T_n)\right)u_n\right)\\
&= -X(p_n)-\mathcal{D}X(p_n)\,T_nH(T_n)(u_n)\\
&= -X(p_n)-\frac{1}{2}\mathcal{D}^2X(p_n)(u_n,H(T_n)\,u_n)\\
&= -X(p_n)-\frac{1}{2}\mathcal{D}^2X(p_n)(u_n,H(T_n)\,u_n)\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathcal{D}^2X(p_n)(v_n,v_n) &= \mathcal{D}^2X(p_n)\left(-\left(I_{T_{p_n}M}+T_nH(T_n)\right)u_n,-\left(I_{T_{p_n}M}+T_nH(T_n)\right)u_n\right)\\
&= \mathcal{D}^2X(p_n)(u_n,u_n)+\mathcal{D}^2X(p_n)(u_n,T_nH(T_n)\,u_n)\\
&+\mathcal{D}^2X(p_n)(T_nH(T_n)\,u_n,u_n)+\mathcal{D}^2X(p_n)(T_nH(T_n)\,u_n,T_nH(T_n)\,u_n)\,,
\end{aligned}$$

so that

$$\begin{aligned}
P_{\gamma_n,1,0}X(p_{n+1}) &= -\frac{1}{2}\mathcal{D}^2X(p_n)(u_n,H(T_n)\,u_n)+\frac{1}{2}\mathcal{D}^2X(p_n)(u_n,u_n)+\frac{1}{2}\mathcal{D}^2X(p_n)(u_n,T_nH(T_n)\,u_n)\\
&+\frac{1}{2}\mathcal{D}^2X(p_n)(T_nH(T_n)\,u_n,u_n)+\frac{1}{2}\mathcal{D}^2X(p_n)(T_nH(T_n)\,u_n,T_nH(T_n)\,u_n)\\
&+\int_0^1(1-t)\left[P_{\gamma_n,t,0}\mathcal{D}^2X(\gamma_n(t))\,P^2_{\gamma_n,0,t}(v_n,v_n)-\mathcal{D}^2X(p_n)(v_n,v_n)\right]dt.
\end{aligned}$$

We conclude that

$$\begin{aligned}
P_{\gamma_n,1,0}X(p_{n+1}) &= \frac{1}{2}\mathcal{D}^2X(p_n)(u_n,u_n)+\frac{1}{2}\mathcal{D}^2X(p_n)(u_n,(T_nH(T_n)-H(T_n))\,u_n)\\
&+\frac{1}{2}\mathcal{D}^2X(p_n)(T_nH(T_n)\,u_n,u_n)+\frac{1}{2}\mathcal{D}^2X(p_n)(T_nH(T_n)\,u_n,T_nH(T_n)\,u_n)\\
&+\int_0^1(1-t)\left[P_{\gamma_n,t,0}\mathcal{D}^2X(\gamma_n(t))\,P^2_{\gamma_n,0,t}-\mathcal{D}^2X(p_n)\right](v_n,v_n)\,dt.
\end{aligned}$$

In this way, taking $\beta_1 = 1$, we obtain

$$\begin{aligned} P_{\gamma_n,1,0}X\left(p_{n+1}\right) &= \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(u_n,u_n\right) \\ &+ \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(u_n,\left(\sum_{k\geq 2}\left(\beta_{k-1}-\beta_k\right)T_n^{k-1}-I_{T_{p_n}M}\right)u_n\right) \\ &+ \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(T_nH\left(T_n\right)u_n,u_n\right)+\frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(T_nH\left(T_n\right)u_n,T_nH\left(T_n\right)u_n\right) \\ &+ \int_0^1\left(1-t\right)\left[P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)P^2_{\gamma_n,0,t}-\mathcal{D}^2X\left(p_n\right)\right]\left(v_n,v_n\right)dt. \end{aligned}$$

Finally, we conclude that

$$\begin{aligned} P_{\gamma_n,1,0}X\left(p_{n+1}\right) &= \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(u_n,\left(\sum_{k\geq 2}\left(\beta_{k-1}-\beta_k\right)T_n^{k-1}u_n\right)\right) \\ &+ \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(T_nH\left(T_n\right)u_n,\left(I_{T_{p_n}M}+T_nH\left(T_n\right)\right)u_n\right) \\ &+ \int_0^1\left(1-t\right)\left[P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)P^2_{\gamma_n,0,t}-\mathcal{D}^2X\left(p_n\right)\right]\left(v_n,v_n\right)dt. \end{aligned}$$

■

Similarly, in the case of fuctions defined on $\mathbb{R}$, we have the following result.

Lemma 42 *If $f(t)$ is the polynomial of the Proposition 38, then*

$$\begin{aligned} f\left(t_{n+1}\right) &= \frac{1}{2}f''\left(t_n\right)\left(\sum_{k\geq 2}\left(2\beta_{k-1}-\beta_k\right)\theta_n^{k-1}+\left(\theta_nh\left(\theta_n\right)\right)^2\right)\left(\frac{f\left(t_n\right)}{f'\left(t_n\right)}\right)^2 \\ &+ \frac{1}{6}c\left(t_n-t_{n+1}\right)^3, \end{aligned}$$

where $h(t) = 1 + \sum_{k\geq 2}\beta_k t^{k-1}$ and $\beta_1 = 1$.

Now we can prove the convergence of the method (3.16). We will begin by showing that $\{t_n\}_{n\in\mathbb{N}}$ is a majorizing sequence for $\{p_n\}_{n\in\mathbb{N}}$.

Theorem 43 *Let us assume $p_0 \in \Omega \subseteq M$ and that $\mathcal{D}X(p_0)$ invertible, and a, b and c positive* *real numbers verifying (3.18), then for any geodesic γ and $\tau_1, \tau_2 \in \mathbb{R}$ with $\gamma[\tau_1, \tau_2] \subseteq \Omega$ it holds that:*

1. $\|X(p_0)\| \leq a,$
2. $\left\|\mathcal{D}X(p_0)^{-1}\right\| \leq 1,$
3. $\left\|\mathcal{D}^2 X(p_0)\right\| \leq b,$
4. $\left\|P_{\gamma,\tau_2,\tau_1}\mathcal{D}^2(\gamma(\tau_2))\, P^2_{\gamma,\tau_1,\tau_2} - \mathcal{D}^2(\gamma(\tau_1))\right\| \leq c\int_a^{\tau_2} \|\gamma'(t)\|\, dt.$

Moreover, if $\{t_k\}_{k\in\mathbb{N}}$ is defined by (3.19), then for all $k \in \mathbb{N}$:
(a) $\left\|\mathcal{D}X(p_k)^{-1}\right\| \leq \frac{1}{f'(t_k)},$
(b) $\left\|\mathcal{D}^2 X(p_k)\right\| \leq f''(t_k),$
(c) $\|X(p_k)\| \leq f(t_k),$
(d) $\|T_k\| \leq \theta_k,$
(e) $\|H(T_k)\| \leq h(\theta_k),$
(f) $d(p_{k+1}, p_k) \leq t_k - t_{k+1}.$

Proof. Proceed by induction.
Items (a)-(e) are clear for $k = 0$. for (f), notice that

$$\begin{aligned} d(p_1, p_0) &\leq \left\| I_{T_{p_0}M} + T_0 + \sum_{k\geq 2} \beta_k T_0^{k-1} \right\| \|u_0\| \\ &\leq \left(\left\|I_{T_{p_0}M}\right\| + \|T_0\| + \sum_{k\geq 2} \beta_k \|T_0\|^{k-1} \right) \left\| \mathcal{D}X(p_0)^{-1} X(p_0) \right\| \\ &\leq \left(1 + \theta_0 + \sum_{k\geq 2} \beta_k^{k-1} \theta_0^{k-1} \right) \frac{f(t_0)}{f'(t_0)} \\ &= t_0 - t_1. \end{aligned}$$

Assume that the seven properties (a)-(f) are true for $k = 0, 1, 2, ..., n$. Let us prove all of them for $k = n + 1$:

(a) Let us consider the geodesic γ_k defined by

$$\gamma_n(t) = \exp_{p_n}(tv_n).$$

Since

$$P_{\gamma_n,1,0}\mathcal{D}X(p_{n+1})P_{\gamma_n,0,1} = \mathcal{D}X(p_n)\left[I_{T_{p_n}M} - \mathcal{D}X(p_n)^{-1}\left(\mathcal{D}X(p_n) - P_{\gamma_n,1,0}\mathcal{D}X(p_{n+1})P_{\gamma_n,0,1}\right)\right],$$

applying (1.22), we obtain

$$\begin{aligned}
\left\|P_{\gamma_n,1,0}\mathcal{D}X(p_{n+1})P_{\gamma_n,0,1} - \mathcal{D}X(p_n)\right\| &\leq \left(\left\|\mathcal{D}^2X(\gamma_n(0))\right\| + \frac{1}{2}c\left\|\gamma_n'(0)\right\|\right)\left\|\gamma_n'(0)\right\| \\
&\leq \left(\left\|\mathcal{D}^2X(p_n)\right\| + \frac{1}{2}c\left\|v_n\right\|\right)\left\|v_n\right\| \\
&\leq \left(\left\|\mathcal{D}^2X(p_n)\right\| + \frac{1}{2}cd(p_{n+1},p_n)\right)d(p_{n+1},p_n) \\
&\leq \left(f''(t_n) + \frac{1}{2}c(t_n - t_{n+1})\right)(t_n - t_{n+1}).
\end{aligned}$$

Given that

$$\begin{aligned}
f'(t_{n+1}) &= f'(t_n) + f''(t_n)(t_{n+1} - t_n) + \int_{t_n}^{t_{n+1}}(t_{n+1} - t)f'''(t)\,dt \\
&= f'(t_n) + f''(t_n)(t_{n+1} - t_n) - \frac{1}{2}c(t_n - t_{n+1})^2 \\
&= f'(t_n) - \left(f''(t_n) + \frac{1}{2}c(t_n - t_{n+1})\right)(t_n - t_{n+1}),
\end{aligned}$$

this yields

$$\begin{aligned}
\left\|\mathcal{D}X(p_n)^{-1}\right\|\left\|P_{\gamma_n,1,0}\mathcal{D}X(p_{n+1})P_{\gamma_n,0,1} - \mathcal{D}X(p_n)\right\| &\leq \frac{1}{f'(t_n)}\left(f'(t_n) - f'(t_{n+1})\right) \\
&\leq \left(1 - \frac{f'(t_{n+1})}{f'(t_n)}\right) \\
&\leq 1.
\end{aligned}$$

Therefore, $\mathcal{D}X\left(p_{n+1}\right)$ is invertible, and

$$\begin{aligned}\left\|\mathcal{D}X\left(p_{n+1}\right)^{-1}\right\| &\leq \frac{\frac{1}{f'(t_n)}}{1-\left(1-\frac{f'(t_{n+1})}{f'(t_n)}\right)} \\ &= \frac{1}{f'\left(t_{n+1}\right)}.\end{aligned}$$

(b) Note that

$$\begin{aligned}\left\|\mathcal{D}^2X\left(p_{n+1}\right)\right\| &= \left\|P_{\gamma_n,1,0}\mathcal{D}^2X\left(p_{n+1}\right)P^2_{\gamma_n,0,1}\right\| \\ &\leq \left\|P_{\gamma_n,1,0}\mathcal{D}^2X\left(p_{n+1}\right)P^2_{\gamma_n,0,1}-\mathcal{D}^2X\left(p_n\right)\right\|+\left\|\mathcal{D}^2X\left(p_n\right)\right\| \\ &\leq cd\left(p_{n+1},p_n\right)+\left\|\mathcal{D}^2X\left(p_n\right)\right\| \\ &\leq c\left(t_n-t_{n+1}\right)+f''\left(t_n\right) \\ &= f''\left(t_n\right)+f'''\left(t_n\right)\left(t_{n+1}-t_n\right) \\ &= f''\left(t_{n+1}\right).\end{aligned}$$

(c) Using the Lemmas 41 and 42, we obtain

$$
\begin{aligned}
\|X(p_{n+1})\| &= \|P_{\gamma_n,1,0}X(p_{n+1})\| \\
&\leq \frac{1}{2}\left\|\mathcal{D}^2X(p_n)\right\|\left(\sum_{k\geq 2}(\beta_{k-1}-\beta_k)\,\theta_n^{k-1}\right)\|u_n\|^2 \\
&+\frac{1}{2}\left\|\mathcal{D}^2X(p_n)\right\|\theta_n h(\theta_n)(1+\theta_n h(\theta_n))\|u_n\|^2 \\
&+\int_0^1\left\|(1-t)\left[P_{\gamma_n,t,0}\mathcal{D}^2X(\gamma_n(t))P^2_{\gamma_n,0,t}-\mathcal{D}^2X(p_n)\right](v_n,v_n)\right\|dt \\
&\leq \frac{1}{2}f''(t_n)\left(\sum_{k\geq 2}(\beta_{k-1}-\beta_k)\,\theta_n^{k-1}\right)\left(\frac{f(t_n)}{f'(t_n)}\right)^2 \\
&+\frac{1}{2}f''(t_n)\,\theta_n h(\theta_n)(1+\theta_n h(\theta_n))\left(\frac{f(t_n)}{f'(t_n)}\right)^2 \\
&+\int_0^1(1-t)\left\|P_{\gamma_n,t,0}\mathcal{D}^2X(\gamma_n(t))P^2_{\gamma_n,0,t}-\mathcal{D}^2X(p_n)\right\|\|v_n\|^2\,dt \\
&\leq \frac{1}{2}f''(t_n)\left(\sum_{k\geq 2}(\beta_{k-1}-\beta_k)\,\theta_n^{k-1}+\theta_n h(\theta_n)(1+\theta_n h(\theta_n))\right)\left(\frac{f(t_n)}{f'(t_n)}\right)^2 \\
&+\int_0^1(1-t)\,cd(p_n,\gamma_n(t))\left[d(p_n,p_{n+1})\right]^2dt \\
&\leq \frac{1}{2}f''(t_{n+1})\left(\sum_{k\geq 2}(\beta_{k-1}-\beta_k)\,\theta_n^{k-1}+\theta_n h(\theta_n)(1+\theta_n h(\theta_n))\right)\left(\frac{f(t_n)}{f'(t_n)}\right)^2 \\
&+\int_0^1(1-t)\,tcd(p_{n+1},p_n)\left[d(p_{n+1},p_n)\right]^2dt \\
&\leq \frac{1}{2}f''(t_{n+1})\left(\sum_{k\geq 2}(\beta_{k-1}-\beta_k)\,\theta_n^{k-1}+\theta_n h(\theta_n)(1+\theta_n h(\theta_n))\right)\left(\frac{f(t_n)}{f'(t_n)}\right)^2 \\
&+\frac{1}{6}c\left[d(p_{n+1},p_n)\right]^3 \\
&\leq \frac{1}{2}f''(t_n)\left(\sum_{k\geq 2}(2\beta_{k-1}-\beta_k)\,\theta_n^{k-1}+(\theta_n h(\theta_n))^2\right)\left(\frac{f(t_n)}{f'(t_n)}\right)^2 \\
&+\frac{1}{6}c(t_n-t_{n+1})^3 \\
&= f(t_{n+1}).
\end{aligned}
$$

(d) It is clear that

$$\begin{aligned}
\|T_{n+1}\| &= \left\| \frac{1}{2}\mathcal{D}X\left(p_{n+1}\right)^{-1}\left(\mathcal{D}^2X\left(p_{n+1}\right)\left(u_{n+1},.\right)\right)\right\| \\
&\leq \frac{1}{2}\left\|\mathcal{D}X\left(p_{n+1}\right)^{-1}\right\|\left\|\mathcal{D}^2X\left(p_{n+1}\right)\right\|\left\|\mathcal{D}X\left(p_{n+1}\right)^{-1}X\left(p_{n+1}\right)\right\| \\
&\leq \frac{1}{2}\frac{f''\left(t_{n+1}\right)f\left(t_{n+1}\right)}{\left[f'\left(t_{n+1}\right)\right]^2} \\
&= \theta_{n+1}.
\end{aligned}$$

(e) It follows inmediatlely true from (d) and from the fact that $\{\beta_k\}_{k\geq 2}$ is a decreasing sequence of positive real numbers.

(f) Finally, let us the geodesic γ_{k+1} be defined by

$$\gamma_{n+1}\left(t\right) = \exp_{p_{n+1}}\left(tv_{n+1}\right).$$

Then

$$\begin{aligned}
d\left(p_{n+2},p_{n+1}\right) &= \|v_{n+1}\| \\
&\leq \left(\left\|I_{T_{p_{n+1}}M}\right\| + \|T_{n+1}\| + \sum_{k\geq 2}\beta_k\|T_{n+1}\|^{k-1}\right)\|u_{n+1}\| \\
&\leq \left(1+\theta_{n+1}+\sum_{k\geq 2}\beta_k\theta_{n+1}^{k-1}\right)\frac{f\left(t_{n+1}\right)}{f'\left(t_{n+1}\right)} \\
&= t_{n+1}-t_{n+2}.
\end{aligned}$$

■

Theorem 44 *Under the same assumptions as in Theorem 44,* $\{p_n\}_{n\in\mathbb{N}} \subseteq B\left(p_0,t_0\right)$ *and* $p_n \longrightarrow p_*$*, where* p_* *is the uique singularity of* X *in* $B\left[p_0,t_0\right]$*. Moreover, if*

$$t_0 \leq -\frac{6}{13c}\left(b-\sqrt{\frac{13}{6}c+b^2}\right),$$

then p_* *is also the unique singularity of* X *in* $B\left(p_0,t_0\right)$*.*

Proof. For all $n \in \mathbb{N}$,

$$\begin{aligned} d(p_n, p_0) &\le \sum_{k=1}^{n} d(p_k, p_{k-1}) \\ &\le \sum_{k=1}^{n} t_{k-1} - t_k \\ &= t_0 - t_n \\ &\le t_0 \end{aligned}$$

so that $\{p_n\}_{n\in\mathbb{N}} \subseteq B(p_0, t_0)$. Moreover

$$\begin{aligned} d(p_{n+1}, p_n) &\le t_n - t_{n+1} \\ &\le t_n. \end{aligned}$$

Thus, $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy secuence, hence it is convergent to $p_* \in B[p_0, t_0]$.

As for all $n \in \mathbb{N}$,

$$\|X(p_n)\| \le f(t_n),$$

passing to the limit as $n \longrightarrow \infty$ we obtain

$$\|X(p_*)\| \le f(0) = 0,$$

thus

$$X(p_*) = 0 \text{ in } T_{p_*}M.$$

It is clear that for $k \ge n$,

$$d(p_k, p_n) \le t_n - t_k.$$

Passing to the limit as $k \longrightarrow \infty$, we get

$$d(p_n, p_*) \le t_n.$$

To show uniqueness, let us assume that there exists a second singularity $q^* \in B(p_0, t_0)$, and let $\gamma : [0,1] \longrightarrow M$ is a minimizing geodesic joining $q^*$and p^* where

$\gamma(0) = p^*$ and $\gamma(1) = q^*$. Then by (1.22),

$$\begin{aligned}
&\left\|\mathcal{D}X(p^*)^{-1}\right\| \left\|\int_0^1 P_{\gamma,t,0}\mathcal{D}X(\gamma(t))\,P_{\gamma,0,t}dt - \mathcal{D}X(p^*)\right\| \\
&\le \left\|\mathcal{D}X(p^*)^{-1}\right\| \int_0^1 \left\|P_{\gamma,t,0}\mathcal{D}X(\gamma(t))\,P_{\gamma,0,t} - \mathcal{D}X(p^*)\right\| dt \\
&= \left\|\mathcal{D}X(\gamma(0))^{-1}\right\| \int_0^1 \left\|P_{\gamma,t,0}\mathcal{D}X(\gamma(t))\,P_{\gamma,0,t} - \mathcal{D}X(\gamma(0))\right\| dt \\
&\le \frac{1}{f'(0)} \int_0^1 \left(\left\|\mathcal{D}^2X(\gamma(0))\right\| + \frac{1}{2}ct\left\|\gamma'(0)\right\|\right) t\left\|\gamma'(0)\right\| dt \\
&\le \frac{1}{f'(0)} \int_0^1 \left(\left\|\mathcal{D}^2X(\gamma(0))\right\| t + \frac{1}{2}ct^2\left\|\gamma'(0)\right\|\right) \left\|\gamma'(0)\right\| dt \\
&= \frac{1}{f'(0)} \left(\frac{1}{2}\left\|\mathcal{D}^2X(p^*)\right\| + \frac{1}{6}cd(p^*, q^*)\right) d(p^*, q^*) \\
&\le \frac{1}{f'(0)} \left(\frac{1}{2}f''(0) + \frac{1}{6}c\left(d(p^*, p_0) + d(q^*, p_0)\right)\right) \left(d(p^*, p_0) + d(q^*, p_0)\right) \\
&\le \frac{2}{f'(0)} \left(\frac{1}{2}f''(0) + \frac{1}{3}ct_0\right) t_0 \\
&= \frac{2}{f'(0)} \left(\frac{1}{2}(b + ct_0) + \frac{1}{3}ct_0\right) t_0 \\
&\le \frac{\left(b + \frac{5}{3}ct_0\right) t_0}{f'(0)} \\
&= \frac{\left(b + \frac{5}{3}ct_0\right) t_0}{-\frac{1}{2}ct_0^2 - bt_0 + 1}.
\end{aligned}$$

Notice that

$$\frac{\left(b + \frac{5}{3}ct_0\right) t_0}{-\frac{1}{2}ct_0^2 - bt_0 + 1} \le 1 \text{ if and only if } \frac{13}{6}ct_0^2 + 2bt_0 - 1 \le 0.$$

The roots of $h(t) = \frac{9}{2}ct^2 + 3bt - 1$ are

$$\left\{-\frac{6}{13c}\left(b - \sqrt{\frac{13}{6}c + b^2}\right), -\frac{6}{13c}\left(b + \sqrt{\frac{13}{6}c + b^2}\right)\right\}.$$

Given that $h(0) = -1$, for $t \geq 0$ we have

$$\frac{13}{6}ct_0^2 + 2bt_0 - 1 \leq 0 \text{ if and only if } t_0 \leq -\frac{6}{13c}\left(b - \sqrt{\frac{13}{6}c + b^2}\right),$$

Since

$$0 < t_0 \leq -\frac{6}{13c}\left(b - \sqrt{\frac{13}{6}c + b^2}\right),$$

we have

$$\frac{\left(b + \frac{5}{3}ct_0\right)t_0}{-\frac{1}{2}ct_0^2 - bt_0 + 1} \leq 1.$$

Therefore

$$\int_0^1 P_{\gamma,t,0}\mathcal{D}X(\gamma(t))\,P_{\gamma,0,t}dt$$

is invertible.

Finally,

$$\begin{aligned} 0 &= X(q^*) - X(p^*) \\ &= \int_0^1 P_{\gamma,t,0}\mathcal{D}X(\gamma(t))\,P_{\gamma,0,t}\left(\gamma'(0)\right)dt. \end{aligned}$$

We conclude that

$$\gamma'(0) = 0,$$

thus

$$0 = \left\|\gamma'(0)\right\| = d(p^*, q^*),$$

hence

$$p^* = q^*.$$

■

Theorem 45 *Let us suppose that X has a single singularity p_* in $B[p_0, t_0]$. If $B(p_0, t_0') \subseteq \Omega$, then p_* is the uique singularity of X in $B[p_0, r]$, where $t_0 < r \leq t_0'$.*

Proof. Let $q_* \in B[p_0, r]$ be a singularity of the vector field X. Let us consider the minimizing geodesic $\gamma : [0,1] \longrightarrow M$ joining p_0 to q_* given by

$$P_{\gamma,s,0}X(\gamma(1)) = X(\gamma(0)) + \mathcal{D}X(\gamma(0))\gamma'(0) + \frac{1}{2}\mathcal{D}^2X(\gamma(0))\left(\gamma'(0),\gamma'(0)\right) + R,$$

where

$$R = \frac{1}{2}\int_0^1 (1-t)^2 P_{\gamma,t,0}\mathcal{D}^3X(\gamma(t))P^3_{\gamma,t,0}\left(\gamma'(0),\gamma'(0),\gamma'(0)\right)dt,$$

thus

$$\gamma'(0) = -\mathcal{D}X(p_0)^{-1}\left(X(p_0) + \frac{1}{2}\mathcal{D}^2X(p_0)\left(\gamma'(0),\gamma'(0)\right) + R\right).$$

Therefore

$$\begin{aligned} d(p_0,q_*) &= \left\|\gamma'(0)\right\| \\ &\le \left\|\mathcal{D}X(p_0)^{-1}\right\|\left(\|X(p_0)\| + \frac{1}{2}\left\|\mathcal{D}^2X(p_0)\right\|\left\|\gamma'(0)\right\|^2 + \|R\|\right) \\ &\le a + \frac{1}{2}bd(p_0,q_*)^2 + \frac{1}{6}cd(p_0,q_*)^3, \end{aligned}$$

hence

$$g(d(p_0,q_*)) \ge 0.$$

Since $d(p_0,q_*) \le r \le t_0'$, then

$$d(p_0,q_*) \le t_0.$$

Finally, by Theorem 45,

$$p_* = q_*.$$

■

3.3 A family of higher order iterative methods under Kantorovich conditions on Riemannian manifolds

Recently, S. Amat, S. Busquier, S. Plaza, C. Bermúdez, M.J. Legaz (see [22]) proposed a new family of higher order methods. which generalizes classical third order methods. Under Kantorovich conditions, they proved the convergence of the methods in Banach spaces. This family of methods is given by:

$$\begin{aligned} y_n &= x_n - DF\left(x_n\right)^{-1} F\left(x_n\right) \\ x_{n+1} &= y_n - \left(I + L_F\left(x_n\right) + L_F^2\left(x_n\right) G\left(x_n\right)\right) DF\left(x_n\right)^{-1} F\left(y_n\right), \end{aligned}$$

where F is a nonlinear operator defined on an open convex subset Ω of a Banach space E into itself, I is the identity operator in E and $L_F\left(x_n\right)$ is defined by

$$L_F\left(x_n\right) = DF\left(x_n\right)^{-1} D^2F\left(x_n\right) DF\left(x_n\right)^{-1} F\left(x_n\right),$$

assuming that $DF\left(x_n\right)^{-1}$ exists and $G\left(x_n\right) : \Omega \subseteq E \longrightarrow E$ is a given nonlinear operator.

Our objective is to extend this method to the context of Riemannian manifolds, and to establish a convergence and uniqueness theorem.

We suppose that M is a compete Riemannian manifold, $\Omega \subseteq M$ an open convex set, $X \in C^2$ and we want to find an approximation to the singularity of the vector field $X : M \longrightarrow TM$. Let us consider the following family of high order iterative methods:

$$\begin{aligned} u_n &= -\mathcal{D}X\left(p_n\right)^{-1} X\left(p_n\right), \\ v_n &= \mathcal{D}X\left(p_n\right)^{-1} P_{\sigma_n,1,0} X\left(q_n\right) P_{\sigma_n,0,1}, \\ q_n &= \exp_{p_n}\left(u_n\right), \\ L_X\left(p_n\right) &= \mathcal{D}X\left(p_n\right)^{-1}\left(\mathcal{D}^2X\left(p_n\right)\left(u_n, .\right)\right), \\ w_n &= -\left(I_{T_{p_n}M} + L_X\left(p_n\right) + L_X\left(p_n\right)^2 G\left(p_n\right)\right) v_n, \\ p_{n+1} &= \exp_{p_n}\left(u_n + w_n\right), \end{aligned} \tag{3.20}$$

where $G : M \longrightarrow TM$ is a vector field (usually depending on the vector field X and its covariants derivatives) and $\left\{\sigma_n\right\}_{n\in\mathbb{N}}$ is a geodesic family defined by

$$\sigma_n\left(t\right) = \exp_{p_n}\left(tu_n\right).$$

Let $p_0 \in \Omega$. Assume that the conditions of Kantorovich holds:

C1. $\Gamma_0 = \mathcal{D}X\left(p_0\right)^{-1}$ exist and $\left\|\Gamma_0\right\| \leq \beta$,

C2. $\left\|\Gamma_0 X\left(p_0\right)\right\| \leq \eta$,

C3. $\left\|\mathcal{D}^2X\left(p\right)\right\| \leq M$ for all $p \in \Omega$, C4. $\left\| P_{\gamma,c,d}\mathcal{D}^2 X\left(\gamma\left(c\right),\cdot\right) P^2_{\gamma,d,c} - \mathcal{D}^2 X\left(\gamma\left(d\right),\cdot\right)\right\| \leq K \int_d^c \left\|\gamma'\left(t\right)\right\| dt$, where γ is an geodesic and $\gamma\left[d,c\right] \subseteq \Omega$.

Under these hypotheses, it is possible to find a cubic polynomial f in an interval $\left[a,b\right]$, $a \geq 0$ such that:

$$f\left(b\right) < 0 < f\left(a\right), \quad f'\left(t\right) < 0 \quad f''\left(t\right) > 0 \quad \text{and} \quad f'''\left(t\right) > 0,$$

in $\left[a,t^*\right]$ with t^* the single simple solution of $f\left(t\right) = 0$, and verifying:

For $t_0 \in \left[a,b\right]$ and $f\left(t_0\right) > 0$:

H1. $\left\|\Gamma_0\right\| \leq -\frac{1}{f'(t_0)}$,

H2. $\left\|\Gamma_0 X\left(p_0\right)\right\| \leq -\frac{f(t_0)}{f'(t_0)}$,

H3. $\left\|\mathcal{D}^2 X\left(p,\cdot\right)\right\| \leq f''\left(t\right)$ for all $p \in \Omega$ such that $d\left(p,p_0\right) \leq t - t_0 \leq t^* - t_0$,

H4. $\left\| P_{\gamma,c,d}\mathcal{D}^2 X\left(\gamma\left(c\right),\cdot\right) P^2_{\gamma,d,c} - \mathcal{D}^2 X\left(\gamma\left(d\right),\cdot\right)\right\| \leq \left|f''\left(u\right) - f''\left(v\right)\right|$, with $\int_d^c \left\|\gamma'\left(t\right)\right\| dt \leq \left|u - v\right|$, $\gamma\left[d,c\right] \subseteq \Omega$ and $u,v \in \left[a,t^*\right]$.

The construction is in ([16]), some properties of the polynomial $f\left(t\right)$ are:

1. $f\left(t\right)$ is decreasing in the interval $\left[a,t^*\right)$,

2. $f\left(t\right) > 0$ in $\left[a,t^*\right]$,

3. $f'\left(t\right)$ is increasing and $f\left(t\right)$ is convex in $\left[a,t^*\right]$,

4. $f''\left(t\right)$ is increasing in $\left[a,t^*\right]$,

5. $N_f\left(t\right) = t - \frac{f(t)}{f'(t)}$ is increasing in $\left[a,t^*\right)$, $N_f\left(t^*\right) = t^*$ and $N'_f\left(t^*\right) = 0$.

6. $L_f\left(t\right) = \frac{f(t)f''(t)}{f'(t)^2} > 0$ in $\left[a,t^*\right)$.

For the validity of the previous statements, it suffices to consider the polynomial

$$f\left(t\right) := \frac{K}{6}t^3 + \frac{M}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta},$$

with

$$\eta \leq \frac{4K + M^2\beta - M\beta\sqrt{M^2 + 2K\beta}}{3\beta K\left(M + \sqrt{M^2 + 2K\beta}\right)}. \tag{3.21}$$

This polynomial has two positive real root t^* and t^{**} (See [16] for more details),

Let us suppose additionally that there exist a function g_f associated to the vector field $L_X(p_n)$ defined in (3.20) satisfying:

C5. $\left\| L_X(p)^2 G(p) \right\| \leq L_f(t)^2 g(t)$ for $d(p, p_0) \leq t - t_0 \leq t^* - t_0$,

C6. $1 + L_f(t) + L_f(t)^2 g(t) \geq 0$ in $[a, t^*]$,

C7. $m'(t) > 0$ in $[a, t^*]$, where

$$m(t) = t - \frac{f(t)}{f'(t)} - \left(1 + L_f(t) + L_f(t)^2 g(t)\right) \frac{f\left(t - \frac{f(t)}{f'(t)}\right)}{f'(t)}.$$

Proposition 46 *If (C6) and (C7) are true, then the sequence*

$$\begin{array}{l} s_n = t_n - \frac{f(t_n)}{f'(t_n)} \\ t_{n+1} = s_n - \left(1 + L_f(t_n) + L_f(t_n)^2 g(t_n)\right) \frac{f(s_n)}{f'(t_n)}, \end{array} \tag{3.22}$$

starting from the above t_0 converges monotonically to t^*. This is the smaller real, simple root of $f(t) = 0$ in $[a, b]$.

Proof. See: [16], [22] ■

Now, we are in conditions to prove the semilocal convergence of (3.20), with the notation of (3.20) we have:

Theorem 47 *Let us assume $p_0 \in \Omega$ and $t_0 \in [a, t^*]$. Suppose that the hypotesis (C1)-(C7) and (3.21) are true. If $B(p_0, t^*) \subseteq \Omega$, then the sequence (3.20) is well defined and it converges to the root p_*, which is the solution of $X(p) = 0$ in $B[p_0, t^*]$.*

Moreover:

i) For all $n \geq 0$,

$$d(p_*, p_n) \leq t^* - t_n,$$

where t_n is defined in (3.22).

ii) If the number t^* also satisfies

$$13K\beta t^* + 6M\beta \leq \sqrt{6}\sqrt{6M^2\beta^2 + 13K\beta},$$

then the root p_* is unique in $B[p_0, t^*]$

Proof. We procced by induction. It suffices to show for all $k \in \mathbb{N}$:

i. $\left\|\mathcal{D}X(p_k)^{-1}\right\| \leq -\frac{1}{f'(t_k)}$,

ii. $\left\|\mathcal{D}^2X(p_k)^{-1}\right\| \leq f''(t_k)$,

iii. $\|X(p_k)\| \leq f(t_k)$,

iv. $\|L_X(p_k)\| \leq L_f(t_k)$,

v. $d(p_{n+1}, p_n) \leq t_{n+1} - t_n$.

The case $k = 0$ follows from the initial conditions on p_0 and t_0.

We assume that all the conditions are valid for $k = n$, and we check them for $k = n + 1$.

Let us consider the family of geodesics

$$\gamma_n(t) = \exp_{p_n}(t(u_n + w_n)), \text{ for all } n \geq 0.$$

i. By (1.22),

$$\|(P_{\gamma_n,1,0}\mathcal{D}X(\gamma_{n+1}(1))P_{\gamma_n,0,1} - \mathcal{D}X(p_n))\| \leq \left(\left\|\mathcal{D}^2X(\gamma_n(0))\right\| + \frac{1}{2}K\left\|\gamma_n'(0)\right\|\right)\left\|\gamma_n'(0)\right\|,$$

and so

$$\begin{aligned}
&\left\|\mathcal{D}X\left(p_n\right)^{-1}\left(P_{\gamma_n,1,0}\mathcal{D}X\left(p_{n+1}\right)P_{\gamma_n,0,1}-\mathcal{D}X\left(p_n\right)\right)\right\| \\
&\leq \left\|\mathcal{D}X\left(p_n\right)^{-1}\right\|\left(\left\|\mathcal{D}^2X\left(p_n\right)\right\|+\frac{1}{2}K\left\|u_n+w_n\right\|\right)\left\|u_n+w_n\right\| \\
&\leq -\frac{1}{f'\left(t_n\right)}\left(f''\left(t_n\right)+\frac{1}{2}Kd\left(p_{n+1},p_n\right)\right)d\left(p_{n+1},p_n\right) \\
&\leq -\frac{1}{f'\left(t_n\right)}\left(f''\left(t_n\right)+\frac{1}{2}K\left(t_{n+1}-t_n\right)\right)\left(t_{n+1}-t_n\right) \\
&= -\frac{1}{f'\left(t_n\right)}\left(f'\left(t_{n+1}\right)-f'\left(t_n\right)\right) \\
&= 1-\frac{f'\left(t_{n+1}\right)}{f'\left(t_n\right)} \\
&\leq 1,
\end{aligned}$$

where the last inequality is due to that $f'(t)$ is increasing. Therefore, using Banach's lemma,

$$\begin{aligned}
\left\|\mathcal{D}X\left(p_{n+1}\right)^{-1}\right\| &= \frac{-\frac{1}{f'(t_n)}}{1-\left(1-\frac{f'(t_{n+1})}{f'(t_n)}\right)} \\
&= -\frac{1}{f'\left(t_{n+1}\right)},
\end{aligned}$$

ii. Is clear that

$$\begin{aligned}
\left\|\mathcal{D}^2X\left(p_{n+1}\right)\right\| &= \left\|P_{\gamma_n,1,0}\mathcal{D}^2X\left(p_{n+1}\right)P^2_{\gamma_n,0,1}\right\| \\
&\leq \left\|P_{\gamma_n,1,0}\mathcal{D}^2X\left(p_{n+1}\right)P^2_{\gamma_n,0,1}-\mathcal{D}^2X\left(p_n\right)\right\|+\left\|\mathcal{D}^2X\left(p_n\right)\right\| \\
&\leq Kd\left(p_{n+1},p_n\right)+\left\|\mathcal{D}^2X\left(p_n\right)\right\| \\
&\leq Kd\left(t_{n+1}-t_n\right)+f''\left(t_n\right) \\
&= f''\left(t_{n+1}\right).
\end{aligned}$$

iii. First we observe that, using the Taylor expansion (1.20),

$$P_{\sigma n,1,0} X(q_n) = X(p_n) + \mathcal{D}X(p_n)\,\sigma_n'(0) + \frac{1}{2}\mathcal{D}^2 X(p_n)\left(\sigma_n'(0), \sigma_n'(0)\right) + R_{\sigma n},$$

where

$$R_{\sigma n} = \int_0^1 (1-t)\left(P_{\sigma_n,t,0}\mathcal{D}^2 X(\sigma_n(t))\, P^2_{\sigma_n,0,t} - \mathcal{D}^2 X(p_n)\right)\left(\sigma_n'(0), \sigma_n'(0)\right) dt.$$

As a consequence,

$$\begin{aligned}
\|X(q_n)\| &\le \left\| X(p_n) + \mathcal{D}X(p_n)\, u_n + \frac{1}{2}\mathcal{D}^2 X(p_n)(u_n, u_n)\right\| + \|R_{\sigma n}\| \\
&\le \left\| X(p_n) + \mathcal{D}X(p_n)\left(-\mathcal{D}X(p_n)^{-1} X(p_n)\right) + \frac{1}{2}\mathcal{D}^2 X(p_n)(u_n, u_n)\right\| + \|R_{\sigma n}\| \\
&\le \left\| \frac{1}{2}\mathcal{D}^2 X(p_n)(u_n, u_n)\right\| + \|R_{\sigma n}\|.
\end{aligned}$$

Since

$$\begin{aligned}
\|R_{\sigma n}\| &\le \int_0^1 (1-t)\left\|P_{\sigma_n,t,0}\mathcal{D}^2 X(\sigma_n(t))\, P^2_{\sigma_n,0,t} - \mathcal{D}^2 X(p_n)\right\| \|u_n\|^2\, dt \\
&\le K\int_0^1 (1-t)\, t\, \|u_n\|^3\, dt \\
&= \frac{K}{6}\left(d(q_n, p_n)\right)^3,
\end{aligned}$$

we have

$$\begin{aligned}
\|X(q_n)\| &\le \frac{1}{2} f''(t_n)(s_n - t_n)^2 + \frac{K}{6}(s_n - t_n)^3 \\
&= f(t_n) + f'(t_n)\left(-\frac{f(t_n)}{f'(t_n)}\right) + \frac{1}{2} f''(t_n)(s_n - t_n)^2 + \frac{K}{6}(s_n - t_n)^3 \\
&= f(t_n) + f'(t_n)(s_n - t_n) + \frac{1}{2} f''(t_n)(s_n - t_n)^2 + \frac{K}{6}(s_n - t_n)^3 \\
&= f(s_n).
\end{aligned}$$

Again, using the Taylor expansion (1.20),

$$P_{\gamma_n,1,0}X\left(p_{n+1}\right) = X\left(p_n\right) + \mathcal{D}X\left(p_n\right)\gamma_n'\left(0\right) + \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right) + R_n,$$

where

$$R_n = \int_0^1 \left(1-t\right)\left(P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)P^2_{\gamma_n,0,t} - \mathcal{D}^2X\left(p_n\right)\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right)dt.$$

Therefore,

$$\begin{aligned}
P_{\gamma_n,1,0}X\left(p_{n+1}\right) &= X\left(p_n\right) + \mathcal{D}X\left(p_n\right)\left(u_n+w_n\right) + \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right) + R_n \\
&= X\left(p_n\right) + \mathcal{D}X\left(p_n\right)\left(-\mathcal{D}X\left(p_n\right)^{-1}X\left(p_n\right)+w_n\right) + \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right) + R_n \\
&= \mathcal{D}X\left(p_n\right)w_n + \frac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right) + R_n.
\end{aligned}$$

Since

$$\begin{aligned}
&\left\|\mathcal{D}X\left(p_n\right)w_n + \tfrac{1}{2}\mathcal{D}^2X\left(p_n\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right)\right\| \\
&\leq \left\|\mathcal{D}X\left(p_n\right)\left(I_{T_{p_n}M} + L_X\left(p_n\right) + L_X\left(p_n\right)^2G\left(p_n\right)\right)v_n\right\| + \tfrac{1}{2}\left\|\mathcal{D}^2X\left(p_n\right)\left(\gamma_n'\left(0\right),\gamma_n'\left(0\right)\right)\right\| \\
&\leq -f'\left(t_n\right)\left(1 + L_f\left(t_n\right) + L_f\left(t_n\right)^2g\left(t_n\right)\right)\tfrac{f\left(s_n\right)}{f'\left(t_n\right)} + \tfrac{1}{2}f'\left(t_n\right)\left(t_{n+1}-t_n\right)^2 \\
&= f'\left(t_n\right)\left(t_{n+1}-s_n\right) + \tfrac{1}{2}f'\left(t_n\right)\left(t_{n+1}-t_n\right)^2,
\end{aligned}$$

we also have

$$\begin{aligned}
\left\|R_n\right\| &\leq \int_0^1 \left(1-t\right)\left\|P_{\gamma_n,t,0}\mathcal{D}^2X\left(\gamma_n\left(t\right)\right)P^2_{\gamma_n,0,t} - \mathcal{D}^2X\left(p_n\right)\right\|\left\|\gamma_n'\left(0\right)\right\|^2dt \\
&\leq \int_0^1 \left(1-t\right)K\int_0^t \left\|\gamma_n'\left(\tau\right)\right\|d\tau dt\left\|\gamma_n'\left(0\right)\right\|^2 \\
&= K\int_0^1 \left(1-t\right)tdt\left\|\gamma_n'\left(0\right)\right\|^3 \\
&= \frac{K}{6}\left(t_{n+1}-t_n\right)^3.
\end{aligned}$$

Therefore

$$
\begin{aligned}
\|X(p_{n+1})\| &\le f'(t_n)(t_{n+1}-s_n)+\frac{1}{2}f'(t_n)(t_{n+1}-t_n)^2+\frac{K}{6}(t_{n+1}-t_n)^3\\
&\le f'(t_n)\left(t_{n+1}+\frac{f(t_n)}{f'(t_n)}-t_n\right)+\frac{1}{2}f'(t_n)(t_{n+1}-t_n)^2+\frac{K}{6}(t_{n+1}-t_n)^3\\
&= f(t_n)+f(t_n)(t_{n+1}-t_n)+\frac{1}{2}f'(t_n)(t_{n+1}-t_n)^2+\frac{K}{6}(t_{n+1}-t_n)^3\\
&= f(t_{n+1}).
\end{aligned}
$$

iv. It is clear that

$$
\begin{aligned}
\|L_X(p_{n+1})\| &\le \left\|\mathcal{D}X(p_{n+1})^{-1}\right\| \left\|\mathcal{D}^2X(p_{n+1})\right\| \|u_{n+1}\|\\
&\le L_f(t_{n+1}).
\end{aligned}
$$

v. Finally, if we let the geodesic γ_{k+1} be defined by

$$
\gamma_{n+1}(t)=\exp_{p_{n+1}}(t(u_{n+1}+w_{n+1})).
$$

Then

$$
\begin{aligned}
d(p_{n+1},p_n) &= \|u_{n+1}+w_{n+1}\|\\
&\le \left\|\mathcal{D}X(p_{n+1})^{-1}X(p_{n+1})\right\|\\
&+\left\|I_{T_{p_{n+1}}M}+L_X(p_{n+1})+L_X(p_{n+1})^2G(p_{n+1})v_{n+1}\right\|\\
&\le -\frac{f(t_{n+1})}{f'(t_{n+1})}-\left(1+L_f(t_{n+1})+L_f(t_{n+1})^2g(t_{n+1})\right)\frac{f(s_{n+1})}{f'(t_{n+1})}\\
&= s_n-t_n+t_{n+1}-s_n\\
&= t_{n+1}-t_n.
\end{aligned}
$$

Using item **v**, we have for $k\ge n$, $n\in\mathbb{N}$,

$$
d(p_k,p_n)\le t_k-t_n.
$$

It follows that $\{t_k\}_{k\in\mathbb{N}}$ is Cauchy sequence. Since M is complete, it converges to the same $p_* \in M$. Moreover, passing to the limit $k \longrightarrow \infty$, for all $n \in \mathbb{N}$ we have

$$d\left(p_*, p_n\right) \leq t^* - t_n,$$

passing to the limit in **iii,** we obtain

$$\left\|X\left(p_*\right)\right\| \leq f\left(t^*\right) = 0,$$

thus

$$X\left(p_*\right) = 0.$$

To show uniqueness, show that the operator

$$\int_0^1 P_{\gamma,t,0}\mathcal{D}X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}dt$$

is invertible. let us assume that there exists a second singularity $q^* \in B\left(p_0, t_0\right)$ and llet $\gamma : [0,1] \longrightarrow M$ be a minimizing geodesic joining $q^*$and p^*, where $\gamma\left(0\right) = p^*$ and $\gamma\left(1\right) = q^*$.

By (1.22),

$$\begin{aligned}
&\left\|\mathcal{D}X\left(p^*\right)^{-1}\right\|\left\|\int_0^1 P_{\gamma,t,0}\mathcal{D}X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}dt-\mathcal{D}X\left(p^*\right)\right\|\\
&\leq\left\|\mathcal{D}X\left(p^*\right)^{-1}\right\|\int_0^1\left\|P_{\gamma,t,0}\mathcal{D}X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}-\mathcal{D}X\left(p^*\right)\right\|dt\\
&=\left\|\mathcal{D}X\left(\gamma\left(0\right)\right)^{-1}\right\|\int_0^1\left\|P_{\gamma,t,0}\mathcal{D}X\left(\gamma\left(t\right)\right)P_{\gamma,0,t}-\mathcal{D}X\left(\gamma\left(0\right)\right)\right\|dt\\
&\leq-\frac{1}{f'\left(t^*\right)}\int_0^1\left(\left\|\mathcal{D}^2X\left(\gamma\left(0\right)\right)\right\|+\frac{1}{2}Kt\left\|\gamma'\left(0\right)\right\|\right)t\left\|\gamma'\left(0\right)\right\|dt\\
&\leq-\frac{1}{f'\left(t^*\right)}\int_0^1\left(\left\|\mathcal{D}^2X\left(\gamma\left(0\right)\right)\right\|t+\frac{1}{2}Kt^2\left\|\gamma'\left(0\right)\right\|\right)\left\|\gamma'\left(0\right)\right\|dt\\
&=-\frac{1}{f'\left(t^*\right)}\left(\frac{1}{2}\left\|\mathcal{D}^2X\left(p^*\right)\right\|+\frac{1}{6}Kd\left(p^*,q^*\right)\right)d\left(p^*,q^*\right)\\
&\leq-\frac{1}{f'\left(0\right)}\left(\frac{1}{2}f''\left(t^*\right)+\frac{1}{6}K\left(d\left(p^*,p_0\right)+d\left(q^*,p_0\right)\right)\right)\left(d\left(p^*,p_0\right)+d\left(q^*,p_0\right)\right)\\
&\leq-\frac{2}{f'\left(t^*\right)}\left(\frac{1}{2}f''\left(t^*\right)+\frac{1}{3}K\left(t^*-t_0\right)\right)\left(t^*-t_0\right)\\
&=-\frac{2}{f'\left(t^*\right)}\left(\frac{1}{2}\left(M+Kt^*\right)+\frac{1}{3}Kt^*\right)t^*\\
&\leq-\frac{\left(M+\frac{5}{3}Kt^*\right)t^*}{f'\left(t^*\right)}.
\end{aligned}$$

Notice that

$$-\frac{\left(M+\frac{5}{3}Kt^*\right)t^*}{f'\left(t^*\right)}\leq 1 \text{ if and only if } 13K\beta(t^*)^2+12M\beta t^*-6\leq 0.$$

The roots of $h\left(t\right)=13K\beta t^2+12M\beta t-6$ are

$$\left\{\frac{1}{13K\beta}\left(-6M\beta+\sqrt{6}\sqrt{6M^2\beta^2+13K\beta}\right),-\frac{1}{13K\beta}\left(6M\beta+\sqrt{6}\sqrt{6M^2\beta^2+13K\beta}\right)\right\}.$$

Given that $h\left(0\right)=-6$, we have for $t\geq 0$

$$13K\beta t^2+12M\beta t-6\leq 0 \text{ if and only if } t\leq\frac{1}{13K\beta}\left(-6M\beta+\sqrt{6}\sqrt{6M^2\beta^2+13K\beta}\right).$$

By hypothesis

$$0 < t^* \leq \frac{1}{13K\beta}\left(-6M\beta + \sqrt{6}\sqrt{6M^2\beta^2 + 13K\beta}\right),$$

hence

$$-\frac{\left(M + \frac{5}{3}Kt^*\right)t^*}{f'(t^*)} \leq 1.$$

Therefore,

$$\int_0^1 P_{\gamma,t,0}\mathcal{D}X(\gamma(t))\,P_{\gamma,0,t}dt$$

is invertible.

Finally,

$$\begin{aligned}
0 &= X(q^*) - X(p^*) \\
&= \int_0^1 P_{\gamma,t,0}\mathcal{D}X(\gamma(t))\,P_{\gamma,0,t}\left(\gamma'(0)\right)dt.
\end{aligned}$$

Therefore,

$$\gamma'(0) = 0.$$

Thus,

$$0 = \left\|\gamma'(0)\right\| = d(p^*, q^*),$$

which yields

$$p^* = q^*.$$

■

Theorem 48 *Without lost of generality, let us assume $t_0 = 0$. Suppose that X has a single singularity p_* in $B[p_0, t^*]$. If $B(p_0, t^{**}) \subseteq \Omega$, then p_* is the unique singularity of X in $B[p_0, r]$, where $t^* < r \leq t^{**}$.*

Proof. Let $q_* \in B[p_0, r]$ be a singularity of a vector field X , Let us consider the minimizing geodesic $\gamma : [0,1] \longrightarrow M$ joining p_0 to q_*. Since

$$P_{\gamma,s,0}X(\gamma(1)) = X(\gamma(0)) + \mathcal{D}X(\gamma(0))\,\gamma'(0) + \frac{1}{2}\mathcal{D}^2X(\gamma(0))\left(\gamma'(0), \gamma'(0)\right) + R,$$

where

$$R=\frac{1}{2}\int_0^1(1-t)^2\,P_{\gamma,t,0}\mathcal{D}^3X\left(\gamma\left(t\right)\right)P_{\gamma,t,0}^3\left(\gamma'\left(0\right),\gamma'\left(0\right),\gamma'\left(0\right)\right)dt,$$

we have

$$\gamma'\left(0\right)=-\mathcal{D}X\left(p_0\right)^{-1}\left(X\left(p_0\right)+\frac{1}{2}\mathcal{D}^2X\left(p_0\right)\left(\gamma'\left(0\right),\gamma'\left(0\right)\right)+R\right).$$

Thus

$$\begin{aligned}d\left(p_0,q_*\right)&=\left\|\gamma'\left(0\right)\right\|\\&\leq\left\|\mathcal{D}X\left(p_0\right)^{-1}\right\|\left(\left\|X\left(p_0\right)\right\|+\frac{1}{2}\left\|\mathcal{D}^2X\left(p_0\right)\right\|\left\|\gamma'\left(0\right)\right\|^2+\left\|R\right\|\right)\\&\leq\eta+\frac{1}{2}\beta Md\left(p_0,q_*\right)^2+\frac{1}{6}\beta Kd\left(p_0,q_*\right)^3.\end{aligned}$$

Therefore,

$$\frac{1}{6}Kd\left(p_0,q_*\right)^3+\frac{1}{2}Md\left(p_0,q_*\right)^2-\frac{1}{\beta}d\left(p_0,q_*\right)+\frac{\eta}{\beta}\geq 0$$

hence

$$f\left(d\left(p_0,q_*\right)\right)\geq 0.$$

Since $d\left(p_0,q_*\right)\leq r\leq t^{**}$, we have

$$d\left(p_0,q_*\right)\leq t^*.$$

Finally, by Theorem 48,

$$p_*=q_*.$$

■

3.4 On a third-order method without bilinear operator under Kantorovich-type condiction on Riemannian manifolds

In 2006, Jisheng Kou, Yitian Li and Xiuhua Wang presented in the case of escalar functions, a new modification of the Newton method cubically convergent for solving nonlinear equations $f\left(x\right)=0$, where $f:\Omega\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ with Ω an open interval (see [11]). The method is described by:

$$\begin{aligned} y_n &= x_n + \tfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \tfrac{f(x_n)}{f'(x_n)}. \end{aligned} \tag{3.23}$$

This new method is preferable if the computational costs of the first derivative are greater or equal to those of the function itself, also it doesn't involve the second derivative. In 2009, S. Amat, C. Bermúdez, S. Busquier, S. Plaza, presented a generalization of (3.23) to Banach spaces (see [23]), so that:

$$\begin{aligned} y_n &= x_n + DF(x_n)^{-1} F(x_n) \\ x_{n+1} &= y_n - DF(x_n)^{-1} F(x_n), \end{aligned}$$

where F is nonlinear operator defined on an open convex subset Ω of a Banach space E.

In [23] was shown, under Kantorovich condictions; its convergence, uniqueness and convergence order. Actually, they studied the equivalent method:

$$x_{n+1} = x_n - DF(x_n)^{-1} [y_n, x_n; F]\, DF(x_n)^{-1} F(x_n),$$

where the operator $[y_n, x_n; F]$ is called divided difference and is defined for

$$[y, x; F](x - y) = F(x) - F(y).$$

Note that if F is Fréchet differentiable in x, then $DF(x) = [x, x; F]$.

The objective now is to extend this method to the context of Riemannian manifolds and to show, under Kantorovich condictions, its convergence and uniqueness. First we define divided differences in this new context.

Definition 49 *Let M be a Riemannian manifold, $\Omega \subseteq M$ an open convex set. Asume that γ is a curve in M , $[a, b] \subset dom(\gamma)$ and $X : M \longrightarrow TM$ a C^0 vector field on M. We define the divided difference of first order for the vector field X on the points $\gamma(s), \gamma(s+h)$ in direction $\gamma'(s)$ by:*

$$[\gamma(s+h), \gamma(s); X]\, \gamma'(s) = \frac{1}{h}\left(P_{\gamma,s+h,s} X(\gamma(s+h)) - X(\gamma(s))\right). \tag{3.24}$$

Note that in the case that M be a Banach space, if γ is the geodesic joining x and y, such that

$$\gamma(s) = x + s(y - x), \ t \in \mathbb{R},$$

then (3.24) implies

$$[y, x; X](y - x) = X(y) - X(x).$$

This is the classic definition of divided difference of first order in Banach spaces (see [23]). Also if there exist $\mathcal{D}X(p)$, then $\mathcal{D}X(p) = [p, p; X]$.

Let us suppose that we have the method

$$\begin{aligned} q_n &= \exp_{p_n}\left(\mathcal{D}X(p_n)^{-1} X(p_n)\right), \\ p_{n+1} &= \exp_{p_n}\left(\mathcal{D}X(p_n)^{-1}(X(p_n) - P_{\gamma,1,0}X(q_n))\right). \end{aligned} \tag{3.25}$$

If

$$\gamma(s) = \exp_{p_n}\left(s\mathcal{D}X(p_n)^{-1} X(p_n)\right),$$

Then, using the definition of divided difference of first order, is clear that

$$[\gamma(1), \gamma(0); X]\gamma'(0) = P_{\gamma,1,0}X(\gamma(1)) - X(\gamma(0)).$$

Thus,

$$[q_n, p_n; X]\mathcal{D}X(p_n)^{-1} X(p_n) = P_{\gamma,1,0}X(q_n) - X(p_n),$$

and therefore

$$\mathcal{D}X(p_n)^{-1}(X(p_n) - P_{\gamma,1,0}X(q_n)) = -\mathcal{D}X(p_n)^{-1}[q_n, p_n; X]\mathcal{D}X(p_n)^{-1} X(p_n).$$

Then (3.25), is transformed in

$$\begin{aligned} q_n &= \exp_{p_n}\left(\mathcal{D}X(p_n)^{-1} X(p_n)\right), \\ p_{n+1} &= \exp_{p_n}\left(-\mathcal{D}X(p_n)^{-1}[q_n, p_n; X]\mathcal{D}X(p_n)^{-1} X(p_n)\right). \end{aligned} \tag{3.26}$$

We will also use the following notations

$$\begin{aligned}\Gamma_n &= \mathcal{D}X(p_n),\\ \Phi_n &= \mathcal{D}X(p_n)\,[q_n,p_n;X]^{-1}\,\mathcal{D}X(p_n).\end{aligned} \tag{3.27}$$

Definition 50 *We say that the divided difference of first order satisfies the ω-condition,if*

$$\|[p_1,q_1;X]-P_{\gamma,1,0}[p_2,q_2;X]\| \le \omega\left(d(p_1,p_2),d(q_1,q_2)\right);\quad p_1,p_2,q_1,q_2\in\Omega, \tag{3.28}$$

where γ is a geodesic joining the points $\gamma(0)=q_1$, $\gamma(1)=q_2$ and $\omega:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a nondecreasing continuous function in two variables, such that $\omega(0,x)=\omega(x,0)=\frac{1}{2}\omega(x,x)$.

Theorem 51 *Let M be a complete Riemannian manifold, $\Omega\subseteq M$ an open convex set, X a C^1 vector field on M. Suppose that the divided difference of first order satisfies the ω-condition, and let $p_0\in\Omega$. Assume that:*

1. $\left\|\Gamma_n^{-1}\right\|\le\beta$.
2. $\max\left\{\left\|\Gamma_0^{-1}X(p_0)\right\|,\left\|\Phi_0^{-1}X(p_0)\right\|\right\}\le\eta$.
3. *The equation*
$$t\left(1-\frac{m}{1-2\beta\omega(t,t)}\right)-\eta=0. \tag{3.29}$$
has a small positive root R, where $m=\beta\omega(\eta,\eta)$.
4. $\beta\omega(R,R)<\frac{1}{3}$ *and* $\overline{B(p_0,R)}\subset\Omega$.

Then,

$$N=\frac{m}{1-2\beta\omega(R,R)}\in(0,1)$$

and the method (3.26) is well defined, $p_n\in B(p_0,R)$ for all $n\in\mathbb{N}$, and it converges to the unique solution of $X(p)=0$ in $B[p_0,R]$.

Proof. First, we note that

$$1-2\beta\omega(R,R)\ge 1-\frac{2}{3}=\frac{1}{3}\ge 0.$$

Thus,

$$\frac{m}{1-2\beta\omega\left(R,R\right)}\geq 0. \tag{3.30}$$

on the another hand, as R is a root of equation (3.29), we obtain

$$\left(1-\frac{m}{1-2\beta\omega\left(R,R\right)}\right)=\frac{\eta}{R},$$

and by (3.30)

$$\left(1-\frac{m}{1-2\beta\omega\left(R,R\right)}\right)\leq 1$$

Therefore

$$\frac{\eta}{R}\leq 1,$$

and then

$$\eta\leq R. \tag{3.31}$$

The previous inequality also implies

$$\omega\left(\eta,\eta\right)\leq\omega\left(R,R\right),$$

so that

$$\begin{aligned}\frac{m}{1-2\beta\omega\left(R,R\right)}&=\frac{\beta\omega\left(\eta,\eta\right)}{1-2\beta\omega\left(R,R\right)}\\&\leq\frac{\beta\omega\left(R,R\right)}{1-2\beta\omega\left(R,R\right)}\\&\leq\frac{\frac{1}{3}}{1-2\frac{1}{3}}\\&=1.\end{aligned}$$

We conclude,by (3.30),

$$0\leq\frac{m}{1-2\beta\omega\left(R,R\right)}\leq 1. \tag{3.32}$$

N*ow, let* us begin considering the families of geodesic

$$\gamma_n(t) = \exp_{p_n}(tu_n),$$
$$\sigma_n(t) = \exp_{p_n}(tv_n);\ \text{for all } n \in \mathbb{N},$$

where

$$u_n = \mathcal{D}X(p_n)^{-1} X(p_n) \ \text{ and } v_n = -\mathcal{D}X(p_n)^{-1}[q_n, p_n; X]\,\mathcal{D}X(p_n)^{-1} X(p_n).$$

Thus

$$\gamma_n(0) = p_n, \quad \gamma_n(1) = q_n$$
$$\text{and}$$
$$\sigma_n(0) = p_n, \quad \sigma_n(1) = p_{n+1}\ .$$

In this way, the functionales $[q_n, p_n; X]$ and $[p_{n+1}, p_n; X]$ are well defined through the geodesics γ_n and σ_n.

Then, using notation (3.27),

Now our objective is to bound $d(p_2, p_1)$. For it, it is necessary to find bounds for

$$\left\|\Gamma_1^{-1} P_{\sigma_0,0,1}\left([p_1, p_0; X] - \Phi_0\right)\right\| \ \text{ and } \ \left\|\Phi_1^{-1}\Gamma_1\right\|\ .$$

(a) For $\left\|\Gamma_1^{-1} P_{\sigma_0,0,1}\left([p_1, p_0; X] - \Phi_0\right)\right\|$.

Let us see that Γ_1 is invertible:

$$\begin{aligned}
\left\|\Gamma_0^{-1}\right\| \left\|\Gamma_0 - P_{\sigma_0,1,0}\Gamma_1\right\| &= \left\|\Gamma_0^{-1}\right\| \left\|\mathcal{D}X(p_0) - P_{\sigma_0,1,0}\mathcal{D}X(p_1)\right\| \\
&= \left\|\Gamma_0^{-1}\right\| \left\|[p_0, p_0; X] - P_{\sigma_0,1,0}[p_1, p_1; X]\right\| \\
&\le \left\|\Gamma_0^{-1}\right\| \omega\left(d(p_0, p_1), d(p_0, p_1)\right) \\
&\le \beta\omega(\eta, \eta) \\
&\le \beta\omega(R, R) \\
&\le 1.
\end{aligned}$$

Thus, by Banach's lemma, Γ_1 is invertible and

$$\begin{aligned}\left\|\Gamma_1^{-1}\right\| &\leq \frac{\left\|\Gamma_0^{-1}\right\|}{1-\left\|\Gamma_0^{-1}\right\|\left\|\Gamma_0-P_{\sigma_0,1,0}\Gamma_1\right\|} \\ &\leq \frac{\beta}{1-\beta\omega\left(\eta,\eta\right)}.\end{aligned} \tag{3.33}$$

In particular, we note that Φ_1^{-1} and p_2 are well defined.

Now, because

$$\begin{aligned}\left\|\Gamma_0^{-1}\right\|\left\|\Gamma_0-[q_0,p_0;X]\right\| &\leq \beta\left\|[p_0,p_0;X]-[q_0,p_0;X]\right\| \\ &\leq \beta\omega\left(d\left(p_0,q_0\right),d\left(p_0,p_0\right)\right) \\ &\leq \beta\omega\left(\left\|\Gamma_0^{-1}X\left(p_0\right)\right\|,0\right) \\ &\leq \beta\omega\left(\eta,0\right) \\ &\leq 1,\end{aligned}$$

we conclude, by Banach's lemma, that $[q_0,p_0;X]$ is invertible and

$$\left\|[q_0,p_0;X]^{-1}\right\| \leq \frac{\beta}{1-\beta\omega\left(\eta,0\right)}. \tag{3.34}$$

Now, is clear that Φ_0 is well defined, and

$$\begin{aligned}\left\|I_{T_{p_0}M}-[q_0,p_0;X]^{-1}\Gamma_0\right\| &= \left\|I_{T_{p_0}M}-[q_0,p_0;X]^{-1}\Gamma_0\right\| \\ &\leq \left\|[q_0,p_0;X]^{-1}\right\|\left\|[q_0,p_0;X]-\Gamma_0\right\| \\ &\leq \left(\frac{\beta}{1-\beta\omega\left(\eta,0\right)}\right)\omega\left(\eta,0\right) \\ &= \frac{\frac{1}{2}\beta\omega\left(\eta,\eta\right)}{1-\frac{1}{2}\beta\omega\left(\eta,\eta\right)} \\ &\leq \frac{\frac{1}{2}}{1-\frac{1}{2}} \\ &= 1.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\left([p_1,p_0;X]-\Phi_0\right)\right\| \\
&=\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\left([p_1,p_0;X]-\Phi_0+\Gamma_0-\Gamma_0\right)\right\| \\
&\le\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\left([p_1,p_0;X]-\Gamma_0\right)\right\|+\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\left(\Gamma_0-\Phi_0\right)\right\| \\
&\le\left\|\Gamma_1^{-1}\right\|\left\|[p_1,p_0;X]-\Gamma_0\right\|+\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\Gamma_0\right\|\left\|\left(I_{T_{p_0}M}-\Gamma_0^{-1}\Phi_0\right)\right\| \\
&\le\left\|\Gamma_1^{-1}\right\|\left\|[p_1,p_0;X]-\Gamma_0\right\|+\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\Gamma_0\right\|\left\|\left(I_{T_{p_0}M}-\Gamma_0^{-1}\Phi_0\right)\right\| \\
&\le\left\|\Gamma_1^{-1}\right\|\left\|[p_1,p_0;X]-[p_0,p_0;X]\right\|+\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\Gamma_0\right\|\left\|\left(I_{T_{p_0}M}-\Gamma_0^{-1}\Phi_0\right)\right\| \\
&\le\left\|\Gamma_1^{-1}\right\|\left\|[p_1,p_0;X]-[p_0,p_0;X]\right\|+\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\Gamma_0\right\|\left\|\left(I_{T_{p_0}M}-[q_0,p_0;X]^{-1}\Gamma_0\right)\right\| \\
&\le\left\|\Gamma_1^{-1}\right\|\left\|[p_1,p_0;X]-[p_0,p_0;X]\right\|+\left\|\Gamma_1^{-1}P_{\sigma_0,0,1}\Gamma_0\right\|\left\|\left(I_{T_{p_0}M}-[q_0,p_0;X]^{-1}\Gamma_0\right)\right\| \\
&\le\tfrac{\beta}{1-\beta\omega(\eta,\eta)}\omega\left(d\left(p_1,p_0\right),0\right)+\tfrac{1}{1-\beta\omega(\eta,\eta)}\tfrac{\beta\omega(\eta,0)}{1-\beta\omega(\eta,0)} \\
&\le\tfrac{\beta}{1-\beta\omega(\eta,\eta)}\omega\left(\eta,0\right)+\tfrac{1}{1-\beta\omega(\eta,\eta)}\tfrac{\beta\omega(\eta,0)}{1-\beta\omega(\eta,0)} \\
&=\tfrac{\beta\omega(\eta,\eta)}{2-2\beta\omega(\eta,\eta)}\tfrac{4-\beta\omega(\eta,\eta)}{2-\beta\omega(\eta,\eta)}.
\end{aligned}
\tag{3.35}
$$

Note that

$$\frac{\beta\omega\left(\eta,\eta\right)}{2-2\beta\omega\left(\eta,\eta\right)}\frac{4-\beta\omega\left(\eta,\eta\right)}{2-\beta\omega\left(\eta,\eta\right)}\le 1.$$

Because $\beta\omega\left(\eta,\eta\right)\le\beta\omega\left(R,R\right)\le\frac{1}{3}$ and $\beta\omega\left(\eta,\eta\right)\le\frac{4}{3}$, we obtain

$$\frac{\beta\omega\left(\eta,\eta\right)}{2-2\beta\omega\left(\eta,\eta\right)}\le\frac{1}{4}\text{ and }\frac{4-\beta\omega\left(\eta,\eta\right)}{2-\beta\omega\left(\eta,\eta\right)}\le 4.$$

(b) For $\left\|\Phi_1^{-1}\Gamma_1\right\|.$

First, let us note that

$$
\begin{aligned}
\Phi_0\left(v_0\right)&=\mathcal{D}X\left(p_0\right)\left[q_0,p_0;X\right]^{-1}\mathcal{D}X\left(p_0\right)\left(-\mathcal{D}X\left(p_0\right)^{-1}\left[q_0,p_0;X\right]\mathcal{D}X\left(p_0\right)^{-1}X\left(p_0\right)\right)\\
&=-X\left(p_0\right).
\end{aligned}
$$

Then

$$X\left(p_0\right)=-\Phi_0\left(v_0\right),$$

and according to the definition of divided differences

$$\left[\sigma_0\left(1\right),\sigma_0\left(0\right);X\right]\sigma_0'\left(0\right)=P_{\sigma_0,1,0}X\left(\gamma\left(1\right)\right)-X\left(\gamma\left(0\right)\right),$$

thus

$$[p_1, p_0; X]\, v_0 = P_{\sigma_0,1,0} X\,(p_1) - X\,(p_0)\,,$$

therefore

$$\begin{aligned} P_{\sigma_0,1,0} X\,(p_1) &= [p_1, p_0; X]\, v_0 - \Phi_0\,(v_0) \\ &= ([p_1, p_0; X] - \Phi_0)\, v_0, \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} d\,(q_1, p_1) &= \|u_1\| \\ &= \left\|\Gamma^{-1} X\,(p_1)\right\| \\ &= \left\|\Gamma^{-1} P_{\sigma_0,0,1}\,([p_1, p_0; X] - \Phi_0)\, v_0\right\| \\ &\leq \left\|\Gamma^{-1} P_{\sigma_0,0,1}\,([p_1, p_0; X] - \Phi_0)\right\| \|v_0\| \\ &\leq \|v_0\| \\ &\leq \eta. \end{aligned}$$

Then

$$\begin{aligned} \left\|I_{T_{p_1}M} - \Gamma_1^{-1}\,[q_1, p_1; X]\right\| &\leq \left\|\Gamma_1^{-1}\right\| \|\Gamma_1 - [q_1, p_1; X]\| \\ &\leq \left\|\Gamma_1^{-1}\right\| \|[p_1, p_1; X] - [q_1, p_1; X]\| \\ &\leq \left\|\Gamma_1^{-1}\right\| \omega\,(d\,(p_1, q_1)\,, 0) \\ &\leq \frac{\beta\omega\,(\eta, 0)}{1 - \beta\omega\,(\eta, \eta)} \\ &\leq 1, \end{aligned}$$

since

$$\begin{aligned}\left\|[q_1,p_1;X]^{-1}\Gamma_1\right\| &= \left\|\Gamma_1^{-1}[q_1,p_1;X]\right\|^{-1}\\ &\leq \frac{1}{1-\left\|I_{T_{p_1}M}-\Gamma_1^{-1}[q_1,p_1;X]\right\|}\\ &\leq \frac{1}{1-\frac{\beta\omega(\eta,0)}{1-\beta\omega(\eta,\eta)}}\\ &= \frac{2-2\beta\omega(\eta,\eta)}{2-3\beta\omega(\eta,\eta)},\end{aligned}$$

given that

$$\begin{aligned}\left\|[q_1,p_1;X]^{-1}\right\| &\leq \left(\frac{\beta}{1-\beta\omega(\eta,\eta)}\right)\left(\frac{2-2\beta\omega(\eta,\eta)}{2-3\beta\omega(\eta,\eta)}\right)\\ &= \frac{2\beta}{2-3\beta\omega(\eta,\eta)},\end{aligned}$$

in this way

$$\begin{aligned}\left\|I_{T_{p_1}M}-\Gamma_1^{-1}\Phi_1\right\| &= \left\|I_{T_{p_1}M}-[q_1,p_1;X]^{-1}\Gamma_1\right\|\\ &\leq \left\|[q_1,p_1;X]^{-1}\right\|\left\|[q_1,p_1;X]-\Gamma_1\right\|\\ &\leq \left\|[q_1,p_1;X]^{-1}\right\|\left\|[q_1,p_1;X]-\Gamma_1\right\|\\ &\leq \frac{2\beta}{2-3\beta\omega(\eta,\eta)}\left\|[q_1,p_1;X]-[p_1,p_1;X]\right\|\\ &\leq \frac{2\beta}{2-3\beta\omega(\eta,\eta)}\omega\left(d\left(q_1,p_1\right),0\right)\\ &= \frac{\beta\omega(\eta,\eta)}{2-3\beta\omega(\eta,\eta)}.\end{aligned}$$

Finally

$$\begin{aligned}\left\|\Phi_1^{-1}\Gamma_1\right\| &\leq \frac{1}{1-\frac{\beta\omega(\eta,\eta)}{2-3\beta\omega(\eta,\eta)}}\\ &= \frac{2-3\beta\omega(\eta,\eta)}{2-4\beta\omega(\eta,\eta)}.\end{aligned} \tag{3.37}$$

Now, let us estimate $d(p_2, p_1)$. Using (3.35) and (3.37), we obtain

$$\begin{aligned} d(p_2, p_1) &= \|v_1\| \\ &= \left\|\Phi_1^{-1} X(p_1)\right\| \\ &= \left\|\Phi_1^{-1}\Gamma_1\Gamma_1^{-1} P_{\sigma_0,0,1}([p_1, p_0; X] - \Phi_0) v_0\right\| \\ &\leq \left\|\Phi_1^{-1}\Gamma_1\right\| \left\|\Gamma_1^{-1} P_{\sigma_0,0,1}([p_1, p_0; X] - \Phi_0)\right\| \|v_0\| \\ &\leq \frac{2-3\beta\omega(\eta,\eta)}{2-4\beta\omega(\eta,\eta)} \frac{\beta\omega(\eta,\eta)}{2-2\beta\omega(\eta,\eta)} \frac{4-\beta\omega(\eta,\eta)}{2-\beta\omega(\eta,\eta)} \eta, \end{aligned}$$

but

$$\frac{2-3\beta\omega(\eta,\eta)}{2-4\beta\omega(\eta,\eta)} \frac{\beta\omega(\eta,\eta)}{2-2\beta\omega(\eta,\eta)} \frac{4-\beta\omega(\eta,\eta)}{2-\beta\omega(\eta,\eta)} \leq \frac{\beta\omega(\eta,\eta)}{1-2\beta\omega(\eta,\eta)}. \tag{3.38}$$

Because (3.38) is equivalent to

$$-2\beta\omega(\eta,\eta) - (\beta\omega(\eta,\eta))^2 \leq 0,$$

then

$$\begin{aligned} d(p_2, p_1) &\leq \frac{\beta\omega(\eta,\eta)}{1-2\beta\omega(\eta,\eta)} \eta \\ &\leq N\eta. \end{aligned}$$

This way

$$\begin{aligned} d(p_2, p_0) &\leq d(p_2, p_1) + d(p_1, p_0) \\ &\leq N\eta + \eta \\ &= (N+1)\eta, \end{aligned}$$

and as R is root of

$$t\left(1 - \frac{m}{1-2\beta\omega(t,t)}\right) - \eta = 0,$$

then

$$R(1-N) - \eta = 0.$$

Thus

$$
\begin{aligned}
(N+1)\,\eta &= \frac{\eta}{R}\,(2R-\eta) \\
&\leq R,
\end{aligned}
$$

therefore

$$d\,(p_2, p_0) \leq R,$$

so that

$$p_2 \in B\,(p_0, R)\,.$$

using similar arguments, in an inductive way, we can prove

$$d\,(p_{n+1}, p_n) \leq N^n d\,(p_1, p_0) \leq N^n \eta \tag{3.39}$$

$$\text{and} \tag{3.40}$$

$$d\,(q_n, p_n) \leq N^n d\,(q_0, p_0) \leq N^n \eta.$$

Then, from the triangular inequality

$$d\,(p_{n+1}, p_0) \leq \sum_{k+0}^{n} N^n \eta,$$

and given that

$$
\begin{aligned}
\sum_{k+0}^{n} N^n \eta &= \left(\frac{1-N^{n+1}}{1-N}\right)\eta \\
&= \left(1-\left(1-\frac{\eta}{R}\right)^{n+1}\right)R \\
&\leq R,
\end{aligned}
$$

then

$$d\,(p_{n+1}, p_0) \leq R,$$

so that,

$$p_{n+1} \in B\,(p_0, R)$$

and (3.39) shows that the sequence $\{p_n\}_{n\in\mathbb{N}}$ is of Cauchy and as M is complete, then it converges to some $p_* \in B[p_0, R]$.

We shall prove that p_* is a singularity of X.

Since

$$\begin{aligned}\|X(p_n)\| &= \left\|\Gamma_n \Gamma_n^{-1} X(p_n)\right\| \\ &\leq \|\Gamma_n\| \, d(p_n, q_n),\end{aligned}$$

by (3.28),

$$\|\Gamma_n\| \leq \|\Gamma_0\| + \omega(R, R)$$

and passing to limit when $n \longrightarrow \infty$ we obtain

$$X(p_*) = 0.$$

Moreover, if q_* is another singularity of X in $B[p_0, R]$, σ is the minimizing geodesic joining the points p_0 and p_* such that $\sigma(0) = p_0$ and $\sigma(1) = p_*$, then

$$\begin{aligned}\left\|\Gamma_0^{-1}\right\| \|\Gamma_0 - P_{\sigma,1,0}[q_*, p_*; X]\| &\leq \omega(d(p_0, p_*), d(p_0, q_*)) \\ &\leq \omega(R, R) \\ &\leq 1.\end{aligned}$$

This shows that the operator $[q_*, p_*; X]$ is invertible. If γ is the minimizing geodesic joining the points p_* and q_* such that $\gamma(0) = p_*$ and $\gamma(1) = q_*$, and because

$$[\gamma(1), \gamma(0); X]\, \gamma'(0) = P_{\gamma,1,0} X(\gamma(1)) - X(\gamma(0)),$$

then

$$[q_*, p_*; X]\, \gamma'(0) = 0.$$

Thus

$$\gamma'(0) = 0,$$

and we conclude

$$d(p_*, q_*) = \left\|\gamma'(0)\right\| = 0,$$

hence

$$p_* = q_*.$$

■

Conclusions

There exists a great interest, among other things, thanks to advances in computational science, of studying the called higher order numerical methods. Recently several mathematicians have been devoted to showing the validity of these methods in spaces more general than Euclidean, such as Banach spaces. In this order of generalization it was easy to expect the interest of showing the validity of those methods on sets even more general than the Banach spaces such as Riemannian manifolds.

The main contribution of this work is to prove the context of Riemannian manifolds, convergence and uniqueness theorems of higher order methods, some of which generalize the classical methods of third order; such as for example, the Chebyshev-Halley.

Developing these methods on manifolds creates new difficulties, which did not exist in Banach spaces. Some of these difficulties are technical; for example, to define a method in the new context. Specifically, in Banach spaces there is not distinction made between the space E and its tangent space T_pE at a point p, since they are isomorphic. Thus it is "legitimate" to sum points and vectors. In manifolds this does not happen, so we must be very careful. For this, the exponential function is used, that is to say, to a point and a vector of a tangent plane is assigned a new point of the manifold. Another difficulty is the definition of the derivative; it might be thought that the ordinary derivative is sufficient to define the methods in the case where the manifold is embed in a Euclidean space, but this not true since the ordinary derivative is not necessary tangent to the manifold at the point considered. Thus it is necessary consider the covariant derivative, which in this particular case is the projection of the ordinary derivative on the tangent plane. For some of the third order methods such as those studied in the third chapter it were necessary to define the second covariant derivative.

Another kind of difficulties found were general, since they depend on the manifold; particularly, one of the problems encountered, is that unlike Banach spaces, where the geodesics are straight lines, the open balls on manifolds are not necessarily geodesically convex, namely, given two points into an open ball, the geodesic joining those points, not necessarily is entirely contained in the ball. This difference with the Banach spaces creates a great difficulty to put in the new context the uniqueness theorems of the classical methods, as could be seen in the proof of the uniqueness of the simplified method of Kantorovich. However, classical techniques for testing the uniqueness can be brought into the context of manifolds and show the uniqueness but with more restrictive assumptions on the constants, which give rise to minor radius of convergence which had for Banach spaces (see methods of chapter three).

A difference of the methods on manifolds with Euclidean space methods is the major computational cost for one iteration. In Banach spaces, the cost usually depends on the number of entries in the array that represents the derivative, although this is also true in manifolds, we must add a new cost, which is derived from the new calculations required to calculate the geodesic that join two consecutive iterations; remember that to find these geodesics we must first calculate the Christoffel symbols (which in some cases it is not an easy task) and with these symbols solve the system of differential equations that give rise to the geodesic.

While the methods of the third chapter are named "Iterative methods of third order" this name is just inherited from the corresponding methods in Banach spaces. It is left as an open problem to prove that they are indeed third order of convergence. Another open problem to analyze the possibility of increases the radius of convergence of the methods of 3.1, 3.2 and 3.3, since as we said before, to apply the techniques used in Banach spaces we had to make additional hypothesis about the constants, which reduced the expected radius of convergence.

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