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Qualitative properties on measure differential and measure functional differential equations. Fixed points on multivalued maps with applications

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QUALITATIVE PROPERTIES ON MEASURE DIFFERENTIAL AND MEASURE FUNCTIONAL DIFFERENTIAL EQUATIONS. FIXED POINTS ON MULTIVALUED MAPS WITH APPLICATIONS

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Abstract

In this work, we investigate the asymptotic behavior for measure differential equations (MDEs, for short) and dynamic equations on time scales via generalized ordinary differential equations (generalized ODEs, for short). We establish new results that guarantee the existence of unbounded solutions for generalized ODEs, and using the known correspondence between generalized ODEs and MDEs, also between MDEs and dynamic equations on time scales, we obtain similar results for these equations. Furthermore, we introduce measure functional differential equations (MFDEs) with infinite time-dependent delay, and we study the correspondence between the solutions of these equations and the solutions of the generalized ODEs in Banach spaces. We obtain an existence–uniqueness result of solutions and continuous dependence on parameters for MFDEs with infinite time-dependent delay. We establish a result of existence of solutions for a MFDE with infinite time-dependent delay in the presence of a perturbation independent of the state. We develop the theory in the context of phase spaces defined axiomatically.

On the other hand, we investigate existence of fixed points for multivalued maps defined on Banach spaces. Using the Banach spaces scale concept, we establish the existence of fixed points of a multivalued map in a vector subspace when the map is only locally Lipschitz continuous. We apply our results to the existence of mild solutions and asymptotically almost periodic solutions of an abstract Cauchy problem governed by a first order differential inclusion. Our results are obtained by using fixed point theory for the measure of non-compactness.

Keywords: Measure differential equations; measure functional differential equations; generalized ordinary differential equations; differential inclusions; multivalued maps.

Resumen

En este trabajo investigamos el comportamiento asintótico para ecuaciones diferenciales en medida (EDMs, abreviado) y ecuaciones dinámicas sobre escalas temporales a través de ecuaciones diferenciales ordinarias generalizadas (EDOs generalizadas, abreviado). Establecemos nuevos resultados que garantizan la existencia de soluciones no acotadas para EDOs generalizadas, y usando la conocida correspondencia entre EDOs generalizadas y EDMs, como también entre EDMs y ecuaciones dinámicas sobre escalas temporales, obtenemos similares resultados para estas ecuaciones. Además, introducimos ecuaciones diferenciales funcionales en medida (EDFMs) con retardo infinito dependiente del tiempo, y estudiamos la correspondencia entre las soluciones de estas ecuaciones y las soluciones de EDOs generalizadas en espacios de Banach. Obtenemos un resultado de existencia y unicidad de soluciones y dependencia continua de parametros para EDFMs con retardo infinito dependiente del tiempo. Establecemos un resultado de existencia de soluciones para una EDFM con retardo infinito dependiente del tiempo en presencia de una perturbación independiente del estado. La teoría se desarrolla en el contexto de espacios de fase definidos axiomáticamente.

Por otra parte, investigamos la existencia de puntos fijos para aplicaciones multivaluadas definidas sobre espacios de Banach. Utilizando el concepto de escalas de espacios de Banach, establecemos la existencia de un punto fijo de una aplicación multivaluada en un subespacio vectorial donde la aplicación es solamente localmente Lipschitz continua. Aplicamos nuestros resultados para la existencia de soluciones débiles y soluciones casi asintóticamente periódicas de un problema de Cauchy abstracto gobernado por una inclusión diferencial de primer orden. Nuestros resultados son obtenidos utilizando la teoría de punto fijo para medidas de no-compacidad.

Palabras clave: Ecuaciones diferenciales en medida; ecuaciones diferenciales funcionales en medida; ecuaciones diferenciales ordinarias generalizadas; inclusiones diferenciales; aplicaciones multivaluadas.

Resumo

Neste trabalho nós investigamos o comportamento assintótico das equações diferenciais em medida (EDF para abreviar) e das equações dinâmicas em escalas temporais por meio das equações diferenciais ordinárias generalizadas (EDOG para abreviar). Estabelecemos novos resultados que garantem a existência de soluções ilimitadas para EDOGs e, usando as conhecidas correspondências entre EDOG e EDM e entre EDM e as equações dinâmicas em escalas temporais, obtemos resultados similares para estas equações. Além disso, introduzimos uma classe de equações chamada equações diferenciais funcionais em medida (EDFM) com retardo infinito dependendo do tempo e estudamos a correspondência entre as soluções dessas equações e as soluções das EDOGs em espaços de Banach. Obtemos um resultado de existência e unicidade e dependência contínua dos parâmetros para EDFMs com retardo infinito dependendo do tempo. Estabelecemos um resultado de existência de soluções para EDFMs com retardo infinito dependendo do tempo na presença de uma perturbação independente do estado. Desenvolvemos a teoria no contexto dos espaços de fase definidos axiomaticamente.

Por outro lado, investigamos a existência de pontos fixos para multifunções definidas em espaços de Banach. Usando o conceito de escalas em espaços de Banach, estabelecemos a existência de um ponto fixo da multifunção em um subespaço vetorial onde a aplicação é apenas localmente Lipschitz contínua. Aplicamos nossos resultados para estabelecer a existência de soluções fracas e de soluções assintoticamente quase-periódicas de um problema de Cauchy abstrato regido por uma inclusão diferencial de primeira ordem. Nossos resultados foram obtidos usando a teoria de ponto fixo para a medida de não-compacidade.

Palavras-chave: Equações diferenciais em medida; equações diferenciais funcionais em medida; equações diferenciais ordinárias generalizadas; inclusões diferenciais; multifunções.

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Introduction

It is a well known fact that if a function $f : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ satisfies the Carathéodory conditions, then the ordinary differential equation

$$x' = f(x, s), \quad x(t_0) = x_0, \tag{1}$$

has a solution in a neighborhood J of the initial condition $t_0 \in [a, b]$, i.e., there exists an absolutely continuous function $x : J \to \mathbb{R}^n$ that satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s, \ t \in J, \tag{2}$$

where the integral in (2) is considered in the sense of Lebesgue. In this case, we said that $x : J \to \mathbb{R}^n$ is a solution in the Carathéodory sense.

Natural questions arise from the integral equation (2). What is a solution when the right–hand side of (2) involves a general concept of integration, for example the Henstock–Kurzweil integral? What kind of conditions on f will we need to obtain an existence result?

Following the successive aproximation to Carathéodory's ideas, Ralph Henstock developed an answer to this problem when the Henstock–Kurzweil integral is considered. He established a result of existence by requiring only "Carathéodory" type conditions on $f : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$, called Henstock conditions. In this case, it is said that $x : J \to \mathbb{R}^n$ is a solution of (2) in the Henstock sense. Note that, since every Lebesgue integrable function is Henstock–Kurzweil integrable, if $x : J \to \mathbb{R}^n$ is a solution in the Carathéodory sense, then $x : J \to \mathbb{R}^n$ is a solution in the Henstock sense as well. Furthermore, a remarkable fact is concerned with the conditions imposed by Henstock. A function $f : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ satisfies the Henstock conditions if and only if $f(x, t) = p(t) + h(x, t)$ for every $(x, t) \in \mathbb{R}^n \times [a, b]$, where the function p is Henstock–Kurzweil integrable and the function h satisfies the Carathéodory conditions. Therefore, the Henstock existence theorem covers the case of a Carathéodory function perturbed by a Henstock–Kurzweil integrable function. This relation shows in which sense the Henstock existence theorem is more general than the Carathéodory.

In 1957, the Czech mathematician Jaroslav Kurzweil [56], motivated by the study of continuous dependence of solutions to ordinary differential equations, introduced in the literature a class of integral equations that he called *generalized ordinary differential equations* (generalized ODEs, for short).

Let X be a Banach space, $\mathcal{O} \subset X$ a nonempty open subset, $t_0 \in \mathbb{R}$, $\Omega = \mathcal{O} \times [t_0, \infty)$, and let $F : \Omega \to X$ be a given X-valued function. Then a function $x : [a, b] \to X$, with $[a, b] \subset [t_0, \infty)$, is called a *solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x, t) \tag{3}$$

on the interval $[a, b]$, if $(x(t), t) \in \Omega$ for every $t \in [a, b]$, and

$$x(d) - x(c) = \int_c^d DF(x(\tau), t), \tag{4}$$

whenever $[c, d] \subseteq [a, b]$. The integral on the right-hand side of (4) is understood in the sense of Kurzweil, and depending of the chosen function F, the Kurzweil integral emcompasses several types of integrals, such as Henstock–Kurzweil integral, see Definition 1.2. It is important to make clear that the equation (3) is symbolical only, this does not mean that the solutions of generalized ODEs should be differentiable. The concept of solution is defined via the integral equation (4).

The word "generalized" in the Kurzweil definition comes from the following simple result holds. Assume that the function $f : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ satisfies the Carathéodory conditions, and define the function

$$F(z, t) = \int_{t_0}^t f(z, s)\mathrm{d}s. \tag{5}$$

Then, $x(t)$ is a solution in the Carathéodory sense if and only if $x(t)$ is a solution of the generalized ODE (3). Therefore, in the definition of a solution of a generalized ODE, with an appropriate function F, we can return to a solution of an ODE. The nature of the Kurzweil integral allows to consider general conditions in which the function F can be integrated, see Definition 1.10.

Although the theory of generalized ODEs was motivated in order to study qualitative properties of ordinary differential equations, over the time, this concept has shown to encompass a wide range of equations, such as impulsive systems, measure differential equations, retarded functional differential equations (finite and infinite delay), dynamic equations on time scales, among others, see e.g. [26–28, 73–75]. One of the principal arguments in which generalized ODEs can encompass several other equations is due to variations of the relation (5), specifically a Henstock–Kurzweil–Stieltjes form is considered for that purpose, and how we will see in the Chapters 2–3, this relation enables to impose bounded and Lipschitz conditions over the integral instead of directly on the function f, see for instance conditions (A1)-(A4) in Section 2.2 or conditions (A)-(C) in Section 3.2.

In the first part of this work, we are interested in developing qualitative properties for some of these classes of equations, specifically for measure differential equations, dynamic equations on time scales and measure functional differential equations with time-dependent delay. The main tool used for our purpose is the theory of generalized ODEs.

Here, we consider the measure differential equation in the integral form

$$x(t) = x(t_0) + \int_{t_0}^t f(x, s)\mathrm{d}g(s), \tag{6}$$

where the integral on the right-hand side is understood in the sense of Henstock-Kurzweil Stieltjes.

Our interest is focused on the asymptotic behaviour of solutions to the measure differential equations of type (6), where the functions f and g satisfy conditions (A1)-(A4) from Section 2.2. Recently in [22], taking into account these considerations for the integral form (6), Federson et al. generalized the known results about the existence and uniqueness of maximal solutions for measure differential equations and dynamic equations on time scales. Actually, this fact was achieved using the theory of generalized ODEs and the relation that exists among generalized ODEs, measure differential equations and dynamic equations on time scales, see [73] and [74]. The relevance of theorems given in [22] lies in the fact that they allow us to investigate asymptotic behaviour of solutions for generalized ODEs and measure differential equations, and as a consequence, for dynamic equations on time scales. Also, Federson et al. proposed new stability results for measure differential equations and dynamic equations on time scales, under more general conditions than the ones found in the literature. Specifically, they obtained uniform stability and uniform asymptotic stability results via Lyapunov functionals without requiring Lipschitz conditions, see [23].

Inspired by these recent papers [22] and [23], we investigate the asymptotic behaviour of solutions to measure differential equations and dynamic equations on time scales, via generalized ODEs and Lyapunov functions. We do not require Lipschitz conditions on the Lyapunov functions, and we work with solutions which are regulated functions. Specifically, we show that the maximal solutions defined on an interval $[t_0, +\infty)$ for this type of equations are unbounded. As far as we know, our results for generalized ODEs were not proved in the literature yet, neither did their analogues to measure differential equations. Also, we apply our results to dynamic equations on time scales.

On the other hand, measure functional differential equations with finite delay

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \tag{7}$$

were introduced by M. Federson, J. Mesquita and A. Slavík in [26], where y and f are functions with values in $\mathbb{R}^n$, the integral on the right-hand side of (7) is the Henstock-Kurzweil integral of the function f with respect to a nondecreasing function g, and as usual in the theory of functional differential equations, y_s represents the “history” or the segment of y at s, i.e., for a fixed $r > 0$, the function $y_s : [-r, 0] \to \mathbb{R}^n$ is defined by $y_s(\theta) = y(s + \theta)$, for $-r \leq \theta \leq 0$. Using the theory of generalized ODEs, they established results about existence–uniqueness of solutions and continuous dependence on parameters for these equations. They also showed that functional dynamic equations on time scales represent a special case of measure functional differential equations.

Later, in [27], the authors studied the relation between measure functional differential equations, impulsive measure functional differential equations, and impulsive functional dynamic equations on time scales. They obtained results on existence–uniqueness of solutions, continuous dependence on parameters, and periodic averaging. Subsequently, M. Federson and J. Mesquita [25] developed a nonperiodic averaging principle for measure functional differential equations, and using the correspondence among measure functional differential equations, functional dynamic equations on time scales, and impulsive measure functional differential equations, they established this type of result for the last mentioned equations.

The case when the equation (7) is considered with infinite delay was later studied by A. Slavík in [75], that means, the function $y_s : (-\infty, 0] \to \mathbb{R}^n$ is defined by $y_s(\theta) = y(s+\theta)$, for $\theta \leq 0$. In this paper, he studied the equation on an appropriate phase space (the space containing the functions y_s), described axiomatically similar to the axiomatic definition of phase space that is used in the classical theory of retarded functional differential equations with infinite delay (the reader can see [40, 50]). Using this framework, he obtained results of existence and uniqueness of solutions. Later, in [66], G. Monteiro and A. Slavík investigated a linear measure functional differential equation which is a special case of (7) with infinite delay. They established existence–uniqueness and continuous dependence of solutions, improving the existing results, even for finite delay. Also, they applied their results to functional differential equations with impulses.

On the other hand, recently many authors have begun to study functional differential equations with time-dependent delay (see [14,29,35,39,59]). These equations have shown to be useful tools for applications, since they can describe more precisely certain environment phenomena. For instance, it is a known fact that the formulation of blowfly equation considering the time of maturation of a population as a delay describes more precisely the size population in the future and this delay is not constant, it changes over time. Therefore, the population dynamic could be better described using functional differential equations with time-dependent delay. Interesting recent papers concerning Nicholson's blowflies systems with time-dependent delays have been studied, see [13, 52] and the references therein. The same happens when we are dealing with some types of disease models that consider incubation period of the virus until the symptoms appears in the patients body. This period is not constant, it changes with the time and can be different depending on the state of the patient. Therefore, equations with time-dependent delays and state-dependent delays are better choice to describe such situation (see [41, 84]). Another example is when we search for data on a computer. This process is not instantaneous, but it takes some time, which we can describe by a delay that clearly is not fixed. It will change accordingly to several variables, but mostly depending how old the data is. Therefore, this type of situation and others which involve memory process are better described using functional differential equations with time-dependent delays.

Motivated by these facts, we are interested in a type of the equation (7) with time-dependent delay. Specifically, we focus our attention on the equation given by

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_{r(s)}, s)\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \tag{8}$$

where r is an appropriate function, y and f are functions taking values in $\mathbb{R}^n$, the function $y_{r(s)} : (-\infty, 0] \to \mathbb{R}^n$ is defined by $y_s(\theta) = y(r(s) + \theta)$, for $\theta \leq 0$, and the integral on the right-hand side is in the sense of Henstock-Kurzweil with respect to a nondecreasing function g.

The name time-dependent delay comes from the fact that the simplest example consists of a

function f defined by
$$f(\varphi,s)=\widetilde{f}(\varphi(-\tau(s))),$$
where $\widetilde{f}:\mathbb{R}^n\to\mathbb{R}^n$ and $\tau:[t_0,t_0+\sigma]\to[0,\infty)$ are functions that satisfy certain technical conditions which allow us to develop a qualitative theory. Hence, with $r(s)=s$, we obtain
$$f(y_{r(s)},s)=\widetilde{f}(y_{r(s)}(-\tau(s)))=\widetilde{f}(y(r(s)-\tau(s)))=\widetilde{f}(y(s-\tau(s))),$$
which represents a difference equation with delay $\tau(s)$.

Our goal is to establish a correspondence between generalized ODEs and measure functional differential equations (MFDEs, for short) with time-dependent delay given by (8). Using this relation, we obtain existence and uniqueness results, and other qualitative properties for MFDEs with time-dependent delay. To establish general results, we will study the equation in the frame on a phase space defined axiomatically. Our approach follow the axiomatic approach of phase space used by many authors in the study of retarded functional differential equations with infinite delay, see [50]. This approach allows us to consider more general phase spaces than the ones considered in [66] or [75]. See the Examples 3.3 to 3.7 in the Section 3.1.

It is worth to note that we can rewrite $f(y_{r(s)},s)$ in simple nonautonomous form by defining a function $h(\cdot,s)$ such that
$$f(y_{r(s)},s)=h(y_s,s),\ s\geq t_0.$$
In fact, this is achieved by defining
$$h(\varphi,s)=f(\varphi_{r(s)-s},s). \tag{9}$$
However, this transformation presents some inconvenient aspects for the development of a theory. At first, it is necessary to ensure that $(\varphi_{r(s)-s},s)$ is included in the domain of f, which is a very strong request, since the phase space $\mathcal{B}$ that arise in the study of retarded functional differential equations with infinite delay usually contain functions φ such that $\varphi_{-t}\notin\mathcal{B}$ for $t>0$. A second aspect refers to the fact that the existing qualitative theory for the equation (7) requires that the function f verify strong conditions of continuity that the function h defined in (9) does not satisfy due to its dependence on $\varphi_{r(s)-s}$ and the fact that this function may not satisfy those properties of continuity. For these reasons, it is preferable to develop the theory based in expressions (8) without reducing to expressions of type (7) with $h(y_s,s)$ instead of $f(y_s,s)$.

On the other hand, in the second part of this work we are concerned with the existence of fixed points of multivalued maps defined on Banach spaces. Using the Banach spaces scale concept, we establish the existence of a fixed point of a multivalued map in a vector subspace where the map is only locally Lipschitz continuous. We apply our results to establish the existence of asymptotically almost periodic mild solutions for a class of abstract Cauchy problem governed by a first order differential inclusion.

The theory of differential inclusions was initiated firstly in 1934-1936 by A. Marchaud and S. K. Zaremba, see [60, 86]. Later, at the beginning of sixties, the elementary theory was developed by the Cracow mathematical school, motivated by Tadeusz Wazewski which proved that each control problem described by an ordinary differential equations of first order can be represented as a differential inclusion. Over the years, differential inclusions was intensively developed and used to describe many phenomena arising from different fields such as physics, chemistry, population dynamics, among others. For this reason, in the last years several researchers have studied various aspects of the theory.

Without the intention to do an exhaustive historical review of differential inclusion, we only mention here those most recent and directly related papers to the topic in which our work is inserted, see e.g. [1, 2, 9, 17, 34, 36, 53, 54, 68, 72, 79] and references therein for the motivation of the theory.

The aim of this work is to establish the existence of mild solutions for the abstract first order differential inclusion

$$\begin{aligned} x'(t) - Ax(t) &\in f(t, x(t)), \quad t \geq 0, && (10)\\ x(0) &= x_0 \in X, && (11) \end{aligned}$$

where X is a Banach space provided with a norm $\|\cdot\|$, $x(t) \in X$, $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $(T(t))_{t\geq 0}$ on X, and f is a set valued map defined on $[0,\infty) \times X$ whose properties will be specified later.

Our goal is to establish a general result of fixed point in scales of Banach spaces, and combining this result with fixed point theory for the measure of non-compactness, we obtain existence of mild solutions and asymptotically almost periodic solutions to the problem (10)-(11).

It is important to mention that several authors have studied strongly nonlinear problems, that is, problems of type (10)-(11) in which A is a m-dissipative operator. The reader can see [80–83] and the references in these works. We wish to emphasize that these works and ours present important differences:

(i) Our aim is to establish the existence of asymptotically almost periodic solutions. For this reason, we need to guarantee the existence of global solutions. We will use the properties of scale of Banach spaces developed in Section 4.2 to obtain existence of solutions defined in $[0,\infty)$.

(ii) Since in our case A is a linear operator, this allows us to decompose f into the form $f = f_1 + f_2$ and to obtain the existence of solutions under different conditions in f_1 and f_2. Specifically, we will show that it is sufficient for f_1 to verify a local Lipschitz condition, while f_2 must verify a compactness property, established in terms of the measure of non-compactness, of global type, that is, in $[0,\infty)$.

(iii) Our results do not require the semigroup $(T(t))_{t\geq 0}$ to be compact. We only need the semigroup $(T(t))_{t\geq 0}$ to be continuous in the norm of operators in $(0,\infty)$. The class of semigroups is very wide, including the differentiable, analytic and compact semigroups, etc. [20], which are the semigroups that frequently arise in applications.

This manuscript is divided in two parts independent from each other. The Part I contains three chapters.

The Chapter 1 is divided in two sections. In the Section 1.1, we present basic notions and results related to Kurzweil integration. In the Section 1.2, we present the notion of generalized ODEs (extensively described in [56, 73]), and results concerned to existence and uniqueness of maximal solutions for generalized ODEs, recently exposed in [22].

The Chapter 2 is divided in three sections. In the Section 2.1, we present our asymptotic behaviour results for generalized ODEs. In Section 2.2, we present asymptotic behaviour results for measure differential equations. In the Section 2.3, using the results of the previous sections, we obtain results about asymptotic behaviour for dynamic equations on time scales. All results exposed in this chapter can be found in [32].

The Chapter 3 is divided in five sections. In the first Section 3.1, we discuss the employment of a convenient phase space for measure functional differential equations with infinite time-dependent delay. In the Section 3.2, we describe the correspondence between the solutions of measure functional differential equations with infinite time-dependent delay and generalized ODEs. In the Section 3.3, we present a theorem concerning the existence and uniqueness of solution of MFDEs with infinite time-dependent delay. In the Section 3.4, we prove continuous dependence results on parameters for MFDEs with time-dependent delay. Finally, in the Section 3.5, we investigate perturbed systems, presenting a correspondence between MFDEs with time-dependent delay and generalized ODEs. Furthermore, an existence–uniqueness theorem. All results exposed in this chapter can be found in [33].

The Part II contains the Chapter 4.

The Chapter 4 is divided in four sections. In the Section 4.1, we develop some properties about the measure of noncompactness, and multivalued analysis which are needed to establish our results. In the Section 4.2, we discuss the existence of fixed points. In the Section 4.3, we apply our results to establish the existence of solutions. Finally, in the Section 4.4, we establish the existence of asymptotically almost periodic solutions to problem (10)-(11). All results exposed in this chapter can be found in [31].

Part I

Measure differential and measure functional differential equations

Chapter 1

Kurzweil integral and generalized ODEs

In this chapter, in order to develop the following Chapters 2 and 3, we define the Kurzweil integral and we include some results from the theory of generalized ordinary differential equations. For more details, the reader can see [56, 58, 73].

1.1 Kurzweil integral

Throughout this chapter, let us assume that X is a Banach space with a norm $\|\cdot\|$.

Consider a function $\delta : [a, b] \to \mathbb{R}^+$, which is called a *gauge* on $[a, b]$. A tagged partition of the interval $[a, b]$ with subdivision points $a = s_0 \leq s_1 \leq \cdots \leq s_k = b$, and *tags* $\tau_i \in [s_{i-1}, s_i]$, $i = 1, ..., k$, is called *δ-fine* if

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \quad i = 1, \ldots, k.$$

The following lemma is fundamental to introduce the notion of Kurzweil integral, and ensures the existence of δ-fine tagged partitions.

Lemma 1.1. ([73, Lemma 1.4], Cousin's lemma) *If $\delta(\cdot)$ is a gauge on $[a, b]$, then every closed sub interval of $[a, b]$ has a δ-fine tagged partition.*

The next definition is due to J. Kurzweil and it was introduced in his work on differential equations, see [56]. Below we present the Banach space valued version.

Definition 1.2. *A function $U : [a, b] \times [a, b] \to X$ is called Kurzweil integrable on $[a, b]$, if there is an element $I \in X$ having the following property: for every $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that*

$$\left\| \sum_{i=1}^{k} [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon,$$

for all δ-fine tagged partition of $[a, b]$. In this case, I is called the Kurzweil integral of U over $[a, b]$ and it will be denoted by $\int_a^b DU(\tau, t)$.

As it should be expected, the Kurzweil integral satisfies the usual properties of linearity, additivity with respect to adjacent intervals, integrability on subintervals, among others.

Notice that the measure $U(\tau,t) - U(\tau,s)$ is not necessarily linear in τ, hence the Kurzweil integral is a non linear integral. If we consider $U(\tau,s) = x(\tau)s$, then we obtain the Kurzweil (linear) integral definition of a function $x : [a,b] \to X$. Independently, R. Henstock in 1961 [47] introduced an equivalent version of the Kurzweil (linear) integral for real valued functions. In the literature, it is known by Henstock-Kurzweil integral due to its equivalence. In particular, for integrable real valued functions $x : [a,b] \to \mathbb{R}$, we have the following proper inclusions

$$\mathcal{R}([a,b];\mathbb{R}) \subset \mathcal{L}_1([a,b];\mathbb{R}) \subset H([a,b];\mathbb{R}) = \mathcal{K}([a,b];\mathbb{R}),$$

where $\mathcal{R}([a,b];\mathbb{R})$ denotes the space of Riemann integrable functions, $\mathcal{L}_1([a,b];\mathbb{R})$ denotes the space of Lebesgue integrable functions, $H([a,b];\mathbb{R})$ denotes the space of Henstock integrable functions and $\mathcal{K}([a,b];\mathbb{R})$ denotes the space of Kurzweil integrable functions.

A remarkable fact concerned with the Henstock–Kurzweil integral is the Fundamental Theorem of Calculus.

Theorem 1.3. ([78, Theorem 10]) *Suppose $x : [a,b] \to \mathbb{R}$ is differentiable at every point of $[a,b]$. Then x' is Henstock–Kurzweil integrable over $[a,b]$ and*

$$\int_a^b x' = x(b) - x(a). \tag{1.1}$$

Note that Theorem 1.3 may be false for both the Riemann and the Lebesgue integrals, depending on the function, since they require the assumption that the derivative x' be integrable in their respective senses in order to obtain (1.1). We recall a classical example of the theory.

Example 1.4. Consider the differentiable function

$$x(t) = \begin{cases} t^2\cos(\pi/t^2), & 0 < t \leq 1, \\ 0, & t = 0, \end{cases}$$

with

$$x'(t) = \begin{cases} 2t\cos(\pi/t^2) + (2\pi/t)\sin(\pi/t^2), & 0 < t \leq 1, \\ 0, & t = 0. \end{cases}$$

Then, x' is not Riemann integrable (is not bounded on $[0,1]$), neither is Lebesgue integrable. However, by Theorem 1.3, x' is Henstock–Kurzweil integrable and $\displaystyle\int_0^1 x' = x(1) - x(0) = -1$.

In contrast to either the Riemann or Lebesgue integrals, the Henstock–Kurzweil integral is non-absolute, that means, there are functions which are integrable but whose absolute values are not integrable. Example 1.4 shows that the function x' is Henstock–Kurzweil integrable, however $|x'|$ is not. The proof of the previous assertion can be found in [77, Example 12]. For an introductory reading concerning Henstock–Kurzweil integral, the reader can see [48, 77, 78].

In the case of an infinite dimensional Banach space X, it is possible to find a X-valued function $x : [a,b] \to X$ which is Kurzweil (linear) integrable and is not Henstock integrable (the X-valued version), see [21, Examples 2.1–3.1]. In particular, we have the following inclusions

$$\mathcal{L}_1([a,b];X) \subset H([a,b];X) \subseteq \mathcal{K}([a,b];X) \ \text{ and } \ \mathcal{R}([a,b];X) \subset \mathcal{K}([a,b];X),$$

where $\mathcal{R}([a,b];X)$ denotes the space of Riemann integrable functions from $[a,b]$ to X, $\mathcal{L}_1([a,b];X)$ denotes the space of Bochner-Lebesgue integrable functions from $[a,b]$ to X, $H([a,b];X)$ denotes the space of Henstock integrable functions from $[a,b]$ to X and $\mathcal{K}([a,b];X)$ denotes the space of Kurzweil integrable functions from $[a,b]$ to X.

On the other hand, if we take $U:[a,b]\times[a,b]\to X$ from the Definition 1.2 as $U(\tau,t) := f(\tau)g(t)$, then we obtain the Kurzweil–Stieltjes integral definition of a X-valued function $f:[a,b]\to X$ with respect to a function $g:[a,b]\to\mathbb{R}$. Note that if $g(t)\equiv t$, then we obtain the Henstock–Kurzweil (linear) integral definition. In the scarce literature about the Kurzweil–Stieltjes integral, this concept appears under different names, such as Henstock–Stieltjes, gauge integral or even Henstock-Kurzweil-Stieltjes integral. In this work, we will use the name *Henstock–Kurzweil–Stieltjes integral* to refer to this definition. The principal reason is that in most of the papers related to our work, its authors refer to this integral concept in this way. However, it is important to remark that this Stieltjes version was firstly used by J. Kurzweil in [57]. The most recent book in this subjet is due to G. Monteiro, A. Slavík, and M. Tvrdý [67], who wrote a monograph about the Kurzweil–Stieltjes integral and its applications, including an exposition of the properties of the Riemann–Stieltjes integral, Moore–Pollard–Stieltjes integral, among others interesting results.

In the following Chapters 2 and 3, we are interested in this type of Stieltjes integral, and from now on, we will denote by $\int_a^b f(s)\mathrm{d}g(s)$, or simply $\int_a^b f\mathrm{d}g$, to refer to the integral of a function f which is Henstock–Kurzweil–Stieltjes integrable with respect to a function g.

In what follows, we recall the notion of X-valued regulated functions.

Definition 1.5. *A function $f:[a,b]\to X$ is called regulated if the limits below exist*

$$\lim_{s\to t^-} f(s) = f(t^-) \ \ \textit{for} \ \ t\in(a,b] \ \ \textit{and} \ \ \lim_{s\to t^+} f(s) = f(t^+) \ \ \textit{for} \ \ t\in[a,b).$$

The space of all regulated functions $f:[a,b]\to X$ will be denoted by $G([a,b],X)$, and it is a Banach space under the usual supremum norm $\|f\|_\infty = \sup\limits_{a\le t\le b}|f(t)|$. Below we present an important characterization of the regulated functions.

Theorem 1.6. ([8, Theorem 4.4]) *A function $f\in G([a,b],X)$ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of step functions on the interval $[a,b]$ which is uniformly convergent on the interval $[a,b]$ to the function f.*

Also, we will denote by $G([t_0,\infty),X)$ the vector space of functions $f:[t_0,\infty)\to X$ such that the restriction $f|_{[a,b]}$ belongs to $G([a,b],X)$ for all $[a,b]\subset[t_0,\infty)$. Further, for our purposes, we consider the following space

$$G_0([t_0,\infty),X) := \left\{ f\in G([t_0,\infty),X) : \sup_{s\in[t_0,\infty)} e^{-\gamma(s-t_0)}\|f(s)\| < \infty \right\},$$

for $\gamma > 0$, endowed with the norm $\|f\|_{[t_0,\infty)} = \sup_{s\in[t_0,\infty)} e^{-\gamma(s-t_0)}\|f(s)\|$. Clearly, $G_0([t_0,\infty), X)$ with the norm $\|\cdot\|_{[t_0,\infty)}$ is a Banach space.

The next result gives us sufficient conditions to ensure the existence of the Henstock–Kurzweil–Stieltjes integral of a function $f : [a, b] \to \mathbb{R}^n$ with respect to a function g.

Theorem 1.7. ([73, Corollary 1.34]) *If $f : [a, b] \to \mathbb{R}^n$ is a regulated function, and $g : [a, b] \to \mathbb{R}$ is a nondecreasing function, then the integral $\int_a^b f\mathrm{d}g$ exists, and*

$$\left\|\int_a^b f(s)\mathrm{d}g(s)\right\| \leq \int_a^b \|f(s)\|\mathrm{d}g(s) \leq \|f\|_\infty[g(b) - g(a)]. \tag{1.2}$$

The previous theorem remains valid if we consider a bounded variation function $g : [a, b] \to \mathbb{R}$, and in this case, the inequality (1.2) becomes

$$\left\|\int_a^b f(s)\mathrm{d}g(s)\right\| \leq \|f\|_\infty var_a^b(g),$$

where $var_a^b(g)$ is the variation of the function g on $[a, b]$.

We point out that the Henstock–Kurzweil–Stieltjes definition allows us to integrate a wide class of functions. For instance, as it was explained previously, if $g(t) \equiv t$ then we obtain the Henstock–Kurzweil case, in which is possible to integrate high oscillatory functions, see Example 1.4. On the other hand, it is well known that if the functions f and g have common points of discontinuity, then it is not possible to integrate in the sense of Riemann–Stieltjes. For example, if we consider the functions $f, g : [-1, 1] \to \mathbb{R}$ defined by

$$f(s) = \begin{cases} 1, & -1 \leq s \leq 0, \\ 0, & 0 < s \leq 1, \end{cases} \qquad \text{and} \qquad g(s) = \begin{cases} 0, & -1 \leq s < 0, \\ 1, & 0 \leq s \leq 1, \end{cases}$$

then for any tagged division $\{(s_i, \tau_i)\}_{i=0}^m$ of the interval $[-1, 1]$ in which $s_i \neq 0$ for every $i = 1, ..., m-1$, the Riemann sum may be 0 or 1, and therefore $\int_{-1}^1 f\mathrm{d}g$ does not exist in the sense of Riemann–Stieltjes. However, by Theorem 1.7, we have that the integral $\int_{-1}^1 f\mathrm{d}g$ exist in the sense of Henstock–Kurzweil–Stieltjes.

The conditions on the functions f and g in Theorem 1.7 to guarantee existence of the Henstock–Kurzweil–Stieltjes integral, are weaker than those required for Riemann–Stieltjes and Moore–Pollard–Stieltjes integral, see [67]. Therefore, it seems most convenient to work with the Henstock–Kurzweil–Stieltjes integral when continuity on the functions f and g is not required.

The following result is a special case of [73, Theorem 1.16] and it states important properties of the Henstock–Kurzweil–Stietltjes integral.

Theorem 1.8. *Let $f : [a, b] \to \mathbb{R}^n$ and $g : [a, b] \to \mathbb{R}$ be a pair of functions such that g is regulated,* *and* $\int_a^b f\mathrm{d}g$ *exists. Then the function* $h:[a,b]\to\mathbb{R}^n$ *given by*

$$h(t)=\int_a^t f(s)\mathrm{d}g(s),\quad t\in[a,b],$$

is well defined, regulated, and satisfies

$$h(t+)=h(t)+f(t)\Delta^+g(t),\quad t\in[a,b),$$
$$h(t-)=h(t)-f(t)\Delta^-g(t),\quad t\in(a,b],$$

where $\Delta^+g(t)=g(t+)-g(t)$ *and* $\Delta^-g(t)=g(t)-g(t-)$.

1.2 Generalized ODEs

We now introduce the concept of generalized ordinary differential equation (generalized ODEs, for short). From now on, we assume that X is a Banach space with norm $\|\cdot\|$, $\Omega=\mathcal{O}\times[t_0,\infty)$, where $\mathcal{O}\subset X$ is an open and nonempty subset, $t_0\geq 0$, and $F:\Omega\to X$ is a given X-valued function defined for $(x,t)\in\Omega$. The following two concepts are taken from [73].

Definition 1.9. *A function* $x:[a,b]\to X$, *with* $[a,b]\subset[t_0,\infty)$, *is called a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau}=DF(x,t) \tag{1.3}$$

on the interval $[a,b]$, *if* $(x(t),t)\in\Omega$ *for every* $t\in[a,b]$, *and*

$$x(d)-x(c)=\int_c^d DF(x(\tau),t), \tag{1.4}$$

whenever $[c,d]\subseteq[a,b]$.

As usual, if $(x_0,s_0)\in\Omega$ is fixed, then we can define the solution of the generalized ODE (1.3) on the interval $[a,b]$ with initial condition $x(s_0)=x_0$ (we are considering that $s_0\in[a,b]$), as a function $x:[s_0,b]\to X$ such that $(x(t),t)\in\Omega$ for all $t\in[s_0,b]$ and satisfies (1.4) for all $[c,d]\subseteq[s_0,b]$. Analogously, we can define a solution of (1.3) for an arbitrary non degenerate interval I with initial condition $x(s_0)=x_0$.

In order to establish existence and uniqueness for generalized ordinary differential equations, we need to require some regularity on the function $F:\Omega\to X$, given by the class $\mathcal{F}(\Omega,h)$.

Definition 1.10. *We say that* $F\in\mathcal{F}(\Omega,h)$ *if there exists a nondecreasing function* $h:[t_0,+\infty)\to\mathbb{R}$ *such that* $F:\Omega\to X$ *satisfies the following conditions*

(F1) *For every* $(x,s_i)\in\Omega$, *with* $i=1,2$, *we have*

$$\|F(x,s_2)-F(x,s_1)\|\leq|h(s_2)-h(s_1)|.$$

(F2) *For every* $(x, s_i), (y, s_i) \in \Omega$, *with* $i = 1, 2$, *it is satisfied*

$$\|F(x, s_2) - F(x, s_1) - F(y, s_2) + F(y, s_1)\| \leq |h(s_2) - h(s_1)|\|x - y\|.$$

The following lemma gives us enough conditions to ensure the existence of the Kurzweil integral on the right hand-side of (1.4). The reader can see the proof for $X = \mathbb{R}^n$ in [73, Corollary 3.16]. This is still valid in this more general case.

Lemma 1.11. *Assume* $F \in \mathcal{F}(\Omega, h)$. *Suppose* $x : [a, b] \to X$ *is a regulated function on* $[a, b]$ *such that* $(x(s), s) \in \Omega$ *for all* $s \in [a, b]$. *Then the Kurzweil integral* $\displaystyle\int_a^b DF(x(\tau), t)$ *exists.*

The next result is an immediate consequence of [73, Lemma 3.9] and describes properties of the solutions of the generalized ODEs when F satisfies the condition (F1).

Lemma 1.12. *Let* $F : \Omega \to X$ *be a function that satisfies condition* (F1). *If* $x : [a, b] \to X$ *is a solution of the generalized ODE* (1.3) *on the interval* $[a, b]$, *then* x *is a regulated function and*

$$\|x(s_2) - x(s_1)\| \leq |h(s_2) - h(s_1)|$$

for each pair $s_1, s_2 \in [a, b]$.

The following property will be important for the results of the Chapter 3.

Definition 1.13. *Let* $I \subset \mathbb{R}$ *be an interval,* $t_0 \in I$, *and let* Y *be a set whose elements are functions* $y : I \to \mathbb{R}^n$. *We say that* Y *has the prolongation property for* $t \geq t_0$, *if for every* $y \in Y$, *and every* $t \in I \cap [t_0, \infty)$, *the function* $\bar{y} : I \to \mathbb{R}^n$ *given by*

$$\bar{y}(s) = \begin{cases} y(s), & s \in (-\infty, t] \cap I \\ y(t), & s \in [t, \infty) \cap I \end{cases}$$

is an element of Y.

It is immediate that $G(I, \mathbb{R}^n)$ and the space of continuous functions $C(I, \mathbb{R}^n)$ have the prolongation property, while the space $C^1(I, \mathbb{R}^n)$ of continuously differentiable functions does not have it.

The following theorem is related with local existence and uniqueness of solutions for an initial value problem of the generalized ODE (1.3).

Theorem 1.14. ([28, Theorem 2.15]) *Let* $F : \Omega \to X$ *be a function which belongs to the class* $\mathcal{F}(\Omega, h)$, *where* $h : [t_0, \infty) \to \mathbb{R}$ *is a nondecreasing and left continuous function. If* $(x_0, s_0) \in \Omega$ *is such that* $x_0 + F(x_0, s_0^+) - F(x_0, s_0) \in \mathcal{O}$, *then there exists* $\Delta > 0$ *and a function* $x : [s_0, s_0 + \Delta] \to X$ *which is the unique solution of the generalized ODE* (1.3) *on the interval* $[s_0, s_0 + \Delta]$ *with initial condition* $x(s_0) = x_0$.

To be able to study the asymptotic behaviour of solutions for generalized ODEs in the Chapter 2, we need the following auxiliary concepts and results concerning the existence of maximal solutions to this class of integral equations. In the next, we will see that the unique solution established in Theorem 1.14 can be extended up to a maximal interval J. For more details, see [22].

At first, for a fixed $(x_0, \tau_0) \in \Omega$ with $x_0 + F(x_0, \tau_0^+) - F(x_0, \tau_0) \in \mathcal{O}$, we define the set

$$S_{x_0,\tau_0} := \left\{ x : I_x \subset [t_0, \infty) \to X \;\middle|\; \begin{array}{l} x \text{ is a solution of the generalized ODE (1.3)}, \\ \text{where } I_x \text{ is an interval s.t. } \tau_0 = \min I_x,\ x(\tau_0) = x_0 \end{array} \right\}.$$

It is possible to provide a total order on S_{x_0,τ_0} by the relation

$$x \preceq z \iff I_x \subset I_z \ \wedge \ z|_{I_x} = x.$$

Definition 1.15. ([22, Definition 3.6]) *Let $\tau_0 \geq t_0$ and let $x : I \to X$, $I \subset [t_0, +\infty)$, be a solution of (1.3) on the interval I, with $\tau_0 = \min I$. The solution $y : J \to X$, $J \subset [t_0, +\infty)$, with $\tau_0 = \min J$, of the generalized ODE (1.3) is called a* prolongation to the right *of x, if $I \subset J$ and $x(t) = y(t)$ for all $t \in I$. If $I \subsetneq J$, then y is called a* proper prolongation *of x to the right.*

Definition 1.16. ([22, Definition 3.7]) *Let $(x_0, s_0) \in \Omega$. We say that $x : J \to X$ is a* maximal solution *of the generalized ODE (1.3) with condition*

$$x(s_0) = x_0, \tag{1.5}$$

if $x \in S_{s_0,x_0}$ and, for every $z : I \to B_c$ in S_{s_0,x_0} such that $x \preceq z$, we have $x = z$. In other words, $x \in S_{s_0,x_0}$ is a maximal solution of (1.3)-(1.5) if there is no proper prolongation of x to the right.

We associate with the function F the set

$$\Omega_F := \{(x, t) \in \Omega : x + F(x, t^+) - F(x, t) \in \mathcal{O}\}.$$

Theorem 1.17. ([22, Theorem 3.9]) *Let $F \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing and left–continuous function, and assume that $\Omega = \Omega_F$. Then for every $(x_0, s_0) \in \Omega$ there exists a unique maximal solution $x : J \to X$ of the generalized ODE (1.3)-(1.5), where J is an interval such that $s_0 = \min J$.*

The condition $\Omega = \Omega_F := \{(x, t) \in \Omega : x + F(x, t^+) - F(x, t) \in \mathcal{O}\}$ ensures that there are not points in Ω for which the solution of the generalized ODE (1.3) can scape from $\mathcal{O}$.

Theorem 1.18. ([22, Theorem 3.10]) *Let $F \in \mathcal{F}(\Omega, h)$, where $h : [t_0, \infty) \to \mathbb{R}$ is a nondecreasing and left continuous function, and assume that $\Omega = \Omega_F$. If $(x_0, s_0) \in \Omega$ and $x : J \to X$ is the maximal solution of the generalized ODE (1.3)-(1.5), then $J = [s_0, \omega)$ with $\omega \leq \infty$.*

Remark 1.19. In the rest of this chapter, we will denote by $\omega(x_0, s_0) \leq \infty$, what we will abbreviate by ω when there is no danger of confusion, the constant that allows us to affirm that the maximal solution corresponding to $(x_0, s_0) \in \Omega$ is defined on the interval $[s_0, \omega)$.

Corollary 1.20. ([22, Corollary 3.12]) *Let $F \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing and left–continuous function, and assume that $\Omega = \Omega_F$. Let $(x_0, s_0) \in \Omega$ and let $x : [s_0, \omega) \to X$ be the maximal solution of the generalized ODE (1.3)-(1.5). If $x(t) \in K$ for every $t \in [s_0, \omega)$, where K is a compact subset of B_c, then $\omega = +\infty$.*

Assuming that $\Omega = X \times [t_0, +\infty)$ and $F \in \mathcal{F}(\Omega, h)$, it is possible to ensure that the maximal solution of the generalized ODE (1.3) is defined on $[s_0, +\infty)$ when $x(s_0) = x_0$.

Corollary 1.21. ([22, Corollary 3.14]) *If $\Omega = X \times [t_0, +\infty)$ and $F \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing and left–continuous function, then for every $(x_0, s_0) \in \Omega$, there exists a unique maximal solution of the generalized ODE (1.3) defined in $[s_0, +\infty)$ with $x(s_0) = x_0$.*

Chapter 2

Growth of solutions for generalized ODEs and applications

In this chapter, we are interested in the asymptotic behaviour of solutions to the measure differential equations of the type
$$x(t) = x(t_0) + \int_{t_0}^{t} f(x, s)\mathrm{d}g(s), \tag{2.1}$$
where the integral on the right-hand side is in the sense of Henstock-Kurzweil-Stieltjes.

We investigate the asymptotic behaviour for measure differential equations and dynamic equations on time scales via generalized ordinary differential equations (generalized ODEs, for short). At first, we establish new results that guarantee the existence of unbounded solutions for generalized ODEs, and after that, using the known correspondence between generalized ODEs and measure differential equations, also between measure differential equations and dynamic equations on time scales, we obtain similar results for these equations.

2.1 Growth of solutions for generalized ODEs

In this section, our goal is to use Lyapunov functions to study the asymptotic behaviour of solutions for generalized ODEs defined in a Banach space.

Throughout this section, X is a Banach space endowed with a norm $\|\cdot\|$, the set B_c denotes the open ball in X centered at zero with radius $c > 0$, and $\Omega = B_c \times [t_0, +\infty)$ with $t_0 \geq 0$.

Now, we consider the following generalized ODE
$$\frac{dx}{d\tau} = DF(x, t), \tag{2.2}$$
where $F : \Omega \to X$ is an X–valued function.

In order to prove our main results, we need the following concept.

Definition 2.1. *An increasing continuous function $W : [0, +\infty) \to [0, +\infty)$ is said to be a* wedge*, if $W(0) = 0$, $W(s) > 0$ for $s > 0$ and $W(s) \to +\infty$ as $s \to +\infty$.*

Remark 2.2. The function introduced by Definition 2.1 can be also called *function of Hahn-class* (see [23, 73]).

We are now in position to prove our first result about asymptotic behaviour of solutions for generalized ODEs. In the following results, we assume that $h : [t_0, +\infty) \to \mathbb{R}$ is a nondecreasing and left–continuous function. Moreover, $x(t, s_0, x_0)$ denotes the maximal solution with $x(s_0, s_0, x_0) = x_0$. The next theorem is inspired by [12, Theorem 4.1.23], and it is new in the setting of generalized ODEs.

Theorem 2.3. *Let $F \in \mathcal{F}(\Omega, h)$ with $\Omega = \Omega_F$. Suppose also that there exist a function $V : [t_0, \infty) \times B_c \to \mathbb{R}$ and wedges $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2, 3$, having the following properties:*

(V1) *For every solution $x : I \to B_c$ of the generalized ODE (2.2), we have*

$$W_1(\|x(t)\|) \leq V(t, x(t)) \leq W_2(\|x(t)\|)$$

for all $t \in I$, where $I \subset [t_0, +\infty)$ is a nondegenerate interval.

(V2) *For every maximal solution $x(t) = x(t, s_0, x_0)$ with $(s_0, x_0) \in [t_0, +\infty) \times B_c$, of the generalized ODE (2.2), the inequality*

$$V(t, x(t)) - V(s, x(s)) \geq \int_s^t W_3(\|x(\xi)\|)\mathrm{d}l(\xi)$$

holds for all $t, s \in [s_0, +\infty)$ with $t \geq s$, where $l : [s_0, +\infty) \to \mathbb{R}$ is a nondecreasing function such that $\lim\limits_{t\to+\infty} l(t) = +\infty$.

Let $(s_0, x_0) \in [t_0, +\infty) \times B_c$. Then $\omega(s_0, x_0) < \infty$.

Proof. Let $s_0 \geq t_0$ and $x_0 \in B_c$. The existence and uniqueness of the maximal solution $x(t, s_0, x_0)$ of the generalized ODE (2.2) is guaranteed by Theorem 1.17. Assume that $\omega = \omega(s_0, x_0) = +\infty$. Let γ be a positive number with $W_2(\gamma) = W_1(\|x_0\|)$. The existence of such γ is ensured by the continuity of the wedge W_2.

By (V2), we infer that the function $[s_0, \omega(s_0, x_0)) \ni t \mapsto V(t, x(t))$ is nondecreasing. Then by (V1), we get that for every $t \in [s_0, \omega)$,

$$W_2(\|x(t)\|) \geq V(t, x(t)) \geq V(s_0, x(s_0)) \geq W_1(\|x(s_0)\|) = W_2(\gamma). \tag{2.3}$$

Since W_2 is an increasing function, it follows from (2.3) that

$$\|x(t)\| \geq \gamma \quad \text{for every} \quad t \in [s_0, +\infty). \tag{2.4}$$

On the other hand, for every $t \in [s_0, +\infty)$, combining condition (V2) and (2.4), we obtain the following inequality

$$\begin{aligned} V(t, x(t)) &\geq V(s_0, x(s_0)) + \int_{s_0}^t W_3(\|x(s)\|)\mathrm{d}l(s) \\ &\geq V(s_0, x(s_0)) + W_3(\gamma)(l(t) - l(s_0)). \end{aligned} \tag{2.5}$$

Therefore, collecting (V1) with (2.5), for every $t \in [s_0, +\infty)$, we have

$$W_2(\|x(t)\|) \geq V(s_0, x(s_0)) + W_3(\gamma)(l(t) - l(s_0)),$$

which implies that $\|x(t)\|$ is large enough when t tends to $+\infty$ which is a contradiction. $\square$

Assume that functions involved in Theorem 2.3 are defined in X instead of B_c, which implies that $\Omega = \Omega_F$. Let $x(\cdot)$ be a maximal solution of (2.2). It follows from Corollary 1.21 that $\omega = \infty$. Therefore, arguing as in the proof of Theorem 2.3, we can state the following property.

Theorem 2.4. *Assume that $\Omega = X \times [t_0, +\infty)$. Let $F \in \mathcal{F}(\Omega, h)$. Suppose that there exist a function $V : [t_0, +\infty) \times X \to \mathbb{R}$ and wedges $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2, 3$, having the properties* (V1)-(V2), *with X instead of B_c. Let $(s_0, x_0) \in [t_0, +\infty) \times X$. Then $\omega(s_0, x_0) = +\infty$ and $\|x(t, s_0, x_0)\| \to +\infty$ as $t \to +\infty$.*

We next establish a similar result under very different conditions on the function V. This result is completely new in the literature associated with the theory of generalized ODEs.

Theorem 2.5. *Let $F \in \mathcal{F}(\Omega, h)$ with $\Omega = \Omega_F$. Suppose there exist a function $V : [t_0, +\infty) \times B_c \to \mathbb{R}$ and wedges $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, having the following properties:*

(G1) *For every solution $x : I \to B_c$ of the generalized ODE* (2.2), *we have*

$$|V(t, x(t))| \leq W_1(\|x(t)\|)$$

for all $t \in I$, where $I \subset [t_0, +\infty)$ is a nondegenerate interval.

(G2) *For every maximal solution $x(t) = x(t, s_0, x_0)$, with $(s_0, x_0) \in [t_0, +\infty) \times B_c$, of the generalized ODE* (2.2), *the following inequality*

$$V(t, x(t)) - V(s, x(s)) \leq -\int_s^t W_2(|V(\xi, x(\xi))|)\mathrm{d}l(\xi)$$

holds for all $t, s \in [s_0, +\infty)$ with $t \geq s$, where $l : [s_0, +\infty) \to \mathbb{R}$ is a nondecreasing function such that $\lim_{t\to+\infty} l(t) = +\infty$.

Let $(s_0, x_0) \in [t_0, +\infty) \times B_c$ such that $V(s_0, x_0) < 0$. Then $\omega(s_0, x_0) < \infty$.

Proof. Since $(s_0, x_0) \in \Omega$, then by Theorem 1.17, there exists a unique maximal solution $x(\cdot)$ of the generalized ODE (2.2) such that $x(s_0) = x_0$. Assume that $\omega(s_0, x_0) = +\infty$. Therefore, condition (G1) and the monotonicity of W_1 imply the following inequality

$$|V(t, x(t))| \leq W_1(\|x(t)\|) \leq W_1(c) \quad \text{for all } t \in [s_0, +\infty). \tag{2.6}$$

By condition (G2), we get

$$V(t, x(t)) - V(s_0, x(s_0)) \leq -\int_{s_0}^t W_2(|V(\xi, x(\xi))|)\mathrm{d}l(\xi) \leq 0,$$

for all $t \in [s_0, +\infty)$. Consequently, we have

$$V(t, x(t)) \leq V(s_0, x(s_0)) = V(s_0, x_0) < 0 \quad \text{for all} \quad t \in [s_0, +\infty),$$

which implies that

$$|V(t,x(t))| \geq |V(s_0,x(s_0))| > 0 \quad \text{for all} \quad t \in [s_0,+\infty). \tag{2.7}$$

From (2.7), and using condition (G2) again, we obtain

$$\begin{aligned} V(t,x(t)) &\leq V(s_0,x(s_0)) - \int_{s_0}^{t} W_2(|V(s,x(s))|)\mathrm{d}l(s) \\ &\leq V(s_0,x(s_0)) - \int_{s_0}^{t} W_2(|V(s_0,x(s_0))|)\mathrm{d}l(s) \\ &= V(s_0,x(s_0)) - W_2(|V(s_0,x(s_0))|)(l(t)-l(s_0)) < 0. \end{aligned}$$

This implies that

$$\begin{aligned} |V(t,x(t))| &\geq -V(s_0,x(s_0)) + W_2(|V(s_0,x(s_0))|)(l(t)-l(s_0)) \\ &> W_2(|V(s_0,x(s_0))|)(l(t)-l(s_0)). \end{aligned} \tag{2.8}$$

Since l is a nondecreasing function and $\lim_{t\to+\infty} l(t) = +\infty$, the inequality (2.8) contradicts (2.6) as t tends to $+\infty$. □

In the case $\Omega = X \times [t_0,+\infty)$, proceeding as in Theorem 2.4, we can establish the following result.

Theorem 2.6. *Assume that $\Omega = X \times [t_0,+\infty)$. Let $F \in \mathcal{F}(\Omega,h)$. Suppose there exist a function $V : [t_0,+\infty) \times X \to \mathbb{R}$ and wedges $W_i : [0,+\infty) \to [0,+\infty)$, $i = 1,2$, having the properties* (G1)-(G2)*, with X instead of B_c. Let $(s_0,x_0) \in [t_0,+\infty) \times X$ such that $V(s_0,x_0) < 0$. Then $\omega(s_0,x_0) = +\infty$ and $\|x(t,s_0,x_0)\| \to +\infty$ as $t \to +\infty$.*

2.2 Growth of solutions for measure differential equations

This section is devoted to study the asymptotic behaviour of solutions for measure differential equations. In order to get this, we will use the well known correspondence between the solutions of generalized ODEs and the solutions of measure differential equations (see [73]).

We are concerned with the integral form of a measure differential equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s),s)\mathrm{d}g(s), \tag{2.9}$$

where $f : B_c \times [t_0,+\infty) \to \mathbb{R}^n$ and $g : [t_0,+\infty) \to \mathbb{R}$ are functions with some properties that we will specify later. The integral on the right hand side is considered in the sense of Henstock–Kurzweil–Stieltjes.

We will use the notation $x \in G([t_0,+\infty),B_c)$ to indicate that x is a function which belongs to the space $G([t_0,+\infty),\mathbb{R}^n)$ such that $x(s) \in B_c$ for every $s \in [t_0,+\infty)$. Similarly, we define the notation $x \in G_0([t_0,+\infty),B_c)$.

Definition 2.7. *We say that a function $M : [t_0,+\infty) \to \mathbb{R}^n$ is locally Henstock–Kurzweil–Stieltjes* *integrable with respect to a function g, if it is Henstock–Kurzweil–Stieltjes integrable with respect to a function g for every subinterval $[a,b] \subset [t_0,+\infty)$.*

We now introduce a set of conditions that $f : B_c \times [t_0,+\infty) \to \mathbb{R}^n$ and $g : [t_0,+\infty) \to \mathbb{R}$ must satisfy to establish our results.

(A1) The function $g : [t_0,+\infty) \to \mathbb{R}$ is nondecreasing and left-continuous on $(t_0,+\infty)$.

(A2) The integral $\displaystyle\int_{u_1}^{u_2} f(x(s),s)\mathrm{d}g(s)$ exists for every $x \in G([t_0,+\infty),B_c)$ and $u_1,u_2 \in [t_0,+\infty)$.

(A3) There exists a function $M : [t_0,+\infty) \to \mathbb{R}_+$, which is locally Henstock–Kurzweil–Stieltjes integrable with respect to g, such that

$$\left\|\int_{u_1}^{u_2} f(x(t),t)\mathrm{d}g(t)\right\| \leq \int_{u_1}^{u_2} M(t)\mathrm{d}g(t),$$

for all $x \in G([t_0,+\infty),B_c)$ and $[u_1,u_2] \subseteq [t_0,+\infty)$ such that $u_2 \geq u_1$.

(A4) There exists a function $L : [t_0,+\infty) \to \mathbb{R}_+$, which is locally Henstock–Kurzweil–Stieltjes integrable with respect to g, such that

$$\left\|\int_{u_1}^{u_2} [f(x(t),t) - f(z(t),t)]\mathrm{d}g(t)\right\| \leq \|x - z\|_{[t_0,+\infty)} \int_{u_1}^{u_2} L(t)\mathrm{d}g(t),$$

for all $y,z \in G_0([t_0,+\infty),B_c)$ and $[u_1,u_2] \subseteq [t_0,+\infty)$ such that $u_2 \geq u_1$.

The next theorem ensures that under assumptions (A1)–(A4), we can define a function F in terms of f and g such that $F \in \mathcal{F}(B_c \times [t_0,+\infty),h)$, for a certain nondecreasing and left–continuous function $h : [t_0,+\infty) \to \mathbb{R}$.

Theorem 2.8. ([22, Theorem 4.2]) *Assume that $f : B_c \times [t_0,+\infty) \to \mathbb{R}^n$ and $g : [t_0,+\infty) \to \mathbb{R}$ satisfy conditions* (A1)–(A4)*. Let $\tau_0 \in [t_0,+\infty)$ and define $F : B_c \times [\tau_0,+\infty) \to \mathbb{R}^n$ by*

$$F(x,t) = \int_{\tau_0}^{t} f(x,s)\mathrm{d}g(s), \quad (x,t) \in B_c \times [\tau_0,+\infty), \tag{2.10}$$

then $F \in \mathcal{F}(\Omega,h)$, where $\Omega = B_c \times [\tau_0,+\infty)$ and $h : [\tau_0,+\infty) \to \mathbb{R}$ is the nondecreasing function given by

$$h(t) = \int_{\tau_0}^{t} [M(s) + L(s)]\mathrm{d}g(s), \quad t \in [\tau_0,+\infty). \tag{2.11}$$

The next result describes the relationship between the solutions of the measure differential equation (2.9) and the solutions of the generalized ODE (2.2) on an interval $I \subset [t_0,+\infty)$.

Theorem 2.9. ([22, Theorem 4.8]) *Assume that $f : B_c \times [t_0,+\infty) \to \mathbb{R}^n$ and $g : [t_0,+\infty) \to \mathbb{R}$ satisfy conditions* (A1)–(A4)*. Then the function $x : I \to \mathbb{R}^n$, where $I \subset [t_0,+\infty)$ is a nondegenerate interval, is a solution of the measure differential equation* (2.9) *on I if, and only if, x is a solution of the generalized ODE* (2.2) *on I, where the function F is given by* (2.10).

The next result will be very important to our purposes. It ensures the existence of a maximal solution to the measure differential equation (2.9) for every $(x_0, s_0) \in B_c \times [t_0, +\infty)$.

Theorem 2.10. ([22, Theorem 4.11]) *Suppose $f : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ and $g : [t_0, +\infty) \to \mathbb{R}$ satisfy conditions* (A1)–(A4)*. Further, assume that for every $(z_0, s_0) \in B_c \times [t_0, +\infty)$, we have $z_0 + f(z_0, s_0)\Delta^+ g(s_0) \in B_c$. Then for every $(x_0, s_0) \in B_c \times [t_0, +\infty)$, there exists a unique maximal solution $x : J \to \mathbb{R}^n$ of the MDE* (2.9) *with $x(s_0) = x_0$ and where J is an interval with $s_0 = \min J$.*

Corollary 2.11. ([22, Corollary 4.15]) *Suppose $f : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ and $g : [t_0, +\infty) \to \mathbb{R}$ satisfy conditions* (A1)–(A4)*. Further, assume that for all $(z_0, s_0) \in B_c \times [t_0, +\infty)$, we have $z_0 + f(z_0, s_0)\Delta^+ g(s_0) \in B_c$. Suppose $(x_0, s_0) \in B_c \times [t_0, +\infty)$ and $x : [s_0, \omega) \to \mathbb{R}^n$ is the maximal solution of the MDE* (2.9) *with $x(s_0) = x_0$. If $x(t) \in N$ for all $t \in [s_0, \omega)$, where N is closed in $\mathbb{R}^n$ and contained in B_c, then $\omega = +\infty$.*

Remark 2.12. Theorem 2.10 and Corollary 2.11 can be considered on an arbitrary open subset $\mathcal{O} \subset \mathbb{R}^n$. However, for our purposes, we are considering the particular case $\mathcal{O} = B_c$.

We present a first result about asymptotic behaviour of solutions for measure differential equations.

Theorem 2.13. *Suppose $f : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ and $g : [t_0, +\infty) \to \mathbb{R}$ satisfy conditions* (A1)–(A4)*. Assume that for every $(x_0, s_0) \in B_c \times [t_0, +\infty)$, we have $x_0 + f(x_0, s_0)\Delta^+ g(s_0) \in B_c$, and that there exist a function $U : [t_0, +\infty) \times B_c \to \mathbb{R}$ and wedges $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2, 3$, having the following properties:*

(U1) *For every solution $x : I \to B_c$ of the measure differential equation* (2.9)*, we have*

$$W_1(\|x(t)\|) \leq U(t, x(t)) \leq W_2(\|x(t)\|),$$

for all $t \in I$, where $I \subset [t_0, +\infty)$ is a nondegenerate interval.

(U2) *For every maximal solution $x(t) = x(t, s_0, x_0)$ with $(s_0, x_0) \in [t_0, +\infty) \times B_c$, of the measure differential equation* (2.9)*, we have*

$$U(t, x(t)) - U(s, x(s)) \geq \int_s^t W_3(\|x(\xi)\|)\mathrm{d}l(\xi),$$

for all $t, s \in [s_0, \omega)$ with $t \geq s$, where $l : [s_0, +\infty) \to \mathbb{R}$ is a nondecreasing function such that $\lim_{t \to +\infty} l(t) = +\infty$.

Let $(s_0, x_0) \in [t_0, +\infty) \times B_c$. Then $\omega(s_0, x_0) < \infty$.

Proof. Let $x(t, s_0, x_0)$ be the unique maximal solution of the measure differential equation (2.9) such that $(s_0, x_0) \in [t_0, +\infty) \times B_c$. The existence of such unique maximal solution is ensured by Theorem 2.10.

Since $f : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies conditions (A2)–(A4), and $g : [t_0, +\infty) \to \mathbb{R}$ satisfies conditions (A1)–(A4), from Theorem 2.8 it follows that the function $F : B_c \times [s_0, +\infty) \to \mathbb{R}^n$ defined by (2.10), with s_0 instead of τ_0, belongs to the class $\mathcal{F}(\Omega, h)$, where $\Omega = B_c \times [s_0, +\infty)$ and $h : [s_0, +\infty) \to \mathbb{R}$ is given by (2.11), with s_0 instead of τ_0. Hence, for each $(x_0, s_0) \in B_c \times [t_0, +\infty)$, we have

$$\begin{aligned} x_0 + F(x_0, s_0^+) - F(x_0, s_0) &= x_0 + \lim_{s \to s_0^+} \int_{s_0}^{s} f(x_0, \tau)\mathrm{d}g(\tau) - \int_{s_0}^{s_0} f(x_0, \tau)\mathrm{d}g(\tau) \\ &= x_0 + \lim_{s \to s_0^+} \int_{s_0}^{s} f(x_0, \tau)\mathrm{d}g(\tau) \\ &= x_0 + f(x_0, s_0)\Delta^+ g(s_0) \in B_c. \end{aligned}$$

Therefore, we get

$$x_0 + F(x_0, s_0^+) - F(x_0, s_0) \in B_c$$

that is, $\Omega = \Omega_F$. On the other hand, if $x : [s_0, \omega) \to B_c$ is a (unique) maximal solution of (2.9), then applying Theorem 1.17 and 2.9, we infer that x is also the unique maximal solution of the generalized ODE (2.2) on the interval $[s_0, \omega)$.

Furthermore, since conditions (U1)–(U2) hold, we can show that conditions (V1)–(V2) involved in the statement of Theorem 2.3 are satisfied. Therefore, since all the conditions of Theorem 2.3 are fulfilled, we infer that $\omega(s_0, x_0) < \infty$, proving the result. □

When $\Omega = \mathbb{R}^n \times [t_0, +\infty)$, proceeding as in the proof of Theorem 2.13 and using Theorem 2.4, we can establish the following result.

Theorem 2.14. *Assume that $\Omega = \mathbb{R}^n \times [t_0, +\infty)$. Suppose $f : \mathbb{R}^n \times [t_0, +\infty) \to \mathbb{R}^n$ and $g : [t_0, +\infty) \to \mathbb{R}$ satisfy conditions* (A1)–(A4)*. Assume that there exist a function $U : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ and wedges $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2, 3$, having the properties* (U1)-(U2)*, for $\mathbb{R}^n$ instead of B_c. Let $(s_0, x_0) \in [t_0, +\infty) \times \mathbb{R}^n$. Then $\omega(s_0, x_0) = +\infty$ and $\|x(t, s_0, x_0)\| \to +\infty$ as $t \to +\infty$.*

Next we present a result of asymptotic behaviour for the solution of a measure differential equation.

Theorem 2.15. *Suppose $f : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ and $g : [t_0, +\infty) \to \mathbb{R}$ satisfy conditions* (A1)–(A4)*. Assume that for every $(x_0, s_0) \in B_c \times [t_0, +\infty)$, we have $x_0 + f(x_0, s_0)\Delta^+ g(s_0) \in B_c$, and that there exist a function $U : [t_0, +\infty) \times B_c \to \mathbb{R}$ and wedges $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, having the following properties:*

(H1) *For every solution $x : I \to B_c$ of the measure differential equation* (2.9)*, we have*

$$|U(t, x(t))| \leq W_1(\|x(t)\|),$$

for all $t \in I$, where $I \subset [t_0, +\infty)$ is a nondegenerate interval.

(H2) *For every maximal solution $x(t) = x(t, s_0, x_0)$, with $(s_0, x_0) \in [t_0, +\infty) \times B_c$, of the measure* *differential equation* (2.9)*, we have*

$$U(t,x(t)) - U(s,x(s)) \leq -\int_s^t W_2(|U(\xi,x(\xi))|)\mathrm{d}l(\xi),$$

for all $t,s \in [s_0,\omega)$ *with* $t \geq s$*, where* $l : [s_0,+\infty) \to \mathbb{R}$ *is a nondecreasing function such that* $\lim\limits_{t\to+\infty} l(t) = +\infty$*.*

Let $(s_0,x_0) \in [t_0,+\infty) \times B_c$ *such that* $U(s_0,x_0) < 0$*. Then* $\omega(s_0,x_0) < \infty$*.*

Proof. Let us fix $(x_0,s_0) \in B_c \times [t_0,+\infty)$. Applying Theorem 2.10 we can affirm that there exists a unique maximal solution of the measure differential equation (2.9).

Since $f : B_c \times [t_0,+\infty) \to \mathbb{R}^n$ satisfies conditions (A2)–(A4), and $g : [t_0,\infty) \to \mathbb{R}$ satisfies conditions (A1)–(A4), the function $F : B_c \times [s_0,+\infty) \to \mathbb{R}^n$, defined by (2.10), with s_0 instead of τ_0, belongs to the class $\mathcal{F}(\Omega,h)$, where $\Omega = B_c \times [s_0,+\infty)$, and $h : [s_0,+\infty) \to \mathbb{R}$ is given by (2.11), with s_0 instead of τ_0. Proceeding as in the proof of Theorem 2.13, we obtain that $\Omega = \Omega_F$.

On the other hand, if $x : [s_0,\omega) \to B_c$ is the unique maximal solution of (2.9), then using Theorem 1.17 and Theorem 2.9, we can conclude that x is also the unique maximal solution of the generalized ODE (2.2) on the interval $[s_0,\omega)$. Also, assumptions (H1)–(H2) imply that $U : [t_0,+\infty) \times B_c \to \mathbb{R}$ satisfies conditions (G1)–(G2) in the statement of Theorem 2.5. Therefore, all hypotheses of Theorem 2.5 are fulfilled, which implies that $\omega(s_0,x_0) < \infty$. □

For the case $\Omega = \mathbb{R}^n \times [t_0,+\infty)$, proceeding as in the proof of Theorem 2.15 and using Theorem 2.6, we can establish the following result.

Theorem 2.16. *Assume that* $\Omega = \mathbb{R}^n \times [t_0,+\infty)$*. Suppose* $f : \mathbb{R}^n \times [t_0,+\infty) \to \mathbb{R}^n$ *and* $g : [t_0,+\infty) \to \mathbb{R}$ *satisfy conditions* (A1)–(A4)*. Suppose there exist a function* $U : [t_0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ *and wedges* $W_i : [0,+\infty) \to [0,+\infty)$*,* $i = 1,2$*, having the properties* (H1)-(H2)*, with* $\mathbb{R}^n$ *instead of* B_c*. Let* $(s_0,x_0) \in [t_0,+\infty) \times \mathbb{R}^n$ *such that* $U(s_0,x_0) < 0$*. Then* $\omega(s_0,x_0) = +\infty$ *and* $\|x(t,s_0,x_0)\| \to +\infty$ *as* $t \to +\infty$*.*

Now, we apply the theory developed in the previous sections to study a measure linear differential equation, described in the next example.

Example 2.17. Consider the measure linear differential equation

$$z(t) = 1 + \int_{t_0}^t z(s)\mathrm{d}P(s), \tag{2.12}$$

for $t_0 \geq 0$. In the equation (2.12), the integral on the right-hand side is considered in the sense of Henstock-Kurzweil-Stieltjes, $P : [t_0,+\infty) \to \mathbb{R}$ is a nondecreasing left-continuous function such that $\lim\limits_{t\to+\infty} P(t) = +\infty$. This implies that $P \in BV_+([t_0,+\infty),\mathbb{R})$ which means that P is a function of locally bounded variation such that $1 + \Delta^+P(t) > 0$ for all $t \in [t_0,+\infty)$ and $1 - \Delta^-P(t) > 0$ for all $t \in (t_0,+\infty)$.

As it was explained in [65], the generalized exponential function $t \mapsto e_{\mathrm{d}P}(t, t_0)$, $t \in [t_0, +\infty)$, is defined as the unique solution $z : [t_0, +\infty) \to \mathbb{R}$ of the equation (2.12). For more details concerning to the generalized exponential function and its properties, we refer to [65].

We consider the functions $f : \mathbb{R} \times [t_0, +\infty) \to \mathbb{R}$ and $g : [t_0, +\infty) \to \mathbb{R}$ defined respectively by $f(y, t) = y$ and $g(t) = P(t)$. It is straightforward to check that f and g satisfy conditions (A1)–(A4) from the Section 2.2. Moreover, since $P \in BV_+([t_0, +\infty), \mathbb{R})$, it follows that $e_{dP}(t, t_0) > 0$ for all $t \in [t_0, +\infty)$, see [65, Theorem 3.6].

We introduce the function $U : [t_0, +\infty) \times \mathbb{R} \to \mathbb{R}$, defined by $U(t, x) = |x|$, and the wedges $W_i(\xi) = \xi$, for $i = 1, 2, 3$, and $\xi \geq 0$. If $z(\cdot)$ is a solution of (2.12), it is immediate that

$$W_1(|z(t)|) = |z(t)| = U(t, z(t)) = W_2(|z(t)|).$$

Similarly, for $t, s \in [t_0, +\infty)$ with $t \geq s$, we have

$$\begin{aligned}\int_s^t W_3(|z(\xi)|)\mathrm{d}P(\xi) = \int_s^t |z(\xi)|\mathrm{d}P(\xi) &= \left[1 + \int_{t_0}^t z(\xi)\mathrm{d}P(\xi)\right] - \left[1 + \int_{t_0}^s z(\xi)\mathrm{d}P(\xi)\right] \\ &= z(t) - z(s) \\ &= |z(t)| - |z(s)| \\ &= U(t, z(t)) - U(s, z(s))\end{aligned}$$

Therefore, by Theorem 2.14, we conclude that $z(t) = e_{dP}(t, t_0) \to +\infty$ as $t \to +\infty$.

Remark 2.18. We mention that if $P(s) = s$, then the equation (2.12) is reduced to the equation

$$z(t) = 1 + \int_{t_0}^t z(s)\mathrm{d}s. \tag{2.13}$$

which has a unique solution, that is the classical exponential function $z(t) = e_{\mathrm{d}P}(t, t_0) = e^{t-t_0}$.

Remark 2.19. We point out that if P is continuously differentiable with $P' = p$, then the equation (2.12) is reduced to

$$z(t) = 1 + \int_{t_0}^t p(s)z(s)\mathrm{d}s, \tag{2.14}$$

which has unique solution $z(t) = e_{\mathrm{d}P}(t, t_0) = e^{\int_{t_0}^t p(s)\mathrm{d}s}$.

2.3 Growth of solutions for dynamic equations on time scales

In this section, we will study the asymptotic behaviour of solutions for dynamic equations on time scales. We will begin by recalling some basic concepts concerning the theory of time scales.

The notion of time scales was introduced by Stefan Hilger [49] in 1988. This theory has been intensively developed during the last decades, see for instance [3, 10, 11, 55]. Some of the reasons for the increasing interest in this theory are that it allows to unify the discrete and continuous analysis or even other cases as quantum analysis (depending on the chosen time scale), its applications to the modelling of strongly non-linear dynamical systems, population and economics models (non-uniform steps), systems with delays, among others (see e.g. [6,10,71]).

First, we remember some basic definitions from [10,11] and a few directly related results in order to prove our main theorems.

A *time scale* is an arbitrary closed nonempty subset $\mathbb{T}$ of the real numbers $\mathbb{R}$. Examples of time scales are the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the set $q^{\mathbb{Z}} = \{q^z : n \in \mathbb{Z}\} \cup \{0\} \subset \mathbb{R}$, where $q > 1$, or the most common time scale $\mathbb{T} = \mathbb{R}$. For $a, b \in \mathbb{T}$, $a \leq b$, we define the time scale interval by $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. The open and half–open intervals are defined in a similar way. On the other hand, $[a, b]$ will be used to denote the usual intervals on the real line. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers.

For all $t \in \mathbb{T}$, we define the *forward jump operator* and the *backward jump operator*, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, we assume that $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, i.e., $\sigma(t) = t$ (respectively, $\rho(t) = t$) if $\mathbb{T}$ contains the maximal (respectively, the minimal) element t. The *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

If $\sigma(t) > t$, we say that t is *right–scattered*; otherwise, t is called *right–dense*. Similarly, if $\rho(t) < t$, then t is called *left–scattered*, while if $\rho(t) = t$, t is said *left–dense*.

If we consider $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$, and $\mu(t) = 0$. For the case $\mathbb{T} = \mathbb{Z}$, we obtain $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = 1$.

In addition to the set $\mathbb{T}$, the set $\mathbb{T}^\kappa$ is defined as follows. If $\mathbb{T}$ contains the left–scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$, and $\mathbb{T}^\kappa = \mathbb{T}$ in the other cases. Therefore,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

In time scale analysis, the usual derivative $f'(t)$ and the integral $\int_a^b f(t)\mathrm{d}t$ of a function $f : [a, b] \to \mathbb{R}$ are replaced by the Δ-derivative $f^\Delta(t)$ and the Δ-integral $\int_a^b f(t)\Delta t$, where $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$. For details, the reader can see [10,11].

Definition 2.20. *The function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable at a point $t \in \mathbb{T}^\kappa$ if there exists $f^\Delta(t) \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a neighborhood W of $t \in \mathbb{T}^\kappa$ (i.e., $W = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$), satisfying*

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| < |\sigma(t) - s|, \quad \text{for all } s \in W.$$

If the function f is Δ-differentiable for any $t \in \mathbb{T}^\kappa$, then $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable on $\mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the Δ-derivative of f on $\mathbb{T}^\kappa$.

If we consider $\mathbb{T} = \mathbb{R}$, then $f^\Delta = f'$, is the usual derivative of a function, and in the case $\mathbb{T} = \mathbb{Z}$, then f^Δ is the standard forward difference operator.

To present our results, we will need to recall some important concepts related to calculus on time scales found in [74], which we establish below.

Given a real number $t \leq \sup \mathbb{T}$, we define

$$t^* = \inf \{s \in \mathbb{T} : s \geq t\},$$

and we define the extension of $\mathbb{T}$ by

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty), & \text{otherwise.} \end{cases}$$

On the other hand, given a function $f : \mathbb{T} \to \mathbb{R}^n$, we define its extension $f^* : \mathbb{T}^* \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*), \; t \in \mathbb{T}^*.$$

In the same way as before, for $B \subset \mathbb{R}^n$ and $f : B \times \mathbb{T} \to \mathbb{R}^n$, we define

$$f^*(x, t) = f(x, t^*), \; x \in B, \; t \in \mathbb{T}^*.$$

The next result can be found in [22, Lemma 5.1].

Lemma 2.21. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$. Let $g : [t_0, +\infty) \to \mathbb{R}$ be given by $g(t) = t^*$ for all $t \in [t_0, +\infty)$. Then g satisfies the following conditions:*

(i) *g is a nondecreasing function.*

(ii) *g is left–continuous on $(t_0, +\infty)$.*

Also, it is possible to define for a function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ the general concept of Riemann Δ-integral and Lebesgue Δ-integral, see for instance [38]. However, to our purpose, we recall the general concept of Henstock-Kurzweil Δ-integral of a function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$, introduced by A. Peterson and B. Thompson in [70], which will be crucial in the rest of this section.

Let $\delta = (\delta_L, \delta_R)$ be a pair of nonnegative functions defined on $[a, b]_{\mathbb{T}}$. We say that δ is a Δ-*gauge* for $[a, b]_{\mathbb{T}}$ provided $\delta_L(t) > 0$ on $(a, b] \cap \mathbb{T}$, $\delta_R(t) > 0$ on $[a, b) \cap \mathbb{T}$, and $\delta_R(t) > \mu(t)$ for all $[a, b) \cap \mathbb{T}$.

A tagged partition of the interval $[a, b]_{\mathbb{T}}$ with subdivision points $s_i \in [a, b]_{\mathbb{T}}$, $i = 1, ..., k$, such that $a = s_0 \leq s_1 \leq \cdots \leq s_k = b$, and *tags* $\tau_i \in [a, b]_{\mathbb{T}}$, with $\tau_i \in [s_{i-1}, s_i]$, $i = 1, ..., k$, is called δ-*fine* if

$$\tau_i - \delta_L(\tau_i) \leq s_{i-1} < s_i \leq \tau_i + \delta_R(\tau_i), \quad i = 1, \ldots, k.$$

Definition 2.22. *A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ is called* Henstock-Kurzweil Δ-integrable*, if there exists a vector $I \in \mathbb{R}^n$ having the following property: for every $\varepsilon > 0$, there is a Δ-gauge on $[a, b]_{\mathbb{T}}$*

such that the inequality

$$\left\| \sum_{i=1}^{k} f(\tau_i)(s_i - s_{i-1}) - I \right\| < \varepsilon,$$

holds for every δ-fine tagged partition of $[a,b]_{\mathbb{T}}$. In this case, I is called Henstock-Kurzweil Δ-integral *of f over $[a,b]_{\mathbb{T}}$ and it will be denoted by* $\displaystyle\int_a^b f(t)\Delta t$.

Remark 2.23. *We say that $M : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}_+$ is* locally Henstock-Kurzweil Δ–integrable *if, and only if, the function $[s_1, s_2]_{\mathbb{T}} \ni t \mapsto M(t)$ is Henstock-Kurzweil Δ–integrable for all $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$.*

We will denote by $G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$ the vector space of functions $x : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ such that the restriction $f|_{[a,b]_{\mathbb{T}}}$ belongs to $G([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ for all $[a,b]_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$, with $a, b \in \mathbb{T}$ and $a \leq b$. Also, we consider the vector space

$$G_0([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n) := \left\{ f \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n) : \sup_{s \in [t_0, +\infty)_{\mathbb{T}}} e^{-\gamma(s-t_0)} \|f(s)\| < \infty \right\},$$

for $\gamma > 0$, endowed with the norm $\|f\|_{[t_0,+\infty)_{\mathbb{T}}} = \displaystyle\sup_{s \in [t_0, +\infty)_{\mathbb{T}}} e^{-\gamma(s-t_0)} \|f(s)\|$. Clearly, this normed vector space is complete.

From now on, we consider the following conditions on a function $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$:

(B1) The Henstock-Kurzweil Δ–integral $\displaystyle\int_{s_1}^{s_2} f(y(t), t)\Delta t$ exists for all $y \in G([t_0, +\infty)_{\mathbb{T}}, B_c)$ and all $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$.

(B2) There exists a locally Henstock-Kurzweil Δ–integrable function $M : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}_+$ such that

$$\left| \int_{s_1}^{s_2} f(y(t), t)\Delta t \right| \leq \int_{s_1}^{s_2} M(t)\Delta t,$$

for all $y \in G([t_0, +\infty)_{\mathbb{T}}, B_c)$ and all $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$, $s_1 \leq s_2$.

(B3) There exists a locally Henstock-Kurzweil Δ–integrable function $L : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}_+$ such that

$$\left| \int_{s_1}^{s_2} [f(y(t), t) - f(w(t), t)]\Delta t \right| \leq \|y - w\|_{[t_0, +\infty)_{\mathbb{T}}} \int_{s_1}^{s_2} L(t)\Delta t,$$

for all $y,\ w \in G_0([t_0, +\infty)_{\mathbb{T}}, B_c)$ and all $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$, $s_1 \leq s_2$.

The next result establishes the relationship between conditions (B1)–(B3) for the function f and their analogues for its extension f^*.

Theorem 2.24. ([22, Theorem 5.8]) *Let $\mathbb{T}$ be a time scale such that* $\sup \mathbb{T} = +\infty$ *and $t_0 \in \mathbb{T}$, and let $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ be a function. Let $g(t) = t^*$ for $t \in [t_0, +\infty)$. The following properties are fulfilled:*

(i) *If $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies the condition (B1), then the Henstock-Kurzweil-Stieltjes integral $\displaystyle\int_{t_1}^{t_2} f^*(x(t), t)\mathrm{d}g(t)$ exists for all $x \in G([t_0, +\infty), B_c)$ and all $t_1, t_2 \in [t_0, +\infty)$.*

(ii) *If $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions (B1)-(B2), then $f^* : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies the estimate*

$$\left|\int_{t_1}^{t_2} f^*(x(t), t)\mathrm{d}g(t)\right| \leq \int_{t_1}^{t_2} M^*(t)\mathrm{d}g(t),$$

for all $t_1, t_2 \in [t_0, +\infty)$, $t_1 \leq t_2$, and all $x \in G([t_0, +\infty), B_c)$.

(iii) *If $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions (B1) and (B3), then $f^* : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ satisfies the estimate*

$$\left|\int_{t_1}^{t_2} [f^*(x(t), t) - f^*(z(t), t)]\mathrm{d}g(t)\right| \leq \|x - z\|_{[t_0, +\infty)} \int_{t_1}^{t_2} L^*(t)\mathrm{d}g(t),$$

for all $t_1, t_2 \in [t_0, +\infty)$, $t_1 \leq t_2$, and all $x,\ z \in G_0([t_0, +\infty), B_c)$.

Next we recall several properties of the Henstock–Kurzweil–Stieltjes integration theory on time scales, which are essential to establish our results.

Theorem 2.25. ([22, Theorem 5.3]) *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$ and $g(t) = t^*$ for all $t \in [t_0, +\infty)$. If $f : [t_0, +\infty) \to \mathbb{R}^n$ is such that the Henstock–Kurzweil–Stieltjes integral $\int_c^d f(t)\mathrm{d}g(t)$ exists for every $c, d \in [t_0, +\infty)$, then*

$$\int_c^d f(t)\mathrm{d}g(t) = \int_{c^*}^{d^*} f(t)\mathrm{d}g(t),$$

for all $t_0 \leq c < d < \infty$.

The next result establishes a relationship between the Henstock–Kurzweil Δ–integral and the Henstock–Kurzweil–Stieltjes integral.

Theorem 2.26. ([22, Theorem 5.4]) *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, $t_0 \in \mathbb{T}$ and $f : [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ be a function such that the Henstock-Kurzweil Δ–integral $\displaystyle\int_a^b f(s)\Delta s$ exists for all $a, b \in [t_0, +\infty)_{\mathbb{T}}$ and $a < b$. Let $a \in [t_0, +\infty)_{\mathbb{T}}$, and define*

$$\begin{aligned} F_1(t) &= \int_a^t f(s)\Delta s,\ t \in [t_0, +\infty)_{\mathbb{T}}, \\ F_2(t) &= \int_a^t f^*(s)\mathrm{d}g(s),\ t \in [t_0, +\infty), \end{aligned}$$

where $g(s) = s^*$, for all $s \in [t_0, +\infty)$. Then $F_2 = F_1^*$. In particular, $F_2(t) = F_1(t)$ for all $t \in [t_0, +\infty)_{\mathbb{T}}$.

The next result ensures that if two functions assume the same value on $\mathbb{T}$, then their respective Henstock–Kurzweil–Stieltjes integrals with respect to $g(t) = t^*$ coincide.

Theorem 2.27. ([26, Theorem 4.2]) *Let $\mathbb{T}$ be a time scale, $[a,b] \subset \mathbb{T}^*$ and let $g : [a,b] \to \mathbb{R}$ be defined by $g(s) = s^*$, for all $s \in [a,b]$. Let $f_1, f_2 : [a,b] \to \mathbb{R}^n$ be functions such that $f_1(t) = f_2(t)$ for all $t \in [a,b] \cap \mathbb{T}$. If the Henstock–Kurzweil–Stieltjes integral $\int_a^b f_1(s)\mathrm{d}g(s)$ exists, then the Henstock–Kurzweil–Stieltjes integral $\int_a^b f_2(s)\mathrm{d}g(s)$ exists as well, and both integrals have the same value.*

The next result describes a correspondence between the measure differential equation (2.9) and dynamic equations on time scales.

Theorem 2.28. ([22, Theorem 5.6]) *Assume that $\mathbb{T}$ is a time scale such that $\sup \mathbb{T} = +\infty$. Let $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$. Assume that for every $x \in G([t_0, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$, the function $t \mapsto f(x(t), t)$ is Henstock-Kurzweil Δ–integrable on $[s_1, s_2]_{\mathbb{T}}$, for all $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$. Define $g : [t_0, +\infty) \to \mathbb{R}$ by $g(s) = s^*$ for all $s \in [t_0, +\infty)$. Let $J \subset [t_0, +\infty)$ be a nondegenerate interval such that $J \cap \mathbb{T}$ is nonempty and for each $t \in J$, we have $t^* \in J \cap \mathbb{T}$. If $x : J \cap \mathbb{T} \to \mathbb{R}^n$ is a solution of the initial value problem*

$$\begin{aligned} x^{\Delta}(t) &= f(x^*(t), t), && (2.15)\\ x(s_0) &= x_0, && (2.16) \end{aligned}$$

for $t \in J \cap \mathbb{T}$, where $s_0 \in J \cap \mathbb{T}$, then $x^* : J \to \mathbb{R}^n$ is a solution of the measure differential equation in integral form

$$y(t) = x_0 + \int_{s_0}^{t} f^*(y(s), s)\mathrm{d}g(s). \tag{2.17}$$

Conversely, if $y : J \to \mathbb{R}^n$ is a solution of the (2.17), then there exists a solution $x : J \cap \mathbb{T} \to \mathbb{R}^n$ of the initial value problem (2.15) such that $y = x^*$.

Next, we recall the concepts of maximal solution and prolongation of solutions for the dynamic equation on time scales (2.15), where $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$. For more details, see [22].

Definition 2.29. ([22, Definition 5.9]) *Assume that $I_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$ and $J_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$ are intervals with $s_0 = \min I_{\mathbb{T}} = \min J_{\mathbb{T}}$. Let $x : I_{\mathbb{T}} \to \mathbb{R}^n$ and $y : J_{\mathbb{T}} \to \mathbb{R}^n$ be solutions of (2.15)–(2.16). The solution $y(\cdot)$ is called a prolongation of $x(\cdot)$ to the right, if $I_{\mathbb{T}} \subseteq J_{\mathbb{T}}$ and $x(t) = y(t)$ for all $t \in I_{\mathbb{T}}$. If $I_{\mathbb{T}} \subsetneqq J_{\mathbb{T}}$, then y is called a proper prolongation of x to the right.*

Definition 2.30. ([22, Definition 5.10]) *Assume that $I_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$ is an interval with $s_0 = \min I_{\mathbb{T}}$. The solution $x : I_{\mathbb{T}} \to \mathbb{R}^n$ of the dynamic equation on time scales (2.15)–(2.16) is called maximal, if there is no proper prolongation of x to the right.*

Next, we recall two results which ensure the existence and uniqueness of maximal solutions of (2.15).

Theorem 2.31. ([22, Theorem 5.11]) *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions* (B1)-(B3). *Assume further that for every $(z_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$ we have $z_0 + f(z_0, s_0)\mu(s_0) \in B_c$. Then, for every $(x_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$*

there exists a unique maximal solution $x : [s_0, \omega)_{\mathbb{T}} \to \mathbb{R}^n$ with $\omega \leq +\infty$ of the problem on time scales (2.15)-(2.16). Furthermore, if $\omega < +\infty$, then $\omega \in \mathbb{T}$ and ω is left–dense.

Theorem 2.32. ([22, Theorem 5.16]) *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions* (B1)-(B3)*. Assume further that for every $(z_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$ we have $z_0 + f(z_0, s_0)\mu(s_0) \in B_c$. Suppose $(x_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$ and $x : [s_0, \omega)_{\mathbb{T}} \to \mathbb{R}^n$ is the unique maximal solution of (2.15)-(2.16). If $x(t) \in N$ for all $t \in [s_0, \omega)_{\mathbb{T}}$, where N is closed in $\mathbb{R}^n$ and contained in B_c, then $\omega = +\infty$.*

In what follows, we present our first result about asymptotic behaviour for solutions of the equation (2.15).

Theorem 2.33. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : B_c \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions* (B1)-(B3) *and for every $(z_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$ we have $z_0 + f(z_0, s_0)\mu(s_0) \in B_c$. Assume further there exist a function $U : [t_0, \infty)_{\mathbb{T}} \times B_c \to [0, +\infty)$ and wedges $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2, 3$, having the following properties:*

(U_1^*) *For every solution $x : I_{\mathbb{T}} \to B_c$ of the dynamic equation on time scales (2.15), we have*

$$W_1(\|x(t)\|) \leq U(t, x(t)) \leq W_2(\|x(t)\|)$$

for all $t \in I_{\mathbb{T}}$, where $I_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$ is a nondegenerate time scale interval.

(U_2^*) *For every maximal solution $x(t) = x(t, s_0, x_0)$ with $(s_0, x_0) \in [t_0, \infty)_{\mathbb{T}} \times B_c$ of the dynamic equation on time scales (2.15), we have*

$$U(t, x(t)) - U(s, x(s)) \geq \int_s^t W_3(\|x(\xi)\|)\Delta\xi$$

for all $t, s \in [s_0, \omega)_{\mathbb{T}}$, with $t \geq s$.

Then for every $(s_0, x_0) \in [t_0, +\infty)_{\mathbb{T}} \times B_c$, $\omega(s_0, x_0) < \infty$.

Proof. Let $x(t, s_0, x_0)$ be the unique maximal solution of the dynamic equation on time scales (2.15) such that $(s_0, x_0) \in [t_0, +\infty)_{\mathbb{T}} \times B_c$. The existence and uniqueness of maximal solution is ensured by Theorem 2.31.

Let $f^* : B_c \times [t_0, +\infty) \to \mathbb{R}^n$ and $g : [t_0, +\infty) \to \mathbb{T}$ be the functions $f^*(x, t) = f(x, t^*)$ for all $x \in B_c$ and all $t \in [t_0, +\infty)$, and $g(t) = t^*$ for all $t \in [t_0, +\infty)$. Since f satisfies conditions (B1)-(B3), it follows from Lemma 2.21 and Theorem 2.24 that f^* and g satisfy conditions (A1)-(A4). Moreover, for $(x_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} x_0 + f^*(x_0, s_0)\Delta^+ g(s_0) &= x_0 + f(x_0, s_0^*)(g(s_0^+) - g(s_0)) \\ &= x_0 + f(x_0, s_0^*)(\sigma(s_0^*) - s_0^*) \\ &= x_0 + f(x_0, s_0^*)\mu(s_0^*) \\ &= x_0 + f(x_0, s_0)\mu(s_0) \in B_c. \end{aligned}$$

On the other hand, if $(x_0, s_0) \in B_c \times [t_0, +\infty)$, but $s_0 \notin \mathbb{T}$, then

$$\begin{aligned} x_0 + f^*(x_0, s_0)\Delta^+ g(s_0) &= x_0 + f(x_0, s_0^*)(g(s_0^+) - g(s_0)) \\ &= x_0 + f(x_0, s_0^*)(s_0^* - s_0^*) \\ &= x_0 \in B_c. \end{aligned}$$

Hence, combining the preceding assertions, for each $(x_0, s_0) \in B_c \times [t_0, +\infty)$, we obtain $x_0 + f^*(x_0, s_0)\Delta^+ g(s_0) \in B_c$.

Furthermore, applying Theorem 2.10 and Theorem 2.28, we can affirm that $x^*(t, s_0, x_0)$ is the unique maximal solution of the measure differential equation (2.9) with $(s_0, x_0) \in [t_0, +\infty) \times B_c$.

On the other hand, for every solution $x^* : I \to B_\rho$ of the measure differential equation (2.9), applying condition (U_1^*), we get

$$\begin{array}{rcl} W_1(\|x^*(t)\|) & = & W_1(\|x(t^*)\|) \\ & \le & U(t^*, x(t^*)) \\ & = & U(t^*, x^*(t)) \\ & = & U^*(t, x^*(t)) \end{array}$$

and also,

$$\begin{array}{rcl} U^*(t, x^*(t)) & = & U(t^*, x(t^*)) \\ & \le & W_2(\|x(t^*)\|) \\ & = & W_2(\|x^*(t)\|). \end{array}$$

This implies that the condition (U1) in the statement of Theorem 2.13 is satisfied. Similarly, for every maximal solution x^* of the measure differential equation (2.9), applying condition (U_2^*), Theorems 2.25, 2.26 and 2.27, we get

$$\begin{array}{rcl} U^*(t, x^*(t)) - U^*(s, x^*(s)) & = & U(t^*, x(t^*)) - U(s^*, x(s^*)) \\ & \ge & \displaystyle\int_{s^*}^{t^*} W_3(\|x(\xi)\|)\Delta\xi \\ & = & \displaystyle\int_{s^*}^{t^*} W_3^*(\|x(\xi)\|)\mathrm{d}g(\xi) \\ & = & \displaystyle\int_{s^*}^{t^*} W_3(\|x(\xi^*)\|)\mathrm{d}g(\xi) \\ & = & \displaystyle\int_{s}^{t} W_3(\|x^*(\xi)\|)\mathrm{d}g(\xi) \end{array}$$

Thus, the condition (U2) in the statement of Theorem 2.13 is fulfilled. Therefore, since all hypotheses of Theorem 2.13 are satisfied, we obtain that $\omega(s_0, x_0) < \infty$. □

Arguing as above in the case $\Omega = \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}}$ and by Theorem 2.14, we can state the following property.

Theorem 2.34. *Let $\mathbb{T}$ be a time scale such that* $\sup\mathbb{T} = +\infty$ *and* $t_0 \in \mathbb{T}$. *Suppose* $f : \mathbb{R}^n \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ *satisfies conditions* (B1)-(B3). *Assume further that there exist a function* $U : [t_0, +\infty)_{\mathbb{T}} \times \mathbb{R}^n \to [0, +\infty)$ *and wedges* $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2, 3$, *having the properties* (U_1^*) *and* (U_2^*), *with* $\mathbb{R}^n$ *instead of* B_c. *Let* $x(t) = x(t, s_0, x_0)$ *be the maximal solution corresponding to* $(s_0, x_0) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^n$ *of the dynamic equation on time scales* (2.15). *Then* $\omega(s_0, x_0) = +\infty$ *and* $\|x(t)\| \to +\infty$ *as* $t \to +\infty$.

We complete this section about asymptotic behaviour of solutions of dynamic equations on time scales with a pair of results similar to Theorem 2.5, Theorem 2.6, Theorem 2.15 and Theorem 2.16. We begin with a result similar to Theorem 2.5 and Theorem 2.15.

Theorem 2.35. *Let $\mathbb{T}$ be a time scale such that* $\sup\mathbb{T} = +\infty$ *and* $t_0 \in \mathbb{T}$. *Suppose* $f : B_c \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ *satisfies conditions* (B1)-(B3) *and for every* $(z_0, s_0) \in B_c \times [t_0, +\infty)_{\mathbb{T}}$, *we have* $z_0 + f(z_0, s_0)\mu(s_0) \in B_c$. *Assume that there exist a function* $U : [t_0, \infty)_{\mathbb{T}} \times B_c \to \mathbb{R}$ *and wedges* $W_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2$, *having the following properties:*

(H_1^*) *For every solution* $x : I_{\mathbb{T}} \to B_c$, *of the dynamic equation on time scales* (2.15), *we have*

$$|U(t, x(t))| \leq W_1(\|x(t)\|)$$

for all $t \in I_{\mathbb{T}}$, *where* $I_{\mathbb{T}} \subset [t_0, +\infty)_{\mathbb{T}}$ *is a nondegenerate time scale interval.*

(H_2^*) *For every maximal solution* $x(t) = x(t, s_0, x_0)$ *with* $(s_0, x_0) \in [t_0, +\infty)_{\mathbb{T}} \times B_c$ *of the dynamic equation on time scales* (2.15), *we have*

$$U(t, x(t)) - U(s, x(s)) \leq -\int_s^t W_2(|U(\xi, x(\xi))|)\Delta\xi$$

for all $t, s \in [s_0, \omega)_{\mathbb{T}}$, *with* $t \geq s$.

Let $(s_0, x_0) \in [t_0, +\infty) \times B_c$ *such that* $U(s_0, x_0) < 0$. *Then* $\omega(s_0, x_0) < \infty$.

Proof. Let $x(t, s_0, x_0)$ be the unique maximal solution of the dynamic equation on time scales (2.15) such that $(s_0, x_0) \in [t_0, +\infty)_{\mathbb{T}} \times B_c$. The existence and uniqueness of maximal solution is ensured by Theorem 2.31.

Let f^* and g be defined as in the proof of Theorem 2.33. Arguing as in the proof of Theorem 2.33, we can affirm that f^* and g satisfy conditions (A1)-(A4), and a direct calculus shows that $x_0 + f^*(x_0, s_0)\Delta^+ g(s_0) \in B_c$ for all $(x_0, s_0) \in B_c \times [t_0, +\infty)$.

Combining Theorem 2.10 and Theorem 2.28, we obtain that $x^*(t, s_0, x_0)$ is the unique maximal solution of the measure differential equation (2.9) with $(s_0, x_0) \in [t_0, +\infty) \times B_c$. Moreover, for every solution $x^* : I \to B_c$ of the measure differential equation (2.9) using (H_1^*), we get

$$\begin{aligned} |U^*(t, x^*(t))| &= |U(t^*, x(t^*))| \\ &\leq W_1(\|x(t^*)\|) \\ &= W_1(\|x^*(t)\|), \end{aligned}$$

which implies that (H1) is satisfied.

In similar way, for every maximal solution x^* of the measure differential equation (2.9), applying condition (H_2^*), Theorems 2.25, 2.26 and 2.27, we get

$$\begin{aligned} U^*(t,x^*(t)) - U^*(s,x^*(s)) &= U(t^*,x(t^*)) - U(s^*,x(s^*)) \\ &\leq -\int_{s^*}^{t^*} W_2(|U(\xi,x(\xi))|)\Delta\xi \\ &= -\int_{s^*}^{t^*} W_2^*(|U(\xi,x(\xi))|)\mathrm{d}g(\xi) \\ &= -\int_{s^*}^{t^*} W_2(|U(\xi^*,x(\xi^*)|)\mathrm{d}g(\xi) \\ &= -\int_{s^*}^{t^*} W_2(|U^*(\xi,x^*(\xi)|)\mathrm{d}g(\xi) \\ &= -\int_{s}^{t} W_2(|U^*(\xi,x^*(\xi)|)\mathrm{d}g(\xi). \end{aligned}$$

This implies that condition (H2) is fulfilled.

Consequently, since all hypotheses of Theorem 2.15 are satisfied, it follows that $\omega(s_0,x_0) < \infty$.

□

Finally, we will finish this section with a similar result to Theorem 2.6 and Theorem 2.16.

Theorem 2.36. *Let $\mathbb{T}$ be a time scale such that $\sup\mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Suppose $f : \mathbb{R}^n \times [t_0,+\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions* (B1)-(B3). *Assume further that there exist a function $U : [t_0,+\infty)_{\mathbb{T}} \times \mathbb{R}^n \to [0,+\infty)$ and wedges $W_i : [0,+\infty) \to [0,+\infty)$, $i = 1,2,3$, having the properties* (H_1^*) *and* (H_2^*), *with $\mathbb{R}^n$ instead of B_c. Let $x(t) = x(t,s_0,x_0)$ be the maximal solution corresponding to $(s_0,x_0) \in [t_0,+\infty)_{\mathbb{T}} \times \mathbb{R}^n$ of the dynamic equation on time scales* (2.15). *Then $\omega(s_0,x_0) = +\infty$ and $\|x(t)\| \to +\infty$ as $t \to +\infty$.*

Chapter 3

Measure functional differential equations with time-dependent delay

In this chapter, we focus our attention on measure functional differential equations with time-dependent delay given by

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_{r(s)}, s)\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma], \tag{3.1}$$

where r is an appropriate function, y and f are functions taking values in $\mathbb{R}^n$, and the integral on the right-hand side is in the sense of Henstock-Kurzweil-Stieltjes.

We study the correspondence between the solutions of this equations and the solutions of the generalized ODEs in Banach spaces. Using the theory of generalized ODEs, we obtain results concerning to existence and uniqueness of solutions and continuous dependence on parameters of measure functional differential equations with infinite time-dependent delay. We also establish a result of existence of solutions for a MFDEs with infinite time-dependent delay in the presence of a perturbation independent of the state. We develop the theory in the context of phase spaces defined axiomatically. Our results in this chapter generalize several previous works on MFDEs with infinite time-independent delay.

3.1 Phase space description

A delicate aspect when we are dealing with equations involving infinite delay lies in the choice of a convenient phase space to develop the theory. As already it was mentioned, in order to develop a general theory for measure functional differential equations with infinite time-dependent delay we need an appropriate concept of phase space. An approximation to this subject was considered in [75]. However, in this work we prefer to adapt the usual definition of phase space for retarded functional differential equations with infinite delay used by many authors (see [40, 50]), which will allow us to work with more general phase spaces. In Examples 3.3 to 3.7 we justify this claim.

In this text we consider as phase space for MFDEs with infinite time-dependent delay a linear space $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ equipped with a norm denoted by $\|\cdot\|_{\mathcal{B}}$, and that satisfies the following axioms:

($\mathcal{B}1$) $\mathcal{B}$ is complete.

($\mathcal{B}2$) If $y : (-\infty, t_0 + \sigma] \to \mathbb{R}^n$, $\sigma > 0$, is regulated on $[t_0, t_0 + \sigma]$ and $y_{t_0} \in \mathcal{B}$, then for every $t \in [t_0, t_0 + \sigma]$, the following conditions hold:

(i) $y_t \in \mathcal{B}$

(ii) There exists a locally bounded continuous function $k_1 : [0, \infty) \to (0, \infty)$ such that

$$\|y(t)\| \leq k_1(t - t_0)\|y_t\|_{\mathcal{B}}.$$

(iii) There exist locally bounded continuous functions $k_2, k_3 : [0, \infty) \to (0, \infty)$ such that

$$\|y_t\|_{\mathcal{B}} \leq k_2(t - t_0)\|y_{t_0}\|_{\mathcal{B}} + k_3(t - t_0) \sup_{t_0 \leq s \leq t} \|y(s)\|,$$

k_1, k_2, k_3 are independent of y, t_0 and σ.

(T) Let $S(t) : \mathcal{B} \to \mathcal{B}$ for $t \geq 0$ be the operator defined by

$$[S(t)\varphi](\theta) = \begin{cases} \varphi(0), & \theta = 0, \\ \varphi(0^-), & -t \leq \theta < 0, \\ \varphi(t + \theta), & \theta < -t. \end{cases}$$

Then there exists a continuous function $k : [0, \infty) \to [0, \infty)$ with $k(0) = 0$, and such that

$$\|S(t)\varphi\|_{\mathcal{B}} \leq (1 + k(t))\|\varphi\|_{\mathcal{B}}$$

for all $\varphi \in \mathcal{B}$.

We now exhibit examples of phase spaces $\mathcal{B}$ for measure functional differential equations with infinite delay.

Example 3.1. Consider the space $\mathcal{B} = BG((-\infty, 0], \mathbb{R}^n)$, which consists of all bounded regulated functions on $(-\infty, 0]$, endowed with the supremum norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in (-\infty, 0]\}, \ y \in \mathcal{B}.$$

It is not difficult to verify that conditions ($\mathcal{B}1$)-($\mathcal{B}2$) are satisfied with $k_1(\xi) = k_2(\xi) = k_3(\xi) = 1$ for all $\xi \geq 0$. It is clear also that $\|S(t)\varphi\|_{\mathcal{B}} = \|\varphi\|_{\mathcal{B}}$, which implies that $k(t) = 0$ for all $t \geq 0$.

In order to study MFDEs with unbounded initial conditions, we can consider the following phase spaces. The next example aims to show that the phase space considered in [66, Example 2.2] also satisfies our axioms.

Example 3.2. Let $\rho : (-\infty, 0] \to (0, \infty)$ be a continuous function such that $\rho(0) = 1$, and that satisfies the condition

(ρ_1) The function $p : [0, \infty) \to (0, \infty)$ given by

$$p(t) = \sup_{\theta \leq -t} \frac{\rho(t + \theta)}{\rho(\theta)}, \ t \geq 0,$$

is locally bounded.

Consider the space

$$\mathcal{B} = BG_\rho((-\infty,0],\mathbb{R}^n) = \left\{\varphi \in G((-\infty,0];\mathbb{R}^n) : \frac{\|\varphi\|}{\rho} \text{ is bounded on } (-\infty,0]\right\},$$

endowed with the norm

$$\|\varphi\|_\rho = \sup_{\theta\le 0} \frac{\|\varphi(\theta)\|}{\rho(\theta)},\ \varphi \in BG_\rho((-\infty,0],\mathbb{R}^n).$$

It is not difficult to verify that conditions ($\mathcal{B}1$)-($\mathcal{B}2$) are satisfied with $k_1(\xi) = \sup_{-\xi\le s\le 0} \rho(s)$, $k_2(\xi) = p(\xi)$ and $k_3(\xi) = \sup_{-\xi\le s\le 0} \frac{1}{\rho(s)}$ for $\xi \ge 0$. In fact, let $y : (-\infty, t_0+\sigma] \to \mathbb{R}^n$ be a regulated function on $[t_0, t_0+\sigma]$, with $\sigma > 0$, and $y_{t_0} \in BG_\rho((-\infty,0],\mathbb{R}^n)$. For a fixed $t \in [t_0, t_0+\sigma]$ and ρ satisfying condition (ρ_1), we have the following estimate

$$\begin{aligned}
\sup_{\theta\le 0}\frac{\|y(t+\theta)\|}{\rho(\theta)} &\le \sup_{\theta\le t_0-t}\frac{\|y(t+\theta)\|}{\rho(\theta)} + \sup_{t_0-t\le\theta\le 0}\frac{\|y(t+\theta)\|}{\rho(\theta)}\\
&\le \sup_{\theta\le t_0-t}\frac{\|y(t+\theta)\|}{\rho(t-t_0+\theta)}\sup_{\theta\le t_0-t}\frac{\rho(t-t_0+\theta)}{\rho(\theta)} + \sup_{t_0-t\le\theta\le 0}\|y(t+\theta)\|\sup_{t_0-t\le\theta\le 0}\frac{1}{\rho(\theta)}\\
&= p(t-t_0)\|y_{t_0}\|_\rho + k_3(t-t_0)\sup_{t_0\le\theta\le t}\|y(\theta)\|.
\end{aligned}$$

Therefore, we conclude that $\|y_t\|_\rho \in BG_\rho((-\infty,0],\mathbb{R}^n)$ and ($\mathcal{B}$2-(iii)) follows directly from the previous inequality. For the axiom ($\mathcal{B}$2-(ii)) it is enough to note that for $t \in [t_0, t_0+\sigma]$, we obtain

$$\|y(t)\| \le \sup_{\theta\in[t_0-t,0]}\|y(t+\theta)\| \le \sup_{\theta\in[t_0-t,0]}\rho(\theta)\cdot \sup_{\theta\in[t_0-t,0]}\frac{\|y(t+\theta)\|}{\rho(\theta)}.$$

On the other hand, the function $f : BG_\rho((-\infty,0],\mathbb{R}^n) \to BG((-\infty,0],\mathbb{R}^n)$ defined by $y \mapsto f(y) = y/\rho$ is an isometric isomorphism, and thus $BG_\rho((-\infty,0],\mathbb{R}^n)$ is a complete space.

Initially, if we consider the case where ρ is nonincreasing, then $p(t) \le 1$ and we can take $k(t) = 0$, which shows that axiom (T) is fulfilled.

The axiom (T) also holds for non monotone functions. We illustrate this statement with the function

$$\rho(\theta) = e^{-\theta+P(\theta)},$$

where $P(\cdot)$ is a 2-periodic and of class C^2 function such that $P([-2,0]) = [0,1]$, $P(0) = 0$, with $P'([-2,-1]) = [0,\beta]$, where $\beta > 1$, $P(\cdot)$ is symmetric with respect to $\theta = -1$, and $P''(\theta) \ne 0$ in those points θ such that $P'(\theta) = 1$. A typical example is

$$P(\theta) = \begin{cases} -(1+2\theta)(\theta+2)^2, & -2 \le \theta \le -1,\\ \theta^2(3+2\theta), & -1 \le \theta \le 0.\end{cases}$$

It is clear that ρ is a continuous function on $(-\infty,0]$, $\rho(0) = 1$, $\rho(\theta) \ge 1$, and ρ has local maximums and minima in the abscissa θ such that $P'(\theta) = 1$. Moreover, for $t \geq 0$, we can write

$$\frac{\rho(t+\theta)}{\rho(\theta)} = e^{-t} e^{P(t+\theta)-P(\theta)} = e^{-t} e^{\xi t},$$

for some $\xi \in [\theta, t+\theta]$. This implies that

$$\sup_{\theta \leq -t} \frac{\rho(t+\theta)}{\rho(\theta)} \leq e^{(\beta-1)t}.$$

Hence, we infer that we can take $k(t) = e^{(\beta-1)t} - 1$. Moreover, we observe that k is a continuous function on $[0, \infty)$. This shows that axiom (T) is fulfilled.

Example 3.3. Let $\rho : (-\infty, 0] \to (0, \infty)$ be a function that satisfies the conditions considered in Example 3.2, and additionally, it satisfies the condition

(ρ_2) $\rho(\theta) \to \infty$ as $\theta \to -\infty$.

We consider the subspace of $BG_\rho((-\infty, 0], \mathbb{R}^n)$ defined as

$$BG^0_\rho((-\infty, 0], \mathbb{R}^n) = \left\{ \varphi \in BG_\rho((-\infty, 0], \mathbb{R}^n) : \frac{\|\varphi(\theta)\|}{\rho(\theta)} \to 0,\ \theta \to -\infty \right\}.$$

It is not difficult to verify that conditions ($\mathcal{B}1$)-($\mathcal{B}2$) are satisfied, so that $BG^0_\rho((-\infty, 0], \mathbb{R}^n)$ is a phase space.

In the following example, we will see that the idea of trying to reduce the time-dependent case into a nonautonomous simple form is not a convenient option. As we mentioned in the introduction, this transformation could be obtained by defining the function $h(\varphi, s) := f(\varphi_{r(s)-s}, s)$, however, if φ belongs to a given phase space $\mathcal{B}$, then the function $\varphi_{r(s)-s}$ is not necessarily an element of $\mathcal{B}$. Therefore, in this case, this transformation is not well-defined.

Example 3.4. Let γ be a positive constant. Consider the function $\rho : (-\infty, 0] \to (0, \infty)$ given by $\rho(\theta) = e^{\gamma \cdot \theta^2}$ for $\theta \leq 0$, and the phase space $\mathcal{B} = BG^0_\rho((-\infty, 0], \mathbb{R})$. It is straightforward to check that conditions (ρ_1) and (ρ_2) are satisfied.

Let $\varphi : (-\infty, 0] \to \mathbb{R}$ be the function defined by

$$\varphi(\theta) = \begin{cases} e^{-\gamma}, & \theta \in [-1, 0], \\ e^{\gamma(\theta^2+2\theta)}, & \theta \in (-\infty, -1]. \end{cases}$$

Since $\dfrac{\varphi(\theta)}{\rho(\theta)} = e^{2\gamma\theta} \to 0, \theta \to -\infty$, we infer that $\varphi \in \mathcal{B}$. On the other hand, note that if we consider the nondecreasing function $r(s) = s - 1$, then $\varphi_{r(s)-s} = \varphi_{-1}$ is not an element of the phase space $\mathcal{B}$. In fact, if θ tends to $-\infty$, then

$$\lim_{\theta \to -\infty} \frac{|\varphi_{-1}(\theta)|}{\rho(\theta)} = \lim_{\theta \to -\infty} \frac{|\varphi(\theta - 1)|}{\rho(\theta)} = e^{-\gamma},$$

which is different of zero. Therefore, it follows that $\varphi_{-1} \notin \mathcal{B}$.

Remark 3.5. The space constructed in the Example 3.3 is not a phase space for the axiomatic approach of [66] or [75]. In fact, for the axiomatic definition considered in these works if $\mathcal{B} \subset G((-\infty,0],\mathbb{R}^n)$ is a phase space, then $\mathcal{B}$ satisfies the following condition (H2):"If $y \in \mathcal{B}$ and $t < 0$, then $y_t \in \mathcal{B}$". However, if we consider the function $\varphi \in BG^0_\rho((-\infty,0],\mathbb{R})$ given in the Example 3.4, then $BG^0_\rho((-\infty,0],\mathbb{R})$ does not satisfy the condition (H2) because $\varphi_{-1} \notin BG^0_\rho((-\infty,0],\mathbb{R})$.

We next exhibit another phase space that does not satisfy the axiomatic approach employed in [66, 75].

Example 3.6. Let $h : (-\infty,0] \to (0,\infty)$ be a continuous function such that $\displaystyle\int_{-\infty}^0 h(s)\mathrm{d}s = L < \infty$. Consider the space

$$\mathcal{B}_h((-\infty,0],\mathbb{R}^n) = \left\{\varphi \in G((-\infty,0],\mathbb{R}^n) : \int_{-\infty}^0 h(s) \sup_{s\leq\xi\leq 0} \|\varphi(\xi)\|\mathrm{d}s < \infty\right\},$$

endowed with the norm

$$\|\varphi\|_h = \int_{-\infty}^0 h(s) \sup_{s\leq\xi\leq 0} \|\varphi(\xi)\|\mathrm{d}s, \ \varphi \in \mathcal{B}_h((-\infty,0],\mathbb{R}^n).$$

The vector normed space $(\mathcal{B}_h((-\infty,0],\mathbb{R}^n), \|\cdot\|_h)$ is complete. In fact, let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}_h((-\infty,0],\mathbb{R}^n)$. For every $a \leq 0$, we have

$$\|\varphi_n - \varphi_m\|_h \geq \int_{-\infty}^a h(s)\mathrm{d}s \sup_{\theta\in[a,0]} \|\varphi_n(\theta) - \varphi_m(\theta)\|.$$

Therefore, $(\varphi_n)_{n\in\mathbb{N}}$ is bounded on $[a,0]$ and it is a Cauchy sequence with the uniform convergence norm. Thus, $(\varphi_n)_{n\in\mathbb{N}}$ converges uniformly on every compact set $[a,0] \subset (-\infty,0]$ to a function φ. Clearly φ is regulated on $(-\infty,0]$. Furthermore, since every Cauchy sequence is bounded, there exists $M > 0$ such that

$$\int_a^0 h(s) \sup_{\theta\in[s,0]} \|\varphi_n(\theta)\|\mathrm{d}s \leq M, \tag{3.2}$$

for all $n \in \mathbb{N}$ and all $a \leq 0$. Now, applying Lebesgue's Dominated Convergence Theorem, we conclude that

$$\int_a^0 h(s) \sup_{\theta\in[s,0]} \|\varphi(\theta)\|\mathrm{d}s \leq M, \tag{3.3}$$

for all $a \leq 0$. Consequently, by Lebesgue's Monotone Convergence Theorem, we can deduce that $\varphi \in \mathcal{B}_h((-\infty,0],\mathbb{R}^n)$.

Using again that $(\varphi_n)_{n\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_h((-\infty,0],\mathbb{R}^n)$, for every $\varepsilon > 0$, there exists a $N_\varepsilon \in \mathbb{N}$ such that

$$\int_a^0 h(s) \sup_{\theta\in[s,0]} \|\varphi_n(\theta) - \varphi_m(\theta)\|\mathrm{d}s \leq \varepsilon,$$

for all $n, m \geq N_\varepsilon$ and all $a \leq 0$. Now, applying Lebesgue's Dominated Convergence Theorem we obtain that

$$\lim_{m\to\infty}\int_a^0 h(s)\sup_{\theta\in[s,0]}\|\varphi_n(\theta)-\varphi_m(\theta)\|\mathrm{d}s=\int_a^0 h(s)\sup_{\theta\in[s,0]}\|\varphi_n(\theta)-\varphi(\theta)\|\mathrm{d}s\le\varepsilon,$$

for all $n\ge N_\varepsilon$ and all $a\le 0$. Hence, by Lebesgue's Monotone Convergence Theorem, we conclude that

$$\int_{-\infty}^0 h(s)\sup_{\theta\in[s,0]}\|\varphi_n(\theta)-\varphi(\theta)\|\mathrm{d}s\le\varepsilon,$$

for all $n\ge N_\varepsilon$. Thus, we have that $\varphi_n\to\varphi$ as $n\to\infty$ in $\mathcal{B}_h((-\infty,0],\mathbb{R}^n)$.

It is straightforward to verify that $\mathcal{B}_h((-\infty,0],\mathbb{R}^n)$ satisfies the condition ($\mathcal{B}2$) with constant functions $k_1(\xi)=\dfrac{1}{L}$, $k_2(\xi)=1$, $k_3(\xi)=L$, for $\xi\ge 0$.

We assume in addition that h is nondecreasing. In this case, for $\varphi\in\mathcal{B}_h((-\infty,0],\mathbb{R}^n)$, we have

$$\int_{-\infty}^0 h(s)\sup_{s\le\xi\le 0}\|\varphi(\xi)\|ds\ge\int_{-\infty}^0 h(s)\|\varphi(0)\|ds=L\|\varphi(0)\|,$$

which implies that

$$\|\varphi(0)\|\le\frac{1}{L}\|\varphi\|_h.$$

In similar way, we can show

$$\int_{-\infty}^0 h(s)\sup_{s\le\xi\le 0}\|\varphi(\xi)\|ds\ge\int_{-\infty}^0 h(s)\|\varphi(0^-)\|ds=L\|\varphi(0^-)\|,$$

so that

$$\|\varphi(0^-)\|\le\frac{1}{L}\|\varphi\|_h.$$

In addition, for $t\ge 0$ and $s<-t$, we have

$$\sup_{s\le\xi\le 0}\|[S(t)\varphi](\xi)\|=\sup_{s\le\xi\le -t}\|\varphi(t+\xi)\|.$$

Combining these estimates, we infer

$$\begin{aligned}
\|S(t)\varphi\|_h &= \int_{-\infty}^0 h(s)\sup_{s\le\xi\le 0}\|[S(t)\varphi](\xi)\|ds\\
&= \int_{-\infty}^{-t} h(s)\sup_{s\le\xi\le 0}\|[S(t)\varphi](\xi)\|ds+\int_{-t}^0 h(s)\sup_{s\le\xi\le 0}\|[S(t)\varphi](\xi)\|ds\\
&\le \int_{-\infty}^{-t} h(s)\sup_{s\le\xi\le -t}\|\varphi(t+\xi)\|ds+\frac{1}{L}\int_{-t}^0 h(s)ds\|\varphi\|_h\\
&= \int_{-\infty}^{-t} h(s)\sup_{s+t\le\eta\le 0}\|\varphi(\eta)\|ds+\frac{1}{L}\int_{-t}^0 h(s)ds\|\varphi\|_h\\
&= \int_{-\infty}^{0} h(\tau-t)\sup_{\tau\le\eta\le 0}\|\varphi(\eta)\|d\tau+\frac{1}{L}\int_{-t}^0 h(s)ds\|\varphi\|_h\\
&= \int_{-\infty}^{0} \frac{h(\tau-t)}{h(\tau)}h(\tau)\sup_{\tau\le\eta\le 0}\|\varphi(\eta)\|d\tau+k(t)\|\varphi\|_h\\
&\le (1+k(t))\|\varphi\|_h,
\end{aligned}$$

where $k(t) = \frac{1}{L}\int_{-t}^{0} h(s)ds$, which shows that $\mathcal{B}_h((-\infty,0],\mathbb{R}^n)$ also satisfies the axiom (T).

Example 3.7. Let $\gamma > 1$. Consider the function $\rho : (-\infty,0] \to (0,\infty)$ as in the Example 3.4, that is, $\rho(\theta) = e^{\gamma\theta^2}$. We take $h(s) = \dfrac{e^s}{\rho(s)}$, for $s \in (-\infty,0]$. It is clear that h is a nondecreasing function. Let $\varphi : (-\infty,0] \to \mathbb{R}$ be the function given by $\varphi(\theta) = e^{\gamma(\theta^2+\theta)}$, for $\theta \leq 0$. Since $\dfrac{\varphi(\theta)}{\rho(\theta)} = e^{\gamma\theta} \to 0$, when $\theta \to -\infty$, we infer that $\varphi \in BG_\rho^0((-\infty,0],\mathbb{R})$. Moreover, $\varphi \in \mathcal{B}_h((-\infty,0],\mathbb{R})$. In fact, from $\int_{-\infty}^{0} h(s)\rho(s)\mathrm{d}s = 1$, we get that $\|\varphi\|_h \leq \|\varphi\|_\rho < \infty$. On the other hand, for $r(s) = s-1$, we have that $\varphi_{r(s)-s} = \varphi_{-1} \notin \mathcal{B}_h((-\infty,0],\mathbb{R})$. In fact

$$\int_{-\infty}^{0} h(s) \sup_{\xi\in[s,0]} \varphi_{-1}(\xi)\mathrm{d}s = \int_{-\infty}^{0} h(s) \sup_{\xi\in[s,0]} \varphi(\xi-1)\mathrm{d}s = \int_{-\infty}^{0} h(s)[e^{\gamma(s^2-s)}]\mathrm{d}s = \int_{-\infty}^{0} e^{s(1-\gamma)}\mathrm{d}s,$$

but the last integral does not converge, since $\gamma > 1$.

The next result generalizes [75, Lemma 2.3].

Lemma 3.8. *Assume that $\mathcal{B}$ is a phase space. If $y : (-\infty,t_0+\sigma] \to \mathbb{R}^n$ is such that $y_{t_0} \in \mathcal{B}$ and $y|_{[t_0,t_0+\sigma]}$ is a regulated function, then $t \mapsto \|y_t\|_{\mathcal{B}}$ is regulated on $[t_0,t_0+\sigma]$.*

Proof. Let $t \in [t_0,t_0+\sigma)$. Initially we will show that there exists $\lim_{s\to t^+}\|y_s\|_{\mathcal{B}}$. Let $t < t_1 < t_2$, and let $z(\cdot)$ be the function given by $z(s) = y(s)$ for $s < t_1$, $z(s) = y(t_1^-)$ for $t_1 \leq s < t_2$ and $z(t_2) = y(t_1)$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $\|y(s) - y(t^+)\| \leq \varepsilon$ for all $s \in (t,t+\delta)$. We take $t_2 \in (t,t+\delta)$. We can decompose

$$\begin{aligned} y_{t_2} &= y_{t_2} - z_{t_2} + z_{t_2} \\ &= y_{t_2} - z_{t_2} + S(t_2-t_1)y_{t_1}. \end{aligned}$$

Applying the axioms of phase space, we obtain

$$\|y_{t_2}\|_{\mathcal{B}} \leq 2k_3(t_2-t_1)\varepsilon + (1+k(t_2-t_1))\|y_{t_1}\|_{\mathcal{B}},$$

which in turn implies that

$$\|y_{t_2}\|_{\mathcal{B}} - \|y_{t_1}\|_{\mathcal{B}} \leq 2k_3(t_2-t_1)\varepsilon + k(t_2-t_1)\|y_{t_1}\|_{\mathcal{B}}. \tag{3.4}$$

On the other hand, the set $\{\|y_s\|_{\mathcal{B}} : s \in (t,t+\delta)\}$ is bounded. Consequently, there exists a decreasing sequence $t_n \to t$ as $n \to \infty$ such that the sequence $\|y_{t_n}\|_{\mathcal{B}}$ converges when n goes to ∞. Hence, for $s \in (t,t+\delta)$ we can assume that $t_n \leq s \leq t_{n-1}$. Applying the estimate (3.4) to the pairs (t_{n-1},s) and (t_n,s), we can affirm that

$$\|y_{t_{n-1}}\|_{\mathcal{B}} - \|y_s\|_{\mathcal{B}} \leq 2k_3(t_{n-1}-s)\varepsilon + k(t_{n-1}-s)\|y_s\|_{\mathcal{B}}, \tag{3.5}$$

$$\|y_s\|_{\mathcal{B}} - \|y_{t_n}\|_{\mathcal{B}} \leq 2k_3(s-t_n)\varepsilon + k(s-t_n)\|y_{t_n}\|_{\mathcal{B}}. \tag{3.6}$$

From (3.5), it follows that

$$\begin{aligned} \|y_{t_n}\|_{\mathcal{B}} - \|y_s\|_{\mathcal{B}} &= \|y_{t_n}\|_{\mathcal{B}} - \|y_{t_{n-1}}\|_{\mathcal{B}} + \|y_{t_{n-1}}\|_{\mathcal{B}} - \|y_s\|_{\mathcal{B}} \\ &\leq \|y_{t_n}\|_{\mathcal{B}} - \|y_{t_{n-1}}\|_{\mathcal{B}} + 2k_3(t_{n-1}-s)\varepsilon + k(t_{n-1}-s)\|y_s\|_{\mathcal{B}}. \end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we obtain that $\lim_{s\to t^+}\|y_s\|_{\mathcal{B}} = \lim_{n\to\infty}\|y_{t_n}\|_{\mathcal{B}}$.

A similar argument shows that there exists $\lim_{s\to t^-}\|y_s\|_{\mathcal{B}}$, which completes the proof. $\square$

Remark 3.9. Let $r:[t_0,t_0+\sigma]\to\mathbb{R}$ be a nondecreasing function such that $r(s)\le s$ for all $s\in[t_0,t_0+\sigma]$. Let $\psi:[r(t_0),r(t_0+\sigma)]\to\mathbb{R}^n$ be a regulated function. It is easy to see that $\psi\circ r$ is also regulated.

Proof. Let $\psi:[r(t_0),r(t_0+\sigma)]\to\mathbb{R}^n$ be a regulated function. The function $\psi\circ r$ is regulated. In fact, if $(t_n)_n$ is an increasing sequence that converges to s_0, then $(r(t_n))_n$ is a nondecreasing and bounded sequence, thus it converges to s_1. Since ψ is regulated, we infer that $(\psi(r(t_n)))_n$ converges to $\psi(s_1^-)$. Consequently, there exists $\lim\limits_{t\to s_0^-}\psi(r(t))$. Proceeding in similar way, we can show that there exists $\lim\limits_{t\to s_0^+}\psi(r(t))$, which implies that $\psi\circ r$ is a regulated function. $\square$

Gathering the Lemma 3.8 with the preceding observation, we obtain the following immediate consequence.

Lemma 3.10. *Let $r:[t_0,t_0+\sigma]\to\mathbb{R}$ be a nondecreasing function such that $r(s)\le s$ for all $s\in[t_0,t_0+\sigma]$. Assume further that $y:(-\infty,r(t_0+\sigma)]\to\mathbb{R}^n$ is such that $y_{r(t_0)}\in\mathcal{B}$ and $y|_{[r(t_0),r(t_0+\sigma)]}$ is a regulated function, then $t\mapsto\|y_{r(t)}\|_{\mathcal{B}}$ is regulated on $[t_0,t_0+\sigma]$.*

In the following sections, we will need to ensure that functions of type $s\mapsto f(y_{r(s)},s)$ are integrable for a wide class of functions $y(\cdot)$. For this purpose, we will now exhibit functions f for which the function $s\mapsto f(y_{r(s)},s)$ is regulated.

Theorem 3.11. *Let $f:\mathcal{B}\times[0,a]\to X$ be a function such that for each $\varphi\in\mathcal{B}$, the function $f(\varphi,\cdot)$ is regulated on $[0,a]$. Let $x:(-\infty,a]\to\mathbb{R}^n$, for $a>0$, be a function such that $x_0\in\mathcal{B}$, $x:[0,a]\to\mathbb{R}^n$ is a regulated function. For each $s\in[0,a]$, we define the function $y(s):[0,a]\to\mathbb{R}^n$ by $y(s)(t)=f(x_t,s)$. Assume that $\{y(s)(\cdot):0\le s\le a\}$ is a equi-continuous set. Then the function $z:[0,a]\to\mathbb{R}^n$ given by $z(t)=f(x_t,t)$ is regulated.*

Further, if $r:[0,a]\to\mathbb{R}$ is a nondecreasing function such that $r(s)\le s$ for all $s\in[0,a]$ and $x_{r(0)}\in\mathcal{B}$, then the function $t\mapsto f(x_{r(t)},t)$ is regulated on $[0,a]$.

Proof. it is immediate that by choosing $r(t)=t$, it is sufficient to establish the second assertion. For each $\varphi\in\mathcal{B}$, we define the function $L(\varphi):[0,a)\to\mathbb{R}^n$ by

$$L(\varphi)(t_0)=\lim_{t\to t_0^+}f(\varphi,t).$$

Let $z:[0,a]\to\mathbb{R}^n$ given by $z(t)=f(x_{r(t)},t)$. Let $t_n\to t_0^+$ as $n\to\infty$. We denote $\alpha=\lim_{n\to\infty}r(t_n)$. We will show that $z(t_n)\to L(x_\alpha,t_0)$ as $n\to\infty$. In fact, we can write

$$z(t_n)-L(x_\alpha)(t_0)=z(t_n)-f(x_\alpha,t_n)+f(x_\alpha,t_n)-L(x_\alpha)(t_0).$$

For $\varepsilon > 0$, we can assume that

$$\|f(x_{r(t_n)}, s) - f(x_\alpha, s)\| \leq \varepsilon/2$$

for all $s \in [0,a]$ and all $n \in \mathbb{N}$ large enough. Moreover, we also can assume that

$$\|f(x_\alpha, t_n) - L(x_\alpha)(t_0)\| \leq \varepsilon/2,$$

and combining these estimates, we obtain

$$\|z(t_n) - L(x_\alpha)(t_0)\| \leq \varepsilon,$$

for $n \in \mathbb{N}$ large enough. This shows that $\lim_{t\to t_0^+} f(x_{r(t)}, t) = L(x_\alpha)(t_0)$. Proceeding in similar way we can prove that there exists $\lim_{t\to t_0^-} f(x_{r(t)}, t)$, which completes the proof of the theorem. □

We next apply these ideas to provide examples of functions f. To simplify the calculations, we only consider linear functions.

Example 3.12. Let $\mathcal{B} = BG^0_\rho$ be the space defined in Example 3.3 where $\rho(\cdot)$ is nonincreasing. Let $G : (-\infty, 0] \to \mathbb{R}$ be a measurable function such that

$$N = \int_{-\infty}^{0} |G(\theta)|\rho(\theta)d\theta < \infty$$

and G is locally q-integrable for some $1 < q \leq \infty$. We define the function $f : \mathcal{B} \to \mathbb{R}^n$ by

$$f(\varphi) = \int_{-\infty}^{0} G(\theta)\varphi(\theta)d\theta,\ \varphi \in \mathcal{B}. \tag{3.8}$$

Let $x : (-\infty, a] \to \mathbb{R}^n$, for $a > 0$, be a function such that $x_0 \in \mathcal{B}$ and $x : [0,a] \to \mathbb{R}^n$ is a regulated function. Then $y : [0,a] \to \mathbb{R}^n$ given by $y(t) = f(x_t)$ is a continuous function.

In order to prove this assertion, we fix $\varepsilon > 0$. For $b > 0$ and $t \leq b$, we have

$$\begin{aligned}\left\|\int_{-\infty}^{-b} G(\theta)x(t+\theta)d\theta\right\| &\leq \int_{-\infty}^{-b} |G(\theta)|\frac{\rho(t+\theta)}{\rho(\theta)}\frac{\|x(t+\theta)\|}{\rho(t+\theta)}\rho(\theta)d\theta \\ &\leq \int_{-\infty}^{-b} |G(\theta)|\rho(\theta)d\theta\|x_0\|_\rho \to 0,\ b \to \infty.\end{aligned}$$

As a consequence, we can assume that

$$\left\|\int_{-\infty}^{-b} G(\theta)x(t+\theta)d\theta\right\| \leq \varepsilon.$$

First we consider the particular case when $x : [-b, a] \to \mathbb{R}^n$ is continuous, hence uniformly continuous. This implies that

$$\|f(x_t) - f(x_s)\| \leq 2\varepsilon + \left\|\int_{-b}^{0} G(\theta)[x(t+\theta) - x(s+\theta)]d\theta\right\|$$

for $s, t \in [0,a]$. Therefore, there exists $\delta > 0$ such that $\|x(t+\theta) - x(s+\theta)\| \leq \varepsilon$ for $|t-s| \leq \delta$, and combining with the previous estimate, it yields

$$\|f(x_t) - f(x_s)\| \leq 2\varepsilon + \varepsilon \int_{-b}^{0} |G(\theta)|\rho(\theta)d\theta \leq (2+N)\varepsilon.$$

Turning to the general case, let $1 \leq p < \infty$ the conjugate exponent of q. Since the space $C([-b,a],\mathbb{R}^n)$ is dense in $L^p([-b,a],\mathbb{R}^n)$, there exists a sequence $x^n \in C([-b,a],\mathbb{R}^n)$ that converges to x for the norm in $L^p([-b,a],\mathbb{R}^n)$. This implies that

$$\begin{aligned}
\left\|\int_{-b}^{0} G(\theta)[x(t+\theta)-x(s+\theta)]d\theta\right\| &\leq \int_{-b}^{0} |G(\theta)|\|x(t+\theta)-x^n(t+\theta)\|d\theta \\
&+ \int_{-b}^{0} |G(\theta)|\|x^n(t+\theta)-x^n(s+\theta)\|d\theta + \int_{-b}^{0} |G(\theta)|\|x^n(s+\theta)-x(s+\theta)\|d\theta \\
&\leq 2\left(\int_{-b}^{0} |G(\theta)|^q d\theta\right)^{1/q} \|x-x^n\|_p + \int_{-b}^{0} |G(\theta)|d\theta \sup_{-b\leq\theta\leq 0} \|x^n(t+\theta)-x^n(s+\theta)\|.
\end{aligned}$$

Using that $x^n(\cdot)$ is continuous on $[-b,a]$ and arguing as above, we complete the proof of the assertion.

Example 3.13. Let $\mathcal{B}_h((-\infty,0],\mathbb{R}^n)$ be the space defined in Example 3.6 with h nondecreasing. Let $G : (-\infty,0] \to \mathbb{R}$ be a measurable function such that $|G(s)| \leq \alpha h(s)$ for some $\alpha > 0$, and $\lim_{\theta\to-\infty} \dfrac{G(\theta)}{h(\theta)} = 0$.

We define the function $f : \mathcal{B}_h((-\infty,0],\mathbb{R}^n) \to \mathbb{R}^n$ by (3.8). Let $x : (-\infty,a] \to \mathbb{R}^n$, for $a > 0$, be a function such that $x_0 \in \mathcal{B}_h((-\infty,0],\mathbb{R}^n)$ and $x : [0,a] \to \mathbb{R}^n$ is a regulated function. Then $y : [0,a] \to \mathbb{R}^n$ given by $y(t) = f(x_t)$ is a continuous function.

In order to prove this assertion, we fix $\varepsilon > 0$. For $b > 0$ and $t \leq b$, we have

$$\begin{aligned}
\left\|\int_{-\infty}^{-b} G(\theta)x(t+\theta)d\theta\right\| &\leq \int_{-\infty}^{-b} \frac{|G(\theta)|}{h(\theta)} h(\theta)\|x(t+\theta)\|d\theta \\
&\leq \sup_{\theta\leq -b} \frac{|G(\theta)|}{h(\theta)} \int_{-\infty}^{-b} h(\theta) \max_{t+\theta\leq\xi\leq 0} \|x(\xi)\|d\theta \\
&\leq \sup_{\theta\leq -b} \frac{|G(\theta)|}{h(\theta)} \int_{-\infty}^{t-b} h(s) \max_{s\leq\xi\leq 0} \|x(\xi)\|ds \\
&\leq \sup_{\theta\leq -b} \frac{|G(\theta)|}{h(\theta)} \|x_0\|_h \to 0,\ b\to\infty.
\end{aligned}$$

As a consequence, we can assume that

$$\left\|\int_{-\infty}^{-b} G(\theta)x(t+\theta)d\theta\right\| \leq \varepsilon.$$

In similar way to Example 3.12, when $x : [-b,a] \to X$ is continuous, hence uniformly continuous, we can write

$$\|f(x_t) - f(x_s)\| \leq 2\varepsilon + \left\|\int_{-b}^{0} G(\theta)[x(t+\theta)-x(s+\theta)]d\theta\right\|$$

for $s,t \in [0,a]$. Therefore, there exists $\delta > 0$ such that $\|x(t+\theta)-x(s+\theta)\| \leq \varepsilon$ for $|t-s| \leq \delta$, and combining with the previous estimate, it yields

$$\|f(x_t) - f(x_s)\| \leq 2\varepsilon + \varepsilon \int_{-b}^{0} |G(\theta)| d\theta.$$

Turning to the general case, and proceeding as in the Example 3.12, in this case we use that the space $C([-b,a],\mathbb{R}^n)$ is dense in $L^1([-b,a],\mathbb{R}^n)$, there exists a sequence $x^n \in C([-b,a],\mathbb{R}^n)$ that converges to x for the norm in $L^1([-b,a],\mathbb{R}^n)$. This implies that

$$\begin{aligned}
\left\| \int_{-b}^{0} G(\theta)[x(t+\theta) - x(s+\theta)] d\theta \right\| &\leq \int_{-b}^{0} |G(\theta)| \|x(t+\theta) - x^n(t+\theta)\| d\theta \\
&+ \int_{-b}^{0} |G(\theta)| \|x^n(t+\theta) - x^n(s+\theta)\| d\theta + \int_{-b}^{0} |G(\theta)| \|x^n(s+\theta) - x(s+\theta)\| d\theta \\
&\leq 2 \int_{-b}^{0} |G(\theta)| d\theta \|x - x^n\|_1 + \int_{-b}^{0} |G(\theta)| d\theta \sup_{-b \leq \theta \leq 0} \|x^n(t+\theta) - x^n(s+\theta)\|.
\end{aligned}$$

Using that $x^n(\cdot)$ is continuous on $[-b,a]$ and arguing as above, we complete the proof of the assertion.

Proceeding as in the above examples, and using Theorem 3.11, we can provide examples of nonlinear functions f such that $t \mapsto f(x_{r(t)}, t)$ is a regulated function for all functions $x(\cdot)$ and $r(\cdot)$ that satisfy the conditions from Theorem 3.11.

3.2 MFDEs with time-dependent delay regarded as generalized ODEs

From now on, we assume that $\mathcal{B} \subset G((-\infty,0],\mathbb{R}^n)$ is a phase space satisfying conditions ($\mathcal{B}1$)-($\mathcal{B}2$)-(T), $t_0 \in \mathbb{R}$, $\sigma > 0$. We assume that r is a nondecreasing function such that $r(s) \leq s$ for all $s \in [t_0, t_0+\sigma]$. Moreover, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function and $f : \mathcal{B} \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ is a function that satisfies appropriate conditions which will be specified later.

We will show that under certain assumptions, a measure functional differential equation with infinite time-dependent delay of the form

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_{r(s)}, s) \mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \tag{3.9}$$

$$y_{t_0} = \phi, \tag{3.10}$$

can be regarded as a generalized ordinary differential equation of the form

$$\frac{dx}{d\tau} = DF(x,t), \quad t \in [t_0, t_0+\sigma]. \tag{3.11}$$

To establish our results, we introduce the space

$$Y := \{y : (-\infty, t_0+\sigma] \to \mathbb{R}^n : y|_{[r(t_0), t_0+\sigma]} \text{ is regulated and } y_{r(t_0)} \in \mathcal{B}\}$$

endowed with the norm

$$\|y\|_Y := \sup\{\|y(s)\| : r(t_0) \leq s \leq t_0+\sigma\} + \|y_{r(t_0)}\|_{\mathcal{B}}, \ y \in Y.$$

It is clear that Y is a Banach space. We assume that $\mathcal{O} \subset Y$, and that the following conditions are fulfilled:

(A) The integral $\displaystyle\int_a^b f(y_{r(s)}, s)\mathrm{d}g(s)$ exists in the sense of Henstock-Kurzweil-Stieltjes for all $y \in \mathcal{O}$ and $a, b \in [t_0, t_0 + \sigma]$.

(B) There exists a function $M : [t_0, t_0+\sigma] \to \mathbb{R}_+$, which is Henstock-Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_a^b f(y_{r(t)}, t)\mathrm{d}g(t) \right\| \leq \int_a^b M(t)\mathrm{d}g(t),$$

for all $y \in \mathcal{O}$ and $[a, b] \subseteq [t_0, t_0 + \sigma]$.

(C) There exists a function $L : [t_0, t_0 + \sigma] \to \mathbb{R}_+$, which is Henstock-Kurzweil-Stieltjes integrable with respect to g, such that

$$\left\| \int_a^b [f(y_{r(t)}, t) - f(z_{r(t)}, t)]\mathrm{d}g(t) \right\| \leq \int_a^b L(t)\|y_{r(t)} - z_{r(t)}\|_{\mathcal{B}}\mathrm{d}g(t), \tag{3.12}$$

for all $y, z \in \mathcal{O}$ and $[a, b] \subseteq [t_0, t_0 + \sigma]$.

Remark 3.14. It follows from Lemma 3.10 that the integral on the right-hand side of (3.12) exists in the sense of Henstock-Kurzweil-Stieltjes.

In the equation (3.11), we assume that x takes values in $\mathcal{O} \subset Y$, and the function $F : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ is given by

$$F(z,t)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} f(z_{r(s)}, s)\mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f(z_{r(s)}, s)\mathrm{d}g(s), & \xi \in [t, t_0 + \sigma], \end{cases} \tag{3.13}$$

for all $z \in \mathcal{O}$ and $t \in [t_0, t_0 + \sigma]$.

Remark 3.15. Under the conditions (B) and (C), the requirements that both M and L are Henstock-Kurzweil-Stieltjes integrable functions with respect to g can be replaced by the requirement of the Lebesgue-Stieltjes integrability with respect to g, since these two classes of functions coincide when we are dealing with nonnegative functions.

In the following lemma we establish that the function F defined by (3.13) satisfies conditions (F1) and (F2) from Definition 1.10, when f satisfies conditions (A)-(C).

Lemma 3.16. *Assume that the function $f : \mathcal{B} \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions* (A)-(C)*. Let $F : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ be the function given by* (3.13)*, then F belongs to the class $\mathcal{F}(\mathcal{O} \times [t_0, t_0 + \sigma], h)$, where*

$$h(t) = \int_{t_0}^{t} (M(s) + K_\sigma L(s))\mathrm{d}g(s), \tag{3.14}$$

for $t \in [t_0, t_0+\sigma]$ *and* $K_\sigma = \max\left\{\sup_{\xi\in[t_0,t_0+\sigma]} k_3(r(\xi)-r(t_0)), \sup_{\xi\in[t_0,t_0+\sigma]} k_2(r(\xi)-r(t_0))\right\}$.

Proof. It is clear that the function h defined by (3.14) is a nondecreasing function. Assume that $t_0 \leq s_1 < s_2 \leq t_0+\sigma$, then we have

$$F(y,s_2)(\xi) - F(y,s_1)(\xi) = \begin{cases} 0, & -\infty < \xi \leq s_1, \\ \int_{s_1}^{\xi} f(y_{r(s)}, s)\mathrm{d}g(s), & s_1 \leq \xi \leq s_2, \\ \int_{s_1}^{s_2} f(y_{r(s)}, s)\mathrm{d}g(s), & s_2 \leq \xi \leq t_0+\sigma \end{cases}$$

for all $y \in Y$. Condition (B) implies that

$$\|F(y,s_2) - F(y,s_1)\|_Y = \sup_{\xi\in[s_1,s_2]} \left\| \int_{s_1}^{\xi} f(y_{r(s)}, s)\mathrm{d}g(s) \right\| \leq \int_{s_1}^{s_2} M(s)\mathrm{d}g(s) \leq h(s_2) - h(s_1),$$

for all $y \in Y$. On the other hand, by using conditions (C) and ($\mathcal{B}2$), for every $y, z \in Y$, we have

$$\begin{aligned} \|F(y,s_2) - F(y,s_1) - F(z,s_2) + F(z,s_1)\|_Y &= \sup_{\xi\in[s_1,s_2]} \left\| \int_{s_1}^{\xi} [f(y_{r(s)}, s) - f(z_{r(s)}, s)]\mathrm{d}g(s) \right\| \\ &\leq \int_{s_1}^{s_2} L(s)\|y_{r(s)} - z_{r(s)}\|_{\mathcal{B}}\mathrm{d}g(s) \\ &\leq \int_{s_1}^{s_2} L(s)\left[k_3(r(s)-r(t_0)) \sup_{r(t_0)\leq\xi\leq t_0+\sigma} \|(y-z)(\xi)\|\right]\mathrm{d}g(s) \\ &\quad + \int_{s_1}^{s_2} L(s)k_2(r(s)-r(t_0))\|(y-z)_{r(t_0)}\|_{\mathcal{B}}\mathrm{d}g(s) \\ &\leq K_\sigma \int_{s_1}^{s_2} L(s)\mathrm{d}g(s)\|y-z\|_Y \\ &\leq (h(s_2) - h(s_1))\|y-z\|_Y, \end{aligned}$$

which completes the proof. □

Remark 3.17. In the rest of this chapter, we will denote by K_σ the constant introduced in the statement of the Lemma 3.16.

The next result establishes a very important property of the solutions of the generalized ordinary differential equations. We will omit its proof because it follows analogously to the one found in [75, Lemma 3.5] or even the corresponding for the case with finite delay, the reader can see [26, Lemma 3.7].

Lemma 3.18. *Let* $\mathcal{O} \subset Y$ *with the prolongation property for* $t \geq t_0$*. Assume* $\phi \in \mathcal{B}$*, and that* $F : \mathcal{O} \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ *is the function given by* (3.13)*. Assume further that* $f : \mathcal{B} \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ *satisfies condition* (A)*. If* $x : [t_0, t_0+\sigma] \to \mathcal{O}$ *is a solution of the generalized ordinary differential equation* (3.11) *on the interval* $[t_0, t_0+\sigma]$ *and* $x(t_0)$ *is a function* *which is constant on* $[t_0, t_0+\sigma]$*, then*

$$x(v)(\xi) = x(v)(v), \quad t_0 \leq v \leq \xi \leq t_0+\sigma, \tag{3.15}$$

$$x(v)(\xi) = x(\xi)(\xi), \quad t_0 \leq \xi \leq v \leq t_0+\sigma. \tag{3.16}$$

The next theorem establishes a relation between solutions of measure functional differential equations with infinite time-dependent delay and solutions of generalized ordinary differential equations.

Theorem 3.19. *Let* $\mathcal{O}$ *be a subset of* Y *having the prolongation property for* $t \geq t_0$*. Assume that* $\phi \in \mathcal{B}$ *and that* $f : \mathcal{B} \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ *satisfies conditions* (A)-(C)*. Let* $F : \mathcal{O} \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ *be the function given by* (3.13)*. If* $y \in \mathcal{O}$ *is a solution of the measure functional differential equation with infinite time-dependent delay* (3.9)-(3.10)*, then the function* $x : [t_0, t_0+\sigma] \to \mathcal{O}$ *given by*

$$x(t)(\xi) = \begin{cases} y(\xi), & \xi \in (-\infty, t], \\ y(t), & \xi \in [t, t_0+\sigma] \end{cases} \tag{3.17}$$

is a solution of the generalized ordinary differential equation (3.11) *on the interval* $[t_0, t_0+\sigma]$*.*

Proof. We need to prove that the integral $\displaystyle\int_{t_0}^{v} DF(x(\tau), t)$ exists, and

$$x(v) - x(t_0) = \int_{t_0}^{v} DF(x(\tau), t),$$

for all $v \in [t_0, t_0+\sigma]$. Fix $\varepsilon > 0$. Since the function $h : [t_0, t_0+\sigma] \to [0, \infty)$ defined by

$$h(t) = \int_{t_0}^{t} M(s)\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma],$$

is nondecreasing, it can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ h(t) \geq \varepsilon$. We denote these points by $t_1, \ldots, t_m$.

We now show that there exists a gauge $\delta : [t_0, t_0+\sigma] \to \mathbb{R}^+$ sufficiently small that satisfies the following four conditions:

(i) $\delta(\tau) < \displaystyle\min_{2 \leq k \leq m} \left\{ \frac{t_k - t_{k-1}}{2} \right\}, \quad \tau \in [t_0, t_0+\sigma],$

(ii) $\delta(\tau) < \displaystyle\min_{1 \leq k \leq m} \{|\tau - t_k|\}, \quad \tau \in [t_0, t_0+\sigma],\ \tau \neq t_k,\ k = 1, \ldots, m.$

These conditions imply that if a point-interval pair $(\tau, [c,d])$ satisfies $[c,d] \subset (\tau - \delta(\tau), \tau + \delta(\tau))$, then $[c,d]$ contains at most one of the points $t_1, \ldots, t_m$, and moreover, $\tau = t_k$ whenever $t_k \in [c,d]$.

(iii) It follows from (3.17) that $y_{r(t_k)} = x(t_k)_{r(t_k)}$, and applying Theorem 1.8 we obtain that

$$\lim_{t \to t_k^+} \int_{t_k}^{t} L(s) \|y_{r(s)} - x(t_k)_{r(s)}\|_{\mathcal{B}} \mathrm{d}g(s) = L(t_k) \|y_{r(t_k)} - x(t_k)_{r(t_k)}\|_{\mathcal{B}} \Delta^+ g(t_k) = 0$$

for all $k \in 1, \ldots, m$. Thus the gauge $\delta(\cdot)$ might be chosen in such a way that

$$\int_{t_k}^{t_k+\delta(t_k)} L(s)\|y_{r(s)} - x(t_k)_{r(s)}\|_{\mathcal{B}} \mathrm{d}g(s) < \frac{\varepsilon}{2m+1}, \quad k \in \{1, \ldots, m\}.$$

(iv) Using the condition (B), we have

$$\|y(t+\tau) - y(\tau)\| = \left\| \int_{\tau}^{\tau+t} f(y_{r(s)}, s)\mathrm{d}g(s) \right\| \leq \int_{\tau}^{\tau+t} M(s)\mathrm{d}g(s) = h(\tau+t) - h(\tau).$$

Therefore

$$\|y(\tau^+) - y(\tau)\| \leq \Delta^+ h(\tau) < \varepsilon, \quad \tau \in [t_0, t_0+\sigma] \setminus \{t_1, \ldots, t_m\}.$$

Thus, we can select the gauge $\delta(\cdot)$ such that

$$\|y(\rho) - y(\tau)\| \leq \varepsilon$$

for all $\tau \in [t_0, t_0+\sigma] \setminus \{t_1, \ldots, t_m\}$ and $\rho \in [\tau, \tau + \delta(\tau))$.

Let $\{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{l}$ be a δ-fine tagged partition of $[t_0, v]$. Then

$$(x(s_i) - x(s_{i-1}))(\xi) = \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \displaystyle\int_{s_{i-1}}^{\xi} f(y_{r(s)}, s)\mathrm{d}g(s), & \xi \in [s_{i-1}, s_i], \\ \displaystyle\int_{s_{i-1}}^{s_i} f(y_{r(s)}, s)\mathrm{d}g(s), & \xi \in [s_i, t_0+\sigma], \end{cases}$$

and

$$(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(\xi) = \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \displaystyle\int_{s_{i-1}}^{\xi} f(x(\tau_i)_{r(s)}, s)\mathrm{d}g(s), & \xi \in [s_{i-1}, s_i], \\ \displaystyle\int_{s_{i-1}}^{s_i} f(x(\tau_i)_{r(s)}, s)\mathrm{d}g(s), & \xi \in [s_i, t_0+\sigma] \end{cases}$$

for all $i \in \{1, \ldots, l\}$. Combining these expressions, we obtain

$$\begin{aligned} &(x(s_i) - x(s_{i-1}))(\xi) - (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(\xi) \\ &= \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \displaystyle\int_{s_{i-1}}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s), & \xi \in [s_{i-1}, s_i], \\ \displaystyle\int_{s_{i-1}}^{s_i} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s), & \xi \in [s_i, t_0+\sigma]. \end{cases} \end{aligned}$$

Consequently,

$$\begin{aligned} &\|(x(s_i) - x(s_{i-1})) - (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))\|_Y \\ &= \sup_{\xi \in [s_{i-1}, s_i]} \left\| \int_{s_{i-1}}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s) \right\|. \end{aligned} \tag{3.18}$$

Using now (3.17), we have that $x(\tau_i)_{r(s)} = y_{r(s)}$ for $s \le \tau_i$. Therefore,

$$\int_{s_{i-1}}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s) = \begin{cases} 0, & \xi \in [s_{i-1}, \tau_i], \\ \displaystyle\int_{\tau_i}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s), & \xi \in [\tau_i, s_i]. \end{cases}$$

We now use condition (C) to estimate

$$\begin{aligned} \left\| \int_{\tau_i}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s) \right\| &\le \int_{\tau_i}^{\xi} L(s)\|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}}\mathrm{d}g(s) \\ &\le \int_{\tau_i}^{s_i} L(s)\|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}}\mathrm{d}g(s). \end{aligned} \tag{3.19}$$

For a point-interval pair $(\tau_i, [s_{i-1}, s_i])$, there are two possibilities:

(a) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, ..., t_m\}$ contains a single point t_k. In this case, it follows from (ii) that $t_k = \tau_i$.

(b) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, ..., t_m\}$ is empty.

In case (a), it follows from the construction of the gauge $\delta(\cdot)$ that

$$\int_{\tau_i}^{s_i} L(s)\|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}}\mathrm{d}g(s) \le \frac{\varepsilon}{2m+1},$$

and combining with (3.18) and (3.19), we get

$$\|(x(s_i) - x(s_{i-1})) - (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))\|_Y \le \frac{\varepsilon}{2m+1}.$$

Assume now case (b), and let $s \in [\tau_i, s_i]$. Let us consider first that $r(s) \in [\tau_i, s_i]$, then

$$\begin{aligned} \|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}} &\le k_3(r(s) - r(t_0)) \sup_{r(t_0) \le \xi \le r(s)} \{\|y(\xi) - x(\tau_i)(\xi)\|\} \\ &\quad + k_2(r(s) - r(t_0))\|y_{r(t_0)} - x(\tau_i)_{r(t_0)}\|_{\mathcal{B}} \\ &\le K_\sigma \sup_{\tau_i \le \xi \le r(s)} \|y(\xi) - y(\tau_i)\| \\ &\le K_\sigma \varepsilon. \end{aligned}$$

Moreover, if $r(s) \le \tau_i$, then $\|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}} = 0$ by (3.17). Thus

$$\int_{\tau_i}^{s_i} L(s)\|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}}\mathrm{d}g(s) \le K_\sigma \varepsilon \int_{\tau_i}^{s_i} L(s)\mathrm{d}g(s),$$

and using (3.18) and (3.19) again, we obtain

$$\|(x(s_i) - x(s_{i-1})) - (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))\|_Y \le K_\sigma \varepsilon \int_{\tau_i}^{s_i} L(s)\mathrm{d}g(s).$$

Combining cases (a) and (b), and using the fact that case (a) occurs at most $2m$ times, it follows that

$$\begin{aligned}\left\| x(v) - x(t_0) - \sum_{i=1}^{l} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})) \right\|_Y &\leq \varepsilon \left(K_\sigma \int_{t_0}^{t_0+\sigma} L(s)\mathrm{d}g(s) + \frac{2m}{2m+1} \right) \\ &< \varepsilon \left(K_\sigma \int_{t_0}^{t_0+\sigma} L(s)\mathrm{d}g(s) + 1 \right),\end{aligned}$$

which completes the proof. □

The next theorem establishes the relation between a solution of a generalized ODE (3.11) and a solution of a measure functional differential equation with infinite time-dependent delay.

Theorem 3.20. *Let $\mathcal{O}$ be a subset of Y having the prolongation property for $t \geq t_0$. Assume that $\phi \in \mathcal{B}$ and that $f : \mathcal{B} \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies conditions* (A)-(C)*. Let $F : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ be the function given by* (3.13)*. If $x : [t_0, t_0 + \sigma] \to \mathcal{O}$ is a solution of the generalized ordinary differential equation* (3.11) *on the interval $[t_0, t_0 + \sigma]$ with the initial condition*

$$x(t_0)(\xi) = \begin{cases} \phi(\xi - t_0), & \xi \in (-\infty, t_0], \\ \phi(0), & \xi \in [t_0, t_0 + \sigma], \end{cases} \tag{3.20}$$

then the function $y \in \mathcal{O}$ defined by

$$y(\xi) = \begin{cases} x(t_0)(\xi), & \xi \in (-\infty, t_0], \\ x(\xi)(\xi), & \xi \in [t_0, t_0 + \sigma] \end{cases} \tag{3.21}$$

is a solution of the measure functional differential equation with infinite time-dependent delay (3.9)-(3.10).

Proof. The equality $y_{t_0} = \phi$ follows directly from the definition of y and $x(t_0)$. It remains to prove that

$$y(v) - y(t_0) = \int_{t_0}^{v} f(y_{r(s)}, s)dg(s)$$

for all $v \in [t_0, t_0 + \sigma]$.

Using Lemma 3.18, we obtain

$$y(v) - y(t_0) = x(v)(v) - x(t_0)(t_0) = x(v)(v) - x(t_0)(v) = \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v). \tag{3.22}$$

Let $\varepsilon > 0$ be fixed. By Lemma 3.16, F belongs to the class $\mathcal{F}(\mathcal{O} \times [t_0, t_0 + \sigma], h)$, where h is given by (3.14). Now, arguing as in the proof of Theorem 3.19. Since h is a nondecreasing function and thus, it can have only a finite numbers of points $t \in [t_0, v]$ such that $\Delta^+ h(t) \geq \varepsilon$. We denote these points by $t_1, \ldots, t_m$. We consider a gauge $\delta : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ that satisfies the following four conditions:

(i) $\delta(\tau) < \min\limits_{2 \leq k \leq m} \left\{ \dfrac{t_k - t_{k-1}}{2} \right\}, \quad \tau \in [t_0, t_0 + \sigma],$

(ii) $\delta(\tau) < \min\limits_{1 \leq k \leq m} \{|\tau - t_k|\}, \quad \tau \in [t_0, t_0 + \sigma], \ \tau \neq t_k, \ k = 1, \ldots, m,$

(iii) $\displaystyle\int_{t_k}^{t_k+\delta(t_k)} L(s)\|y_{r(s)} - x(t_k)_{r(s)}\|_{\mathcal{B}} \mathrm{d}g(s) < \frac{\varepsilon}{2m+1}$, $k \in \{1, \dots, m\}$,

(iv) $\|h(\rho) - h(\tau)\| \leq \varepsilon$, $\tau \in [t_0, t_0 + \sigma] \setminus \{t_1, \dots, t_m\}$, $\rho \in [\tau, \tau + \delta(\tau))$.

Let $\{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{l}$ be a δ-fine tagged partition of $[t_0, v]$ such that

$$\left\| \int_{t_0}^{v} DF(x(\tau), t) - \sum_{i=1}^{l} [F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})] \right\|_Y < \varepsilon.$$

Using (3.22), axiom ($\mathcal{B}2$) and the previous inequality, we obtain

$$\begin{aligned} \left\| y(v) - y(t_0) - \int_{t_0}^{v} f(y_{r(s)}, s)\mathrm{d}g(s) \right\| &= \left\| \left(\int_{t_0}^{v} DF(x(\tau), t) \right)(v) - \int_{t_0}^{v} f(y_{r(s)}, s)\mathrm{d}g(s) \right\| \\ &\leq k_1(v - r(t_0)) \left\| \left(\int_{t_0}^{v} DF(x(\tau), t) \right)_v - \left(\sum_{i=1}^{l} [F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})] \right)_v \right\|_{\mathcal{B}} \\ &\quad + \left\| \sum_{i=1}^{l} (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(v) - \int_{t_0}^{v} f(y_{r(s)}, s)\mathrm{d}g(s) \right\| \\ &\leq C_\sigma K_\sigma \varepsilon + \sum_{i=1}^{l} \left\| (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(v) - \int_{s_{i-1}}^{s_i} f(y_{r(s)}, s)\mathrm{d}g(s) \right\|, \end{aligned} \tag{3.23}$$

where $C_\sigma = \sup_{\xi \in [t_0, t_0+\sigma]} k_1(\xi - r(t_0))$.

On the other hand, the definition of F yields

$$(F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(v) = \int_{s_{i-1}}^{s_i} f(x(\tau_i)_{r(s)}, s)\mathrm{d}g(s),$$

which implies

$$\begin{aligned} &\left\| (F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}))(v) - \int_{s_{i-1}}^{s_i} f(y_{r(s)}, s)\mathrm{d}g(s) \right\| \\ &\qquad = \left\| \int_{s_{i-1}}^{s_i} [f(x(\tau_i)_{r(s)}, s) - f(y_{r(s)}, s)]\mathrm{d}g(s) \right\|. \end{aligned} \tag{3.24}$$

We affirm that, for every $i \in \{1, \dots, l\}$, we have $x(\tau_i)_{r(s)} = x(s)_{r(s)} = y_{r(s)}$ for $s \in [s_{i-1}, \tau_i]$, and $y_{r(s)} = x(s)_{r(s)} = x(s_i)_{r(s)}$ for $s \in [\tau_i, s_i]$. In fact, suppose that $s \in [s_{i-1}, \tau_i]$. If $t_0 \leq r(s) + \theta \leq s$, where $\theta \in (-\infty, 0]$, then by Lemma 3.18 we can write

$$x(\tau_i)_{r(s)}(\theta) = x(\tau_i)(r(s) + \theta) = x(r(s) + \theta)(r(s) + \theta) = x(s)(r(s) + \theta) = x(s)_{r(s)}(\theta).$$

The equality $x(s)_{r(s)} = y_{r(s)}$ follows directly from (4.18). Now, if we consider $r(s) + \theta \leq t_0$, since $x(\cdot)$ is a solution of (3.11), it follows from (3.13) that $x(\tau_i)_{r(s)} - x(s)_{r(s)} = 0$, and

$$x(\tau_i)_{r(s)} - y_{r(s)} = x(\tau_i)_{r(s)} - x(t_0)_{r(s)} = 0.$$

The case when $s \in [\tau_i, s_i]$ can be proved analogously. Therefore,

$$\begin{aligned}\left\|\int_{s_{i-1}}^{s_i}[f(x(\tau_i)_{r(s)},s)-f(y_{r(s)},s)]\mathrm{d}g(s)\right\| &= \left\|\int_{\tau_i}^{s_i}[f(x(\tau_i)_{r(s)},s)-f(y_{r(s)},s)]\mathrm{d}g(s)\right\| \\ &= \left\|\int_{\tau_i}^{s_i}[f(x(\tau_i)_{r(s)},s)-f(x(s_i)_{r(s)},s)]\mathrm{d}g(s)\right\| \\ &\leq \int_{\tau_i}^{s_i} L(s)\|x(\tau_i)_{r(s)}-x(s_i)_{r(s)}\|_{\mathcal{B}}\mathrm{d}g(s)\end{aligned}$$

where the last inequality follows from condition (C).

To estimate (3.24), we distinguish two cases:

(a) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, ..., t_m\}$ contains a single point $t_k = \tau_i$.

(b) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, ..., t_m\}$ is empty.

In case (a), it follows from the definition of $\delta(\cdot)$ that

$$\int_{\tau_i}^{s_i} L(s)\|x(\tau_i)_{r(s)}-x(s_i)_{r(s)}\|_{\mathcal{B}}\mathrm{d}g(s) \leq \frac{\varepsilon}{2m+1},$$

which implies that

$$\left\|(F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1}))(v)-\int_{s_{i-1}}^{s_i} f(y_{r(s)},s)\mathrm{d}g(s)\right\| \leq \frac{\varepsilon}{2m+1}.$$

We now consider the case (b). It follows from the definition of F that $x(s_i)(\xi)-x(\tau_i)(\xi)=0$, $\xi \in (-\infty, t_0]$. Using Lemma 1.12 and axiom ($\mathcal{B}2$), for $s \in [\tau_i, s_i]$ we obtain the estimate

$$\begin{aligned}\|x(s_i)_{r(s)}-x(\tau_i)_{r(s)}\|_{\mathcal{B}} \leq{}& k_3(r(s)-r(t_0)) \sup_{r(t_0)\leq\xi\leq r(s)} \|x(s_i)(\xi)-x(\tau_i)(\xi)\| \\ &+ k_2(r(s)-r(t_0))\|x(s_i)_{r(t_0)}-x(\tau_i)_{r(t_0)}\|_{\mathcal{B}}.\end{aligned}$$

It implies that

$$\begin{aligned}\|x(s_i)_{r(s)}-x(\tau_i)_{r(s)}\|_{\mathcal{B}} &\leq K_\sigma\left[\sup_{\xi\in[r(t_0),t_0+\sigma]}\|x(s_i)(\xi)-x(\tau_i)(\xi)\|+\|x(s_i)_{r(t_0)}-x(\tau_i)_{r(t_0)}\|_{\mathcal{B}}\right] \\ &= K_\sigma\|x(s_i)-x(\tau_i)\|_Y \\ &\leq K_\sigma(h(s_i)-h(\tau_i)) \leq K_\sigma\varepsilon.\end{aligned}$$

Consequently,

$$\left\|(F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1}))(v)-\int_{s_{i-1}}^{s_i} f(y_{r(s)},s)\mathrm{d}g(s)\right\| \leq K_\sigma\varepsilon\int_{\tau_i}^{s_i} L(s)dg(s).$$

Combining the cases (a) and (b), and using the fact that the case (a) occurs at most $2m$ times, we obtain

$$\sum_{i=1}^{l}\left\|(F(x(\tau_i),s_i)-F(x(\tau_i),s_{i-1}))(v)-\int_{s_{i-1}}^{s_i} f(y_{r(s)},s)\mathrm{d}g(s)\right\|$$
$$\le K_\sigma\varepsilon\int_{t_0}^{t_0+\sigma} L(s)\mathrm{d}g(s)+\frac{2m\varepsilon}{2m+1}$$
$$<\varepsilon\left(K_\sigma\int_{t_0}^{t_0+\sigma} L(s)\mathrm{d}g(s)+1\right),$$

and substituting in (3.23) yields

$$\left\|y(v)-y(t_0)-\int_{t_0}^{v} f(y_{r(s)},s)\mathrm{d}g(s)\right\|<\varepsilon\left(C_\sigma K_\sigma+K_\sigma\int_{t_0}^{t_0+\sigma} L(s)\mathrm{d}g(s)+1\right).$$

Which completes the proof. □

Remark 3.21. By considering the function $r(t) := \rho(t, x_t)$ with $t \in [t_0, t_0+\sigma]$, we can obtain results concerning to measure functional differential equations with state-dependent delays. Certainly, we have to take account conditions over the function ρ and f which will allow us to develop an appropriate theory.

Remark 3.22. Using similar arguments as the ones found in [26,27], it is also possible to establish a correspondence between measure functional differential equations with time-dependent delay and functional dynamic equations on time scales with time-dependent delay and impulsive measure functional differential equations with time-dependent delay.

3.3 Existence and uniqueness of solutions

In this section, our goal is to present a result concerning to existence and uniqueness of solutions of measure functional differential equations with infinite time-dependent delay. We begin by regarding the following local existence and uniqueness theorem for generalized ODEs which was exposed in the Chapter 1 and proved in [28, Theorem 2.15].

Theorem 3.23. *Assume that X is a Banach space, $\mathcal{O} \subset X$ is an open set and the function $F : \mathcal{O} \times [t_0, t_0+\sigma] \to X$ belongs to the class $\mathcal{F}(\mathcal{O} \times [t_0, t_0+\sigma]; h)$, where $h : [t_0, t_0+\sigma] \to \mathbb{R}$ is a left continuous nondecreasing function. If $x_0 \in \mathcal{O}$ is such that $x_0 + F(x_0, t_0^+) - F(x_0, t_0) \in \mathcal{O}$, then there exists a $\beta > 0$ and a function $x : [t_0, t_0+\beta] \to X$ which is a unique solution of the generalized ODE (3.11) with initial condition $x(t_0) = x_0$.*

We now present a result of local existence and uniqueness of solutions for measure functional differential equations with infinite time-dependent delay.

Theorem 3.24. *Assume that $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function and r is a nondecreasing regulated function such that $r(s) \le s$ for all $s \in [t_0, t_0+\sigma]$. Let $\mathcal{O} \subset Y$ be an open subset having the prolongation property for $t \ge t_0$. Assume that $f : \mathcal{B} \times [t_0, t_0+\sigma] \to \mathbb{R}^n$*

satisfies conditions (A)-(C). Let $F : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ *be the function given by* (3.13) *and let* $\phi \in \mathcal{B}$*. Assume further that the function* $x_0 = x(t_0)$ *defined by* (3.20) *and*

$$z(t) = \begin{cases} \phi(t - t_0), & t \in (-\infty, t_0], \\ \phi(0) + f(\varphi, t_0)\Delta^+ g(t_0), & t \in (t_0, t_0 + \sigma] \end{cases}$$

are elements of $\mathcal{O}$*, where* φ *is defined by* $\varphi(\theta) = \phi(\theta + r(t_0) - t_0)$*, for* $\theta \in (-\infty, 0]$*. Then there exists* $\beta > 0$*, and a function* $y : (-\infty, t_0 + \beta] \to \mathbb{R}^n$ *which is the unique solution of the initial value problem* (3.9)-(3.10) *on* $(-\infty, t_0 + \beta]$*.*

Proof. Since $f : \mathcal{B} \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ satisfies the conditions (A)-(C), and $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is a left-continuous nondecreasing function, it follows from Lemma 3.16 that F satisfies conditions (F1) and (F2). Let

$$x_0(\xi) = \begin{cases} \phi(\xi - t_0), & \xi \in (-\infty, t_0], \\ \phi(0), & \xi \in [t_0, t_0 + \sigma]. \end{cases}$$

We affirm that $x_0 + F(x_0, t_0^+) - F(x_0, t_0) \in \mathcal{O}$. In fact, we first point out that $F(x_0, t_0) = 0$. The limit $F(x_0, t_0^+)$ is taken with respect to the supremum norm and we know it must exist since F is regulated with respect to the second variable. By the definition of F and Theorem 1.8, we have

$$F(x_0, t_0^+)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ f(\varphi, t_0)\Delta^+ g(t_0), & \xi \in (t_0, t_0 + \sigma]. \end{cases}$$

Therefore, it follows that $x_0 + F(x_0, t_0^+) - F(x_0, t_0) \in \mathcal{O}$. Consequently, all hypotheses of Theorem 3.23 are satisfied, which implies that there exists a $\beta > 0$ and a unique solution $x : [t_0, t_0 + \beta] \to X$ of the generalized ODE

$$\frac{dx}{d\tau} = DF(x, t), \quad x(t_0) = x_0. \tag{3.25}$$

Defining the function $y : (-\infty, t_0 + \beta] \to \mathbb{R}^n$ by

$$y(\xi) = \begin{cases} x(t_0)(\xi), & \xi \in (-\infty, t_0], \\ x(\xi)(\xi), & \xi \in [t_0, t_0 + \beta], \end{cases}$$

we obtain by Theorem 3.20 that $y(\cdot)$ is the unique solution of the initial value problem (3.9)-(3.10). □

3.4 Continuous dependence on parameters

In this section, our goal is to prove a continuous dependence result for measure functional differential equations with time-dependent delays, using the existing results for generalized ODEs and the correspondences established in Section 3.2.

We initiate this section by regarding the following result of continuous dependence on parameters for generalized ODEs which can be found in [73, Theorem 2.4]. We point out that in [73] the result is proved for the case when $X = \mathbb{R}^n$, but it is not difficult to extend the result for the general case, that is, when X is a general Banach space. The proof follows as the same way. Therefore, we will omit its proof here.

Theorem 3.25. *Let X be a Banach space, $\mathcal{O} \subset X$ be an open subset, and $h_k : [a,b] \to \mathbb{R}$, $k \in \mathbb{N}$, be a sequence of nondecreasing left-continuous functions such that $h_k(b) - h_k(a) \leq c$, for some $c > 0$ and all $k \in \mathbb{N}_0$. Assume that for every $k \in \mathbb{N}_0$, $F_k : \mathcal{O} \times [a,b] \to X$ belongs to the class $\mathcal{F}(\mathcal{O} \times [a,b], h_k)$ and that*

$$\lim_{k\to\infty} F_k(x,t) = F_0(x,t),\ x \in \mathcal{O},\ t \in [a,b],$$
$$\lim_{k\to\infty} F_k(x,t^+) = F_0(x,t^+),\ x \in \mathcal{O},\ t \in [a,b).$$

For every $k \in \mathbb{N}$, let $x_k : [a,b] \to \mathcal{O}$ be a solution of the generalized ODE

$$\frac{dx}{d\tau} = DF_k(x,t),\ t \in [a,b].$$

If there exists a function $x_0 : [a,b] \to \mathcal{O}$ such that $\lim_{k\to\infty} x_k(t) = x_0(t)$ uniformly for $t \in [a,b]$, then x_0 is a solution of

$$\frac{dx}{d\tau} = DF_0(x,t),\ t \in [a,b].$$

We mention that in the statement of Theorem 3.25 the functions F_k are defined in $\mathcal{O} \times [a,b]$, while in [73], the author consider that F_k are defined on $\mathcal{O} \times (-T,T)$, where $[a,b] \subset (-T,T)$ and h_k are defined on $(-T,T)$. However, as pointed out in [26], it is not difficult to adapt the proof in order that the Theorem 3.25 is still valid.

Next we recall an auxiliary result, which is essential to prove the continuous dependence result for MFDEs with time-dependent delay. It can be found in [30, Theorem 2.18].

Theorem 3.26. *The following conditions are equivalent:*

(i) A set $\mathcal{A} \subset G([\alpha,\beta],\mathbb{R}^n)$ is relatively compact.

(ii) The set $\{x(\alpha) : x \in \mathcal{A}\}$ is bounded and there is an increasing continuous function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(0) = 0$ and an increasing function $K : [\alpha,\beta] \to \mathbb{R}$ such that

$$\|x(t_2) - x(t_1)\| \leq \eta(K(t_2) - K(t_1)),$$

for all $x \in \mathcal{A}$, and $\alpha \leq t_1 \leq t_2 \leq \beta$.

We are in position to present our main result of this section, that is, the continuous dependence result for measure functional differential equations with time-dependent delay. Our proof follows analogously as the proof of [26, Theorem 6.3].

In the next statement we denote by $g : [t_0, t_0+\sigma] \to \mathbb{R}$ a left-continuous nondecreasing function, r is a nondecreasing function such that $r(s) \leq s$ for all $s \in [t_0, t_0 + \sigma]$, and the space Y is defined as in Section 3.2.

Theorem 3.27. *Suppose that $f_k : \mathcal{B} \times [t_0, t_0+\sigma] \to \mathbb{R}^n$, $k \in \mathbb{N}_0$, is a sequence of functions which satisfy conditions* (A)-(C) *with the same functions M and L. Assume that for every $y \in Y$,*

$$\lim_{k\to\infty} \int_{t_0}^{t} f_k(y_{r(s)}, s)\mathrm{d}g(s) = \int_{t_0}^{t} f_0(y_{r(s)}, s)\mathrm{d}g(s),$$

uniformly with respect to $t \in [t_0, t_0+\sigma]$. Let $\mathcal{O}$ be an open subset of Y having the prolongation property for $t \geq t_0$. For every $k \in \mathbb{N}$, let $F_k : \mathcal{O} \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ be the function given by

$$F_k(z,t)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} f_k(x_{r(s)}, s)\mathrm{d}g(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} f_k(x_{r(s)}, s)\mathrm{d}g(s), & \xi \in [t, t_0+\sigma]. \end{cases}$$

Let $\phi_k \in \mathcal{B}$, $k \in \mathbb{N}_0$ be a sequence of functions such that $\lim\limits_{k\to\infty} \phi_k = \phi_0$ uniformly on $(-\infty, 0]$. Let $y_k \in \mathcal{O}$, $k \in \mathbb{N}$, be the solution of

$$\begin{aligned} y_k(t) &= y_k(t_0) + \int_{t_0}^{t} f((y_k)_{r(s)}, s)\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ (y_k)_{t_0} &= \phi_k. \end{aligned}$$

If there exists a function $y_0 \in Y$ such that $\lim\limits_{k\to\infty} y_k = y_0$ on $(-\infty, t_0+\sigma]$, then y_0 is a solution of

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^{t} f((y_0)_{r(s)}, s)\mathrm{d}g(s), \quad t \in [t_0, t_0+\sigma], \\ (y_0)_{t_0} &= \phi_0. \end{aligned}$$

Proof. By hypotheses, for every $x \in \mathcal{O}$ we have $\lim\limits_{k\to\infty} F_k(x,t) = F_0(x,t)$ uniformly with respect to $t \in [t_0, t_0+\sigma]$. By the Moore-Osgood Theorem ([67, Lemma 4.2.3.]), we obtain $\lim\limits_{k\to\infty} F_k(x,t^+) = F_0(x,t^+)$ for all $x \in \mathcal{O}$ and $t \in [t_0, t_0+\sigma]$. Moreover, since Y is a complete space, we have that F_0 takes values in Y. Also, it follows from Lemma 3.18 that F_k belongs to the class $\mathcal{F}(\mathcal{O} \times [t_0, t_0+\sigma]; h)$, for all $k \in \mathbb{N}$, where the function h is given by (3.14). Since $\lim\limits_{k\to\infty} F_k(x,t) = F_0(x,t)$ uniformly, it follows that F_0 belongs to the class $\mathcal{F}(\mathcal{O} \times [t_0, t_0+\sigma], h)$. For every $k \in \mathbb{N}_0$ and $[t_0, t_0+\sigma]$, we define

$$x_k(t)(\xi) = \begin{cases} y_k(\xi), & \xi \in (-\infty, t], \\ y_k(t), & \xi \in [t, t_0+\sigma]. \end{cases}$$

It follows from Theorem 3.19 that the function $x_k(\cdot)$ is a solution of the generalized ODE

$$\frac{dx}{d\tau} = DF_k(x,t), \; t \in [t_0, t_0+\sigma].$$

Let $k \in \mathbb{N}$ and $t_0 \leq t_1 \leq t_2 \leq t_0+\sigma$, then

$$\|y_k(t_2) - y_k(t_1)\| = \left\| \int_{t_1}^{t_2} f_k((y_k)_{r(s)}, s)\mathrm{d}g(s) \right\| \leq \int_{t_1}^{t_2} M(s)\mathrm{d}g(s) \leq K(t_2) - K(t_1),$$

where $K(t) = t + \int_{t_0}^{t} M(s)\mathrm{d}g(s)$ is an increasing function. Moreover, the sequence $\{y_k(t_0)\}_{k\in\mathbb{N}}$ is bounded. Thus, we see that condition (ii) in Theorem 3.26 is satisfied. It follows that the sequence $\{y_k|_{[t_0,t_0+\sigma]}\}_{k\in\mathbb{N}}$ contains a subsequence which is uniformly convergent on $[t_0, t_0+\sigma]$. Without loss of generality, we denote this subsequence again by $\{y_k\}_{k\in\mathbb{N}}$. Since $(y_k)_{t_0} = \phi_k$ for $\theta \in (-\infty, 0]$, we see that $\{y_k\}_{k\in\mathbb{N}}$ is in fact uniformly convergent on $(-\infty, t_0+\sigma]$. Using the definition of x_k, we obtain that $\lim_{k\to\infty} x_k(t) = x_0(t)$ uniformly with respect to $t \in [t_0, t_0+\sigma]$. It follows from Theorem 3.25 that x_0 is a solution of

$$\frac{dx}{d\tau} = DF_0(x,t),\ t \in [t_0, t_0+\sigma].$$

The proof is finished by applying Theorem 3.20, which ensures that y_0 satisfies

$$\begin{aligned} y_0(t) &= y_0(t_0) + \int_{t_0}^{t} f((y_0)_{r(s)}, s)\mathrm{d}g(s),\ \ t \in [t_0, t_0+\sigma], \\ (y_0)_{t_0} &= \phi_0, \end{aligned}$$

which completes the proof. □

3.5 Existence and uniqueness of solutions for the perturbed system

In this section, our objective is to study a perturbed measure functional differential equations with time-dependent delay. These results will be very important to study stability results in the future.

In the rest of this section, we will assume that $t_0 \in \mathbb{R}$, $\sigma > 0$, $\mathcal{B} \subset G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is a phase space that satisfies axioms $(\mathcal{B}1)$-$(\mathcal{B}2)$-(T) and $g : [t_0, t_0+\sigma] \to \mathbb{R}$ is a nondecreasing function. Further, we assume that the function $f : \mathcal{B} \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ satisfies conditions (A)-(C), and r is a nondecreasing function such that $r(s) \leq s$ for all $s \in [t_0, t_0+\sigma]$.

Let $p : [t_0, t_0+\sigma] \to \mathbb{R}^n$ be a regulated function and $u : [t_0, t_0+\sigma] \to \mathbb{R}$ be a nondecreasing function. We assume that the function p satisfies:

(D) There exists a function $K : [t_0, t_0+\sigma] \to \mathbb{R}_+$, which is Henstock-Kurzweil-Stieltjes integrable with respect to u, such that

$$\left\| \int_a^b p(s)\mathrm{d}u(s) \right\| \leq \int_a^b K(s)\mathrm{d}u(s),$$

for all $[a,b] \subseteq [t_0, t_0+\sigma]$.

In this section, we will consider the perturbed measure functional differential equation with infinite time-dependent delay

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_{r(s)}, s)\mathrm{d}g(s) + \int_{t_0}^{t} p(s)\mathrm{d}u(s),\ \ t \in [t_0, t_0+\sigma], \tag{3.26}$$

We point out that under our previous assumptions a solution $y(\cdot)$ of equation (3.26) is a regulated function. We keep the notations introduced in the previous sections. In particular, $\mathcal{O}$ is a subset of Y having the prolongation property for $t \geq t_0$.

Let $Q : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ be the function defined by

$$Q(y, t) = F(y, t) + P(t), \ t \in [t_0, t_0 + \sigma], \tag{3.27}$$

where F is given by (3.13), and

$$P(t)(\xi) = \begin{cases} 0, & \xi \in (-\infty, t_0], \\ \displaystyle\int_{t_0}^{\xi} p(s)\mathrm{d}u(s), & \xi \in [t_0, t], \\ \displaystyle\int_{t_0}^{t} p(s)\mathrm{d}u(s), & \xi \in [t, t_0 + \sigma]. \end{cases} \tag{3.28}$$

The following result is similar to the Lemma 3.16. For this reason, we omit its proof.

Lemma 3.28. *If $Q : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ is the function defined by (3.27), then the function Q satisfies conditions $(F1)$ and $(F2)$ with*

$$h(t) = \left(\int_{t_0}^{t} (M(s) + K_\sigma L(s))\mathrm{d}g(s) + \int_{t_0}^{t} K(s)\mathrm{d}u(s) \right),$$

for all $t \in [t_0, t_0 + \sigma]$.

The next two theorems are modified versions of Theorem 3.19 and Theorem 3.20. The following results represent the correspondence between solutions of perturbed measure functional differential equations with infinite time-dependent delay and solutions of generalized ordinary differential equations.

Theorem 3.29. *Assume that $Q : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ is the function defined by (3.27) If $y \in \mathcal{O}$ is a solution of the equation (3.26) with initial condition (3.10), then the function $x : [t_0, t_0 + \sigma] \to \mathcal{O}$ given by (3.17) is a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DQ(x, t), \tag{3.29}$$

on the interval $[t_0, t_0 + \sigma]$.

Proof. Our aim is to show that the integral $\displaystyle\int_{t_0}^{v} DQ(x(\tau), t)$ exists and

$$x(v) - x(t_0) = \int_{t_0}^{v} DQ(x(\tau), t),$$

for all $v \in [t_0, t_0 + \sigma]$.

Let an arbitrary $\varepsilon > 0$ be given. We define the function

$$h(t) = \left(\int_{t_0}^{t} [M(s) + K_\sigma L(s)]\mathrm{d}g(s) + \int_{t_0}^{t} K(s)\mathrm{d}u(s) \right), \ \ t \in [t_0, t_0 + \sigma].$$

Since the functions g and u are nondecreasing, they can have only a finite number of points $t \in [t_0, v]$ such that $\Delta^+ g(t) > 0$ and $\Delta^+ u(t) > 0$. The same assertion remains true for the function

h. Thus h can have only a finite numbers of points $t \in [t_0, v]$ such that $\Delta^+ h(t) \geq \varepsilon$. Let us denote these points by $t_1, ..., t_m$.
We now select a gauge $\delta : [t_0, t_0 + \sigma] \to \mathbb{R}^+$ that satisfies the following four conditions:

(i) $\delta(\tau) < \min_{2 \leq k \leq m} \left\{ \dfrac{t_k - t_{k-1}}{2} \right\}, \ \ \tau \in [t_0, t_0 + \sigma],$

(ii) $\delta(\tau) < \min_{1 \leq k \leq m}\{|\tau - t_k|\}, \ \tau \in [t_0, t_0 + \sigma], \ \tau \neq t_k, \ k = 1, \ldots, m.$

These conditions assure that if a point-interval pair $(\tau, [c, d])$ satisfies $[c, d] \subset (\tau - \delta(\tau), \tau + \delta(\tau))$, then $[c, d]$ contains at most one of the points $t_1, \ldots, t_m$. Moreover, if $t_k \in [c, d] \subset (\tau - \delta(\tau), \tau + \delta(\tau))$, then $\tau = t_k$.

(iii) Since $y_{r(t_k)} = x(t_k)_{r(t_k)}$, it follows from Theorem 1.8 that

$$\lim_{t \to t_k^+} \int_{t_k}^{t} L(s) \|y_{r(s)} - x(t_k)_{r(s)}\|_{\mathcal{B}} \mathrm{d}g(s) = L(t_k) \|y_{r(t_k)} - x(t_k)_{r(t_k)}\|_{\mathcal{B}} \Delta^+ g(t_k) = 0$$

for all $k \in 1, \ldots, m$. Thus the gauge δ might be chosen in such a way that

$$\int_{t_k}^{t_k + \delta(t_k)} L(s) \|y_{r(s)} - x(t_k)_{r(s)}\|_{\mathcal{B}} \mathrm{d}g(s) < \frac{\varepsilon}{2m + 1}, \ \ k \in \{1, \ldots, m\}.$$

(iv) From conditions (B) and (D), we infer

$$\begin{aligned}
\|y(t + \tau) - y(\tau)\| &= \left\| \int_{\tau}^{\tau + t} f(y_{r(s)}, s) \mathrm{d}g(s) + \int_{\tau}^{\tau + t} p(s) \mathrm{d}u(s) \right\| \\
&\leq \int_{\tau}^{\tau + t} M(s) \mathrm{d}g(s) + \int_{\tau}^{\tau + t} K(s) \mathrm{d}u(s) \\
&\leq h(\tau + t) - h(\tau),
\end{aligned}$$

which implies that

$$\|y(\tau^+) - y(\tau)\| \leq \Delta^+ h(\tau) < \varepsilon, \ \ \tau \in [t_0, t_0 + \sigma] \setminus \{t_1, \ldots, t_m\}.$$

Hence, we can select the gauge $\delta(\cdot)$ such that

$$\|y(\rho) - y(\tau)\| \leq \varepsilon$$

for all $\tau \in [t_0, t_0 + \sigma] \setminus \{t_1, \ldots, t_m\}$ and $\rho \in [\tau, \tau + \delta(\tau))$.

Let $\{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{l}$ be a δ-fine tagged partition of $[t_0, v]$. Then, by definition of x and G we have

$$(x(s_i) - x(s_{i-1}))(\xi) = \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \int_{s_{i-1}}^{\xi} f(y_{r(s)}, s) \mathrm{d}g(s) + \int_{s_{i-1}}^{\xi} p(s) \mathrm{d}u(s), & \xi \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} f(y_{r(s)}, s) \mathrm{d}g(s) + \int_{s_{i-1}}^{s_i} p(s) \mathrm{d}u(s), & \xi \in [s_i, t_0 + \sigma], \end{cases}$$

and

$$(Q(x(\tau_i), s_i) - Q(x(\tau_i), s_{i-1}))(\xi) = \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \int_{s_{i-1}}^{\xi} f(x(\tau_i)_{r(s)}, s)\mathrm{d}g(s) + \int_{s_{i-1}}^{\xi} p(s)\mathrm{d}u(s), & \xi \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} f(x(\tau_i)_{r(s)}, s)\mathrm{d}g(s) + \int_{s_{i-1}}^{s_i} p(s)\mathrm{d}u(s), & \xi \in [s_i, t_0+\sigma], \end{cases}$$

for all $i \in \{1, \dots, l\}$. Combining these expressions, we obtain that

$$\begin{aligned} &[(x(s_i) - x(s_{i-1}))(\xi) - (Q(x(\tau_i), s_i) - Q(x(\tau_i), s_{i-1}))(\xi)] \\ &= \begin{cases} 0, & \xi \in (-\infty, s_{i-1}], \\ \int_{s_{i-1}}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s), & \xi \in [s_{i-1}, s_i], \\ \int_{s_{i-1}}^{s_i} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s), & \xi \in [s_i, t_0+\sigma]. \end{cases} \end{aligned}$$

Consequently,

$$\begin{aligned} &\|(x(s_i) - x(s_{i-1})) - (Q(x(\tau_i), s_i) - Q(x(\tau_i), s_{i-1}))\|_Y \\ &= \sup_{\xi \in [s_{i-1}, s_i]} \left\| \int_{s_{i-1}}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s) \right\|. \end{aligned}$$

Since $x(\tau_i)_{r(s)} = y_{r(s)}$ for $s \leq \tau_i$, we have

$$\int_{s_{i-1}}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s) = \begin{cases} 0, & \xi \in [s_{i-1}, \tau_i], \\ \int_{\tau_i}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s), & \xi \in [\tau_i, s_i]. \end{cases}$$

We now use condition (C) to obtain the estimate

$$\begin{aligned} \left\| \int_{\tau_i}^{\xi} [f(y_{r(s)}, s) - f(x(\tau_i)_{r(s)}, s)]\mathrm{d}g(s) \right\| &\leq \int_{\tau_i}^{\xi} L(s) \|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}} \mathrm{d}g(s) \\ &\leq \int_{\tau_i}^{s_i} L(s) \|y_{r(s)} - x(\tau_i)_{r(s)}\|_{\mathcal{B}} \mathrm{d}g(s). \end{aligned}$$

Given a particular point-interval pair$(\tau_i, [s_{i-1}, s_i])$, there are two possibilities:

(a) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ contains a single point $t_k = \tau_i$.

(b) The intersection of $[s_{i-1}, s_i]$ and $\{t_1, \dots, t_m\}$ is empty.

We conclude this proof proceeding as in the proof of Theorem 3.19. □

By considering appropriate modifications for the perturbed system, the proof of the following theorem follows analogously to the one performed for Theorem 3.20. For that reason, we omit its proof.

Theorem 3.30. *Assume that $Q : \mathcal{O} \times [t_0, t_0+\sigma] \to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is the function defined by (3.27) and $\phi \in \mathcal{B}$. If $x : [t_0, t_0+\sigma] \to \mathcal{O}$ is a solution of the generalized ordinary differential equation (3.29) on the interval $[t_0, t_0+\sigma]$ with initial condition $x(t_0)$ given in (3.20), then the function $y \in \mathcal{O}$ defined by (3.21) is a solution of the measure functional differential equation with infinite time-dependent delay (3.26) and initial condition (3.10).*

In the next result, we establish the existence and uniqueness of solutions for the perturbed measure functional differential equations with time-dependent delay. We omit the proof because it is similar to the one performed for Theorem 3.24.

Theorem 3.31. *Let $\mathcal{O} \subset Y$ be an open subset having the prolongation property for $t \geq t_0$. Assume that $Q : \mathcal{O} \times [t_0, t_0 + \sigma] \to G((-\infty, t_0 + \sigma], \mathbb{R}^n)$ is the function defined by (3.27), $\phi \in \mathcal{B}$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ and $u : [t_0, t_0+\sigma] \to \mathbb{R}$ are left-continuous and nondecreasing functions. Assume further that the function $x(t_0)$ defined by (3.20) and*

$$z(t) = \begin{cases} \phi(t - t_0), & t \in (-\infty, t_0], \\ \phi(0) + f(\varphi, t_0)\Delta^+ g(t_0) + p(t_0)\Delta^+ u(t_0), & t \in (t_0, t_0 + \sigma] \end{cases}$$

are elements of $\mathcal{O}$, where φ is defined by $\varphi(\theta) = \phi(\theta + r(t_0) - t_0)$, for $\theta \in (-\infty, 0]$. Then there exists $\beta > 0$, and a function $y : (-\infty, t_0 + \beta] \to \mathbb{R}^n$ which is the unique solution of the initial value problem (3.26)-(3.10) on $(-\infty, t_0 + \beta]$.

Part II

Multivalued maps and applications to Cauchy problems governed by a first order differential inclusion

Chapter 4

Fixed points of multivalued maps under local Lipschitz conditions and applications

Throughout this chapter, we denote by X a Banach space provided with a norm $\|\cdot\|$. We assume that $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $(T(t))_{t\geq 0}$ on X. We will consider the abstract first order differential inclusion

$$\begin{aligned} x'(t) - Ax(t) &\in f(t, x(t)), \quad t \geq 0, && (4.1) \\ x(0) &= x_0 \in X, && (4.2) \end{aligned}$$

where $x(t) \in X$ and f is a set valued map defined on $[0, \infty) \times X$ whose properties will be specified later.

As a model, we consider a general heat equation described by a first order differential inclusion

$$\begin{aligned} \frac{\partial u(t,\xi)}{\partial t} - \frac{\partial^2 u(t,\xi)}{\partial \xi^2} &\in f(t, u(t,\cdot)), && (4.3) \\ u(t,0) &= u(t,\pi) = 0, && (4.4) \\ u(0,\xi) &= \varphi(\xi), && (4.5) \end{aligned}$$

for $t \geq 0$ and $\xi \in (0, \pi)$. In this system, we assume that f is a multivalued map, and the inclusion indicated in (4.3) will be explained in Section 4.3. Moreover, φ is an appropriate function.

The goal of this chapter is to establish a general result of fixed point in scales of Banach spaces, and combinig these results with the theory of measure of noncompactness, we establish the existence of solutions to the problem (4.1)-(4.2) and the existence of asymptotically almost periodic solutions to the problem (4.1)-(4.2).

4.1 Multivalued maps and measure on noncompactness

In this section, we will present the basic concepts and properties of the abstract Cauchy problem and the theory of multivalued functions on which this work is based. The terminology and notations that will be used throughout this chapter are those generally used in functional analysis. In particular, if $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are Banach spaces, we denote by $\mathcal{L}(Y, Z)$ the Banach space of bounded linear operators from Y into Z, and we abbreviate this notation to $\mathcal{L}(Y)$ whenever

$Z = Y$. If $y \in Y$, then $B_r(y, Y)$ denotes the closed ball with center at y and radius $r > 0$. When the space Y is clear from the context, we abbreviate this notation to $B_r(y)$. Moreover, for a compact interval $J \subseteq \mathbb{R}$, we denote by $C(J, Y)$ the space of continuous functions from J into Y endowed with the norm of uniform convergence. Similarly, $C_b([0,\infty), Y)$ is the space of bounded continuous functions from $[0,\infty)$ into Y provided with the norm of uniform convergence and $C_0([0,\infty), Y)$ is the subspace of $C_b([0,\infty), Y)$ consisting of functions that vanishes at infinite. Furthermore, $L^p(J, Y)$, $1 \le p \le \infty$, denotes the space of p-integrable functions in the Bochner sense from J into Y.

We next only mention a few concepts and properties concerning to the first order abstract Cauchy problem. We denote by $M \ge 1$ and $\omega_1 \in \mathbb{R}$ some constants such that $\|T(t)\| \le Me^{\omega_1 t}$ for $t \ge 0$.

The existence of solutions of the first order abstract Cauchy problem

$$x'(t) = Ax(t) + f(t), \ t \ge 0, \tag{4.6}$$

$$x(0) = x_0, \tag{4.7}$$

where $f : [0,\infty) \to X$ is a locally integrable function, and the existence of solutions for the semilinear first order abstract Cauchy have been discussed in many works [5,69]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \ t \ge 0, \tag{4.8}$$

is called mild solution of (4.6)-(4.7).

In Section 4.3 we will use the following uniqueness property.

Remark 4.1. *Let $f \in L^1([0,a], X)$ be a function that satisfies $\int_0^t T(t-s)f(s)ds = 0$ for all $0 \le t \le a$. Then $f(t) = 0$ a.e. $t \in [0,a]$.*

Proof. It follows from [69, Theorem 4.2.9] that the function $v(t) = \int_0^t T(t-s)f(s)ds$ is a strong solution of (4.6) with initial condition $x_0 = 0$. □

We recall some facts concerning multivalued analysis, which will be used later. Let (Ω, d) be a metric space. Throughout this chapter $\mathcal{P}(\Omega)$ denotes the collection of all nonempty subsets of Ω, $\mathcal{P}_b(\Omega)$ (respectively, $\mathcal{P}_c(\Omega)$) stands for the collection of all bounded (respectively, closed) nonempty subsets of Ω, and $\mathcal{P}_{cb}(\Omega)$ denotes the collection of all closed bounded nonempty subsets of Ω. The Hausdorff metric d_H on $\mathcal{P}_{cb}(\Omega)$ is given by

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$.

Let $F : \Omega \to \mathcal{P}(\Omega)$ be a multivalued map. A point $x \in \Omega$ is said to be a fixed point of F if

$x \in F(x)$. We denote $Fix(F)$ the set consisting of fixed points of F.

Let (Ω^1, d^1) be a metric space. A multivalued map $F : \Omega \to \mathcal{P}_{cb}(\Omega^1)$ is said to be k-contraction, where $0 \leq k < 1$, if

$$d_H^1(Fx, Fy) \leq kd(x, y), \ \forall x, y \in \Omega.$$

The following result relates these concepts and extends the Banach principle to multivalued mappings [37, Theorem I.2.3.1].

Theorem 4.2. *Let (Ω, d) be a complete metric space and let $F : \Omega \to \mathcal{P}_{cb}(\Omega)$ be a k-contraction map. Then F has a fixed point.*

We also need to relate the notion of fixed point with the concept of measure of non-compactness. For this reason, we next recall a few properties of this concept. For general information, see [4, 7, 15, 42, 53]. In this chapter, we use the notion of Hausdorff measure of noncompactness on the corresponding working space.

Definition 4.3. *Let B be a bounded subset of a metric space Ω. The Hausdorff measure of noncompactness of B is defined by*

$$\eta(B) = \inf\{\varepsilon > 0 : B \text{ has a finite cover by closed balls of radius } < \varepsilon\}.$$

Remark 4.4. *Let $B, B_1, B_2 \subseteq \Omega$ be bounded sets. The Hausdorff measure of non-compactness has the following properties.*

(a) If $B_1 \subseteq B_2$, then $\eta(B_1) \leqslant \eta(B_2)$.

(b) $\eta(B) = \eta(\overline{B})$.

(c) $\eta(B) = 0$ if and only if B is totally bounded.

(d) $\eta(B_1 \cup B_2) = \max\{\eta(B_1), \eta(B_2)\}$.

(e) The function $\eta : \mathcal{P}_{cb}(\Omega) \to [0, \infty)$ is d_H-continuous.

In what follows, we assume that Y is a normed space. For a set $B \subseteq Y$, we denote by $\overline{co}(B)$ the closed convex hull of the set B.

Remark 4.5. *Let $B, B_1, B_2 \subseteq Y$ be bounded sets. The following properties hold.*

(a) For $\lambda \in \mathbb{R}$, $\eta(\lambda B) = |\lambda| \eta(B)$.

(b) $\eta(B_1 + B_2) \leqslant \eta(B_1) + \eta(B_2)$, where $B_1 + B_2 = \{b_1 + b_2 : b_1 \in B_1, b_2 \in B_2\}$.

(c) $\eta(B) = \eta(\overline{co}(B))$.

For the proof of these properties, we refer the reader to the already mentioned references. Moreover, in these references the reader will find the development of the abstract concept of "measure of noncompactness" as well as numerous concrete examples of measure of noncompactness.

Henceforth we use the notations $\mathcal{K}(Y)$, $\upsilon(Y)$ and $\mathcal{K}\upsilon(Y)$ to denote the following sets

(s0) $\mathcal{K}(Y) = \{D \in \mathcal{P}(Y) : \text{D is compact}\}$.

(s1) $\upsilon(Y) = \{D \in \mathcal{P}(Y) : \text{D is convex}\}$.

(s2) $\mathcal{K}\upsilon(Y) = \mathcal{K}(Y) \cap \upsilon(Y)$.

Moreover, for a multivalued map $F : \Omega \to \mathcal{P}(Y)$, we denote

$$\begin{aligned} F^{-1}(V) &= \{w \in \Omega : F(w) \subseteq V\}, \\ F_+^{-1}(V) &= \{w \in \Omega : F(w) \cap V \neq \emptyset\}. \end{aligned}$$

Definition 4.6. *Let Ω be a metric space. A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is said to be:*

(i) Upper semi-continuous (u.s.c. for short) if $F^{-1}(V)$ is an open subset of Ω for all open set $V \subseteq Y$.

(ii) Closed if its graph $G_F = \{(w, y) : y \in F(w)\}$ is a closed subset of $\Omega \times Y$.

(iii) Compact if its range $F(\Omega)$ is relatively compact in Y.

(iv) Lower semi-continuous (l.s.c. for short) if $F_+^{-1}(V)$ is an open subset of Ω for all open set $V \subseteq Y$.

The following result is an inmediate consequence to the fact that $\Omega \setminus F_+^{-1}(V) = F^{-1}(Y \setminus V)$ for every $V \subset Y$.

Proposition 4.7. *A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is u.s.c. (respectively, l.s.c.), if and only if for every closed set $V \subset Y$, the set $F_+^{-1}(V)$ (respectively, $F^{-1}(V)$) is a closed subset of Ω.*

Definition 4.8. *Let $F : \Omega \to \mathcal{P}(Y)$ be a multivalued map and $f : \Omega \to Y$ be a singlevalued map. We shall say that f is a selection of F provided $f(x) \in F(x)$, for every $x \in \Omega$.*

The problem concerned to the existence of appropriate selections for multivalued mappings is very important in the fixed point theory. In what follows we introduce some selection theorems which will be crucial in the Section 4.3.

Theorem 4.9. ([53, Theorem 1.3.5.]) *Let X, Y be Banach spaces, $I \subset \mathbb{R}$ be a compact set, and let $F : I \times X \to \mathcal{K}(Y)$ be a multivalued map such that*

(i) for every $x \in X$, the multimap $F(\cdot, x) : I \to \mathcal{K}(Y)$ has a strongly measurable selection;

(ii) for every $t \in I$, the multimap $F(t,\cdot) : X \to \mathcal{K}(Y)$ is u.s.c.

Then for every strongly measurable function $q : I \to X$ there exists a strongly measurable selection $\varphi : I \to Y$ of the multivalued map $F(\cdot, q(\cdot)) : I \to \mathcal{K}(Y)$.

Definition 4.10. *Let $(\Omega, \mathcal{A})$ be a measurable space and let Y be a Banach space. A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is said to be measurable if $F_+^{-1}(V) \in \mathcal{A}$ for all open set $V \subseteq Y$.*

Below we shall present the Kuratowski–Ryll–Nardzewski selection theorem.

Theorem 4.11. ([36, Theorem 19.7]) *Let $(\Omega, \mathcal{A})$ be a measurable space and let Y be a separable complete space. Then every measurable multivalued map $F : \Omega \to \mathcal{P}(Y)$ has a measurable selection.*

It is clear that if Ω is a metric space and $\mathcal{A}$ is the Borel σ-algebra in Ω, then every l.s.c. map $F : \Omega \to \mathcal{P}(Y)$ is measurable.

In what follows, we assume that Y is a Banach space and Ω is a closed subset of Y. We denote by η any measure of noncompactness on Y that satisfies the properties mentioned in Remark 4.4 and Remark 4.5. The following concept is taken from [53, Definition 2.2.6].

Definition 4.12. *A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is said to be a condensing map with respect to η (abbreviated, η-condensing) if for every set $D \subset \Omega$ that is not relatively compact we have that $\eta(F(D)) \ngeq \eta(D)$.*

The next result is essential for the development of the rest of our work. We point out that if $F : \Omega \to Kv(Y)$ is u.s.c., then F is closed. This allows us to establish the following version of the fixed point theorem [53, Corollary 3.3.1].

Theorem 4.13. *Let M be a convex closed subset of Y, and let $F : M \to Kv(M)$ be a u.s.c. η-condensing multivalued map. Then $Fix(F)$ is a nonempty compact set.*

Next we study some properties of the measure of noncompactness on a space of functions with values in X. To establish some properties, in what follows we denote by χ the Hausdorff measure of noncompactness in X, and by β the Hausdorff measure of noncompactness in a space of continuous functions with values in X. We next collect some properties of measure β which are needed to establish our results. Let $J = [0, a]$.

Lemma 4.14. *Let $G : J \to \mathcal{L}(X)$ be a strongly continuous operator valued map. Let $D \subset X$ be a bounded set. Then $\beta(\{G(\cdot)x : x \in D\}) \le \sup_{0\le t\le a} \|G(t)\|\, \chi(D)$.*

The proof of this property is an immediate consequence from Definition 4.3.

Lemma 4.15. ([7]) *Let $W \subseteq C(J; X)$ be a bounded set. Then $\chi(W(t)) \leqslant \beta(W)$ for all $t \in J$. Furthermore, if W is equicontinuous on J, then $\chi(W(\cdot))$ is continuous on J, and*

$$\beta(W) = \sup\{\chi(W(t)) \ : \ t \in J\}.$$

Lemma 4.16. ([46, Lemma 2.9]) *Let $W \subseteq C(J;X)$ be a bounded set. Then there exists a countable set $W_0 \subseteq W$ such that $\beta(W_0) = \beta(W)$.*

Lemma 4.17. *Let $G : J \to \mathcal{L}(X)$ be a strongly continuous semigroup of linear operators, and $\Lambda : L^1(J,X) \to C(J,X)$ be the map defined by $\Lambda u(t) = \int_0^t G(t-s)u(s)ds$, for every $t \in J$. Then Λ satisfies the following conditions*

(S1) *There exists a constant $D \geq 0$ such that*

$$\|\Lambda u(t) - \Lambda v(t)\| \leq D \int_0^t \|u(t) - v(t)\| ds$$

for all $u, v \in L^1(J,X)$ and $t \in J$.

(S2) *For any compact $K \subset X$ and sequence $(u_n)_{n\in\mathbb{N}} \subset L^1(J,X)$ such that $(u_n(t))_{n\in\mathbb{N}} \subset K$ for a.e. $t \in J$, the weak convergence $u_n \xrightarrow{w} u$ implies that $\Lambda u_n \to \Lambda u$.*

Proof. The condition (S1) easily follows due to $(G(t))_{t\geq 0}$ is a strongly continuous semigroup of linear operators. Let $K \subset X$ be a compact set and let $(u_n)_{n\in\mathbb{N}} \subset L^1(J,X)$ be a sequence such that $(u_n(t))_{n\in\mathbb{N}} \subset K$ for a.e. $t \in J$. Suppose that $u_n \xrightarrow{w} u$ weakly. Since Λ is a continuous operator, it follows that $\Lambda u_n \xrightarrow[n\to\infty]{w} \Lambda u$ weakly. To conclude the condition (S2), it is enough to prove that $\{\Lambda u_n : n \in \mathbb{N}\}$ is relatively compact in $C(J,X)$. In fact, if this last assertion is true, then every sequence $(\Lambda u_{n_k})_{k\in\mathbb{N}} \subset \{\Lambda u_n : n \in \mathbb{N}\}$ has a convergent subsequence $(\Lambda u_{n_{k_j}})_{j\in\mathbb{N}}$ in $C(J,X)$, that means $\Lambda u_{n_{k_j}} \xrightarrow[j\to\infty]{\|\cdot\|_\infty} v$, with $v \in C(J,X)$. Consequently, $\Lambda u_{n_{k_j}} \xrightarrow[j\to\infty]{w} v$ weakly, and thus $v = \Lambda u$. This fact implies that $\Lambda u_n \xrightarrow[n\to\infty]{\|\cdot\|_\infty} \Lambda u$, obtaining the desired result. Therefore, we will prove that $\{\Lambda u_n : n \in \mathbb{N}\}$ is relatively compact in $C(J,X)$.

Let $t \in J$. Firstly note that for each $n \in \mathbb{N}$, the function $s \mapsto G(t-s)u_n(s)$, $s \in [0,t]$ takes values in a compact set $\hat{K} \subset X$. Then, by the Mean Value Theorem for the Bochner integral, we have

$$\Lambda u_n(t) \in t \cdot \overline{co}(\hat{K}).$$

Now, since the closed convex hull of a compact set is compact, we conclude that $\{\Lambda u_n(t) : n \in \mathbb{N}\} \subset X$ is relatively compact for $t \in J$.

On the other hand, for $u \in \{\Lambda u_n : n \in \mathbb{N}\}$ and $t \in J$, we have

$$\begin{aligned}
\Lambda u(t+h) - \Lambda u(t) &= \int_0^{t+h} G(t+h-s)u(s)ds - \int_0^t G(t-s)u(s)ds \\
&= \int_0^t [G(t+h-s) - G(t-s)]u(s)ds - \int_t^{t+h} G(t+h-s)u(s)ds \\
&= (G(h) - I)\int_0^t G(t-s)u(s)ds - \int_t^{t+h} G(t+h-s)u(s)ds \\
&\xrightarrow[h\mapsto 0]{} 0.
\end{aligned}$$

Therefore, $\{\Lambda u_n : n \in \mathbb{N}\}$ is an equicontinuous family of $C(J,X)$. Thus, by Arzelá–Ascoli Theorem we conclude that $\{\Lambda u_n : n \in \mathbb{N}\}$ is relatively compact in $C(J,X)$.

□

Definition 4.18. *A set $W \subseteq L^1(J,X)$ is said to be uniformly integrable if there exists a positive function $\mu \in L^1(J)$ such that $\|w(t)\| \le \mu(t)$ a.e. for $t \in J$ and all $w \in W$.*

Proposition 4.19. ([53, Proposition 4.2.1.]) *Assume that $\Omega \subset L^1(J,X)$ is uniformly integrable and the sets $\Omega(t)$ are relatively compact for a.e. $t \in J$. Then Ω is weakly compact in $L^1(J,X)$.*

Theorem 4.20. ([53, Theorem 4.2.2.]) *Let $W \subset L^1(J,X)$ be a uniformly integrable set. Assume that there is a positive function $q \in L^1(J)$ such that $\chi(W(t)) \le q(t)$ for a.e. $t \in J$. If $\Lambda : L^1(J,X) \to C(J,X)$ is an operator satisfying properties* (S1) *and* (S2)*, then*

$$\chi(\Lambda(W)(t)) \le 2D \int_0^t q(s)ds,$$

for all $t \in J$, where $D \ge 0$ is the constant introduced in condition (S1)*.*

Lemma 4.21. *Let $G : J \to \mathcal{L}(X)$ be a strongly continuous operator valued map such that G is continuous for the norm of operators on $(0,a]$, and $\Lambda : L^1(J,X) \to C(J,X)$ be the map defined by*

$$\Lambda(u)(t) = \int_0^t G(t-s)u(s)ds.$$

Let $W \subset L^1(J,X)$ be a uniformly integrable set. Assume that there is a positive function $q \in L^1(J)$ such that $\chi(W(t)) \le q(t)$ for a.e. $t \in J$. Then

$$\beta(\Lambda(W)) \le 2 \sup_{0\le t\le a} \|G(t)\| \int_0^a q(t)dt.$$

Proof. Applying Theorem 4.20, we can affirm that

$$\chi(\Lambda(W)(t)) \le 2 \sup_{0\le t\le a} \|G(t)\| \int_0^t q(s)ds$$

for every $t \in J$. Moreover, we will prove that $\Lambda(W)$ is an equicontinuous set of continuous functions. Let $M \ge 0$ be a constant such that $\|G(t)\| \le M$ for all $t \in J$, and $\mu \in L^1(J)$ such that $\|w(t)\| \le \mu(t)$ a.e. for $t \in J$ and all $w \in W$.

Initially we study the equicontinuity of $\Lambda(W)$ at $t = 0$. For $u \in W$, we estimate

$$\begin{aligned} \|\Lambda(u)(h)\| &= \left\| \int_0^h G(t-s)u(s)ds \right\| \\ &\le M \int_0^h \mu(s)ds \to 0,\ h \to 0, \end{aligned}$$

independent of $u \in W$, which implies that $\Lambda(W)$ is equicontinuous at $t = 0$.

We next study the equicontinuity of $\Lambda(W)$ at $t > 0$. Let $\varepsilon > 0$. We select $0 < \delta_1 < t/2$. Then $G : [\delta_1, a] \to \mathcal{L}(X)$ is uniformly continuous for the norm of operators. Consequently there exists

$0 < \delta < \delta_1$ such that

$$\|G(\xi + h) - G(\xi)\| \leq \varepsilon,$$

for all $\xi \in [2\delta_1, a]$ and $|h| < \delta$ such that $\xi + h \leq a$. To simplify the writing, we consider $h \geq 0$. We have that

$$\begin{aligned}
\Lambda(u)(t+h) - \Lambda(u)(t) &= \int_0^{t+h} G(t+h-s)u(s)ds - \int_0^t G(t-s)u(s)ds \\
&= \int_0^t [G(t+h-s) - G(t-s)]u(s)ds + \int_t^{t+h} G(t+h-s)u(s)ds \\
&= \int_0^{t-2\delta_1} [G(t+h-s) - G(t-s)]u(s)ds + \int_{t-2\delta_1}^t [G(t+h-s) - G(t-s)]u(s)ds \\
&\quad + \int_t^{t+h} G(t+h-s)u(s)ds.
\end{aligned}$$

From this decomposition, we can estimate

$$\|\Lambda(u)(t+h) - \Lambda(u)(t)\| \leq \varepsilon \int_0^{t-2\delta_1} \mu(s)ds + 2M \int_{t-2\delta_1}^t \mu(s)ds + M \int_t^{t+h} \mu(s)ds,$$

which shows that $\Lambda(u)(t+h) - \Lambda(u)(t) \to 0$ as $h \to 0$ independent of $u \in W$, which in turn implies that $\Lambda(W)$ is equicontinuous at t.

Using now Lemma 4.15 we obtain the assertion. □

4.2 Existence of fixed points

Let $(Y, \|\cdot\|)$ be a Banach space and let $F : Y \to \mathcal{P}_b(Y)$ be a map. Let Z be a closed vector subspace of Y which is invariant under F, that is to say, $F : Z \to \mathcal{P}_b(Z)$. In this section we establish the existence of fixed points of F in Z. We assume that F only satisfies certain local conditions on Y. To represent the idea of local conditions, we assume that there exists a scale of Banach spaces

$$(Y, \|\cdot\|) \ldots \hookrightarrow \ldots (Y_n, \|\cdot\|_n) \overset{R_{n-1,n}}{\hookrightarrow} (Y_{n-1}, \|\cdot\|_{n-1}) \hookrightarrow \ldots (Y_1, \|\cdot\|_1),$$

where $(Y_n, \|\cdot\|_n)$ are Banach spaces for $n \in \mathbb{N}$, $R_{n-1,n} : (Y_n, \|\cdot\|_n) \to (Y_{n-1}, \|\cdot\|_{n-1})$ are bounded surjective linear maps, and there exist bounded surjective linear maps $R_n : (Y, \|\cdot\|) \to (Y_n, \|\cdot\|_n)$, and u.s.c. maps $F_n : Y_n \to \mathcal{P}_{cb}(Y_n)$ for all $n \in \mathbb{N}$. We assume that F, F_n, $R_{n-1,n}$ and R_n are related as follows.

(H1) *Uniqueness property.* Let $y, z \in Y$ such that $R_n y \in F_n(R_n z)$ for all $n \in \mathbb{N}$, then $y \in F(z)$.

(H2) *Extension property.* For every $n \in \mathbb{N}$, and for every $y \in Y_{n+1}$ such that $y^n = R_{n,n+1}y \in F_n(y^n)$ there exists $z \in F_{n+1}(y)$ such that $R_{n,n+1}y = R_{n,n+1}z$.

(H3) *Inclusion property.* If $(y^n)_n$ is a sequence such that $y^n \in Y_n$, $y^n = R_{n,n+1}y^{n+1}$, and $\{\|y^n\|_n : n \in \mathbb{N}\}$ is a bounded set, then there exists $y \in Y$ such that $y^n = R_n y$ for all $n \in \mathbb{N}$.

(H4) *Concatenation property.* For every $n \in \mathbb{N}$,

$$R_{n,n+1}F_{n+1} \subseteq F_n R_{n,n+1}.$$

Remark 4.22. *It follows from (H4) that if $u^{n+1} \in Fix(F_{n+1})$, then $R_{n,n+1}u^{n+1} \in Fix(F_n)$. In fact,*

$$R_{n,n+1}u^{n+1} \in R_{n,n+1}F_{n+1}u^{n+1} \subseteq F_n R_{n,n+1}u^{n+1},$$

which shows that $R_{n,n+1}u^{n+1} \in Fix(F_n)$.

In what follows, to abbreviate the text, we represent by β a generic measure of noncompactness on Y_n that satisfies the properties mentioned in Remark 4.4 and Remark 4.5.

Lemma 4.23. *Assume that $F_n : Y_n \to Kv(Y_n)$, $n \in \mathbb{N}$, satisfy the conditions (H1)-(H4). Assume further that F_n is an u.s.c. β-condensing multivalued map for all $n \in \mathbb{N}$. If $y^n \in Fix(F_n)$, then there exists $y^{n+1} \in Fix(F_{n+1})$ such that $y^n = R_{n,n+1}y^{n+1}$.*

Proof. Let $C_{n+1} = \{y \in Y_{n+1} : y^n = R_{n,n+1}y\}$. Since $R_{n,n+1}$ is a linear bounded surjective map, we infer that C_{n+1} is a nonempty closed convex set. We define the map G_{n+1} by

$$G_{n+1}(y) = \{z \in F_{n+1}(y) : y^n = R_{n,n+1}z\},\ y \in Y_{n+1}.$$

It follows from (H2) that $G_{n+1}y \neq \emptyset$, and $G_{n+1} : C_{n+1} \to Kv(C_{n+1})$ is an u.s.c. β-condensing multivalued map. In fact, let $V \subset C_{n+1}$ be a closed subset. Since $G^{-1}_{n+1_+}(V) = R^{-1}_{n,n+1}(y^n) \cap F^{-1}_{n+1_+}(V)$, and F_{n+1} is an u.s.c. map, we conclude that G_{n+1} is an u.s.c. map. On the other hand, let $D \subset C_{n+1}$ be a bounded set that is not relatively compact. Now, since F_{n+1} is a β-condensing map, we can estimate

$$\begin{aligned}\beta\left(\{G_{n+1}(w) : w \in D\}\right) &= \beta(\{R^{-1}_{n,n+1}(y^n)\} \cap \{F_{n+1}(w) : w \in D\}) \\ &\leq \min\{\beta(R^{-1}_{n,n+1}(y^n)), \beta(\{F_{n+1}(w) : w \in D\})\} \\ &< \beta(D),\end{aligned}$$

and thus, G_{n+1} is a β-condensing map.

It follows from Theorem 4.13 that $Fix(G_{n+1})$ is a nonempty compact set. Therefore, there exists $y^{n+1} \in G_{n+1}(y^{n+1})$. This implies that $y^{n+1} \in F_{n+1}(y^{n+1})$ and $y^n = R_{n,n+1}y^{n+1}$. □

Theorem 4.24. *Assume that $F : Y \to \mathcal{P}_b(Y)$ and $F_n : Y_n \to Kv(Y_n)$ for $n \in \mathbb{N}$, satisfy conditions (H1)-(H4), and $\{\|z\|_n : z \in F_n(y), y \in Y_n, n \in \mathbb{N}\}$ is a bounded set. Assume further that F_n is an u.s.c. β-condensing multivalued map for all $n \in \mathbb{N}$. Then $Fix(F)$ is a nonempty set.*

Proof. It follows from Theorem 4.13 that $Fix(F_n)$ is a nonempty compact set. Proceeding inductively by using Lemma 4.23, we can construct a sequence $(y^n)_n$ such that $y^n \in Fix(F_n)$ and $y^n = R_{n,n+1}y^{n+1}$. It follows from our hypotheses that $\{\|y^n\|_n : n \in \mathbb{N}\}$ is a bounded set. Applying condition (H3), we infer that there exists $y \in Y$ such that $y^n = R_n y$ for all $n \in \mathbb{N}$. Since

$$y^n = R_n y \in F_n(R_n y),$$

for all $n \in \mathbb{N}$, using now (H1), we obtain that $y \in F(y)$. □

We next keep the notation y for the fixed point of F whose existence was established in Theorem 4.24.

Corollary 4.25. *Assume that $F : Y \to \mathcal{P}_b(Y)$ and $F_n : Y_n \to K\upsilon(Y_n)$, $n \in \mathbb{N}$, satisfy the conditions of Theorem 4.24. Let Z be a closed vector subspace of Y such that $F : Z \to \mathcal{P}_c(Z)$. Assume further that F satisfies the local Lipschitz condition*

$$d_H(Fx_2, Fx_1) \leq L(r, x)\|x_2 - x_1\|,$$

for all $x \in Y$, $r > 0$, and $x_1, x_2 \in B_r(x)$. If there exists $r_0 > 0$ such that $B_{r_0}(y) \cap Z \neq \emptyset$ and $L(r_0, y) < 1$, then F has a fixed point in Z.

Proof. Let $x \in B_{r_0}(y)$. Then

$$d(y, F(x)) \leq d_H(F(x), F(y)) \leq L(r_0, y)\|x - y\| < \|x - y\|,$$

which implies that $F(x) \subseteq B_{r_0}(y)$. Consequently, $F : B_{r_0}(y) \cap Z \to \mathcal{P}_c(B_{r_0}(y) \cap Z)$ and F is a $L(r_0, y)$-contraction. It follows from Theorem 4.2 that F has a fixed point in $B_{r_0}(y) \cap Z$. □

Corollary 4.26. *Assume that $F : Y \to \mathcal{P}_b(Y)$ and $F_n : Y_n \to K\upsilon(Y_n)$, $n \in \mathbb{N}$, satisfy the conditions of Theorem 4.24. Let Z be a closed vector subspace of Y such that $F : Z \to \mathcal{P}_c(Z)$. Assume further that there exists $r > 0$ such that $B_r(y) \cap Z \neq \emptyset$ and $F : B_r(y) \to \mathcal{K}\upsilon(B_r(y))$ is β-condensing. Then F has a fixed point in Z.*

Proof. Since $F : B_r(y) \cap Z \to \mathcal{K}\upsilon(B_r(y) \cap Z)$ is β-condensing, it follows from Theorem 4.13 that F has a fixed point in $B_r(y) \cap Z$. □

4.3 Applications to the abstract Cauchy problem

In this section, we establish some results of the existence of mild solutions of problem (4.1)-(4.2). Initially we will establish the general framework of conditions under which we will study this problem. Throughout this section, χ denotes the Hausdorff measure of noncompactness in X and β denotes the Hausdorff measure of noncompactness in any space $C([0, a], X)$ for $a > 0$. We assume that the semigroup $(T(t))_{t \geq 0}$ is uniformly asymptotically stable, that is, there exist constants $M \geq 1$ and $\omega > 0$ such that

$$\|T(t)\| \leq Me^{-\omega t},\ t \geq 0. \tag{4.9}$$

Moreover, in what follows, we assume that $f = f_1 + f_2$, where $f_1 : [0, \infty) \times X \to X$ and f_2 is a multivalued map from $[0, \infty) \times X$ into $\upsilon(X)$. We assume that f_1 satisfies the following conditions.

(F1) The function $f_1(\cdot, x) : [0, \infty) \to X$ is strongly measurable for each $x \in X$ and the function $f_1(\cdot, 0)$ is bounded on $[0, \infty)$.

(F2) For each $t \geq 0$, the function $f_1(t, \cdot) : X \to X$ is continuous.

(F3) There is a function $\nu \in L^1_{loc}([0, \infty))$ such that

$$\|f_1(t, x_2) - f_1(t, x_1)\| \leq \nu(t)\|x_2 - x_1\|, \text{ a.e. } t \geq 0,$$

for all $x_2, x_1 \in X$.

4.3.1 Existence under compactness conditions

In this subsection we will study the case characterized by $f_2 : [0, \infty) \times X \to \mathcal{K}v(X)$. We assume that f_2 satisfies the following properties:

(F4) The function $f_2(\cdot, x) : [0, \infty) \to \mathcal{K}v(X)$ admits a strongly measurable selection for each $x \in X$.

(F5) For each $t \geq 0$, the function $f_2(t, \cdot) : X \to \mathcal{K}v(X)$ is u.s.c.

(F6) For each $r > 0$, there is a function $\mu_r \in L^1_{loc}([0, \infty))$ such that $\sup_{t \geq 0} \int_0^t e^{-\omega(t-s)} \mu_r(s) ds < \infty$ and

$$\|f_2(t, x)\| := \sup\{\|v\| : v \in f_2(t, x)\} \leq \mu_r(t), \text{ a.e. } t \geq 0,$$

for all $x \in X$ with $\|x\| \leq r$.

(F7) There exists a positive function $k \in L^1([0, \infty))$ such that

$$\chi(f_2(t, \Omega)) \leq k(t)\chi(\Omega), \text{ a.e. } t \geq 0,$$

for all bounded set $\Omega \subseteq X$.

Remark 4.27. Let $x(\cdot) \in C_b([0, \infty), X)$. From conditions (F4)-(F6), and applying Theorem 4.9, we infer that the function $f_2(\cdot, x(\cdot)) : [0, \infty) \to \mathcal{K}v(X)$, $t \mapsto f_2(t, x(t))$, admits a Bochner locally integrable selection. As a consequence, the set

$$\mathcal{S}_{f_2,x} = \{u \in L^1_{loc}([0, \infty), X) : u(t) \in f_2(t, x(t)),\ t \in [0, \infty)\} \neq \emptyset,$$

and $\mathcal{S}_{f_2,x}$ is convex.

Motivated by expression (4.8), we introduce the following concept of mild solution to problem (4.1)-(4.2).

Definition 4.28. *A function $x(\cdot) \in C_b([0, \infty), X)$ is said to be a mild solution of problem (4.1)-(4.2) if the integral equation*

$$x(t) = T(t)x_0 + \int_0^t T(t-s) f_1(s, x(s)) ds + \int_0^t T(t-s) u(s) ds$$

is satisfied for $u \in \mathcal{S}_{f_2,x}$ and all $t \in [0, \infty)$.

When there is no possibility of confusion, we denote by $(L^1_{loc}([0,\infty),X),|||\cdot|||)$ the Banach space consisting of the equivalence classes of locally integrable functions $u:[0,\infty)\to X$ such that $\sup_{t\geq 0}\int_0^t e^{-\omega(t-s)}\|u(s)\|ds<\infty$, endowed with the norm

$$|||u|||=\sup_{t\geq 0}\int_0^t e^{-\omega(t-s)}\|u(s)\|ds.$$

We introduce now the operator $\Lambda : L^1_{loc}([0,\infty),X)\to C_b([0,\infty),X)$ given by

$$\Lambda u(t)=\int_0^t T(t-s)u(s)ds,\ t\geq 0. \tag{4.10}$$

It is clear that Λ is a bounded linear operator. Using Λ we can construct the multivalued map $\widetilde{\Lambda} : C_b([0,\infty),X)\to \upsilon(C_b([0,\infty),X))$ given by

$$\widetilde{\Lambda}x=\Lambda(\mathcal{S}_{f_2,x}).$$

We next define the solution map for problem (4.1)-(4.2) as follows. Let $x\in C_b([0,\infty),X)$. We define $\Gamma(x)$ to be the set formed by all functions v given by

$$v(t)=T(t)x_0+\int_0^t T(t-s)f_1(s,x(s))ds+\int_0^t T(t-s)u(s)ds,\ t\geq 0,$$

for $u\in\mathcal{S}_{f_2,x}$. It follows from our hypotheses that $v\in C_b([0,\infty),X)$. Hence, Γ is a multivalued map from $C_b([0,\infty),X)$ into $\mathcal{P}(C_b([0,\infty),X))$. Furthermore, it is clear that $x(\cdot)$ is a mild solution of problem (4.1)-(4.2) if and only if $x(\cdot)$ is a fixed point of Γ.

In order to apply our results of Section 4.2, we take $Y=C_b([0,\infty),X)$ and $Y_n=C([0,n],X)$ for $n\in\mathbb{N}$. The maps $R_n : Y\to Y_n$ are defined by $R_ny=y|_{[0,n]}$, and $R_{n,n+1} : Y_{n+1}\to Y_n$ are defined by $R_{n,n+1}y=y|_{[0,n]}$ for $n\in\mathbb{N}$. Proceeding as above, for $n\in\mathbb{N}$ and $x\in C([0,n],X)$, we define

$$\mathcal{S}^n_{f_2,x}=\{u\in L^1([0,n],X) : u(t)\in f_2(t,x(t)),\ t\in[0,n]\}\neq\emptyset.$$

We introduce the operator $\Lambda_n : L^1([0,n],X)\to C([0,n],X)$ given by

$$\Lambda_n u(t)=\int_0^t T(t-s)u(s)ds,\ 0\leq t\leq n. \tag{4.11}$$

It is clear that Λ_n is a bounded linear operator. Using Λ_n we can construct the multivalued map $\widetilde{\Lambda_n} : C([0,n],X)\to\upsilon(C([0,n],X))$ given by

$$\widetilde{\Lambda_n}x=\Lambda_n(\mathcal{S}^n_{f_2,x}).$$

In the following Proposition 4.29, Lemma 4.30 and Lemma 4.31 we will consider a compact interval $[0,d]\subset\mathbb{R}$, for $d>0$. It is worth to note that previous definitions are the same with $d>0$ instead of a fixed $n\in\mathbb{N}$.

Proposition 4.29. *Let $d>0$. Let $f_2 : [0,d]\times X\to\mathcal{K}\upsilon(X)$ be a multivalued map satisfying the* *properties* (F4)–(F7) *with* $[0,d]$ *instead of* $[0,\infty)$. *If* $x \in C([0,d],X)$, *then the set*

$$\mathcal{S}^d_{f_2,x} = \{u \in L^1([0,d],X) : u(t) \in f_2(t,x(t)),\ t \in [0,d]\} \neq \emptyset,$$

is convex, closed and weakly compact.

Proof. Since f_2 takes convex values, it is immediate that $\mathcal{S}^d_{f_2,x}$ is convex. We prove now that $\mathcal{S}^d_{f_2,x}$ is closed. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{S}^d_{f_2,x}$ be a sequence with $u_n \xrightarrow[n\to\infty]{\|\cdot\|_1} u$. Then, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}} \subset (u_n)_{n\in\mathbb{N}}$ such that $u_{n_k} \to u$ pointwise a.e. Moreover, for every $k \in \mathbb{N}$, $u_{n_k}(t) \in f_2(t,x(t))$ which is a closed set, hence $u(t) \in f_2(t,x(t))$ and thus, $u \in \mathcal{S}^d_{f_2,x}$. On the other hand, since f_2 takes compact values and satisfies condition (F6), it follows from Proposition 4.19 that $\mathcal{S}^d_{f_2,x}$ is weakly compact. □

Lemma 4.30. ([53, Lemma 5.1.1.]) *Let* $d > 0$. *Let* $f_2 : [0,d] \times X \to \mathcal{K}v(X)$ *be a multivalued map satisfying the properties* (F4)–(F7) *with* $[0,d]$ *instead of* $[0,\infty)$. *Assume the sequences* $(x_n)_{n\in\mathbb{N}} \subset C([0,d],X)$, $(u_n)_{n\in\mathbb{N}} \subset L_1([0,d],X)$, $u_n \in \mathcal{S}^d_{f_2,x_n}$, *for* $n \geq 1$ *are such that* $x_n \xrightarrow{\|\cdot\|_\infty} x$ *and* $u_n \xrightarrow{w} u$ *weakly. Then* $u \in \mathcal{S}^d_{f_2,x}$.

Since $T(\cdot)$ is a strongly continuous operator valued function, the assertion from Lemma 4.17 remains valid for Λ_d, with $d > 0$. Hence, combining our previous remarks, we have the following property.

Lemma 4.31. *Let* $d > 0$. *Let* $f_2 : [0,\infty) \times X \to \mathcal{K}v(X)$ *be a multivalued map satisfying conditions* (F4)–(F7). *Then* $\widetilde{\Lambda_d}$ *is an u.s.c. map with convex compact values.*

Proof. Notice that $\widetilde{\Lambda_d}(x) = \Lambda_d(\mathcal{S}^d_{f_2,x})$ is $\|\cdot\|$-closed and weakly compact. In fact, by the linearity of Λ_d, we have that $\Lambda_d(\mathcal{S}^d_{f_2,x})$ is a convex set. Further, since Λ_d is norm continuous and $\mathcal{S}^d_{f_2,x}$ is weakly compact, we have that $\Lambda_d(\mathcal{S}^d_{f_2,x})$ is weakly compact. Thus, by Mazur's Theorem, we obtain the desired result.

We prove now that $\widetilde{\Lambda_d}$ is an u.s.c. map. Let V be a closed subset of $C([0,d],X)$. We claim that $\widetilde{\Lambda_{d+}}^{-1}(V) = \{x \in C([0,d],X) : \widetilde{\Lambda_d}(x) \cap V \neq \emptyset\}$ is closed. In fact, let $(x_n)_{n\in\mathbb{N}} \subset C([0,d],X)$ be a sequence such that $x_n \xrightarrow{\|\cdot\|_\infty} x$ and $\widetilde{\Lambda_d}(x_n) \cap V \neq 0$. We take $y_n \in \widetilde{\Lambda_N}(x_n) \cap V$ and $u_n \in \mathcal{S}^d_{f_2,x_n}$, i.e. $u_n(t) \in f_2(t,x_n(t))$ for $t \in [0,d]$, such that $y_n = \Lambda_d(u_n)$.

Since $\{x_n(t) : n \in \mathbb{N}\}$ is a bounded set, it follows from (F7) that $\{f_2(t,x_n(t)) : n \in \mathbb{N}\}$ is relatively compact. Hence, the set $\{u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact. Furthermore, by (F6), we obtain that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded. Now, using Proposition 4.19 we conclude that $\{u_n\}_{n\in\mathbb{N}}$ is weakly compact, and as a consequence, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $u_{n_k} \xrightarrow{w} u$ weakly in $L_1([0,d],X)$. Therefore, by Lemma 4.30 we have that $u \in \mathcal{S}^d_{f_2,x}$. On the other hand, since $\Lambda_d(u_{n_k}) \xrightarrow{w} \Lambda_d(u)$ weakly in $C([0,d],X)$ and the operator Λ_d satisfies the condition (S2) of Lemma 4.17, we obtain that $\Lambda_d(u_{n_k}) = y_{n_k} \xrightarrow{\|\cdot\|_\infty} \Lambda_d(u)$. Now, since V is closed, we have that $\Lambda_d(u) \in \widetilde{\Lambda_d}(x) \cap V$, and thus $x \in \widetilde{\Lambda_{d+}}^{-1}(V)$. □

Proceeding in a similar way, we can establish the following property.

Lemma 4.32. *Let $f_2 : [0,\infty) \times X \to \mathcal{K}v(X)$ be a multivalued map satisfying conditions (F4)-(F7). Then $\widetilde{\Lambda}$ is an u.s.c. map with convex closed values.*

Proof. Following the proof of Lemma 4.31, we note that the assertion is a consequence from the fact that $\mathcal{S}^n_{f_2,x}$ is weakly compact. Since the Lebesgue's measure of $[0,\infty)$ is σ-finite, using a standard diagonal selection process and [19, Corollary 2.6] we conclude that $\mathcal{S}_{f_2,x}$ is weakly compact. □

For $x \in C([0,n],X)$, we define $\Gamma_n(x)$ as the set formed by all functions v given by

$$v(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,x(s))ds + \int_0^t T(t-s)u(s)ds,\ t \in [0,n],$$

for $u \in \mathcal{S}^n_{f_2,x}$.

Proposition 4.33. *Assume that conditions* (F1)-(F7) *are satisfied. Then the multivalued map $\Gamma_n : C([0,n],X) \to Kv(C([0,n],X))$ is u.s.c. for all $n \in \mathbb{N}$, and the scheme $(\Gamma, \Gamma_n, R_n, R_{n,n+1})_{n\in\mathbb{N}}$ satisfies conditions* (H1)-(H4) *from Section 4.2.*

Proof. (i) To prove (H1), we consider $y, z \in C_b([0,\infty),X)$ such that $R_n y \in \Gamma_n(R_n z)$ for all $n \in \mathbb{N}$. This means that

$$y(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,z(s))ds + \int_0^t T(t-s)u^n(s)ds,\ t \in [0,n],\ n \in \mathbb{N},$$

where $u^n \in \mathcal{S}^n_{f_2,z}$. Applying Remark 4.1, we have that $u^n = u^{n+1}|_{[0,n]}$. This allows us to define $u(t) = u^n(t)$ for $0 \le t \le n$. Hence,

$$y(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,z(s))ds + \int_0^t T(t-s)u(s)ds,\ t \ge 0.$$

Moreover, from (F6) it follows that $u \in (L^1_{loc}([0,\infty),X), |||\cdot|||)$, and combining this assertion with the previous expression we conclude that $y \in \Gamma(z)$.

(ii) We now consider $n \in \mathbb{N}$ and $y \in C([0,n+1],X)$ such that $y^n = R_{n,n+1}y \in \Gamma_n(y^n)$. This implies that

$$y(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,y(s))ds + \int_0^t T(t-s)u^n(s)ds,\ t \in [0,n],$$

for some $u^n \in \mathcal{S}^n_{f_2,y^n}$. It follows from Remark 4.27 that there exists a locally integrable function $\tilde{u}$ defined on $[0,\infty)$ such that $\tilde{u} \in \mathcal{S}_{f_2,y}$. Defining

$$v(t) = \begin{cases} u^n(t), & 0 \le t \le n, \\ \tilde{u}(t), & n < t \le n+1, \end{cases}$$

we obtain that $v(\cdot)$ is an extension of u^n such that $v \in \mathcal{S}^{n+1}_{f_2,y}$.

We define $z(\cdot)$ by

$$z(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,y(s))ds + \int_0^t T(t-s)v(s)ds,\ t \in [0,n+1].$$

It is clear that $z \in \Gamma_{n+1}(y)$ and $y(t) = z(t)$ for all $t \in [0, n]$. This shows that (H2) is fulfilled.

(iii) To prove that (H3) holds, we take a sequence $(y^n)_n$ such that $y^n \in C([0, n], X)$, $y^n = R_{n,n+1}y^{n+1}$, and $\|y^n\| \le r$ for some $r \ge 0$ and all $n \in \mathbb{N}$. This allows us to define $y(t) = y^n(t)$ for $0 \le t \le n$. It is clear that $\|y(t)\| \le r$ for all $t \ge 0$, which implies that $y \in C_b([0, \infty), X)$ and $y^n = R_n y$ for all $n \in \mathbb{N}$.

(iv) Condition (H4) arises easily from the construction.

Finally, the fact that $\Gamma_n : C([0, n], X) \to Kv(C([0, n], X))$ is an u.s.c. map follows as a direct consequence of Lemma 4.31 □

We are now in position to prove our first result of this section. A strongly continuous semigroup of bounded linear operators $(T(t))_{t\ge0}$ is said to be immediately norm continuous if the function $T : (0, \infty) \to \mathcal{L}(X)$ is continuous for the norm of operators in $\mathcal{L}(X)$ ([20, Definition II.4.17]).

Theorem 4.34. *Let $(T(t))_{t\ge0}$ be an immediately norm continuous semigroup. Assume conditions* (F1)-(F7) *and* (4.9) *hold. Assume further the following conditions are fulfilled:*

$$M \sup_{t\ge0} \int_0^t e^{-\omega(t-s)}\nu(s)ds + M \liminf_{r>0} \sup_{t\ge0} \int_0^t e^{-\omega(t-s)} \frac{\mu_r(s)}{r} ds \quad < \quad 1, \tag{4.12}$$

$$M\Big[\sup_{0\le t<\infty} \int_0^t e^{-\omega(t-s)}\nu(s)ds + 2\int_0^\infty k(t)dt\Big] \quad < \quad 1. \tag{4.13}$$

Then there exists a mild solution $y(\cdot)$ of problem (4.1)-(4.2).

Proof. Let $n \in \mathbb{N}$. It follows from our hypotheses and Proposition 4.33 that Γ_n is an u.s.c. multivalued map with convex compact values.

Using (4.12) we can prove that there exists $R > 0$ such that $\Gamma_n(B_R(0, Y_n)) \subseteq B_R(0, Y_n)$. In fact, it follows from (4.12) that there exists $R > 0$ large enough such that

$$M\left(\frac{\|x_0\|}{R} + \frac{1}{R}\sup_{t\ge0}\int_0^t e^{-\omega(t-s)}\|f_1(s, 0)\|ds + \sup_{t\ge0}\int_0^t e^{-\omega(t-s)}\nu(s)ds + \sup_{t\ge0}\int_0^t e^{-\omega(t-s)}\frac{\mu_R(s)}{R}ds\right)$$

is smaller than 1.

Let $x \in B_R(0, Y_n)$ and $v \in \Gamma_n(x)$. This implies that

$$v(t) = T(t)x_0 + \int_0^t T(t - s)f_1(s, x(s))ds + \int_0^t T(t - s)u(s)ds$$

for $u \in \mathcal{S}^n_{f_2,x}$. Hence

$$\begin{aligned} \|v(t)\| &\le Me^{-\omega t}\|x_0\| + M\int_0^t e^{-\omega(t-s)}\|f_1(s, 0)\|ds \\ &+ M\int_0^t e^{-\omega(t-s)}\nu(s)dsR + M\int_0^t e^{-\omega(t-s)}\mu_R(s)ds \\ &\le R, \end{aligned}$$

for all $t \ge 0$.

Next we show that Γ_n is β-condensing on $B_R(0, Y_n)$. Let $\Omega \subset B_R(0, Y_n)$. It follows from Lemma 4.16 that there exists a sequence $(v_k)_k$ in $\Gamma_n(\Omega)$ such that $\beta(\Gamma_n(\Omega)) = \beta(\{v_k : k \in \mathbb{N}\})$. We can write $v_k \in \Gamma_n(x_k)$ for some $x_k \in \Omega$. Using (4.11) we have

$$v_k(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x_k(s))ds + \Lambda_n(u_k)(t),\ 0 \le t \le n, \tag{4.14}$$

for $u_k \in \mathcal{S}^n_{f_2,x_k}$.

Using Lemma 4.14, we obtain

$$\begin{aligned}\beta(\{v_k(\cdot) : k \in \mathbb{N}\}) \le M \max_{0\le t\le n} \int_0^t e^{-\omega(t-s)}\nu(s)ds\beta(\{x_k(\cdot) : k \in \mathbb{N}\})\\ + \beta(\{\Lambda_n(u_k)(\cdot) : k \in \mathbb{N}\}).\end{aligned} \tag{4.15}$$

On the other hand, since $u_k \in \mathcal{S}^n_{f_2,x_k}$, for $t \in [0, n]$ we have that $u_k(t) \in f_2(t, x_k(t))$ a.e. This implies that $\{u_k : k \in \mathbb{N}\}$ is uniformly integrable and, applying condition (F7), we obtain

$$\chi(\{u_k(t) : k \in \mathbb{N}\}) \le k(t)\chi(\{x_k(t) : k \in \mathbb{N}\}),\ a.e.\ t \in [0, n].$$

Combining this estimate with Lemma 4.21, we infer that

$$\beta(\{\Lambda_n(u_k)(\cdot) : k \in \mathbb{N}\}) \le 2M\beta(\{x_k : k \in \mathbb{N}\}) \int_0^n k(t)dt.$$

Substituting in (4.15), and using (4.13), we obtain

$$\begin{aligned}\beta(\{v_k(\cdot) : k \in \mathbb{N}\}) \le& M \max_{0\le t\le n} \int_0^t e^{-\omega(t-s)}\nu(s)ds\beta(\{x_k(\cdot) : k \in \mathbb{N}\}) + 2M\beta(\{x_k : k \in \mathbb{N}\}) \int_0^n k(t)dt\\ \le& M\Big[\max_{0\le t\le n} \int_0^t e^{-\omega(t-s)}\nu(s)ds + 2\int_0^n k(t)dt\Big]\beta(\Omega),\end{aligned}$$

which implies that Γ_n is a β-condensing map.

Finally, taking $Z = C_b([0, \infty), X)$ and applying Theorem 4.24, we infer the existence of a mild solution $y(\cdot)$ of problem (4.1)-(4.2). □

We point out that the constant $R > 0$ defined in the proof of Theorem 4.34 is independent of $n \in \mathbb{N}$.

Corollary 4.35. *Assume that X is a reflexive space. Let $(T(t))_{t\ge0}$ be a compact semigroup. Assume conditions* (F1)-(F6), (4.9) *and* (4.12) *hold. Then there exists a mild solution $y(\cdot)$ of problem (4.1)-(4.2).*

Proof. Using the reflexivity of X and the compactness of Λ_n, we can establish that the assertions from Lemma 4.31 hold. Let $R > 0$ be the constant defined in the proof of Theorem 4.34. Using that $T(t)$ is a compact operator for all $t > 0$, we can show that $\Gamma_n(B_R(0, Y_n))$ is a relatively compact set. We conclude the proof arguing as in the proof of Theorem 4.34. □

4.3.2 Existence under measurability conditions

We can avoid the condition that f_2 has compact values used in the Theorem 4.34 applying the Kuratowski-Ryll-Nardzewski Theorem, see Theorem 4.11.

Initially, we recall the concept of Lipschitzian map ([76]). Let (Ω, d) be a metric space.

Definition 4.36. *A multivalued map $f : \Omega \to \mathcal{P}(X)$ is said to be:*

(a) Lipschitzian if there exists $L \geq 0$ such that

$$f(\omega_1) \subseteq f(\omega_2) + Ld(\omega_1, \omega_2)B_1(0, X), \tag{4.16}$$

for all $\omega_1, \omega_2 \in \Omega$.

(b) Locally Lipschitzian if for every $\omega_0 \in \Omega$ there exist $\varepsilon > 0$ and $L \geq 0$ such that (4.16) holds for all $\omega_1, \omega_2 \in B_\varepsilon(\omega_0)$.

In particular, we can adapt the previous concept for maps defined on product spaces.

Definition 4.37. *We say that a multivalued map $f : [0, d] \times X \to \mathcal{P}(X)$ is locally Lipschitzian if for every $(t_0, x_0) \in [0, d] \times X$ there exist $\varepsilon > 0$ and $L_1, L_2 \geq 0$ such that*

$$f(t_2, x_2) \subseteq f(t_1, x_1) + (L_1|t_2 - t_1| + L_2\|x_2 - x_1\|)B_1(0, X),$$

for all $|t_i - t_0| \leq \varepsilon$ and $\|x_i - x_0\| \leq \varepsilon$ for $i = 1, 2$.

We next denote by m the Lebesgue measure on $[0, d]$. We say that a function $u : [0, d] \to X$ is m-measurable if it is strongly measurable in the Bochner sense. The reader can see [45, 61] for properties of m-measurable functions.

Proposition 4.38. *Assume that X is a separable Banach space. Let $f_2 : [0, d] \times X \to \mathcal{P}(X)$ be a locally Lipschitzian map with closed values. Let $x : [0, d] \to X$ be a continuous function. Then there exists a m-measurable function $u : [0, d] \to X$ such that $u(t) \in f_2(t, x(t))$ for all $t \in [0, d]$.*

Proof. Let $g : [0, d] \to \mathcal{P}(X)$ be given by $g(t) = f_2(t, x(t))$. We first show that g is l.s.c. In fact, let $V \subseteq X$ be an open set and $I = \{t \in [0, d] : g(t) \cap V \neq \emptyset\}$. If $t_0 \in I$, then there exists $u_0 \in g(t_0) \cap V$. Since V is an open set, there exists $\varepsilon > 0$ such that $B_\varepsilon(u_0) \subset V$. Moreover, there exists $\varepsilon_1 > 0$ such that

$$u_0 \in f_2(t_0, x(t_0)) \subseteq f_2(t, x(t)) + (L_1|t - t_0| + L_2\|x(t) - x(t_0)\|)B_1(0, X),$$

for all t such that $|t - t_0| \leq \varepsilon_1$ and $\|x(t) - x(t_0)\| \leq \varepsilon_1$. We can take $\varepsilon_1 > 0$ small enough so that $(L_1 + L_2)\varepsilon_1 < \varepsilon$. Since $x(\cdot)$ is continuous, there exists $0 < \delta < \varepsilon_1$ such that $\|x(t) - x(t_0)\| < \varepsilon_1$ for $|t - t_0| < \delta$. Combining these assertions, if $|t - t_0| < \delta$, we infer that there exists $u \in f_2(t, x(t))$ that satisfies

$$u_0 = u + \alpha B_1(0, X),$$

where $0 \leq \alpha < \varepsilon$. Consequently, $u \in B_\varepsilon(u_0) \subset V$ and $g(t) \cap V \neq \emptyset$. Hence we can affirm that $t \in I$ and I is open.

It follows from the Kuratowski-Ryll-Nardzewski theorem (Theorem 4.11) that there exists a measurable function $u : [0, d] \to X$ such that $u(t) \in f_2(t, x(t))$ for all $t \in [0, d]$. Applying now [61, Proposition 2.2.6], we infer that u is m-measurable. $\square$

We next weaken the concept of u.s.c map.

Definition 4.39. *Let Ω be a metric space and let Y be a Banach space. A multivalued map $F : \Omega \to \mathcal{P}(Y)$ is said to be weakly upper semi-continuous (w.u.s.c. for short) if $F^{-1}(V)$ is an open subset of Ω for all weakly open set $V \subseteq Y$.*

It is clear from Definition 4.39 that $F : \Omega \to \mathcal{P}(Y)$ is w.u.s.c. if and only if $F_+^{-1}(V)$ is a closed subset of Ω for all weakly closed set $V \subseteq Y$. We use this equivalence in the proof of the next Proposition.

Proposition 4.40. *Assume X is a reflexive space. Let $F : X \to v(X)$ be a locally Lipschitzian map with closed values that takes bounded sets into bounded sets. Then F is w.u.s.c.*

Proof. Let $V \subset X$ be a weakly closed set and $x_n \in F_+^{-1}(V)$, $n \in \mathbb{N}$, such that $x_n \to x$ as $n \to \infty$. Let $y_n \in F(x_n) \cap V$. Since $\{y_n : n \in \mathbb{N}\}$ is a bounded set, there is a subsequence $(y_{n_k})_k$ of $(y_n)_n$ which converges to y weakly. Since

$$F(x_{n_k}) \subseteq F(x) + w_k$$

where $w_k \to 0$ as $k \to \infty$, we can affirm that there exists $z_k \in F(x)$ such that $y_{n_k} = z_k + w_k$ and $z_k \to y$ as $k \to \infty$ weakly. Since $F(x)$ is a closed convex set, we infer that $y \in F(x)$. Hence $y \in F(x) \cap V$, which implies that $x \in F_+^{-1}(V)$. $\square$

Example 4.41. *Let $X = L^2([0, \pi])$, $a, b : \mathbb{R} \to \mathbb{R}$, $a \leq b$, functions that satisfy the Lipschitz conditions*

$$\begin{aligned} |a(x_2) - a(x_1)| &\leq a_0|x_2 - x_1|, \\ |b(x_2) - b(x_1)| &\leq b_0|x_2 - x_1|, \end{aligned}$$

for some positive constants a_0, b_0, and all $x_1, x_1 \in \mathbb{R}$. Let $f : X \to v(X)$ be the multivalued function given by

$$f(x) = \{u \in X : a(x(\xi)) \leq u(\xi) \leq b(x(\xi)),\ 0 \leq \xi \leq \pi\}.$$

Then f is a w.u.s.c. Lipschitzian map with closed bounded values.

Proof. It follows from [62, Chapter 5] that $a(x(\cdot)), b(x(\cdot)) \in X$ for all $x \in X$. This implies that $f(x) \neq \emptyset$. Moreover, it is easy to see that $f(x)$ is a closed bounded set in X. Let $L = \max\{a_0, b_0\}$. Then

$$f(x_2) \subset f(x_1) + L\|x_2 - x_1\|B_1(0, X),$$

for all $x_1, x_2 \in X$. In fact, if $u \in f(x_2)$, for every $\xi \in [0,\pi]$, we have

$$\begin{aligned} u(\xi) &\in [a(x_2(\xi)), b(x_2(\xi))] \\ &\subseteq [a(x_1(\xi)) - a_0|x_2(\xi) - x_1(\xi)|, b(x_1(\xi)) + b_0|x_2(\xi) - x_1(\xi)|] \\ &\subseteq [a(x_1(\xi)) - L|x_2(\xi) - x_1(\xi)|, b(x_1(\xi)) + L|x_2(\xi) - x_1(\xi)|]. \end{aligned}$$

We define the sets

$$\begin{aligned} E_1 &= \{\xi \in [0,\pi] : u(\xi) < a(x_1(\xi))\}, \\ E_2 &= \{\xi \in [0,\pi] : a(x_1(\xi)) \leq u(\xi) \leq b(x_1(\xi))\}, \\ E_3 &= \{\xi \in [0,\pi] : b(x_1(\xi)) < u(\xi)\}. \end{aligned}$$

Let $v(\xi) = u(\xi) + L|x_2(\xi) - x_1(\xi)|(\chi_{E_1} - \chi_{E_3})$, where χ_E denotes the characteristic function of E. Let $w(\xi) = L|x_2(\xi) - x_1(\xi)|(\chi_{E_1} - \chi_{E_3})$. It follows from this construction that $v \in f(x_1)$, $w \in X$, and

$$\|w\| = \left(\int_0^\pi |w(\xi)|^2 d\xi\right)^{1/2} \leq L\|x_2 - x_1\|,$$

which shows that f is a Lipschitzian map.

Finally, we infer from Proposition 4.40 that f is w.u.s.c. $\square$

We next modify slightly conditions (F6)-(F7).

(F6') For each $r > 0$, there is a function $\mu_r \in L^2_{loc}([0,\infty))$ such that

$$\|f_2(t,x)\| := \sup\{\|v\| : v \in f_2(t,x)\} \leq \mu_r(t),\ a.e.\ t \geq 0,$$

for all $x \in X$ with $\|x\| \leq r$.

(F7') There exists a positive function $k \in L^2([0,\infty))$ such that

$$\chi(f_2(t,\Omega)) \leq k(t)\chi(\Omega),\ a.e.\ t \geq 0,$$

for all bounded set $\Omega \subseteq X$.

The following consequence is immediate.

Corollary 4.42. *Assume that the hypotheses from Proposition 4.38 are fulfilled, and that condition* (F6') *holds. Let u be the function whose existence was established in Proposition 4.38. Then $u \in L^2([0,d],X)$.*

Next we assume that the hypotheses from Corollary 4.42 hold. Proceeding as in the previous part, for $d > 0$ and $x \in C([0,d],X)$, we define

$$\mathcal{S}^d_{f_2,x} = \{u \in L^2([0,d],X) : u(t) \in f_2(t,x(t)),\ t \in [0,d]\} \neq \emptyset.$$

We consider Λ_d given by (4.11) on $L^2([0,d],X)$. In similar way, for $x \in C_b([0,\infty),X)$ we define

$$\mathcal{S}_{f_2,x} = \{u \in L^2_{loc}([0,\infty),X) : u(t) \in f_2(t,x(t)),\ t \in [0,\infty)\} \neq \emptyset.$$

Without possibility of confusion, in this case we denote by $(L^2_{loc}([0,\infty),X), |||\cdot|||)$ the Banach space consisting of the equivalence classes of locally integrable functions $u:[0,\infty)\to X$ that satisfy the condition $\sup\limits_{t\geq 0}\int_0^t e^{-\omega(t-s)}\|u(s)\|^2 ds<\infty$, endowed with the norm

$$|||u|||=\sup_{t\geq 0}\left(\int_0^t e^{-\omega(t-s)}\|u(s)\|^2 ds\right)^{1/2}.$$

We introduce now the operator $\Lambda: L^2_{loc}([0,\infty),X)\to C_b([0,\infty),X)$ given by (4.10). It is clear that Λ is a bounded linear operator.

Remark 4.43. *The assertion in Lemma 4.30 remains valid for w.u.s.c. maps. Specifically, let $F:[0,d]\times X\to \upsilon(X)$ be a map with closed values such that $F(t,\cdot)$ is w.u.s.c. for each $t\in[0,d]$. Assume that $x_n\in C([0,d],X)$, $x_n\to x_0$, $n\to\infty$, and $u_n\in \mathcal{S}^d_{F,x_n}$ is a sequence that converges weakly to u_0. Then $u_0\in\mathcal{S}^d_{F,x_0}$.*

Proposition 4.44. *Assume that X is a separable reflexive Banach space and that $(T(t))_{t\geq 0}$ is an immediately norm continuous semigroup. Let $f_2:[0,\infty)\times X\to\upsilon(X)$ be a locally Lipschitzian map with closed values such that* (F6') *holds. Then $\widetilde{\Lambda}_d: C([0,d],X)\to\mathcal{K}\upsilon(C([0,d],X))$ is an u.s.c. map.*

Proof. We separate the proof in three steps.

(i) Initially we prove that $\widetilde{\Lambda}_d(x)=\Lambda_d(\mathcal{S}^d_{f_2,x})$ is closed for any $x\in C([0,d],X)$. Let $u_n\in\mathcal{S}^d_{f_2,x}$ such that $y_n=\Lambda_d(u_n)\to y$ as $n\to\infty$. Since $L^2([0,d],X)$ is a reflexive space, there exists a subsequence $(u_{n_k})_k$ that converges weakly to some function u. Moreover, $\mathcal{S}^d_{f_2,x}$ is a convex and norm closed set. This implies that $\mathcal{S}^d_{f_2,x}$ is a weakly closed set and $u\in\mathcal{S}^d_{f_2,x}$. Since Λ_d is norm continuous, it is also weakly continuous, and $\Lambda_d(u_{n_k})\to\Lambda_d(u)=y$ as $k\to\infty$.

(ii) We now prove that $\widetilde{\Lambda}_d(x)$ is a relatively compact set. Let $(u_n)_n$ be a sequence in $\mathcal{S}^d_{f_2,x}$. Proceeding as in (i), there exists a subsequence $(u_{n_k})_k$ that converges weakly to some function u and $\Lambda_d(u_{n_k})\to\Lambda_d(u)$ as $k\to\infty$ weakly in $C([0,d],X)$. This implies that $\Lambda_d(u_{n_k})(t)\to\Lambda_d(u)(t)$ as $k\to\infty$ for all $t\in[0,d]$. Moreover, proceeding as in the proof of Lemma 4.21, we obtain that $\{\Lambda_d(u_n):n\in\mathbb{N}\}$ is an equicontinuous set.

(iii) We can argue as in [53, Corollary 5.1.2] to conclude that $\widetilde{\Lambda}_d$ is u.s.c. Specifically, let V be a closed subset of $C([0,d],X)$ and $x_n\in C([0,d],X)$ such that $x_n\to x$ as $n\to\infty$ and $\widetilde{\Lambda}_d(x_n)\cap V\neq\emptyset$. We take $y_n\in\widetilde{\Lambda}_d(x_n)\cap V$ and $u_n\in\mathcal{S}^d_{f_2,x_n}$ such that $y_n=\Lambda_d(u_n)$. Since $\{u_n:n\in\mathbb{N}\}$ is a bounded set in $L^2([0,d],X)$, there exists a subsequence $(u_{n_k})_k$ that converges weakly to some function u. This implies that $y_{n_k}\to y=\Lambda_d(u)$, $k\to\infty$, weakly. Furthermore, proceeding as in (ii), we can see that the set $\cup_{n=1}^{\infty}\widetilde{\Lambda}_d(x_n)$ is relatively compact. Therefore, there is a subsequence of $(y_{n_k})_k$ which converges uniformly. This implies that $y\in V$. Using now Remark 4.43, we obtain that $u\in\mathcal{S}^d_{f_2,x}$, which in turn implies that $y\in\widetilde{\Lambda}_d(x)$. $\square$

Using Proposition 4.44 in the space $C([0,n];X)$ for $n\in\mathbb{N}$, we can argue as in the proofs carried out from Proposition 4.33 and Theorem 4.34 in order to establish the following property.

Theorem 4.45. *Assume that X is a separable reflexive Banach space and that $(T(t))_{t\geq 0}$ is an immediately norm continuous semigroup. Let $f_1 : [0,\infty)\times X \to X$ be a function that satisfies* (F1)-(F3)*, and let $f_2 : [0,\infty)\times X \to \upsilon(X)$ be a locally Lipschitzian map with closed values such that conditions* (F6')-(F7') *are fulfilled. Assume further that conditions* (4.9)*,* (4.12) *and* (4.13) *hold. Then there exists a mild solution $y(\cdot)$ of problem (4.1)-(4.2).*

In similar way, modifying slightly the proof of Corollary 4.35, by using Theorem 4.45 instead of Theorem 4.34, we obtain the following property.

Corollary 4.46. *Assume that X is a separable reflexive Banach space and that $(T(t))_{t\geq 0}$ is a compact semigroup. Let $f_1 : [0,\infty)\times X \to X$ be a function that satisfies* (F1)-(F3)*, and let $f_2 : [0,\infty)\times X \to \upsilon(X)$ be a locally Lipschitzian map with closed values such that the condition* (F6') *is fulfilled. Assume further that conditions* (4.9) *and* (4.12) *hold. Then there exists a mild solution $y(\cdot)$ of problem* (4.1)-(4.2)*.*

In what follows, we will reserve the notation $y(\cdot)$ to denote the solution of problem (4.1)-(4.2) established in any of Theorem 4.34, Theorem 4.45, Corollary 4.35 or Corollary 4.46. It follows from Theorem 4.24 and its corollary that $\|y\|_\infty \leq R$. From the definition of solutions, we have

$$y(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,y(s))ds + \int_0^t T(t-s)u(s)ds,\ 0\leq t<\infty, \tag{4.17}$$

for some $u \in \mathcal{S}_{f_2,y}$. Moreover, combining with Remark 4.1 we can affirm that there is a unique $u \in \mathcal{S}_{f_2,y}$ that verifies (4.17).

4.4 Existence of asymptotically almost periodic solutions

We next study the existence of asymptotically almost periodic solutions of (4.1)-(4.2). For general properties of almost periodic and asymptotically almost periodic functions with values in abstract spaces, we refer the reader to [18, 43, 85]. We only recall here the basic definitions. In the first place, we remember that a set P is called relatively dense in $\mathbb{R}$ (respectively, in $[0,\infty)$) if there exists $L>0$ so that for any interval $I\subset\mathbb{R}$ (respectively, $I\subset[0,\infty)$) with length greater than or equal to L we have $I\cap P\neq\emptyset$.

Definition 4.47. *A function $x\in C(\mathbb{R},X)$ is called almost periodic (in short, a.p.) if for every $\varepsilon>0$ there exists a relatively dense subset P_ε of $\mathbb{R}$ such that*

$$\|x(t+\tau)-x(t)\|\leq\varepsilon,\ t\in\mathbb{R},\ \tau\in P_\varepsilon.$$

Definition 4.48. *A function $z\in C_b([0,\infty),X)$ is called asymptotically almost periodic (abbreviated, a.a.p.) if there exists $w\in C_0([0,\infty),X)$ and an almost periodic function $x(\cdot)$ such that $z(t)=x(t)+w(t)$ for all $t\geq 0$.*

Remark 4.49. ([85, Theorem 5.5]) *A function $f \in C([0,\infty), X)$ is asymptotically almost periodic if and only if for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ and a relatively dense subset P_ε of $[0,\infty)$ such that*

$$\|f(t+\xi) - f(t)\| \leq \varepsilon,$$

for all $t \geq t_\varepsilon$ and $\xi \in P_\varepsilon$.

In this chapter, $AP(X)$ and $AAP(X)$ denote the spaces consisting of a.p. (respectively, a.a.p.) functions endowed with the norm of the uniform convergence. The following property is well known ([44, Lemma 3.1]).

Lemma 4.50. *Let $(T(t))_{t\geq 0}$ be a uniformly asymptotically stable C_0-semigroup on X, and let $u : [0,\infty) \to X$ be an a.a.p. function. Then the function $v : [0,\infty) \to X$ given by*

$$v(t) = \int_0^t T(t-s)u(s)ds,\ t \geq 0,$$

also is a.a.p.

Definition 4.51. *A function $f \in C([0,\infty) \times X, X)$ is called uniformly asymptotically almost periodic (abbreviated, u.a.a.p.) on compact sets if for every $\varepsilon > 0$ and every compact $K \subset X$ there exists a relatively dense subset $P_{K,\varepsilon}$ in $[0,\infty)$ and $t_{K,\varepsilon} > 0$ such that*

$$\|f(t+\tau, x) - f(t,x)\| \leq \varepsilon,\ t \geq t_{K,\varepsilon},\ (\tau, x) \in P_{K,\varepsilon} \times K.$$

To establish our results, we need some properties of a.a.p. functions. We begin with the following remark.

Remark 4.52. *(a) Assume that $(T(t))_{t\geq 0}$ is a uniformly asymptotically stable C_0-semigroup on X. Let $f_1 \in C([0,\infty) \times X, X)$ be a function that satisfies* (F3) *and $f_1(\cdot, 0)$ is bounded on $[0,\infty)$. Let $K \subset X$ be a compact set. Then*

$$\int_a^t T(s) f_1(t-s, z)ds \to 0,\ a \to \infty,$$

uniformly for $t \geq a$ and $z \in K$.

Proof. We can assume that $(T(t))_{t\geq 0}$ satisfies (4.9). This implies that

$$\begin{aligned}
\Big\| \int_a^t T(s) f_1(t-s,z)ds \Big\| &\leq M \int_a^t e^{-\omega s}\nu(t-s)\|z\|ds + M\int_a^t e^{-\omega s}\|f_1(t-s,0)\|ds \\
&\leq M\|z\| \int_0^\infty \nu(\xi)d\xi e^{-\omega a} + \frac{M}{\omega}\sup_{t\geq 0}\|f_1(t,0)\|e^{-\omega a} \\
&\to 0,\ a \to \infty,
\end{aligned}$$

uniformly for $z \in K$ and $t \geq a$. □

(b) Let $x : [0,\infty) \to X$ be an a.a.p. function. Then the range of $x(\cdot)$ is a relatively compact set in X.

This is a consequence of the fact that both the range of an a.p. function, and the range of a function that vanishes at infinity are relatively compact sets. The reader can see [18, Proposition 3.9] or [85, Proposition 5.3].

Using Remark 4.52, we can establish an important property of a.a.p. functions.

Lemma 4.53. *Let $f_1 \in C([0,\infty)\times X, X)$ be a uniformly asymptotically almost periodic on compact sets function that satisfies condition* (F3). *Let $x : [0,\infty) \to X$ be an a.a.p. function. Then the function $v : [0,\infty) \to X$ given by*

$$v(t) = \int_0^t T(t-s) f_1(s, x(s))ds, \ t \geq 0,$$

also is a.a.p.

Proof. Since the range $Im(x)$ of $x(\cdot)$ is a relatively compact set, for every $\varepsilon > 0$, there exist a set P relatively dense in $[0,\infty)$ and $t_\varepsilon^1 > 0$ such that

$$\|x(t+\tau) - x(t)\| \leq \varepsilon, \tag{4.18}$$

$$\|f_1(t+\tau, z) - f_1(t, z)\| \leq \varepsilon, \tag{4.19}$$

for all $\tau \in P$, $t \geq t_\varepsilon^1$ and $z \in K = Im(x)$.

Let $a_\varepsilon > 0$. For $t \geq a_\varepsilon$, we can write

$$\begin{aligned} v(t+\tau) - v(t) &= \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t+\tau-s)) - f_1(t+\tau-s, x(t-s))]ds \\ &+ \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t-s)) - f_1(t-s, x(t-s))]ds \\ &+ \int_{a_\varepsilon}^{t+\tau} T(s) f_1(t+\tau-s, x(t+\tau-s))ds - \int_{a_\varepsilon}^{t} T(s) f_1(t-s, x(t-s))ds. \end{aligned}$$

Now we estimate each term on the right hand side of the above expression separately. We select $a_\varepsilon > 0$ appropriately as follows. For the third and fourth terms, using Remark 4.52 we can assume that

$$\begin{aligned} \left\| \int_{a_\varepsilon}^{t+\tau} T(s) f_1(t+\tau-s, x(t+\tau-s))ds \right\| &\leq \varepsilon, \\ \left\| \int_{a_\varepsilon}^{t} T(s) f_1(t-s, x(t-s))ds \right\| &\leq \varepsilon, \end{aligned}$$

for all $t \geq a_\varepsilon$ and $\tau \in P$. Let $t_\varepsilon = t_\varepsilon^1 + a_\varepsilon$. For $t \geq t_\varepsilon$, using (4.18) we can estimate the first term as

$$\begin{aligned} &\left\| \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t+\tau-s)) - f_1(t+\tau-s, x(t-s))]ds \right\| \\ &\leq M \int_0^{a_\varepsilon} e^{-\omega s} \nu(t+\tau-s) \|x(t+\tau-s) - x(t-s)\| ds \\ &\leq M \int_0^{\infty} \nu(\xi) d\xi \varepsilon. \end{aligned}$$

Proceeding in similar way, using (4.19) instead of (4.18), the second term yields

$$\Big\| \int_0^{a_\varepsilon} T(s)[f_1(t+\tau-s, x(t-s)) - f_1(t-s, x(t-s))]ds \Big\| \leq M \int_0^{a_\varepsilon} e^{-\omega s} ds \varepsilon \leq \frac{M}{\omega}\varepsilon.$$

Combining these estimates, we obtain that

$$\|v(t+\tau) - v(t)\| \leq \Big(2 + M\int_0^\infty \nu(\xi)d\xi + \frac{M}{\omega}\Big)\varepsilon$$

for all $\tau \in P$, $t \geq t_\varepsilon$. Using Remark 4.49, we can affirm that $v(\cdot)$ is an a.a.p. function. □

In the following statement, assuming that the hypotheses from Theorem 4.34, Theorem 4.45, Corollary 4.35 or Corollary 4.46 are fulfilled, we denote

$$r = M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s)R + \mu_R(s)]ds.$$

Theorem 4.54. *Assume the hypotheses from Theorem 4.34, Theorem 4.45, Corollary 4.35 or Corollary 4.46 hold. Assume further that the following conditions are satisfied:*

(i) *The function f_1 is uniformly asymptotically almost periodic on compact sets.*

(ii) *For every $\delta > 0$ there exists a measurable function $\sigma_\delta : [0,\infty) \to [0,\infty)$ such that $\sigma_{\delta_1} \leq \sigma_{\delta_2}$ for $\delta_1 \leq \delta_2$ and having the following property: for every $x_1, x_2 \in C_b([0,\infty), X)$ with $\|x_2 - x_1\|_\infty \leq \delta$, for every $u_2 \in \mathcal{S}_{f_2,x_2}$, there exists $u_1 \in \mathcal{S}_{f_2,x_1}$ such that*

$$\|u_2(t) - u_1(t)\| \leq \sigma_\delta(t)\|x_2(t) - x_1(t)\|, \ t \geq 0.$$

(iii) *For every $x \in AAP(X)$ the set $\widetilde{\mathcal{S}}_{f_2,x} = \mathcal{S}_{f_2,x} \cap AAP(X) \neq \emptyset$.*

If

$$M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s) + \sigma_{2r}(s)]ds < 1, \tag{4.20}$$

then there exists an a.a.p. mild solution of problem (4.1)-(4.2).

Proof. As it was previously explained in this section, there is a fixed point y of Γ with $\|y\|_\infty \leq R$.

On the other hand, for $x \in AAP(X)$, we define $\widetilde{\Gamma}(x)$ as the set formed by all functions v given by

$$v(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s, x(s))ds + \int_0^t T(t-s)u(s)ds, \ t \geq 0,$$

with $u \in \widetilde{\mathcal{S}}_{f_2,x}$.

Since $\widetilde{\Gamma}$ is a restriction of Γ on $AAP(X)$, we infer that $\widetilde{\Gamma} : AAP(X) \to \mathcal{K}v(C_b([0,\infty), X))$. Moreover, $\widetilde{\Gamma}(AAP(X)) \subseteq \mathcal{K}v(AAP(X))$. In fact, this is an immediate consequence of Lemma 4.50 and Lemma 4.53. As a consequence, we can affirm that $\widetilde{\Gamma}(AAP(X) \cap B_R(0)) \subseteq \mathcal{K}v(AAP(X) \cap B_R(0))$.

We next estimate $d(y, AAP(X))$. Using (4.17) we have

$$\begin{aligned} y(t) &= T(t)x_0 + \int_0^t T(t-s)f_1(s,y(s))ds + \int_0^t T(t-s)u(s)ds \\ &= T(t)x_0 + \int_0^t T(t-s)[f_1(s,y(s)) - f_1(s,0)]ds + \int_0^t T(t-s)f_1(s,0)ds + \int_0^t T(t-s)u(s)ds \end{aligned}$$

for $u \in \mathcal{S}_{f_2,y}$. Since the function $z(\cdot)$ given by

$$z(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,0)ds, \ t \geq 0,$$

is a.a.p., we obtain

$$\begin{aligned} d(y, AAP(X)) &\leq \Big\| \int_0^t T(t-s)[f_1(s,y(s)) - f_1(s,0)]ds + \int_0^t T(t-s)u(s)ds \Big\|_\infty \\ &\leq M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s)\|y(s)\| + \mu_R(s)]ds \\ &\leq M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s)R + \mu_R(s)]ds \\ &= r. \end{aligned}$$

Consequently, $AAP(X) \cap B_r(y)$ is a nonempty closed set.

Let $x_i \in AAP(X) \cap B_r(y)$, $i = 1, 2$. We now estimate $d_H(\widetilde{\Gamma}(x_2), \widetilde{\Gamma}(x_1))$. Let $z_2 \in \widetilde{\Gamma}(x_2)$ and $u_2 \in \widetilde{\mathcal{S}}_{f_2,x_2}$ be such that

$$z_2(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,x_2(s))ds + \int_0^t T(t-s)u_2(s)ds, \ t \geq 0.$$

Using (ii), we can take $u_1 \in \mathcal{S}_{f_2,x_1}$ such that

$$\|u_2(t) - u_1(t)\| \leq \sigma_{2r}(t)\|x_2(t) - x_1(t)\|, \ t \geq 0,$$

and define $z_1(\cdot)$ by

$$z_1(t) = T(t)x_0 + \int_0^t T(t-s)f_1(s,x_1(s))ds + \int_0^t T(t-s)u_1(s)ds, \ t \geq 0.$$

We obtain

$$\begin{aligned} \|z_2(t) - z_1(t)\| &\leq M \int_0^t e^{-\omega(t-s)}\nu(s)\|x_2(s) - x_1(s)\|ds + M \int_0^t e^{-\omega(t-s)}\|u_2(s) - u_1(s)\|ds \\ &\leq M \int_0^t e^{-\omega(t-s)}\nu(s)\|x_2(s) - x_1(s)\|ds + M \int_0^t e^{-\omega(t-s)}\sigma_{2r}(s)\|x_2(s) - x_1(s)\|ds \\ &\leq M \int_0^t e^{-\omega(t-s)}[\nu(s) + \sigma_{2r}(s)]ds \max_{0\leq s\leq t} \|x_2(s) - x_1(s)\| \end{aligned}$$

for $t \geq 0$. This implies that

$$d(z_2, \widetilde{\Gamma}(x_1)) \leq M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s) + \sigma_{2r}(s)]ds\|x_2 - x_1\|_\infty.$$

Since the right-hand side of the above inequality only depends on x_1 and x_2, we conclude that

$$d_H(\widetilde{\Gamma}(x_2), \widetilde{\Gamma}(x_1)) \leq M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s) + \sigma_{2r}(s)]ds\|x_2 - x_1\|_\infty.$$

In addition, $\widetilde{\Gamma}(AAP(X) \cap B_r(y)) \subseteq AAP(X) \cap B_r(y)$. In fact, if $x \in AAP(X) \cap B_r(y)$ and $z \in \widetilde{\Gamma}(x)$, using (4.17) and proceeding as above, we have that

$$\begin{aligned}
\|z - y\|_\infty &\leq M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s) + \sigma_r(s)]ds\|x - y\|_\infty \\
&\leq M \sup_{t\geq 0} \int_0^t e^{-\omega(t-s)}[\nu(s) + \sigma_{2r}(s)]dsr \\
&\leq r
\end{aligned}$$

which shows that $z \in B_r(y)$.

Combining (4.20) and Theorem 4.2, we infer that $\widetilde{\Gamma}$ has a fixed point x in $AAP(X) \cap B_r(y)$. It is clear that $x(\cdot)$ is an a.a.p. mild solution of (4.1)-(4.2). □

Next we use our previous results to study the existence of solutions of problem (4.3)-(4.5). To model this problem in the abstract form (4.1)-(4.2), we consider the space $X = L^2([0, \pi])$. We define the operator $A : D(A) \subset X \to X$ by

$$Az(\xi) = z''(\xi),\ 0 \leq \xi \leq \pi,$$

on $D(A) = \{z \in X : z'' \in X, z(0) = z(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a compact semigroup $(T(t))_{t\geq 0}$ that satisfies the estimate (4.9) with $M = \omega = 1$. We assume that $\varphi \in X$.

Example 4.55. Let $f_1 : [0, \infty) \times X \to X$ be a function given by $f_1(t, x)(\xi) = \tilde{f}_1(t, x(\xi))$ for $0 \leq \xi \leq \pi$, where $\tilde{f}_1 : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a function that satisfies the Carattheódory conditions [62] and

$$|\tilde{f}_1(t, w_2) - \tilde{f}_1(t, w_1)| \leq \nu(t)|w_2 - w_1|,\ t \geq 0,\ w_1, w_2 \in \mathbb{R},$$

where $\nu \in L^1_{loc}([0, \infty))$.

This implies that f_1 satisfies conditions (F1)-(F3). Let $f_2 : X \to \upsilon(X)$ be the map given by

$$f_2(x) = \{u \in X : a(x(\xi)) \leq u(\xi) \leq b(x(\xi)),\ 0 \leq \xi \leq \pi\},$$

where a, b are functions that satisfy the conditions considered in Example 4.41.

Under these conditions, problem (4.3)-(4.4) is modeled as (4.1)-(4.2) with $x_0 = \varphi$. Moreover, for $u \in f_2(x)$ we obtain

$$\|u\| \leq c_0 + c_1\|x\|,$$

where $c_0 = \sqrt{\pi} \max\{|a(0)|, |b(0)|\}$ and $c_1 = \max\{a_0, b_0\}$. It shows that f_2 satisfies (F6') with $\mu_r(t) = c_1 r + c_0$.

We assume that

$$\sup_{t\geq 0}\int_0^t e^{-(t-s)}\nu(s)ds + c_1 < 1. \tag{4.21}$$

Consequently, it follows from Corollary 4.46 that there exists a mild solution $u(\cdot)$ of problem (4.3)-(4.5).

We assume further that $\tilde{f}_1$ satisfies the condition

$$|\tilde{f}_1(t,x) - \tilde{f}_1(s,x)| \leq |q(t)-q(s)||x|,\ s,t\geq 0,\ x\in\mathbb{R}, \tag{4.22}$$

where $q : [0,\infty) \to \mathbb{R}$ is an a.a.p. function. Hence we infer that f_1 is uniformly a.a.p. on compact sets, which implies that condition (i) from Theorem 4.54 is fulfilled.

On the other hand, for every $x_1, x_2 \in C_b([0,\infty), X)$ and $u_2 \in \mathcal{S}_{f_2,x_2}$, proceeding as in the proof of Example 4.41 we can select $u_1 \in \mathcal{S}_{f_2,x_1}$ such that

$$|u_2(t,\xi) - u_1(t,\xi)| \leq c_1|x_2(t,\xi) - x_1(t,\xi)|\ t\geq 0,\ \xi\in[0,\pi],$$

which implies that condition (ii) from Theorem 4.54 is satisfied with $\sigma_\delta = c_1$.

Finally, since the function $a(\cdot)$ satisfies a Lipschitz condition, for every $x \in AAP(X)$ the function $\tilde{a} : [0,\infty) \to X$ given by $\tilde{a}(t)(\xi) = a(x(t,\xi))$ also is a.a.p., which shows that $\mathcal{S}_{f_2,x} \cap AAP(X) \neq \emptyset$. Combining with (4.21), and applying Theorem 4.54, we infer that there exists an a.a.p. mild solution $u(\cdot)$ of problem (4.3)-(4.5).

Example 4.56. Let $f_1 : [0,\infty)\times X \to X$ be a function as in Example 4.55, and let $\tilde{f}_2 : X \to X$ be the map given by

$$\tilde{f}_2(x)(\xi) = a(x(\xi)),\ 0\leq\xi\leq\pi,$$

where a is a function that satisfies the conditions considered in Example 4.41, and for every $s_0 \in \mathbb{R}$ there exist $\delta, k > 0$ such that

$$|a(s) - a(s_0)| \geq k|s - s_0| \tag{4.23}$$

for all $s \in \mathbb{R}$ such that $|s - s_0| < \delta$.

For fixed $\varepsilon > 0$, we define

$$f_2(x) = \tilde{f}_2(x + C_\varepsilon(0)) = \{\tilde{f}_2(x+z) : z \in C_\varepsilon(0)\},$$

where $C_\varepsilon(0) = \{z \in X : |z(\xi)| \leq \varepsilon,\ \text{a.e. } 0\leq\xi\leq\pi\}$. It is clear that $C_\varepsilon(0)$ is a closed convex subset of X. Under these conditions, problem (4.3)-(4.4) is modeled as (4.1)-(4.2) with $x_0 = \varphi$.

As a consequence of the Intermediate Value Theorem, the map f_2 has convex values. Moreover, it follows from (4.23) that $f_2(x)$ is closed for all $x \in X$. In addition, for $u \in f_2(x)$ we obtain

$$\|u\| \leq c_0 + c_1\|x\|,$$

where $c_0 = \sqrt{\pi}|a(0)| + a_0\varepsilon$ and $c_1 = a_0$. This shows that f_2 satisfies (F6') with $\mu_r(t) = c_1 r + c_0$. We assume that (4.21) holds. It is also clear that f_2 is a Lipschitz continuous map. Collecting these assertions, we can apply again Corollary 4.46 to conclude that there exists a mild solution

$u(\cdot)$ of problem (4.3)-(4.5).

On the other hand, assume further that f_1 satisfies (4.22). In similar way, using that $a(\cdot)$ is Lipschitz continuous, we obtain that condition (ii) from Theorem 4.54 is satisfied with $\sigma_\delta = c_1$, and proceeding as in Example 4.55, we infer that $\mathcal{S}_{f_2,x} \cap AAP(X) \neq \emptyset$. Combining with (4.21), and applying Theorem 4.54, we infer that there exists an a.a.p. mild solution $u(\cdot)$ of problem (4.3)-(4.5).

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