Universidad de Santiago de Chile
Facultad de Ciencia
Departamento de Matemática y Ciencia de la Computación

Spectral view of Hartman-Grobman's theorem for nonuniform and unbounded hyperbolic flows.

Por
Ignacio Roberto Huerta Navarro

Profesor Guía:
Álvaro Castañeda
Cristóbal Rivas

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Abstract

In this thesis we will consider a nonautonomous linear system which admits nonuniform contraction and a nonlinear perturbation bounded at the origin. We search to establish an equivalence between the solutions of the systems before mentioned, being our main objective to construct a topological equivalence between them.

In order to obtain such a result we review the spectral theory associated with the nonuniform hyperbolicity, specifically we consider nonuniform exponential dichotomy of the linear system. In addition we highlight some fundamental results of the spectral theory such as: i) under certain hypotheses the nonuniform spectrum of the linear system can be written as the finite union of compact intervals, ii) the linear system is equivalent by means of a nonuniform kinematic similarity to a new linear system composed of blocks, where the spectrum of each of these blocks corresponds to one of the connected components of the original linear system spectrum.

On the other hand and thanks to the aforementioned spectral theory, we will show that the initial linear system is nonuniformly contracted to its spectrum when it is nonuniformly kinemically similar to a linear system composed of the sum of one diagonal matrix, where its elements are functions whose images belong to the spectrum, and a matrix whose norm can be chosen sufficiently small. We will call this property almost nonuniform reducibility.

Finally, if the linear system admits nonuniform contraction, we use the previous results combined with Lyapunov's theory of functions to establish the existence of homeomorphism that relates the solutions of both systems.

Keywords: Systems of nonautonomous differential equations; nonuniform exponential dichotomy; nonuniform spectrum; Lyapunov functions; topological equivalence.

Resumen

En esta tesis consideraremos un sistema lineal no autonomo el cual admite contracción no uniforme y una perturbación no lineal acotada en el origen. Buscamos establecer una equivalencia entre las soluciones de los sistemas anteriormente mencionados, siendo nuestro principal objetivo construir una equivalencia topológica entre ellos.

Con la finalidad de obtener tal resultado repasamos la teoría espectral asociada a la hiperbolicidad no uniforme, especificamente la dicotomía exponencial no uniforme del sistema lineal. Además destacamos algunos resultados fundamentales de la teoría espectral tales como: i) bajo ciertas hipótesis el espectro no uniforme del sistema lineal se puede escribir como la unión finita de intervalos compactos, ii) el sistema lineal es equivalente por medio de una similaridad cinemática no uniforme a un nuevo sistema lineal compuesto por bloques, donde el espectro de cada uno de esos bloques corresponde a una de las componentes conexas del espectro del sistema lineal original.

Por otro lado y gracias a la teoría espectral anteriormente mencionada, el sistema lineal inicial es no uniformemente contraído a su espectro cuando es no uniformemente cinemáticamente similar a un sistema lineal compuesto por la suma de una matriz diagonal, donde sus elementos son funciones cuyas imágenes pertenecen al espectro, y una matriz cuya norma puede ser escogida suficientemente pequeña. A esta propiedad la llamaremos casi reducibilidad no uniforme.

Finalmente, si el sistema lineal admite contracción no uniforme, usamos los resultados anteriores combinado con la teoría de funciones de Lyapunov para establecer la existencia del homeomorfismo que relaciona las soluciones de ambos sistemas.

Palabras clave: Sistema de ecuaciones diferenciales no autónomo; dicotomía exponencial no uniforme; espectro no uniforme; funciones de Lyapunov; equivalencia topológica.

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Introduction

In this thesis we will prove the existence of a topological equivalence between the nonautonomous linear system

$$\dot{x} = A(t)x \tag{1}$$

and a nonlinear perturbation

$$\dot{x} = A(t)x + f(t,x), \tag{2}$$

where $A : \mathbb{R}_0^+ \to M_n(\mathbb{R})$, $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and x is a column vector of $\mathbb{R}^n$. The hypotheses with which we will work are that the system (1) admits nonuniform contraction and the function f of the equation (2) has a Lipschitz condition and it is bounded at the origin.

This work is based on different studies of nonautonomous differential equations, which we will detail further; and in order to do this we will recall the tools which we study in a basic course of autonomous differential equations. We will consider an autonomous linear system

$$\dot{x} = Ax \tag{3}$$

where A is an $n \times n$ matrix and x as a column vector of $\mathbb{R}^n$. It is known that the solution of the linear system (3) with the initial condition $x(0) = x_0$ is given by

$$x(t) = \exp(At)x_0$$

where $\exp(At)$ is an $n \times n$ matrix function defined by the Taylor series.

An important fact in the study of (3) is the algebraic technique of diagonalizing a square matrix A, which can be used to reduce the linear system to an uncoupled linear system [36]. However, it is worth mentioning that a matrix A is not always diagonalizable, so by using the eigenvalues and the generalized eigenvectors we find a form to describe the matrix A by means of submatrices. These submatrices are composed by the sum of a nilpotent matrix and a diagonal matrix, whose diagonal element is one of the eigenvalues of A; this method is called the Jordan forms. The exponential of a linear operator will help us find the solution to the equation (3), therefore the sign that has the real part of each eigenvalues allowing us to construct the stable and unstable invariant manifolds associated to the system. From the above mention we know that the eigenvalues of the matrix A give us information about the qualitative behavior of (3).

We take note that $x_0 = 0$ is the unique equilibrium point of (3) and when the real part of the eigenvalues of A are not equal to 0, we have x_0 as a hyperbolic equilibrium point. This fact about the equilibrium point is known as *hyperbolicity* and from this definition we begin the study of nonlinear autonomous system

$$\dot{x} = f(x) \tag{4}$$

with $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ a vector field of class C^1 and x_0 an equilibrium point of f. In this context we mention two kinds of studies of these systems: local and global. For the local study, there is a classic differential equation theorem that establishes equivalences between the solutions of the systems (3) and (4), namely:

Theorem 0.1. ([36]) [Hartman-Grobman Theorem] *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^1 and let x_0 an equilibrium point. Suppose that $A = Df(x_0)$ (the Jacobian matrix of f in the point x_0) has its eigenvalues with real part not null, then there exist open neighborhoods $V, U \subset \mathbb{R}^n$ of x_0 and 0 respectively, such that the maps $f|_V$ and $A|_U$ are topologically conjugated.*

The global behavior study starts in 1969, when C. Pugh [40] studied a particular case of the Hartman–Grobman's Theorem focused on linear systems (3) and

$$\dot{x} = Ax + f(x). \tag{5}$$

If we consider $\mu > 0$ small, then for each f bounded by μ and with Lipschitz constant $\omega \leq \mu$, we have that the systems (3) and (5) are globally topologically conjugated.

The work cited above inspired K.J. Palmer [34] to achieve the first result of global linearization in the nonautonomous framework. In fact, Palmer considered the linear system (1) which admits exponential dichotomy. Moreover, when we have the nonlinear system (2) where the perturbation f is bounded and Lipschitz, and if we assume that there exists a relation between the bound of f, the Lipschitz constant and the parameters that depend on the exponential dichotomy, then there exists a topological equivalence between the systems (1) and (2).

The seminal article of K.J. Palmer and its extensions [24,42] considers vector fields whose linear component inherits the hyperbolicity property of the autonomous case, while the nonlinear part satisfies boundedness and Lipschitzness properties. As well, a remarkable extension of the previous work was made by F. Lin [31], who considered this problem by dropping the boundedness of the nonlinear perturbations, opening new ideas and methods. The work of Lin is mainly based in three steps:

(i) If the linear system (1) is supposed to be uniformly asymptotically stable, then it can be reduced to the linear system

$$\dot{x} = [C(t) + B(t)]x \tag{6}$$

where $C(t)$ is diagonal, $B(t)$ is small enough, and the diagonal part is contained in the spectrum associated to nonautonomous hyperbolicity.

(ii) From the system (6), he consider a nonlinear system as follows

$$\dot{x} = [C(t) + B(t)]x + g(t, x) \tag{7}$$

where $g(t,x)$ has an equilibrium point at the origin for any $t \in \mathbb{R}$ and its Lipschitz constant dependent of the smallness of $B(t)$. The system (7) is topologically equivalent to an autonomous linear system, which is uniformly asymptotically stable. The construction of this topological equivalence is made by using the concept of *crossing times* with respect to the unit sphere. Notice that a suitable Lyapunov function is used to find these crossing times.

(iii) Based on the steps (i) and (ii), a chain of homeomorphisms let him establish the topological equivalence between the systems (1) and (2).

As in nonautonomous framework the hyperbolicity condition does not have an univocal definition, it should be noted that in step (i) we will use nonautonomous hyperbolicity defined by L. Barreira and C. Valls [5] with its associated spectrum [17, 47]; and with these tools we will construct a topological equivalence generalizing the work of F. Lin in [31].

Structure of the thesis. This work is divided into three chapters, we will summarize each one of them as follows.

In the first chapter we show the definition of nonuniform exponential dichotomy, which is denoted $\Sigma(A)$; its associated spectral theory; an application of the nonuniform spectrum of the linear system (1), which is establish by means of a linear nonautonomous coordinate change, also known as nonuniform kinematically similarity, that the linear system (1) is equivalent to a diagonal block system noted below

$$\dot{y} = \begin{pmatrix} B_1(t) & & \\ & \ddots & \\ & & B_m(t) \end{pmatrix} y \tag{8}$$

and the spectrum of each block corresponds to connected components of the spectrum.

Moreover, on the articles [4, 17, 47] spectrum properties are described and we will adapt these proofs to the continuous context and on the half line, since that in [17, 47] the properties of the spectrum are verified in the continuous context but for the case $\mathbb{R}$; and in [4] the discrete case was studied on the half line. The most remarkable property of the spectrum is that under certain conditions this can be the finite union of compact intervals.

On the other hand, the second chapter presents our first main contribution obtained in this investigation. We introduce the concept of *nonuniform almost reducibility*, which is a very important development for us. This concept establishes that between systems (1) and a system of the form

$$\dot{y} = (C(t) + B(t))y$$

there exists a nonuniform kinematical similarity, where the norm of $B(t)$ can be chosen small enough and the images of diagonal matrix $C(t)$ is contained in the compact set $\Sigma(A)$.

Some of the technical results of this chapter are based on the books of L. Y. Adrianova [1] and

W. Coppel [18], while some of the results proved in chapter one are mentioned to contextualize the tools that will be used in this chapter, which is guided by the work done by F. Lin [30]. At the end of this chapter, three examples of this application are presented, two for them are scalar linear systems and an example of a planar system, formed by the previous ones is also given.

Finally, in the third chapter the main objective of the thesis comes afloat, which consists of establishing a theorem of the Hartman-Grobman in the nonautonomous context, between the systems (1) and the nonlinear system (2) in order to construct a topological equivalence between both systems. Some hypotheses that stand out in order to obtain the result shown are: (i) that the system (1) supports nonuniform contraction, *i.e.*, a kind of stability associated with nonuniform exponential dichotomy, also (ii) the nonlinear perturbation (2) is bounded at the origin.

Starting with Lyapunov functions and quadratic form theory, we can make a relationship the nonuniform contraction and Lyapunov functions. On the other hand, we use the main result obtained in chapter two, which allows us to relate the Lyapunov function associated to (6) with the behavior of the solutions of the perturbed system

$$\dot{x} = [C(t) + B(t)]x + g(t, x) \tag{9}$$

where $g(t, x)$ has an equilibrium point at the origin for any $t \in \mathbb{R}_0^+$ and satisfies certain properties with respect to its Lipschitz condition. It is possible to construct the topological equivalence between (6) and (9), using the concept of *crossing times* for the Lyapunov function.

We will work with the systems (6) and (9) as auxiliary systems in order to conclude that (1) and (2) are topologically equivalent.

Chapter 1

Spectral Theory

The next step in the study of the ordinary differential equations is to consider the nonautonomous context. Namely, to study the system

$$\dot{x} = A(t)x, \tag{1.1}$$

where matrix function $t \mapsto A(t) \in M_n(\mathbb{R})$ and to use our knowledge of eigenvalues to obtain qualitative information about system (1.1).

In the following example we will realize that the eigenvalues do not always allow to conclude on the stability of the solutions.

Example 1.1 ([32]). *Consider the nonautonomous linear system in its matrix form*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(t) & 1 - \frac{3}{2}\cos(t)\sin(t) \\ -1 - \frac{3}{2}\cos(t)\sin(t) & -1 + \frac{3}{2}\sin^2(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.2}$$

A simple calculation show that $tr(A(t)) = -\frac{1}{2}$ *and* $det(A(t)) = \frac{1}{2}$*, this implies that the eigenvalues of* $A(t)$ *(which are independents of* t*) have real part negative and as consequence the origin is stable. However, a fundamental matrix for* (1.2) *denoted* $X(t)$*, which satisfies the equation* $\dot{X} = A(t)X(t)$*, is*

$$X(t) = \begin{pmatrix} \exp\left(\frac{t}{2}\right)\cos(t) & \exp(-t)\sin(t) \\ -\exp\left(\frac{t}{2}\right) & \exp(-t)\cos(t) \end{pmatrix}, \tag{1.3}$$

and if we look at the first column of the (1.3)*, the solutions diverge when* t *tends to* $+\infty$ *or* $-\infty$*.*

In view of the previous example, thus an alternative approach must be considered in order to study the qualitative behavior of the system (1.1). A first approach in this direction was given by G. Floquet [21], which established that a periodic system as (1.1) can be transformed into a constant coefficients system. The Floquet's result can be seen as an example of the properties of *kinematical similarity* and *reducibility*, which refers that a linear system (1.1) can be transformed into

$$\dot{y} = B(t)y \tag{1.4}$$

through a Lyapunov transformation $x = L(t)y$.

The problem to obtain a simpler form to (1.1) has been tackled by using the concept of *reducibility* by O. Perron in [37], which proves that (1.1) can be reduced via unitary transformation to a system (1.4) where $B(t)$ has a triangular form whose diagonal coefficients are real. Moreover, under subtle technical conditions it can be proved that $B(t)$ has a block–triangular form consisting of blocks whose diagonal coefficients are real.

We have mentioned that an eigenvalues–based approach has several shortcomings and is not an adequate tool to cope with stability issues in the nonautonomous framework. A tool that emulates the role of the eigenvalues in this context was developed in terms of the property of *uniform exponential dichotomy* (a type of nonautonomous hyperbolicity), namely, the Sacker–Sell spectrum associated to (1.1), which is the set

$$\sigma(A) = \{\lambda \in \mathbb{R} \colon \dot{x} = (A(t) - \lambda I)x \quad \text{has not N.E.D on } J \subset \mathbb{R}.\}$$

where the acronym N.E.D. means *nonuniform exponential dichotomy*. On the other hand and retouching the concept of nonautonomous hyperbolicity, in this thesis we consider a more general type of hyperbolic behavior than the uniform exponential dichotomy, which is the *nonuniform exponential dichotomy* defined by L. Barreira and C. Valls [5]. They proved that in a finite-dimensional space, essentially any equation as (1.1) with nonzero Lyapunov exponents has a nonuniform exponential dichotomy (see [5]). In relation to nonuniform part of the dichotomy, from the point of view of ergodic theory, this can be arbitrarily small for almost every trajectory, as a consequence of Oseledets multiplicative ergodic theorem in [33].

As well as in the case of exponential dichotomy a spectrum is defined, for nonuniform exponential dichotomy the spectrum will be defined in a similar form.

1.1 Preliminaries.

To start, we first consider the nonautonomous linear differential system (1.1) with $A : \mathbb{R}_0^+ \to M_n(\mathbb{R})$ square matrix function. We assumme that each solution of system (1.1) is defined on $\mathbb{R}_0^+$. We denote by $\Phi(t,s)$ the evolution operator associated to system (1.1). Then we have

$$x(t) = \Phi(t,s)x(s), \quad \Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau) \ \text{ for all } t,s,\tau \in \mathbb{R}_0^+$$

where $x(t)$ is a solution of (1.1).

Definition 1.1. ([5], [17], [47]) *The system* (1.1) *has a nonuniform exponential dichotomy on $J \subset \mathbb{R}$ if there exist an invariant projector $P(\cdot)$, constants $K \geq 1$, $\alpha > 0$ and $\mu \geq 0$, with such that*

$$\left\{ \begin{array}{rcl} \|\Phi(t,s)P(s)\| & \leq & K\exp(-\alpha(t-s) + \mu\,|s|), \quad t \geq s, \quad t,s \in J, \\ \\ \|\Phi(t,s)(I-P(s))\| & \leq & K\exp(\alpha(t-s) + \mu\,|s|), \quad t \leq s, \quad t,s \in J. \end{array} \right. \tag{1.5}$$

In our context, we consider the interval $J = \mathbb{R}_0^+$.

Remark 1.1. *We have the following comments with respect to this nonuniform dichotomy:*

1. *In the definition of nonuniform exponential dichotomy the condition $\mu < \alpha$ appears, for technical reasons, in* [17] *and* [47].

2. *It is considered a projector $P(t)$ that satisfies the equation*

$$P(t)\Phi(t,s) = \Phi(t,s)P(s)$$

and it is invariant in the next sense

$$dim(Ker(P(t))) = dim(Ker(P(s))),$$

for all $t, s \in \mathbb{R}_0^+$. In fact, for any fixed $t \in \mathbb{R}_0^+$, there exists an invertible matrix J_1 such that

$$J_1^{-1}P(t)J_1 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

with $k = dim\ (Im(P(t)))$. Now, since for any fixed $s \in \mathbb{R}_0^+$ we have $P(t) = \Phi(t,s)P(s)\Phi(s,t)$, then

$$J_1^{-1}\Phi(t,s)P(s)\Phi(s,t)J_1 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

If we define $J_2 = \Phi(s,t)J_1$, then B is an invertible matrix, with $B^{-1} = J_1^{-1}\Phi(t,s)$ and

$$J_2^{-1}P(s)J_2 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

which implies that $k = dim\ (Im(P(s)))$.

The *nonuniform dichotomy spectrum* of system (1.1) is the set

$$\Sigma(A) = \{\lambda \in \mathbb{R} \colon \dot{x} = (A(t) - \lambda I)x \text{ has not nonuniform exp. dichotomy on } J \subset \mathbb{R}\}$$

and its complement $\rho(A) = \mathbb{R} \setminus \Sigma(A)$ is called the *resolvent set* of system (1.1).

The following result shows that the images of projector $P(\tau)$ are uniquely determined, that is, are independent of τ. We note that the same does not happen to the images of projector $Q(\tau) = Id - P(\tau)$.

Proposition 1.1. *For each $\tau \in \mathbb{R}_0^+$, we have*

$$Im(P(\tau)) = \left\{ v \in \mathbb{R}^n \colon \sup_{t \geq \tau} \|\Phi(t,\tau)v\| < +\infty \right\}.$$

Proof. This proof follows the lines of the articles [4, 47]. It follows from (1.5) that if $v \in Im(P(\tau))$, then

$$\sup_{t \geq \tau} \|\Phi(t,\tau)v\| < +\infty. \tag{1.6}$$

Now take a vector $v \in \mathbb{R}^n$ satisfying (1.6). Since $v = P(\tau)v + Q(\tau)v$, it follows from (1.5) that

$$\sup_{t\geq\tau} \|\Phi(t,\tau)Q(\tau)v\| < +\infty. \tag{1.7}$$

On the other hand, by (1.5) for $t \geq \tau$ we have

$$\|Q(\tau)v\| = \|\Phi(\tau,t)\Phi(t,\tau)Q(\tau)v\| \leq K\exp(-\alpha(t-\tau)+\mu t)\,\|\Phi(t,\tau)Q(\tau)v\|$$

and so

$$\frac{1}{K}\exp(\alpha(t-\tau)-\mu t)\,\|Q(\tau)v\| \leq \|\Phi(t,\tau)Q(\tau)v\|\,.$$

Hence, if $Q(\tau)v \neq 0$, and if $\alpha > \mu$ we obtain

$$\sup_{t\geq\tau} \|\Phi(t,\tau)Q(\tau)v\| = +\infty,$$

but this contradicts to (1.7). Therefore, $Q(\tau)v = 0$ and so $v \in Im(P(\tau))$. □

The following statement specifies the freedom that is allowed when choosing the projector $P(\tau)$.

Proposition 1.2. *Assume that the system (1.1) admits nonuniform exponential dichotomy with projector $P(\cdot)$. Moreover, let $\tilde{P}(\cdot)$ be projector such that*

$$\tilde{P}(t)\Phi(t,s) = \Phi(t,s)\tilde{P}(s) \quad \text{for} \quad t,s \in \mathbb{R}_0^+. \tag{1.8}$$

Then the system (1.1) admits nonuniform exponential dichotomy with projector $\tilde{P}(\cdot)$ if and only if $ImP(0) = Im\tilde{P}(0)$.

Proof. This proof follows the ideas of the discrete case proved by Barreira and Vall in [4]. If the system (1.1) admits nonuniform exponential dichotomy with projector $\tilde{P}(\cdot)$, it follows from Proposition 1.1 that

$$Im\tilde{P}(0) = \left\{ v \in \mathbb{R}^n \colon \sup_{t\geq 0} \|\Phi(t,0)v\| < +\infty \right\} = ImP(0).$$

Now assume that $ImP(0) = Im\tilde{P}(0)$, then

$$P(0)\tilde{P}(0) = \tilde{P}(0) \quad \text{and} \quad \tilde{P}(0)P(0) = P(0).$$

In particular,

$$P(0) - \tilde{P}(0) = P(0)(P(0) - \tilde{P}(0)) = (P(0) - \tilde{P}(0))Q(0)$$

and so it follows from (1.1) that

$$\begin{aligned}
\left\|\Phi(t,0)(P(0)-\tilde{P}(0))v\right\| &= \left\|\Phi(t,0)P(0)(P(0)-\tilde{P}(0))v\right\|, \\
&\leq K\exp(-\alpha(t-0)+\mu 0)\left\|(P(0)-\tilde{P}(0))v\right\|, \\
&= K\exp(-\alpha t)\left\|(P(0)-\tilde{P}(0))Q(0)v\right\|, \\
&\leq K\exp(-\alpha t)\left\|(P(0)-\tilde{P}(0))\right\|\|Q(0)v\|, \\
&= K\exp(-\alpha t)\left\|(P(0)-\tilde{P}(0))\right\|\|\Phi(0,\tau)\Phi(\tau,0)Q(0)v\|, \\
&= K\exp(-\alpha t)\left\|(P(0)-\tilde{P}(0))\right\|\|\Phi(0,\tau)Q(\tau)\Phi(\tau,0)v\|, \\
&\leq K^2\exp(-\alpha t+(-\alpha+\mu)\tau)\left\|(P(0)-\tilde{P}(0))\right\|\|\Phi(\tau,0)v\|
\end{aligned}$$

for $t,\tau\in\mathbb{R}_0^+$ and $v\in\mathbb{R}^n$. Therefore, for $t\geq\tau$

$$\begin{aligned}
\left\|\Phi(t,\tau)\tilde{P}(\tau)v\right\| &\leq \|\Phi(t,\tau)P(\tau)v\|+\left\|\Phi(t,\tau)(P(\tau)-\tilde{P}(\tau))v\right\|, \\
&= \|\Phi(t,\tau)P(\tau)v\|+\left\|\Phi(t,0)(P(0)-\tilde{P}(0))\Phi(0,\tau)v\right\|, \\
&\leq K\exp(-\alpha(t-\tau)+\mu\tau)\|v\|+K^2\exp(-\alpha t+(-\alpha+\mu)\tau)\left\|(P(0)-\tilde{P}(0))\right\|\|v\|, \\
&\leq K\exp(-\alpha(t-\tau)+\mu\tau)\|v\|+K^2\exp(-\alpha(t-\tau)+\mu\tau)\left\|(P(0)-\tilde{P}(0))\right\|\|v\|, \\
&= \tilde{K}\exp(-\alpha(t-\tau)+\mu\tau)\|v\|
\end{aligned}$$

where

$$\tilde{K}=K+K^2\left\|P(0)-\tilde{P}(0)\right\|.$$

Similarly, letting $Q(\tau)=Id-P(\tau)$, for $t\leq\tau$ we obtain

$$\begin{aligned}
\left\|\Phi(t,\tau)\tilde{Q}(\tau)v\right\| &\leq \|\Phi(t,\tau)Q(\tau)v\|+\left\|\Phi(t,\tau)(P(\tau)-\tilde{P}(\tau))v\right\|, \\
&= \|\Phi(t,\tau)Q(\tau)v\|+\left\|\Phi(t,0)(P(0)-\tilde{P}(0))\Phi(0,\tau)v\right\|, \\
&\leq K\exp(\alpha(t-\tau)+\mu\tau)\|v\|+K^2\exp(-\alpha t+(-\alpha+\mu)\tau)\left\|P(0)-\tilde{P}(0)\right\|\|v\|, \\
&\leq K\exp(\alpha(t-\tau)+\mu\tau)\|v\|+K^2\exp(\alpha(t-\tau)+\mu\tau)\left\|P(0)-\tilde{P}(0)\right\|\|v\|, \\
&= \tilde{K}\exp(\alpha(t-\tau)+\mu\tau)\|v\|.
\end{aligned}$$

This shows that the system (1.1) admits nonuniform exponential dichotomy with projector $\tilde{P}(\cdot)$. $\square$

1.2 Properties and Characteristics of Nonuniform Spectrum.

In this section, using the previous propositions, we will provide results that will allow us to give properties and characteristics of the nonuniform spectrum.

A *linear integral manifold* (see [47]) of (1.1) is a nonempty set W of $\mathbb{R}_0^+\times\mathbb{R}^n$ satisfying $\left\{(t,\Phi(t,\tau)v);t\in\mathbb{R}_0^+\right\}\subset W$ for each $(\tau,v)\in W$, and for any given $\tau\in\mathbb{R}_0^+$ the fiber $W(\tau)=\{v\in\mathbb{R}^n;(\tau,v)\in W\}$ is a linear subspace of $\mathbb{R}^n$. We note that all the fibers $W(\tau)$ have the same dimension, denoted by $\dim W$.

For each $\gamma \in \mathbb{R}$ and $\tau \in \mathbb{R}_0^+$ we define

$$\mathcal{U}_\gamma = \left\{(\tau, v) \in \mathbb{R}_0^+ \times \mathbb{R}^n\colon \sup_{t\geq\tau} \|\Phi(t,\tau)v\| \exp(-\gamma(t-\tau)) < +\infty\right\}$$

An immediate result of the definition of $\mathcal{U}_\gamma$ is the following

Proposition 1.3 ([47]). *The following statements hold.*

(i) $\mathcal{U}_\gamma$ *is linear manifold of system* (1.1)*.*

(ii) If $\gamma_1 \leq \gamma_2$*, then* $\mathcal{U}_{\gamma_1}(\tau) \subseteq \mathcal{U}_{\gamma_2}(\tau)$*.*

Proof. We will prove the two statements:

i) For any $v \in \mathcal{U}_\gamma(\tau)$, by definition we only need to prove that $\Phi(s,\tau)v \in \mathcal{U}_\gamma(\tau)$ for all $s \in \mathbb{R}_0^+$. In fact, it follows from the fact that

$$\sup_{t\geq\tau} \|\Phi(t,s)\Phi(s,\tau)v\| \exp(-\gamma(t-\tau)) \quad = \sup_{t\geq\tau} \|\Phi(t,\tau)v\| \exp(-\gamma(t-\tau)) < +\infty,$$

ii) Let $v \in \mathcal{U}_{\gamma_1}(\tau)$, then

$$\sup_{t\geq\tau} \|\Phi(t,\tau)v\| \exp(-\gamma_2(t-\tau)) \leq \sup_{t\geq\tau} \|\Phi(t,\tau)v\| \exp(-\gamma_1(t-\tau)) < +\infty,$$

so $v \in \mathcal{U}_{\gamma_2}(\tau)$.

□

The following proposition is a characterization for $\mathcal{U}_\gamma(\tau)$ using the projector $P(\cdot)$.

Proposition 1.4 ([47]). *For* $\gamma \in \mathbb{R}$*, if*

$$\dot{x} = (A(t) - \gamma I)x \tag{1.9}$$

admits a nonuniform exponential dichotomy with an invariant projector $P(\tau)$*, then* $\mathcal{U}_\gamma(\tau) = ImP(\tau)$*, for any* $\tau \in \mathbb{R}_0^+$*.*

Proof. Let $\Phi(t,s)$ be a evolution operator of (1.1), some easy calculate show that $\Phi_\gamma(t,s) = \exp(-\gamma(t-s))\Phi(t,s)$ is an evolution operator of (1.9), and that $P(\cdot)$ is an invariant projector of $\Phi(t,s)$ if and only if is an invariant projector of $\Phi_\gamma(t,s)$. By the assumption there exist K_γ, $\alpha_\gamma > 0$ and $\mu_\gamma \geq 0$ such that

$$\begin{aligned} \|\Phi_\gamma(t,s)P(s)\| &\leq K_\gamma \exp(-\alpha_\gamma(t-s) + \mu_\gamma s), \quad t \geq s, \\ \|\Phi_\gamma(t,s)(I-P(s))\| &\leq K_\gamma \exp(\alpha_\gamma(t-s) + \mu_\gamma s), \quad t \leq s. \end{aligned}$$

First we prove that $ImP(\tau) \subset \mathcal{U}_\gamma(\tau)$. Considering the characterization for $ImP(\tau)$ of the Proposition 1.1, if $v \in ImP(\tau)$, then

$$\sup_{t\geq\tau} \exp(-\gamma(t-\tau)) \|\Phi(t,\tau)v\| \leq \sup_{t\geq\tau} \|\Phi(t,\tau)v\| < +\infty.$$

On the other hand, to prove $\mathcal{U}_\gamma(\tau) \subset ImP(\tau)$, consider $v \in \mathcal{U}_\gamma(\tau)$, i.e., there exists a constant c_γ such that

$$\sup_{t\geq\tau} \|\Phi_\gamma(t,s)\| \leq c_\gamma.$$

Now we write $v = P(\tau)v + Q(\tau)v$. Since $P(\cdot)$ is an invariant projector of $\Phi_\gamma(t,s)$, we have

$$Q(\tau)v = \Phi_\gamma(\tau,t)Q(t)\Phi_\gamma(t,\tau)v,$$

then for $t \geq \tau$, we obtain the estimation

$$\|Q(\tau)v\| \leq K_\gamma \exp(\alpha_\gamma(\tau - t) + \mu_\gamma t) \|\Phi_\gamma(t,\tau)v\|.$$

If $\alpha_\gamma > \mu_\gamma$ and $t \to +\infty$, we have $Q(\tau)v = 0$ thus $v = P(\tau)v \in ImP(\tau)$. □

Now we establish a topological result to nonuniform spectrum and a property of linear manifold $\mathcal{U}_\gamma(\tau)$.

Proposition 1.5 ([47]). *The nonuniform spectrum $\Sigma(A)$ is closed set. Moreover, for $\gamma \in \rho(A)$ we have*

$$\mathcal{U}_\gamma(\tau) = \mathcal{U}_\eta(\tau)$$

for all $\tau \in \mathbb{R}_0^+$ and η in some neighborhood J of γ.

Proof. For $\gamma \in \rho(A)$, then the system (1.9) admits nonuniform exponential dichotomy with an invariant projection $P(\cdot)$. So there exist $K \geq 1$, $\alpha > 0$ and $\mu \geq 0$ such that

$$\begin{aligned}
\|\Phi_\gamma(t,s)P(s)\| &\leq K\exp(-\alpha(t-s)+\mu s), \quad t \geq s,\\
\|\Phi_\gamma(t,s)(I-P(s))\| &\leq K\exp(\alpha(t-s)+\mu s), \quad t \leq s.
\end{aligned}$$

If we consider $\mu < \alpha$, set $\sigma = \frac{\alpha-\mu}{2} > 0$. For $\eta \in (\gamma-\sigma, \gamma+\sigma)$, it is easy to see that $P(\cdot)$ is an invariant projection of the evolution operator $\Phi_\eta(t,s) = \exp(-\eta(t-s))\Phi(t,s)$ of the system

$$\dot{x} = (A(t) - \eta I)x. \tag{1.10}$$

Moreover we have

$$\begin{aligned}
\|\Phi_\eta(t,s)P(s)\| &= \exp((\gamma-\eta)(t-s)) \|\Phi_\gamma(t,s)P(s)\|,\\
&\leq K\exp((\gamma-\eta-\alpha)(t-s)+\mu s)
\end{aligned}$$

for $t \geq s$ and

$$\begin{aligned}
\|\Phi_\eta(t,s)(I-P(s))\| &= \exp((\gamma-\eta)(t-s)) \|\Phi_\gamma(t,s)(I-P(s))\|,\\
&\leq K\exp((\gamma-\eta+\alpha)(t-s)+\mu s)
\end{aligned}$$

for $t \leq s$. It follows from the choice of σ and η that in first place

$$-\sigma-\alpha < \gamma-\eta-\alpha < \sigma-\alpha = -\frac{(\alpha+\mu)}{2},$$

so with $\alpha^* = \frac{\alpha+\mu}{2} > \mu$, we have

$$\|\Phi_\eta(t,s)P(s)\| \leq K\exp(-\alpha^*(t-s)+\mu s)$$

for $t \geq s$. In second place

$$\frac{\alpha+\mu}{2} = -\frac{(\alpha-\mu)}{2} + \alpha < \gamma - \eta + \alpha < \sigma + \alpha,$$

then we have

$$\|\Phi_\eta(t,s)(I-P(s))\| \leq K\exp(\alpha^*(t-s)+\mu s)$$

for $t \leq s$. This proves that the systems (1.10) admits nonuniform exponential dichotomy with $\alpha* > \mu$, for all $\eta \in (\gamma-\sigma, \gamma+\sigma)$, thus $(\gamma-\sigma, \gamma+\sigma) \subset \rho(A)$ and in particular the nonuniform spectrum $\Sigma(A)$ is closed. Moreover, it follows from Proposition 1.1 that

$$\mathcal{U}_\eta(\tau) = \mathcal{U}_\gamma(\tau) = ImP(\tau).$$

□

The following Theorem give us a complete description of the structure of the nonuniform spectrum.

Theorem 1.1 ([17,47]). *The nonuniform spectrum $\Sigma(A)$ of system (1.1) is the union of m disjoint closed intervals in $\mathbb{R}$ (called spectral intervals) with $0 \leq m \leq n$. Precisely, if $m = 0$ then $\Sigma(A) = \emptyset$; if $m = 1$ then $\Sigma(A) = \mathbb{R}$ or $(-\infty, b_1]$ or $[a_1, b_1]$ or $[a_1, +\infty)$; if $m > 1$ then*

$$\Sigma(A) = I_1 \cup [a_2, b_2] \cup \ldots [a_{m-1}, b_{m-1}] \cup I_m \tag{1.11}$$

with $I_1 = [a_1, b_1]$ or $(-\infty, b_1]$ and $I_m = [a_m, b_m]$ or $[a_m, +\infty)$, where $a_i \leq b_i < a_{i+1}$ for $i \in \{1, \ldots, m-1\}$.

Proof. We first establish an auxiliary result.

Lemma 1.1 ([47]). *For each $\gamma_1, \gamma_2 \in \rho(A)$ with $\gamma_1 < \gamma_2$, the following statements are equivalent:*

(i) $\mathcal{U}_{\gamma_1}(\tau) = \mathcal{U}_{\gamma_2}(\tau)$ *for some* $\tau \in \mathbb{R}_0^+$ *(and so for all* $\tau \in \mathbb{R}_0^+$*);*

(ii) $[\gamma_1, \gamma_2] \subset \rho(A)$.

Proof. Assume that $\mathcal{U}_{\gamma_1}(\tau) = \mathcal{U}_{\gamma_2}(\tau)$ for all $\tau \in \mathbb{R}_0^+$. It follows from Proposition (1.1) and (1.2) that the systems

$$\dot{x} = (A(t) - \gamma_1 I)x$$

and

$$\dot{x} = (A(t) - \gamma_2 I)x$$

admit nonuniform exponential dichotomies with the same projection $P(\cdot)$. Hence, there exist

$K_1, K_2 \geq 1$, $\alpha_1, \alpha_2 > 0$ and $\mu_1, \mu_2 \geq 0$ such that

$$\|\Phi_{\gamma_i}(t,s)P(s)\| \leq K_i \exp(-\alpha_i(t-s) + \mu_i s) \quad \text{for} \quad t \geq s \tag{1.12}$$

and

$$\|\Phi_{\gamma_i}(t,s)(I-P(s))\| \leq K_i \exp(\alpha_i(t-s) + \mu_i s) \quad \text{for} \quad t \leq s. \tag{1.13}$$

For each $\gamma \in [\gamma_1, \gamma_2]$, by (1.12) and if we take $\alpha = \gamma - \gamma_1 + \alpha_1$, $\beta = \gamma_2 - \gamma + \alpha_2$, we have

$$\begin{aligned} \|\Phi_\gamma(t,s)P(s)\| &= \exp(-(\gamma-\gamma_1)(t-s))\, \|\Phi_{\gamma_1}(t,s)P(s)\|\,, \\ &\leq K_1 \exp(-\alpha(t-s) + \mu_1 s), \quad \text{for} \quad t \geq s \end{aligned}$$

and similary, by (1.13),

$$\begin{aligned} \|\Phi_\gamma(t,s)(I-P(s))\| &= \exp((\gamma_2-\gamma)(t-s))\, \|\Phi_{\gamma_2}(t,s)(I-P(s))\|\,, \\ &\leq K_2 \exp(\beta(t-s) + \mu_2 s), \quad \text{for} \quad t \leq s. \end{aligned}$$

Taking the constants $\alpha^* = \min\{\alpha, \beta\}$, $\mu^* = \max\{\mu_1, \mu_2\}$ and $K^* = \max\{K_1, K_2\}$, we conclude that $[\gamma_1, \gamma_2] \subset \rho(A)$.

Now we assume that $[\gamma_1, \gamma_2] \subset \rho(A)$ and we proceed by contradiction. Namely, assume that, in addition, $\mathcal{U}_{\gamma_1}(\tau) \neq \mathcal{U}_{\gamma_2}(\tau)$ for some $\tau \in \mathbb{R}_0^+$. Let

$$b = \inf\left\{\gamma \in \rho(A) \colon \mathcal{U}_\gamma(\tau) = \mathcal{U}_{\gamma_2}(\tau) \quad \text{for some} \quad \tau \in \mathbb{R}_0^+\right\}.$$

Since $\mathcal{U}_{\gamma_1}(\tau) \neq \mathcal{U}_{\gamma_2}(\tau)$, it follows from Proposition 1.5 that $\gamma_1 < \gamma < \gamma_2$. We will show that $\gamma \in \Sigma(A)$. Otherwise, we consider two possibilities: either $\mathcal{U}_\gamma(\tau) = \mathcal{U}_{\gamma_2}(\tau)$ or $\mathcal{U}_\gamma(\tau) \neq \mathcal{U}_{\gamma_2}(\tau)$. In the first case, by Proposition 1.5 we have $\mathcal{U}_{\gamma'}(\tau) = \mathcal{U}_{\gamma_2}(\tau)$ and $\gamma' \in \rho(A)$ for all $\gamma' \in (\gamma-\varepsilon, \gamma]$ and some $\varepsilon > 0$. But this contradicts to the definition of γ. In the second case, again by Proposition 1.5 we have $\mathcal{U}_{\gamma'}(\tau) \neq \mathcal{U}_{\gamma_2}(\tau)$ and $\gamma' \in \rho(A)$ for all $\gamma' \in [\gamma, \gamma+\varepsilon)$ and some $\varepsilon > 0$, that again contradicts to the definition of γ. Hence, $\gamma \in \Sigma(A)$ but this contradicts to the assumption that $[\gamma_1, \gamma_2] \subset \rho(A)$.

□

We proceed with the proof of the Theorem 1.1. By Proposition 1.5, the set $\Sigma(A)$ is a disjoint union of (possibly infinite) closed intervals. Assume that $\Sigma(A)$ is composed of $n+1$ disjoint closed intervals. Then there exist $\gamma_1, \ldots, \gamma_n \in \rho(A)$ such that the intervals

$$(-\infty, \gamma_1), (\gamma_1, \gamma_2), \ldots, (\gamma_{n-1}, \gamma_n), (\gamma_n, +\infty)$$

intersect $\Sigma(A)$. By Lemma 1.1, we have

$$0 \leq \dim \mathcal{U}_{\gamma_1} < \dim \mathcal{U}_{\gamma_2} < \cdots < \dim \mathcal{U}_{\gamma_n} \leq n. \tag{1.14}$$

Now we show that

$$\dim \mathcal{U}_{\gamma_n} < n \quad \text{and} \quad \dim \mathcal{U}_{\gamma_1} > 0. \tag{1.15}$$

If $\dim \mathcal{U}_{\gamma_1} = 0$ then $\mathcal{U}_{\gamma_1}(\tau) = \{0\}$ for $\tau \in \mathbb{R}_0^+$. Since $\gamma_1 \in \rho(A)$, there exist constants

$K \geq 1, \alpha > 0, \mu \geq 0$ and projection $P(\tau) = 0$ such that

$$\|\exp(-\gamma_1(t-s))\Phi(t,s)\| \leq K\exp(\alpha(t-s)+\mu s) \quad \text{for} \quad t \leq s.$$

Hence, for $\gamma < \gamma_1$ we have

$$\|\exp(-\gamma(t-s))\Phi(t,s)\| \leq K\exp(\alpha(t-s)+\mu s) \quad \text{for} \quad t \leq s.$$

This shows that $(-\infty,\gamma_1) \subset \rho(A)$, which is impossible since $(-\infty,\gamma_1)$ intersects $\Sigma(A)$. Now we assume that $\dim \mathcal{U}_{\gamma_n} = n$. Then $\mathcal{U}_{\gamma_n}(\tau) = \mathbb{R}^n$ for $\tau \in \mathbb{R}_0^+$. Since $\gamma_n \in \rho(A)$, there exist a constants $K \geq 1, \alpha > 0, \mu \geq 0$ and projection $P(\tau) = Id$ such that

$$\|\exp(-\gamma_n(t-s))\Phi(t,s)\| \leq K\exp(-\alpha(t-s)+\mu s) \quad \text{for} \quad t \geq s.$$

Hence, for $\gamma > \gamma_n$, we have

$$\|\exp(-\gamma(t-s))\Phi(t,s)\| \leq K\exp(-\alpha(t-s)+\mu s) \quad \text{for} \quad t \geq s.$$

This shows that $(\gamma_n,+\infty) \subset \rho(A)$, which is impossible since $(\gamma_n,+\infty)$ intersects $\Sigma(A)$. Finally, it follows from (1.15) that (1.14) cannot hold and so there are at most n disjoint closed intervals on the right-hand side of (1.11).

□

Theorem 1.1 allows us to know the structure that the spectrum has. Next, we will present a sufficient condition so that the nonuniform spectrum to be nonempty and bounded.

Definition 1.2 ([17, 47]). *The evolution operator $\Phi(t,s)$ of the system (1.1) has nonuniformly bounded growth if there exist $K \geq 1, \bar{\alpha} \geq 0$ and $\bar{\mu} \geq 0$ such that*

$$\|\Phi(t,s)\| \leq K\exp(\bar{\alpha}|t-s|+\bar{\mu}s), \quad t,s \in \mathbb{R}_0^+. \tag{1.16}$$

Remark 1.2. *If $\bar{\mu} = 0$ the evolution operator has bounded growth, which is a particular case of the nonuniformly bounded growth (see [43] and [45]).*

Theorem 1.2 ([47]). *Assume that the evolution operator of system (1.1) has a nonuniformly bounded growth. Then the nonuniform spectrum $\Sigma(A)$ is nonempty and bounded, i.e.,*

$$\Sigma(A) = [a_1,b_1] \cup \cdots \cup [a_m,b_m], \tag{1.17}$$

with $m \geq 1$ and $-\infty < a_1 \leq b_1 < \cdots < a_m \leq b_m < \infty$.

Proof. By the assumption the evolution operator of system (1.1) satisfies (1.16) with constants $K \geq 1, \bar{\alpha} \geq 0$ and $\bar{\mu} \geq 0$. First we affirm that $\Sigma(A) \subset [-\bar{\alpha},\bar{\alpha}]$, an so it is bounded.

For $\gamma > \bar{\alpha}$, we get from (1.16) that

$$\|\Phi_\gamma(t,s)\| \leq K\exp(-(\gamma-\bar{\alpha})(t-s)+\bar{\mu}s), \quad \text{for} \quad t \geq s,$$

then the system $\dot{x} = (A(t)-\gamma I)x$ admits a nonuniform exponential dichotomy with the invariant projector $P(\tau) = Id$. This proves that $\gamma \in \rho(A)$ and consequently $(\bar{\alpha},+\infty) \subset \rho(A)$.

For $\gamma < -\bar{\alpha}$, we have

$$\|\Phi_\gamma(t,s)\| \leq K \exp(-(\gamma+\bar{\alpha})(t-s)+\bar{\mu}s), \quad \text{for} \quad t \leq s,$$

then the system $\dot{x} = (A(t)-\gamma I)x$ admits a nonuniform exponential dichotomy with the invariant projector $P(\tau)=0$. Hence we have $(-\infty,-\bar{\alpha}) \subset \rho(A)$. Consequently $\Sigma(A) \subset [-\bar{\alpha},\bar{\alpha}]$.

Now we will prove that $\Sigma(A) \neq \emptyset$. The above proof implies that for $\gamma > \bar{\alpha}, \mathcal{U}_\gamma = ImP(\tau) = \mathbb{R}^n$ because $P(\tau) = Id$, and that for $\gamma < -\bar{\alpha}, \mathcal{U}_\gamma = ImP(\tau) = \{0\}$ because $P(\tau)=0$. Set

$$\gamma_0 = \inf\left\{\gamma \in \rho(A) \colon \mathcal{U}_\gamma(\tau) = \mathbb{R}^n\right\}.$$

From the definition and previous comments, we have $\gamma_0 \in [-\bar{\alpha},\bar{\alpha}]$. To get a contradiction, let us assume that $\gamma_0 \in \rho(A)$. There are two cases to consider: $\mathcal{U}_{\gamma_0}(\tau) = \mathbb{R}^n$ or $\mathcal{U}_{\gamma_0}(\tau) \neq \mathbb{R}^n$. In the first case, by Proposition 1.5 there is $\varepsilon > 0$ such that for $\gamma \in (\gamma_0-\varepsilon,\gamma_0+\varepsilon)$, we have $\mathcal{U}_\gamma(\tau) = \mathbb{R}^n$, which contradicts the definition of γ_0. In the second case, Proposition 1.5 implies $\mathcal{U}_\gamma(\tau) \neq \mathbb{R}^n$ for $\gamma \in (\gamma_0-\varepsilon,\gamma_0+\varepsilon)$, which contradicts the definition of γ_0. So $\Sigma(A) \neq \emptyset$.

□

Our new objective will be to use the decomposition of the nonuniform spectrum (see equation (1.17)) to write the system (1.1) as a new system, which will depend on the spectral intervals of the spectrum $\Sigma(A)$.

1.3 An application of the nonuniform spectrum for linear systems.

As first defined in [17], we say that system (1.1) and the system

$$\dot{y} = B(t)y \tag{1.18}$$

are *nonuniformly kinematically similar* if there exists a matrix function $S : \mathbb{R}_0^+ \to GL_n(\mathbb{R})$ satisfying

$$\|S(t)\| \leq M_\upsilon \exp(\upsilon t) \quad \text{and} \quad \left\|S^{-1}(t)\right\| \leq M_\upsilon \exp(\upsilon t), \tag{1.19}$$

with $M = M_\upsilon > 0, \upsilon \geq 0$ constants, such that the change of coordinates $y(t) = S^{-1}(t)x(t)$ transform the system (1.1) into (1.18) and S is called a *nonuniform Lyapunov transformation.* Moreover, $B(t)$ satisfies

$$B(t) = S^{-1}(t)A(t)S(t) - S^{-1}(t)\dot{S}(t). \tag{1.20}$$

Remark 1.3. *It is important to mention that the nonuniform kinematic similarity is an equivalence relation between the linear systems. In this context, we will assume that at 0 the S function has a derivative on the right.*

The following result characterizes the normal forms of nonautonomous linear differential systems via their nonuniform dichotomy spectrums.

Theorem 1.3. *Suppose that the evolution operator of system* (1.1) *has a nonuniformly bounded growth. Let*

$$\Sigma(A) = [a_1, b_1] \cup \cdots \cup [a_m, b_m],$$

with $m \geq 1$ *and* $-\infty < a_1 \leq b_1 < \cdots < a_m \leq b_m < \infty$ *be the nonuniform spectrum. Then system (1.1) is nonuniformly kinematically similar to*

$$\dot{y} = \begin{pmatrix} B_1(t) & & \\ & \ddots & \\ & & B_m(t) \end{pmatrix} y \tag{1.21}$$

where $B_i : \mathbb{R}_0^+ \to \mathbb{R}^{n_i \times n_i}$ *and* $\Sigma(B_i) = [a_i, b_i]$ *for* $i \in \{1, \ldots, m\}$.

For proving this Theorem we need some preliminary results, which will be presented below.

Lemma 1.2. *If the systems* (1.1) *and* (1.18) *are nonuniformly kinematically similar, then* $\Sigma(A) = \Sigma(B)$.

Proof. In first place, we will prove that if (1.1) and (1.18) are nonuniform kinematically similar with matrix function S, $\Phi_A(t,s)$ and $\Phi_B(t,s)$ their respective evolution operator, then we have

$$\Phi_A(t,s)S(s) = S(t)\Phi_B(t,s) \tag{1.22}$$

for $t, s \in \mathbb{R}_0^+$. In fact, we know that $x(t) = \Phi_A(t,s)x(s)$ is a solution of (1.1) and $y(t) = \Phi_B(t,s)y(s)$ is a solution of (1.18). Moreover , $x(t) = S(t)y(t)$ is a nonuniform Lyapunov transformation, for all $t \in \mathbb{R}_0^+$, so on the one hand we have

$$x(t) = S(t)y(t) = S(t)\Phi_B(t,s)y(s) = S(t)\Phi_B(t,s)S^{-1}(s)x(s)$$

and on the other hand we have

$$x(t) = \Phi_A(t,s)x(s)$$

and the two previous equations imply

$$\Phi_A(t,s) = S(t)\Phi_B(t,s)S^{-1}(s).$$

Now we continue the proof of Lemma. Let $\lambda \in \rho(A)$ then the system

$$\dot{x} = [A(t) - \lambda I]x$$

have a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with invariant projector $P(\cdot)$.

By (1.22) we have $\exp(-\lambda(t-s))\Phi_B(t,s) = S^{-1}(t)\exp(-\lambda(t-s))\Phi_A(t,s)S(s)$, which is the evolution operator associated to system

$$\dot{y} = [B(t) - \lambda I]y$$

and its invariant projection is $Q(t) = S^{-1}(t)P(t)S(t)$.

This fact combined with the submultiplicative property of norms and the estimates for S and

S^{-1} allows to prove that if $t \geq s$ (the case $t \leq s$ can be proved similarly),

$$\begin{aligned}\|\Phi_B(t,s)\exp(-\lambda(t-s))Q(s)\| &\leq \left\|S^{-1}(t)\right\| \|\Phi_A(t,s)\exp(-\lambda(t-s))P(s)\| \|S(s)\|,\\ &\leq M_\upsilon^2 K_\lambda \exp(\upsilon t)\exp(-\alpha_\lambda(t-s)+\mu_\lambda s)\exp(\upsilon s),\\ &\leq M_\upsilon^2 K_\lambda \exp(-(\alpha_\lambda-\upsilon)(t-s)+(\mu_\lambda+2\upsilon)s).\end{aligned}$$

Finally, if $\alpha_\lambda > \upsilon$, then $\lambda \in \rho(B)$. To prove the other contention, we use the fact that nonuniform kinematic similarity is an equivalence relation. $\square$

Lemma 1.3 ([47]). *Let $P_0 \in \mathbb{R}^{n\times n}(M_n(\mathbb{R}))$ be a symmetric projection and $X(t) \in GL_n(\mathbb{R})$ with $t \in \mathbb{R}_0^+$. Set $Q(t) = P_0X(t)^TX(t)P_0 + (I-P_0)X(t)^TX(t)(I-P_0)$. Then*

(i) $Q(t)$ is positively definite and symmetric.

(ii) There exists a unique positively definite and symmetric matrix function $R(t)$ such that $R^2(t) = Q(t)$ and $P_0R(t) = R(t)P_0$.

(iii) $S(t) = X(t)R^{-1}(t)$ is invertible and satisfies $S(t)P_0S^{-1}(t) = X(t)P_0X^{-1}(t)$ and

$$\|S(t)\| \leq \sqrt{2}, \quad \left\|S^{-1}(t)\right\| \leq \sqrt{\|X(t)P_0X^{-1}(t)\|^2 + \|X(t)(I-P_0)X^{-1}(t)\|^2}. \tag{1.23}$$

Proof. We will prove each statement:

(i) The next proof is for any fixed $t \in \mathbb{R}_0^+$. In first place, the matrix $Q(t)$ is symmetric and positively definite. In fact, deduce $Q(t) = Q(t)^T$ is a direct consequence of $P_0^T = P_0$ combinated with the property $(CD)^T = D^TC^T$. In order to prove that $Q(t)$ is positively definite, let $\xi \in \mathbb{R}^n \setminus \{0\}$ and notice that

$$\begin{aligned}\xi^TQ(t)\xi &= \xi^TP_0X(t)^TX(t)P_0\xi + \xi^T(I-P_0)X(t)^TX(t)(I-P_0)\xi,\\ &= \|X(t)P_0\xi\|^2 + \|X(t)(I-P_0)\xi\|^2 \geq 0\end{aligned}$$

and we will see that the inequality is strict. Indeed, otherwise if $\xi^TQ(t)\xi = 0$, it will be equivalent to

$$X(t)P_0\xi = 0 \quad \text{and} \quad X(t)(I-P_0)\xi = 0,$$

which implies that $X(t)\xi = 0$ and $\xi = 0$ (since $X(t)$ is invertible), obtaining a contradiction.

(ii) As $Q(t)$ is symmetric and positively definite, we know that there exist a unique matrix $R(t) = R(t)^T$ such that

$$R^2(t) = Q(t),$$

which is a property of any real symmetric matrix. Now we will verify that $R(t)$ and P_0 commute. As $P_0^2 = P_0$, it is easy to see that

$$R^2(t)P_0 = P_0R^2(t) = P_0X(t)^TX(t)P_0.$$

As $R^2(t)$ and P_0 are symmetric matrices they are diagonalizable. In addition, as they commute the have simultaneous diagonalization, that is, there exist an invertible matrix $W(t)$ such that

$$R^2(t) = W^{-1}(t)\tilde{A}(t)W(t) \quad \text{and} \quad P_0 = W^{-1}(t)\tilde{B}(t)W(t),$$

where A and B are diagonal matrices containing the (positive) eigenvalues of $R^2(t)$ and P_0 respectively.

As the diagonal matrix $\tilde{A}(t)$ has only positive terms, we can consider $\tilde{A}(t) = \tilde{D}(t)\tilde{D}(t)$, where $\tilde{a}_{ii} = \sqrt{\tilde{d}_{ii}}$. Then, it follows that

$$R^2(t) = W^{-1}(t)\tilde{D}(t)\tilde{D}(t)W(t) = (W^{-1}(t)\tilde{D}(t)W(t))(W^{-1}(t)\tilde{D}(t)W(t))$$

and we have that $R(t) = W^{-1}(t)\tilde{D}(t)\tilde{D}(t)W(t)$.

As $\tilde{B}(t)\tilde{D}(t) = \tilde{D}(t)\tilde{B}(t)$ since are diagonal matrices, we have

$$W^{-1}(t)\tilde{B}(t)\tilde{D}(t)W(t) = W^{-1}(t)\tilde{D}(t)\tilde{R}(t)W(t),$$

which is equivalent to

$$(W^{-1}(t)\tilde{B}(t)W(t))(W^{-1}(t)\tilde{D}(t)W(t)) = (W^{-1}(t)\tilde{D}(t)W(t))(W^{-1}(t)\tilde{B}(t)W(t)),$$

which is equivalent to

$$P_0R(t) = R(t)P_0.$$

(iii) Finally, as $P_0R(t) = R(t)P_0$, we also have that

$$\begin{aligned} R(t)(I-P_0) &= R(t) - R(t)P_0, \\ &= R(t) - P_0R(t) = (I-P_0)R(t). \end{aligned}$$

Now let us consider the transformation $S(t) = X(t)R^{-1}(t)$. Note that

$$\begin{aligned} S(t)P_0S^{-1}(t) &= X(t)R^{-1}(t)P_0R(t)X^{-1}(t), \\ &= X(t)R^{-1}(t)R(t)P_0X^{-1}(t), \\ &= X(t)P_0X^{-1}(t) \end{aligned}$$

since $P_0R(t) = R(t)P_0$.

By using $S(t)R(t) = X(t)$ and $R^T(t) = R(t)$, we have that

$$\begin{aligned} R^2(t) &= P_0X^T(t)X(t)P_0 + (I-P_0)X^T(t)X(t)(I-P_0), \\ &= P_0R^T(t)S^T(t)S(t)R(t)P_0 + (I-P_0)R^T(t)S^T(t)S(t)R(t)(I-P_0), \\ &= P_0R(t)S^T(t)S(t)R(t)P_0 + (I-P_0)R(t)S^T(t)S(t)R(t)(I-P_0), \\ &= R(t)P_0S^T(t)S(t)P_0R(t) + R(t)(I-P_0)S^T(t)S(t)(I-P_0)R(t). \end{aligned}$$

We multiply by $R^{-1}(t)$ by the right and the left, obtaining

$$I = P_0S^T(t)S(t)P_0 + (I-P_0)S^T(t)S(t)(I-P_0).$$

Now, let $\xi \in \mathbb{R}^n$ and note that

$$\begin{aligned}
\|S(t)\xi\|^2 &= \|S(t)P_0\xi + S(t)(I-P_0)\xi\|^2, \\
&\leq \{\|S(t)P_0\xi\| + \|S(t)(I-P_0)\xi\|\}^2, \\
&\leq 2\left\{\|S(t)P_0\xi\|^2 + \|S(t)(I-P_0)\xi\|^2\right\}, \\
&\leq 2\left\{\xi^T P_0 S^T(t)S(t)P_0\xi + \xi^T(I-P_0)S^T(t)S(t)(I-P_0)\xi\right\}, \\
&\leq 2\xi^T\left\{P_0 S^T(t)S(t)P_0\xi + (I-P_0)S^T(t)S(t)(I-P_0)\right\}\xi, \\
&\leq 2\xi^T\xi = 2\|\xi\|^2,
\end{aligned}$$

the we can conclude that $\|S(t)\| \leq \sqrt{2}$. On the other hand

$$\begin{aligned}
(S^{-1}(t))^T S^{-1}(t) &= (X^{-1}(t))^T R^T(t)R(t)X^{-1}(t), \\
&= (X^{-1}(t))^T R^2(t)X^{-1}(t), \\
&= (X^{-1}(t))^T[P_0X(t)^TX(t)P_0 + (I-P_0)X(t)^TX(t)(I-P_0)]X^{-1}(t),
\end{aligned}$$

and hence

$$\left\|S^{-1}(t)\right\|^2 \leq \left\|X(t)P_0X^{-1}(t)\right\|^2 + \left\|X(t)(I-P_0)X^{-1}(t)\right\|^2.$$

Moreover, if $X(t)$ is continuous , or continuously differentiable, function of t in $\mathbb{R}_0^+$, then from the fact that the positive squart root of a continuous, or continuously differentiable, positive symmetric matrix function $R(t)$ is again continuous, or continuously differentiable, $S(t)$ to be too.

□

Lemma 1.4 ([47]). *Assume that system* (1.1) *has an invariant projector $P(\cdot)$ with $P(t) \neq 0$ or I, for all $t \in \mathbb{R}_0^+$. Then there exists a nonuniform Lyapunov matrix function $S : \mathbb{R}_0^+ \to GL_n(\mathbb{R})$ such that*

$$S^{-1}(t)P(t)S(t) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. Since $P(\cdot)$ is an invariant projection associated with the evolution operator $\Phi(t,s)$ of system (1.1), i.e., $P(t)\Phi(t,s) = \Phi(t,s)P(s)$ for $t,s \in \mathbb{R}_0^+$ and for what is mentioned in the Remark 1.1, we have $dim\ (ImP(t)) = dim\ (ImP(s))$. the fact that $P(\cdot)$ is a projection implies that for any given $s \in \mathbb{R}_0^+$ there exists a $T(s) \in GL_n(\mathbb{R})$ such that

$$T(s)P(s)T^{-1}(s) = \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & 0_{n_2\times n_2} \end{pmatrix} = P_0 \tag{1.24}$$

where $n_1 = dim\ (ImP(t))$ and $n_2 = dim\ (KerP(t))$. Applying Lemma 1.3 to $X(t) = \Phi(t,s)T^{-1}(s)$ and P_0, we get an $R(t)$ satisfying $P_0R(t) = R(t)P_0$ for $t \in \mathbb{R}_0^+$. Set $S(t) = \Phi(t,s)T^{-1}(s)R^{-1}(t)$, we have

$$
\begin{aligned}
S^{-1}(t)P(t)S(t) &= R(t)T(s)\Phi(s,t)P(t)\Phi(t,s)T^{-1}(s)R^{-1}(t),\\
&= R(t)T(s)P(s)\Phi(s,t)\Phi(t,s)T^{-1}(s)R^{-1}(t),\\
&= R(t)P_0R^{-1}(t),\\
&= P_0,
\end{aligned}
\tag{1.25}
$$

where we used the fact that $\Phi(t,s)\Phi(s,t) = I$, the invariance of the projection $P(\cdot)$ with respect to $\Phi(t,s)$ and the equation (1.24).

By Lemma 1.3 the matrix function S satisfies $S(t)P_0S^{-1}(t) = X(t)P_0X^{-1}(t)$ and using (1.25) in the estimates of (1.23), we have

$$
\begin{aligned}
\|S(t)\| \leq \sqrt{2}, \quad \left\|S^{-1}(t)\right\| &\leq \sqrt{\|S(t)P_0S^{-1}(t)\|^2 + \|S(t)(I-P_0)S^{-1}(t)\|^2},\\
\|S(t)\| \leq \sqrt{2}, \quad \left\|S^{-1}(t)\right\| &\leq \sqrt{\|P(t)\|^2 + \|(I-P(t))\|^2}
\end{aligned}
$$

and by (1.5), in the last expression we obtain

$$
\|S(t)\| \leq \sqrt{2}, \quad \left\|S^{-1}(t)\right\| \leq \sqrt{2}K\exp(\mu t)
$$

and if we define $M = \sqrt{2}K$, we have

$$
\|S(t)\| \leq M\exp(\mu t), \quad \left\|S^{-1}(t)\right\| \leq M\exp(\mu t)
$$

which allow us conclude that S is a nonuniform Lyapunov transformation.

□

Lemma 1.5. *The system* (1.1) *is nonuniform kinematically similar to an equation*

$$
\dot{y} = B(t)y \tag{1.26}
$$

where $B(t)$ commute with P_0 and satisfies

$$
\|B(t)\| \leq \|A(t)\|, \quad \text{for every} \quad t \in \mathbb{R}_0^+.
$$

In particular, if there exist $\mathcal{M} > 0$ and $\upsilon \geq 0$ such that $\|A(t)\| \leq \mathcal{M}\exp(\upsilon t)$ for all $t \in \mathbb{R}_0^+$, then

$$
\|B(t)\| \leq \mathcal{M}\exp(\upsilon t),
$$

for all $t \in \mathbb{R}_0^+$.

Proof. This proof follows the line of [18], (see Lemma 2, page 40). Under the present circumstances the functions $R(t)$ and $S(t)$ of the Lemma 1.3 are continuously differentiable in $\mathbb{R}_0^+$. Then change of variable $x(t) = S(t)y(t)$ transforms (1.1) into

$$
\dot{z} = C(t)z, \tag{1.27}
$$

where $C(t) = S^{-1}(t)A(t)S(t) - S^{-1}(t)\dot{S}(t)$. From Lemma 1.3, $X(t) = S(t)R(t)$ and if we derive this equation, then

$$\begin{aligned}
A(t)X(t) &= \dot{S}(t)R(t) + S(t)\dot{R}(t),\\
A(t)S(t)R(t) &= A(t)S(t)R(t) - S(t)C(t)R(t) + S(t)\dot{R}(t),\\
C(t) &= \dot{R}(t)R^{-1}(t),
\end{aligned}$$

and as P_0 is constant and commute with $R(t)$, then also commute with $C(t)$. Let $\mathcal{U}(t)$ be the fundamental matrix for the equation

$$\dot{u} = \frac{1}{2}[C(t) - C^T(t)]u \tag{1.28}$$

such that $\mathcal{U}(0) = I$. Then $\mathcal{U}(t)$ is unitary for every $t \in \mathbb{R}_0^+$, since the coefficient matrix is symmetric. Moreover $\mathcal{U}(t)$ commutes with P_0 for every $t \in \mathbb{R}_0^+$, since $B^T(t)$ commute with $P_0^T = P_0$ and the solutions of (1.28) are uniquely determined by their initial values. It is easily verified that the further change of variables $z(t) = \mathcal{U}(t)y(t)$ transform (1.27) into (1.18), where

$$B(t) = \frac{1}{2}\mathcal{U}^{-1}(t)[C(t) + C^T(t)]\mathcal{U}(t)$$

is symmetric and commutes with P_0 for every $t \in \mathbb{R}_0^+$.

For each fixed $t \in \mathbb{R}_0^+$, there exist $\mathcal{M} > 0$, $\upsilon \geq 0$ such that

$$-2\mathcal{M}\exp(\upsilon t)I \leq A(t) + A^T(t) \leq 2\mathcal{M}\exp(\upsilon t)I$$

and since $\left\|A^T(t)\right\| = \|A(t)\|$, we have that

$$\left\|A(t) + A^T(t)\right\| \leq 2\mathcal{M}\exp(\upsilon t).$$

From the definition of $R(t)$, we have

$$\begin{aligned}
R(t)\dot{R}(t) + \dot{R}(t)R(t) = \quad & P_0 X^T(t)[A(t) + A^T(t)]X(t)P_0\\
& + (I - P_0)X^T(t)[A(t) + A^T(t)]X(t)(I - P_0).
\end{aligned}$$

It follows that

$$-2\mathcal{M}\exp(\upsilon t)R^2(t) \leq R(t)\dot{R}(t) + \dot{R}(t)R(t) \leq 2\mathcal{M}\exp(\upsilon t)R^2(t)$$

and hence

$$-2\mathcal{M}\exp(\upsilon t) \leq \dot{R}(t)R^{-1}(t) + R^{-1}(t)\dot{R}(t) \leq 2\mathcal{M}\exp(\upsilon t).$$

Therefore

$$\left\|C(t) + C^T(t)\right\| \leq \left\|A(t) + A^T(t)\right\|.$$

Since $\mathcal{U}(t)$ is unitary, this implies that

$$\|B(t)\| = \frac{1}{2}\left\|C(t) + C^T(t)\right\| \leq \|A(t)\| \leq \mathcal{M}\exp(\upsilon t).$$

Finally, we have $x(t) = T(t)y(t)$ that transform (1.1) into (1.26), where $T(t) = S(t)U(t)$ and satifies

$$\|T(t)\| \leq \sqrt{2}, \quad \left\|T^{-1}(t)\right\| \leq \sqrt{2} K \exp(\mu t).$$

□

Now we are able to prove Theorem 1.3.

Proof. By hypothesis and Theorem 1.2 we have the nonuniform spectrum $\Sigma(A)$ as in (1.17) with $m \geq 1, a_1 > -\infty$ and $b_m < +\infty$.

Next we will prove the following technical Lemma:

Lemma 1.6 ([30]). *If the evolution operator of system* (1.1) *has a nonuniform bounded growth and if* $\Sigma(A) \subset [a, b]$ *and* $\lambda > b$ *(resp. or* $\lambda < a$*) the system*

$$\dot{x} = (A(t) - \lambda I)x$$

has a nonuniform exponential dichotomy with projector $P(t) = I$ *(resp. with projector* $P(t) = 0$*).*

Proof. We have that the evolution operator satisfies

$$\|\Phi(t, s)\| \leq K \exp(\bar{\alpha}\,|t - s| + \bar{\mu} s), \quad t,\ s \in \mathbb{R}_0^+.$$

We consider $\lambda > b$. Let $h = \max\{\bar{\alpha} + 1 + \bar{\mu}, \lambda + 1 + \bar{\mu}\}$ and

$$\Phi_h(t, s) = \Phi(t, s)\exp(-h(t - s)).$$

Then

$$\|\Phi_h(t, s)\| = \|\Phi(t, s)\| \exp(-h(t - s)) \leq K \exp(\bar{\alpha}(t - s) + \bar{\mu} s - h(t - s)), \quad (t \geq s).$$

Now we define $\alpha = h - \bar{\alpha} > \bar{\mu}$ and the previous equation becomes

$$\|\Phi_h(t, s)\| \leq K \exp(-\alpha(t - s) + \bar{\mu} s) \quad (t \geq s),$$

which implies that the system

$$\dot{x} = (A(t) - hI)x$$

has a nonuniform exponential dichotomy with projector $P(t) = I$ and $[\lambda, h] \subset \rho(A)$.

By Lemma 1.1, the system

$$\dot{x} = (A(t) - \lambda I)x$$

also has a nonuniform exponential dichotomy with projector $P(t) = I$.

For $\lambda < a$ the proof is similar considering $h = \min\{-(\bar{\alpha} + 1 + \bar{\mu}), (\lambda - 1 - \bar{\mu})\}$.

□

In what follows we call the open intervals $(-\infty, a_1), (b_1, a_2), \ldots, (b_{m-1}, a_m)$ and $(b_m, +\infty)$ the *spectral gaps*. As $\Sigma(A) \subset [a_1, b_m]$, the previous Lemma implies that for $\gamma \in (-\infty, a_1)$, the system

$\dot{x} = (A(t) - \gamma I)x$ has a nonuniform exponential dichotomy with projector $P(t) = 0$ and for $\gamma \in (b_m, +\infty)$, the system $\dot{x} = (A(t) - \gamma I)x$ has a nonuniform exponential dichotomy with projector $P(t) = I$, so we choose $\gamma_i \in (b_i, a_{i+1})$ for $i \in \{1, \ldots, m-1\}$.

For $\gamma_1 \in (b_1, a_2)$, the system

$$\dot{x} = (A(t) - \gamma_1 I)x$$

admits a nonuniform exponential dichotomy with an invariant projector $P_1(\cdot)$ associated to evolution operator $\Phi_{\gamma_1}(t,s) = \exp(-\gamma_1(t-s))\Phi(t,s)$. Then we have

$$\begin{cases} \|\Phi_{\gamma_1}(t,s)P_1(s)\| & \leq & K_1 \exp(-\alpha_1(t-s) + \mu_1 s), \quad t \geq s, \\ \|\Phi_{\gamma_1}(t,s)(I - P_1(s))\| & \leq & K_1 \exp(\alpha_1(t-s) + \mu_1 s), \quad t \leq s. \end{cases} \tag{1.29}$$

where $K_1 \geq 1, \alpha_1 > 0$ and $\mu_1 \geq 0$.

We claim that system (1.1) is nonuniformly kinematically similiar to

$$\dot{y} = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_{22}(t) \end{pmatrix} y \tag{1.30}$$

with $B_1 : \mathbb{R}_0^+ \to M_{n_1}(\mathbb{R})$ and $B_{22} : \mathbb{R}_0^+ \to M_{m_2}(\mathbb{R})$, where $n_1 = dim\ (ImP_1(t))$ and $m_2 = dim\ (KerP_1(t))$. Moreover, $\Sigma(B_1) = [a_1, b_1]$ and $\Sigma(B_{22}) = [a_2, b_2] \cup \cdots \cup [a_m, b_m]$.

Now we prove this claim. By Lemma 1.4 there exists a nonuniform Lyapunov matrix function $S_1 : \mathbb{R}_0^+ \to GL_n(\mathbb{R})$ such that

$$S_1^{-1}(t)P_1(t)S_1(t) = \begin{pmatrix} I_{n_1 \times n_1} & 0_{n_1 \times m_2} \\ 0_{m_2 \times n_1} & 0_{m_2 \times m_2} \end{pmatrix} = \tilde{P}_1 \tag{1.31}$$

We define

$$B(t) = S_1^{-1}(t)A(t)S_1(t) - S_1^{-1}(t)\dot{S}_1(t), \quad \text{for} \quad t \in \mathbb{R}_0^+,$$

so the system (1.1) is nonuniformly kinematically similar to

$$\dot{y} = B(t)y, \tag{1.32}$$

via the transformation $x(t) = S_1(t)y(t)$ and by (1.22) the evolution operator of (1.32) is

$$\Phi_B(t,s) = S_1^{-1}(t)\Phi(t,s)S_1(s). \tag{1.33}$$

Set $R(t) = S_1^{-1}(t)\Phi(t,s)T^{-1}(s)$. From the proof of Lemma 1.4, we have $\tilde{P}_1 R(t) = R(t)\tilde{P}_1$. This implies that $R^{-1}(t)$ and $\dot{R}(t)$ both commute with $\tilde{P}_1$, since $\tilde{P}_1$ is constant. By (1.33) and definition of $R(t)$, we have

$$\Phi_B(t,s) = R(t)T(s)S_1(s) \tag{1.34}$$

so if we derive (1.34) with respect to t, on the one hand we have

$$\frac{d\Phi_B(t,s)}{dt} = \dot{R}(t)T(s)S_1(s)$$

and on the other hand

$$\frac{d\Phi_B(t,s)}{dt} = B(t)\Phi_B(t,s) = B(t)R(t)T(s)S_1(s),$$

therefore, from the two previous equations it can be affirmed that

$$B(t) = \dot{R}(t)R^{-1}(t) \quad \text{and} \quad \tilde{P}_1 B(t) = B(t)\tilde{P}_1. \tag{1.35}$$

Now we write $B(t)$ in the block form, i.e.,

$$B(t) = \begin{pmatrix} B_1(t) & C_1(t) \\ C_{22}(t) & B_{22}(t) \end{pmatrix}$$

where $B_1 : \mathbb{R}_0^+ \to M_{n_1}(\mathbb{R}), B_{22} : \mathbb{R}_0^+ \to M_{m_2}(\mathbb{R}), C_1 : \mathbb{R}_0^+ \to M_{n_1 \times m_2}(\mathbb{R})$ and $C_{22} : \mathbb{R}_0^+ \to M_{m_2 \times n_1}(\mathbb{R})$. From (1.31) and the second expression of (1.35) we get that $C_1(t) = 0$ and $C_{22}(t) = 0$.

By Lemma 1.2, the systems (1.1) and (1.30) have the same nonuniform dichotomy spectrum.

Before continuing we will prove the following lemma.

Lemma 1.7. *Consider the system* (1.18) *where B is written as*

$$B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix} \tag{1.36}$$

where $B_1 : \mathbb{R}_0^+ \to M_{n_1}(\mathbb{R}), B_2 : \mathbb{R}_0^+ \to M_{n_2}(\mathbb{R})$, then we have that

$$\Sigma(B) = \Sigma(B_1) \cup \Sigma(B_2). \tag{1.37}$$

Proof. If $\lambda \in (\rho(B_1) \cap \rho(B_2))$, then the systems

$$\dot{x}_1 = (B_1(t) - \lambda I)x_1$$

and

$$\dot{x}_2 = (B_2(t) - \lambda I)x_2$$

with evolution operators $\Phi_{\lambda,1}(t,s)$ and $\Phi_{\lambda,2}(t,s)$ respectively, have a nonuniform exponential dichotomy, i.e., there exist constants $K_1, K_2 \geq 1, \alpha_i > 0, \mu_i \geq 0$, for $i \in \{1,2\}$ and invariant projectors $P_1 : \mathbb{R}_0^+ \to M_{n_1}(\mathbb{R}), P_2 : \mathbb{R}_0^+ \to M_{n_2}(\mathbb{R})$ respectively, satisfying

$$\begin{cases} \|\Phi_{\lambda,i}(t,s)P_i(s)\| & \leq & K_i \exp(-\alpha_i(t-s) + \mu_i s), \quad t \geq s, \\ \|\Phi_{\lambda,i}(t,s)(I - P_i(s))\| & \leq & K_i \exp(\alpha_i(t-s) + \mu_i s), \quad t \leq s. \end{cases}$$

If we consider the evolution operator of system $\dot{x} = (B(t) - \lambda I)x$ and invariant projection as

$$\Phi_\lambda(t,s) = diag(\Phi_{\lambda,1}(t,s), \Phi_{\lambda,2}(t,s)) \quad \text{and} \quad P(t) = diag(P_1(t), P_2(t)) \tag{1.38}$$

and $K = \max\{K_1, K_2\}, \alpha = \min\{\alpha_1, \alpha_2\}$ and $\mu = \max\{\mu_1, \mu_2\}$, then we have

$$\begin{cases} \|\Phi_\lambda(t,s)P(s)\| & \leq & K \exp(-\alpha(t-s) + \mu s), \quad t \geq s, \\ \|\Phi_\lambda(t,s)(I - P(s))\| & \leq & K \exp(\alpha(t-s) + \mu s), \quad t \leq s, \end{cases} \tag{1.39}$$

so $\lambda \in \rho(B)$.

If $\lambda \in \rho(B)$, then the system $\dot{x} = (B(t) - \lambda I)x$ admits nonuniform exponential dichotomy, which implies that it is satisfied (1.39). We can write $\Phi_\lambda(t,s)$ and $P(t)$ as in (1.38) and we can get the following estimates:

$$\begin{aligned} \|\Phi_{\lambda,i}(t,s)P_i(s)\| \leq \|\Phi_\lambda(t,s)P(s)\| &\leq K\exp(-\alpha(t-s)+\mu s), \quad t \geq s, \\ \|\Phi_{\lambda,i}(t,s)(I-P_i(s))\| \leq \|\Phi_\lambda(t,s)(I-P(s))\| &\leq K\exp(\alpha(t-s)+\mu s), \quad t \leq s, \end{aligned}$$

which implies that $\lambda \in (\rho(B_1) \cap \rho(B_2))$.

□

Now we continue with the proof of the Theorem 1.3. Using the previous Lemma and by (1.30), we have that

$$\Sigma(B) = \Sigma(B_1) \cup \Sigma(B_{22}) = [a_1, b_1] \cup \cdots \cup [a_m, b_m].$$

Moreover the evolution operator of system (1.30) has the invariant projection $\tilde{P}_1$ given in (1.31). Since the first inequality of (1.29) also holds for all $\gamma \geq a_2$, so we get that $[a_2, +\infty) \subset \rho(B_1)$, then $\Sigma(B_1) \subset (-\infty, a_2)$. Similarly from the second inequality of (1.29) also holds for all $\gamma \leq b_1$, so we get that $(-\infty, b_1] \subset \rho(B_{22})$, then $\Sigma(B_{22}) \subset (b_1, +\infty)$. These last arguments imply that $\Sigma(B_1) = [a_1, b_1]$ and $\Sigma(B_{22}) = [a_2, b_2] \cup \cdots \cup [a_m, b_m]$.

Now we work with the system

$$\dot{y}_{22} = B_{22}(t) y_{22} \tag{1.40}$$

and its evolution operator $\Phi_2(t,s)$, so for $\gamma_2 \in (b_2, a_3)$, the system

$$\dot{y}_{22} = (B_{22}(t) - \gamma_2 I) y_{22}$$

admits a nonuniform exponential dichotomy with an invariant projector $P_2 : \mathbb{R}_0^+ \to GL_{n_2}(\mathbb{R})$ associated to evolution operator $\Phi_{\gamma_2}(t,s) = \exp(\gamma_2(t-s))\Phi_2(t,s)$. Then we have

$$\left\{ \begin{aligned} \|\Phi_{\gamma_2}(t,s)P_2(s)\| &\leq K_2\exp(-\alpha_2(t-s)+\mu_2 s), \quad t \geq s, \\ \|\Phi_{\gamma_2}(t,s)(I-P_2(s))\| &\leq K_2\exp(\alpha_2(t-s)+\mu_2 s), \quad t \leq s. \end{aligned} \right. \tag{1.41}$$

In the same way as before, it is proved that the system (1.40) is nonuniformly kinematically similar to

$$\dot{y}_{23} = \begin{pmatrix} B_2(t) & 0 \\ 0 & B_{33}(t) \end{pmatrix} y_{23}$$

where $B_2 : \mathbb{R}_0^+ \to M_{n_2}(\mathbb{R})$, $B_{33} : \mathbb{R}_0^+ \to M_{m_3}(\mathbb{R})$ and $m_2 = n_2 + m_3$, via a nonuniformly Lyapunov transformation $y_{22}(t) = S_{22}(t) y_{23}(t)$. Take

$$S_2(t) = \begin{pmatrix} I_{n_1 \times n_1} & 0 \\ 0 & S_{22}(t) \end{pmatrix} S_1(t).$$

Then the system (1.1) is nonuniformly kinematically similar to

$$\dot{z} = C(t), \qquad C(t) = \begin{pmatrix} B_1(t) & 0 & 0 \\ 0 & B_2(t) & 0 \\ 0 & 0 & B_{33}(t) \end{pmatrix}, \tag{1.42}$$

via the nonuniformly Lyapunov transformation $x(t) = C(t)z(t)$. Since the first inequality of (1.41) also holds for all $\gamma \geq a_3$, taking into account equation (1.42) we get that $\Sigma(diag(B_1, B_2)) \subset (-\infty, a_3)$. Similarly from the second inequality of (1.41) we have $\Sigma(B_{33}) \subset (b_2, +\infty)$. The last arguments and Lemma 6 imply that $\Sigma(B_2) = [a_2, b_2]$ and $\Sigma(B_{33}) = [a_3, b_3] \cup \cdots \cup [a_m, b_m]$.

According to the above process, we get a nonuniform Lyapunov transformation $x(t) = S_{m-1}(t)w(t)$, which send system (1.1) to

$$\dot{w} = D(t)w, \qquad D(t) = \begin{pmatrix} B_1(t) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & B_{m-2}(t) & 0 \\ 0 & 0 & 0 & B_{m-1,m-1}(t) \end{pmatrix},$$

with $\Sigma(B_i) = [a_i, b_i]$ for $i \in \{1, 2, \ldots, m-2\}$. Take $\gamma_{m-1} \in (b_{m-1}, a_m)$ and the system

$$\dot{w} = (D(t) - \gamma_{m-1} I)w$$

admits a nonuniform exponential dichotomy with an invariant projection $P_{m-1} : \mathbb{R}_0^+ \to GL_{n_{m-1}}(\mathbb{R})$. Using the same arguments as before, the system (1.1) is nonuniformly kinematically similar to system (1.21). Again as before we get that $\Sigma(diag(B_1, \ldots, B_{m-1})) \subset (-\infty, a_m)$ and $\Sigma(B_m) \subset (b_{m-1}, +\infty)$. This implies that $\Sigma(B_i) = [a_i, b_i]$ for $i \in \{1, 2, \ldots, m\}$. □

Remark 1.4. *In the articles [4] and [47] the normal forms of nonautonomous differential systems and properties of the Lyapunov exponents with respect to the nonuniform spectrum are studied respectively.*

Chapter 2

Nonuniform Almost Reducibility of nonautonomous linear differential equations.

In the previous chapter we learned about the context that revolves around the nonuniform spectrum, from its definition to the properties of its structure.

This spectrum plays a fundamental role in a better localization of diagonal terms when the system (1.1) can be transformed to a diagonal one. In fact, in 1967 B. F. Bylov in [8] introduced the notion of *almost reducibility*, *i.e.*, reducibility with a negligible error and proved that any linear system is almost reducible to some diagonal system with real coefficients. Later in 1999, F. Lin in [30] improves the Bylov's result by proving that the diagonal coefficients are contained in the Sacker–Sell Spectrum. Moreover, F. Lin proved that this spectrum is the minimal compact set where the diagonal terms belong, this phenomenon is known as the *contractibility* of a linear system.

We emphasize that these concepts of reducibility and almost reducibility also have a vast literature as well as in the uniform hyperbolicity ([18], [26]) or in Schrödinger operators [23].

2.1 Preliminaries.

We consider the linear system (1.1) with x as a column vector of $\mathbb{R}^n$ and the matrix function $t \mapsto A(t) \in M_n(\mathbb{R})$ with the following properties:

(P1) For $\theta, \mathcal{M} > 0$, $\|A(t)\| \leq \mathcal{M} \exp(\theta t)$ for any $t \in \mathbb{R}_0^+$.

(P2) The evolution operator $\Phi(t,s)$ of (1.1) has a nonuniformly bounded growth ([47]), namely, there exist constants $K \geq 1$, $\bar{\alpha} \geq 0$ and $\bar{\mu} \geq 0$ such that

$$\|\Phi(t,s)\| \leq K \exp(\bar{\alpha}\,|t-s| + \bar{\mu} s), \quad t,\ s \in \mathbb{R}_0^+.$$

The purpose of this chapter is to study the nonuniform contractibility or nonuniform almost reducibility to a diagonal system. Namely, the δ-nonuniform kinematical similarity of (1.1) to

$$\dot{y} = U(t)y, \tag{2.1}$$

when $U(t) = C(t) + B(t)$, $C(t)$ is a diagonal matrix and $B(t)$ has smallness properties which will be explained later.

Within this chapter you will find several of the definitions of the previous chapter, we will do this in order to contextualize what will be the topics that we will use to prove the main result of this chapter. To begin we will give the definition of nonuniform kinematical similarity.

Definition 2.1 ([17], [47]). *The system* (1.1) *is nonuniformly kinematically similar (resp. $\delta-$nonuniformly kinematically similar with a fixed $\delta > 0$) to* (2.1) *if there exist an invertible transformation $S(t)$ (resp. $S_\delta(t)$) and $\upsilon \geq 0$ satisfying*

$$\|S(t)\| \leq M_\upsilon \exp(\upsilon t) \quad \textit{and} \quad \left\|S^{-1}(t)\right\| \leq M_\upsilon \exp(\upsilon t)$$

or respectively

$$\|S(\delta, t)\| \leq M_{\upsilon,\delta} \exp(\upsilon t) \quad \textit{and} \quad \left\|S^{-1}(\delta, t)\right\| \leq M_{\upsilon,\delta} \exp(\upsilon t),$$

such that the change of coordinates $y(t) = S^{-1}(t)x(t)$ (resp. $y(t) = S_\delta^{-1}(t)x(t)$) transforms (1.1) *into* (2.1)*, where*

$$U(t) = S^{-1}(t)A(t)S(t) - S^{-1}(t)\dot{S}(t), \tag{2.2}$$

for any $t \in \mathbb{R}_0^+$.

Remark 2.1. *Nonuniform kinematical similarity preserves nonuniformly growth bounded. In fact, if* (1.1) *and* (2.1) *are nonuniform kinematically similar through of the function $S(\cdot)$ and their respective evolution operators are $\Phi_1(t, s)$ and $\Phi_2(t, s)$, then by the proof of Lemma* 1.2 *(see equation* (1.22)*) we have the equality*

$$\Phi_1(t, s)S(s) = S(t)\Phi_2(t, s) \quad \textit{for} \quad \textit{all} \quad t, s \in \mathbb{R}_0^+,$$

and if $\|\Phi_1(t, s)\| \leq K \exp(\bar{\alpha} |t - s| + \bar{\mu}s)$, then we have

$$\begin{aligned} \|\Phi_2(t, s)\| &\leq \left\|S^{-1}(t)\right\| \|S(s)\| \|\Phi_1(t, s)\|, \\ &\leq M_\upsilon \exp(\upsilon t) M_\upsilon \exp(\upsilon s) K \exp(\bar{\alpha}|t - s| + \bar{\mu}s), \end{aligned}$$

and finally, we obtain that

$$\|\Phi_2(t, s)\| \leq M_\upsilon^2 K \exp((\upsilon + \bar{\alpha})|t - s| + (2\upsilon + \bar{\mu})s).$$

As we said previously, the concept of almost reducibility was introduced by B. F. Bylov in the continuous context. A discrete version of this notion was given by Á. Castañeda and G. Robledo (see [15]).

Now we introduce the definition of *nonuniformly almost reducible* which is a version of the previous concept in the nonuniform framework.

Definition 2.2 ([9]). *The system* (1.1) *is nonunifomly almost reducible to*

$$\dot{y} = C(t)y,$$

if for any $\delta > 0$ *and* $\varepsilon \geq 0$, *there exists a constant* $K_{\delta,\varepsilon} \geq 1$ *such that (1.1) is* $\delta-$*nonuniformly kinematically similar to*

$$\dot{y} = [C(t) + B(t)]y, \quad \textit{with} \quad \|B(t)\| \leq \delta K_{\delta,\varepsilon}$$

for any $t \in \mathbb{R}_0^+$.

In the case when $C(t)$ is a diagonal matrix, if $K_{\delta,\varepsilon} = 1$ it is said that (1.1) is almost reducible to a diagonal system and it was proved in [8] that any continuous linear system satisfies this property and the components of $C(t)$ are real numbers.

The concept of almost reducibility to diagonal system was rediscovered and improved by F. Lin in [30], who introduces the concept of contractibility in the continuous context, while in the discrete case was proposed by Á. Castañeda and G. Robledo in [15]. In this thesis we introduce its nonuniform version.

Definition 2.3. *The system* (1.1) *is nonuniformly contracted to the compact subset* $E \subset \mathbb{R}$ *if is nonuniformly almost reducible to a diagonal system*

$$\dot{y} = Diag(C_1(t), \dots, C_n(t))y,$$

where $C_i(t) \in E$, *for any* $t \in \mathbb{R}_0^+$.

It is worth emphasize that while Bylov's result only says that the diagonal components are real numbers, Lin's definition provides explicit localization properties, as the fact that a compact set is contractible if it is the minimal compact set such that the system (1.1) can be contracted.

In the continuous and discrete cases, the concept of contractibility has been applied in some results of topological equivalence and almost topological equivalence respectively (see [31], [16]). The major contribution of [30] is to prove that the contractible set of a linear system (1.1) is its Sacker and Sell spectrum (see [41]). Mimicing the construction of the Sacker and Sell spectrum, J. Chu, et.al. in [17] and X. Zhang [47] defined the nonuniform spectrum $\Sigma(A)$. To the best of knowledge there no exists result in the nonuniform framework and the purpose of this chapter is to obtain condition for the nonuniform contractibility of (1.1) to $\Sigma(A)$ by following some lines of Lin's and Castañeda–Robledo's works.

2.2 Main result: Nonuniform almost reducibility to diagonal systems.

2.2.1 Dichotomy, nonuniform spectrum and properties.

We begin this section by remembering the definition of nonuniform exponential dichotomy and the nonuniform spectrum.

Definition 2.4. ([5], [17], [47]) *The system* (1.1) *has a nonuniform exponential dichotomy on*

$J \subset \mathbb{R}$ if there exist an invariant projector $P(\cdot)$, constants $K \geq 1$, $\alpha > 0$ and $\mu \geq 0$ such that

$$\begin{cases} \|\Phi(t,s)P(s)\| \leq K\exp(-\alpha(t-s)+\mu\,|s|), & t \geq s, \quad t,s \in J, \\ \|\Phi(t,s)(I-P(s))\| \leq K\exp(\alpha(t-s)+\mu\,|s|), & t \leq s, \quad t,s \in J. \end{cases} \tag{2.3}$$

Definition 2.5. ([17] , [47]) *The nonuniform spectrum (also called nonuniform exponential dichotomy spectrum) of* (1.1) *is the set $\Sigma(A)$ of $\lambda \in \mathbb{R}$ such that the systems*

$$\dot{x} = [A(t) - \lambda I]x \tag{2.4}$$

have not nonuniform exponential dichotomy on $\mathbb{R}_0^+$.

Now we rewrite Theorem 3 of the previous chapter, which allows us to give a better description of the spectrum if the evolution operator has a nonuniformly bounded growth.

Theorem 2.1. ([4], [28], [43], [47]) *If the evolution operator of* (1.1) *satisfies* **(P2)**, *its nonuniform spectrum $\Sigma(A)$ is the union of m compact intervals where $0 < m \leq n$, namely,*

$$\Sigma(A) = \bigcup_{i=1}^{m}[a_i, b_i], \tag{2.5}$$

with $-\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty$.

The following result allows characterizing the nonuniformly bounded growth of the evolution operator associated to (1.1) from subtle hypothesis about its nonuniform spectrum.

Proposition 2.1. *Suppose that the system* (1.1) *has spectrum $\Sigma(A) = [a,b]$, then its evolution operator $\Phi(t,s)$ satisfies* **(P2)**.

Proof. Let $\gamma, \lambda \in \rho(A)$ such that $\gamma < a \leq b < \lambda$, then we have the system

$$\dot{x} = (A(t) - \gamma I)x$$

has a nonuniform exponential dichotomy with projector $P(t) = 0$. On the other hand, the system

$$\dot{x} = (A(t) - \lambda I)x$$

has a nonuniform exponential dichotomy with projector $P(t) = I$.

Then there exist $\alpha_1, \alpha_2, > 0$, $\mu_1, \mu_2 \geq 0$, $K_1, K_2 \geq 1$ such that satisfies

$$\|\Phi(t,s)\| \leq K_1 \exp((\gamma + \alpha_1)(t-s) + \mu_1 s) \quad t \leq s,$$

$$\|\Phi(t,s)\| \leq K_2 \exp((\lambda - \alpha_2)(t-s) + \mu_2 s) \quad t \geq s.$$

Now we define $\bar{\alpha} = \max\{0, -\gamma - \alpha_1, \lambda - \alpha_2\}$, $\bar{\mu} = \max\{\mu_1, \mu_2\}$ and $K = \max\{K_1, K_2\}$ then we conclude that

$$\|\Phi(t,s)\| \leq K\exp(\bar{\alpha}|t-s| + \bar{\mu}s) \quad t,s \in \mathbb{R}_0^+.$$

$\square$

2.2.2 Main result of chapter.

The main goal of this chapter is prove the following result.

Theorem 2.2. *If* **(P1)-(P2)** *are satisfied, then* (1.1) *is nonuniformly contracted to* $\Sigma(A)$.

2.3 Preparatory Results.

The nonuniform kinematical similarity between (1.1) and (2.1) will be denoted by $A \cong U$. Let us recall that nonuniform kinematical similarity is an equivalence relation having several properties, many of which were already demonstrated in chapter 1.

Lemma 2.1. *If* $A \cong B$*, then* $A - \lambda I \cong B - \lambda I$ *for any* $\lambda \in \mathbb{R}$

Proof. If $A \cong B$ by the transformation $y(t) = S^{-1}(t)x(t)$, then $S(t)$ satisfies

$$B(t) = S^{-1}(t)A(t)S(t) - S^{-1}(t)\dot{S}(t).$$

It is straightforward see that

$$(B(t) - \lambda I) = S^{-1}(t)(A(t) - \lambda I)S(t) - S^{-1}(t)\dot{S}(t),$$

then $A(t) - \lambda I \cong B(t) - \lambda I$.

□

Lemma 2.2. *If* $A \cong B$*, then* $\Sigma(A) = \Sigma(B)$.

Proof. See Lemma 1.2 □

Proposition 2.2. *If* $\Sigma(A) \subset [a, b]$ *and* $\lambda > b$ *(resp. or* $\lambda < a$*) the system*

$$\dot{x} = (A(t) - \lambda I)x$$

has a nonuniform exponential dichotomy with projector $P(t) = I$ *(resp. with projector* $P(t) = 0$*).*

Proposition 2.3. ([17, 47]) *If the system* (1.1) *satisfies* **(P1)–(P2)** *then its spectrum is as in* (2.5) *and there exist* m *matrix functions* $B_i : \mathbb{R} \to M_{n_i}(\mathbb{R})$*such that*

$$\|B_i(t)\| \leq \mathcal{M}_i \exp(\mu_i t) \text{ with } \mu_i \geq 0, \mathcal{M}_i > 0 \tag{2.6}$$

where $\Sigma(B_i) = [a_i, b_i]$ *with* $i \in \{1, \ldots, m\}$*, such that* (1.1) *is nonuniformly kinematically similar to*

$$\dot{y} = Diag(B_1(t), \ldots, B_m(t))y. \tag{2.7}$$

We point out that in [7] the concept of diagonal significance is studied in the continuous framework. In our case this condition it is not necessary. Moreover, in the case of nonuniform exponential dichotomy the condition of diagonal significance is still open.

Proposition 2.4. *Let $C(t)$ be an upper triangular $n \times n$-matrix function such that $\Sigma(C) = [a, b]$, then*

$$\bigcup_{i=1}^{n} \Sigma(c_{ii}) \subset \Sigma(C),$$

where $c_{ii}(t)$ are the diagonal coefficients of $C(t)$.

Proof. We will prove that $\bigcup_{i=1}^{n} \Sigma(c_{ii}) \subset \Sigma(C)$. Let $\lambda \notin \Sigma(C) = [a, b]$ such that $\lambda > b$. By Proposition 2.2, we have that the upper triangular system

$$\dot{x} = (C(t) - \lambda I)x, \tag{2.8}$$

has nonuniform exponential dichotomy with projector $P(t) = I$. That is, the evolution operator of (2.8), namely $\Phi_\lambda(t, s)$, satisfies

$$\|\Phi_\lambda(t, s)\| \leq K_\lambda \exp(-\alpha_\lambda(t - s) + \mu_\lambda s) \quad (t \geq s).$$

Now for each $i \in \{1, \ldots, n\}$, we have the following estimate

$$\begin{aligned} \exp\left(\int_s^t (c_{ii}(r) - \lambda)dr\right) & \leq \|\Phi_\lambda(t, s)\|, \\ & \leq K_\lambda \exp(-\alpha_\lambda(t - s) + \mu_\lambda s) \quad t \geq s, \end{aligned}$$

and we conclude that the diagonal systems

$$\dot{x}_i = (c_{ii}(t) - \lambda)x_i$$

has a nonuniform exponential dichotomy with projector $P(t) = 1$ (scalar systems), which implies that $\lambda \notin \bigcup_{i=1}^{n} \Sigma(c_{ii})$.

The case $\lambda < a$ can be proved analogously, thus $\bigcup_{i=1}^{n} \Sigma(c_{ii}) \subset \Sigma(C)$.

□

2.4 Proof of Main Result of chapter.

2.4.1 Proof of Theorem 2.2.

The proof will be made in several steps:

Step 1): (1.1) is nonuniform kinematically similar to an upper triangular system: By Theorem 2.1, there exists a positive integer $m \leq n$ such that:

$$\Sigma(A) = \bigcup_{i=1}^{m} [a_i, b_i], \quad \text{with } -\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty.$$

The Proposition 2.3 says that (1.1) is nonuniform kinematically similar to (2.7), where $B_i(t)$ are matrix function of order $n_i \times n_i$ satisfying (2.6) and $\Sigma(B_i) = [a_i, b_i]$ with $i \in \{1, \ldots, m\}$. Now, we will prove two preliminary result.

Lemma 2.3 ([1]). *Any fundamental matrix $X(t)$ of system (1.1) can be represented in the form of the product of two continuously differentiable matrices: a unitary one $U(t)$ and an upper triangular one $R(t)$ with positive diagonal.*

Proof. Let

$$X(t) = \{x_1(t), \ldots, x_n(t)\}.$$

We apply the Schmidt orthogonalization process to the basis vectors:

$$\begin{aligned} \xi_1 &= x_1, & e_1 &= \tfrac{\xi_1}{\|\xi_1\|}, \\ \xi_2 &= x_2 - \langle x_2, e_1 \rangle e_1, & e_2 &= \tfrac{\xi_2}{\|\xi_2\|}, \\ &\vdots & &\vdots \\ \xi_n &= x_n - \sum_{j=1}^{n-1} \langle x_n, e_j \rangle e_j, & e_n &= \tfrac{x_n}{\|\xi_n\|}. \end{aligned}$$

Obviously, $\langle e_i, e_j \rangle = \delta_{ij}$, i.e., the matrix

$$U(t) = \{e_1, \ldots, e_n\}$$

is unitary since $U^T(t)U(t) = I$. At the same time

$$\begin{aligned} x_1 &= \|\xi_1\| e_1, \\ x_2 &= \langle x_2, e_1 \rangle e_1 + \|\xi_2\| e_2, \\ &\vdots \\ x_n &= \sum_{j=1}^{n-1} \langle x_n, e_j \rangle e_j + \|\xi_n\| e_n. \end{aligned}$$

This implies that the equality

$$X(t) = U(t)R(t) \tag{2.9}$$

holds, where

$$R(t) = \begin{pmatrix} \|\xi_1\| & \langle x_2, e_1 \rangle & \ldots & \langle x_n, e_1 \rangle \\ 0 & \|\xi_2\| & \ldots & \langle x_n, e_2 \rangle \\ \vdots & & \vdots & \\ 0 & \ldots & \ldots & \|\xi_n\| \end{pmatrix}, \tag{2.10}$$

i.e.,

$$r_{ij}(t) = 0, \quad i > j, \quad r_{ii}(t) > 0, \quad \text{for} \quad t \in \mathbb{R}_0^+, \quad i, j \in \{1, \ldots, n\}.$$

□

Lemma 2.4 ([1]). *[Perron's theorem on the triangulation of a linear system]. By means of a unitary transformation any linear system (1.1) can be reduce to a system with an upper triangu-*

lar matrix. If the initial system has exponential growth coefficients, then the coefficients of the triangular system are also exponential growth, and the unitary matrix is a nonuniform Lyapunov transformation.

Proof. We show that there exists a transformation

$$x(t) = U(t)y(t), \quad U^T(t)U(t) = I, \tag{2.11}$$

such that

$$\dot{y} = (U^{-1}(t)A(t)U(t) - U^{-1}(t)\dot{U}(t))y = B(t)y, \tag{2.12}$$

where

$$b_{kj}(t) = 0, \quad k > j.$$

Let us take the fundamental matrix $X(t)$ of system (1.1) and choose the unitary matrix defined by previous Lemma as $U(t)$. The transformation (2.11) for the matrix $X(t)$ uniquely defines a fundamental matrix $Y(t)$ of system (2.12),

$$X(t) = U(t)Y(t).$$

From condition (2.9) we obtain that $Y(t) = R(t)$ is upper triangular with positive diagonal. From system (2.12) we have $\dot{Y} = B(t)Y(t)$, or

$$B(t) = \dot{Y}(t)Y^{-1}(t) = \dot{R}(t)R^{-1}(t),$$

i.e., $B(t)$ is upper triangular and

$$\begin{aligned} b_{kk}(t) &= \dot{r}_{kk}(t)r_{kk}^{-1}(t), \\ &= \frac{\left\|\dot{\xi}_k(t)\right\|}{\|\xi_k(t)\|}, \\ &= \frac{d}{dt}\ln(\|\xi_k(t)\|) \end{aligned}$$

which follows from (2.10). Thus the first part of the Lemma is proved. Now we show that $\|B(t)\|$ has exponential growth for $t \in \mathbb{R}_0^+$ if $\|A(t)\|$ has exponential growth. Indeed,

$$B(t) = U^{-1}(t)A(t)U(t) - U^{-1}(t)\dot{U}(t) = \tilde{A}(t) - V(t).$$

If there exist $M > 0$ and $\upsilon \geq 0$ such that $\|A(t)\| \leq M\exp(\upsilon t)$, then

$$\left\|\tilde{A}\right\| \leq \left\|U^{-1}(t)\right\| \cdot \|A(t)\| \cdot \|U(t)\| \leq M\exp(\upsilon t), \quad \text{for} \quad t \in \mathbb{R}_0^+.$$

Note that $V^T(t) = -V(t)$, i.e., $V(t)$ is an antisymmetric matrix. Indeed, let us verify the equality

$$(U^{-1}(t)\dot{U}(t))^T = -U^{-1}(t)\dot{U}(t). \tag{2.13}$$

By differentiating the identity $U^T(t)U(t) = I$, we obtain

$$\dot{U}^T(t)U(t) + U^T(t)\dot{U}(t) = 0,$$

therefore,

$$\dot{U}(t) = -(U^T(t))^{-1}\dot{U}^T(t)U(t).$$

Substituting $\dot{U}(t)$ in the left-hand side of equation (2.13) and taking into account that $U^T(t)U(t) = I$ we find that (2.13) holds. The diagonal elements of antisymmetric matrices are 0, therefore, $V_{kk}(t) = 0$. $B(t)$ is upper triangular, thus

$$V_{kj}(t) = \tilde{a}_{kj}(t) \quad \text{for} \quad k > j,$$

and, since $V(t)$ is antisymmetric, we have

$$V_{kj}(t) = -V_{jk}(t) \quad \text{for} \quad k < j.$$

Hence, $\|V(t)\| \leq M \exp(\upsilon t)$, and, therefore,

$$\|B(t)\| \leq \left\|\tilde{A}(t)\right\| + \|V(t)\| \leq 2M \exp(\upsilon t), \quad t \in \mathbb{R}_0^+.$$

We claim that $\|U(t)\|$ and $\left\|U^{-1}(t)\right\|$ has exponential growth since $U(t)$ is unitary, which prove that the matrix $U(t)$ is a nonuniform Lyapunov transformation.

□

By using the previous Lemma, we know that, for each $i \in \{1, \dots, m\}$, the systems

$$\dot{x}_i = B_i(t)x_i \tag{2.14}$$

are kinematically similar (see Definition 2.1 with $\upsilon = 0$) to

$$\dot{y}_i = D_i(t)y_i, \tag{2.15}$$

where $D_i(t)$ is a upper triangular $n_i \times n_i$-matrix function such that

$$\|D_i(t)\| \leq \mathcal{N}_i \exp(\varsigma_i t) \text{ and } \Sigma(D_i) = [a_i, b_i]$$

where the last estimate is obtained from the lemma 1.5.

Step 2): Nonuniform exponential dichotomy of scalar differential equation: From now on, the diagonal terms of the upper triangular matrix D_i described in (2.15) will be denoted by $\left\{d_{rr}^{(i)}\right\}_{r=1}^{n_i}$ where i is a fixed element of $\{1, \dots, m\}$. Now, by Proposition 2.4, we have

$$\bigcup_{r=1}^{n_i} \Sigma(d_{rr}^{(i)}) \subset \Sigma(D_i).$$

By Proposition 2.2, for any $\delta > 0$ there exists $M_\delta = \frac{\delta}{m} > 0$ such that the scalar differential equation

$$\dot{x} = \left[d_{rr}^{(i)}(t) - (a_i - M_\delta)\right] x \tag{2.16}$$

has a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projector $P(t) = 0$ and

$$\dot{x} = \left[d_{rr}^{(i)}(t) - (b_i + M_\delta)\right] x \tag{2.17}$$

has a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projector $P(t) = 1$. In consequence, there exist $K \geq 1$, $\alpha > 0$, $\mu \geq 0$ and in this case we need the condition $\alpha > \mu$, such that

$$\left\{ \begin{array}{lcll} |\exp(\Phi(t,s))| & \leq & K\exp(\alpha(t-s)+\mu s) & t \leq s, \\ |\exp(\Psi(t,s))| & \leq & K\exp(-\alpha(t-s)+\mu s) & t \geq s, \end{array} \right. \tag{2.18}$$

where

$$\exp(\Phi(t,s)) = \exp\left(\int_s^t (d_{rr}^{(i)}(\tau) - (a_i - M_\delta))d\tau\right),$$

and

$$\exp(\Psi(t,s)) = \exp\left(\int_s^t (d_{rr}^{(i)}(\tau) - (b_i + M_\delta))d\tau\right),$$

are the evolution operators of (2.16) and (2.17) respectively.

Step 3): Upper and lower bounds for (2.18)*:* For any fixed $i \in \{1,\dots,m\}$, there exist two functions $c^{(i)}$ and $\lambda^{(i)}$ such that

$$a_i \leq c_r^{(i)}(t) \leq b_i \quad \text{and} \quad |\lambda_r^{(i)}(t)| \leq M_\delta \quad \text{for any } t \in \mathbb{R}_0^+ \tag{2.19}$$

and there exist $\bar{\Delta}, \upsilon \geq 0$ verifying

$$\left| \int_0^t [d_{rr}^{(i)}(\tau) - (c_r^{(i)}(\tau) + \lambda_r^{(i)}(\tau))]d\tau \right| \leq \bar{\Delta} + \upsilon t, \quad \text{if } t \geq 0 \tag{2.20}$$

for any $r \in \{1,\dots,n_i\}$.

We will construct a strictly increasing and unbounded sequence of real numbers $\left\{T_l^{(i)}\right\}_{l=0}^{+\infty}$ satisfying $T_0^{(i)} = 0$ such that the function $c_r^{(i)}, \lambda_r^{(i)} : \mathbb{R}_0^+ \to \mathbb{R}$ defined by:

$$c_r^{(i)}(t) = \left\{ \begin{array}{lll} a_i & \text{if} & t \in [T_q^{(i)}, T_{q+1}^{(i)}) \quad (q = 0,2,4,\dots) \\ b_i & \text{if} & t \in [T_{q+1}^{(i)}, T_{q+2}^{(i)}) \end{array} \right.$$

and

$$\lambda_r^{(i)}(t) = \left\{ \begin{array}{lll} -M_\delta & \text{if} & t \in [T_q^{(i)}, T_{q+1}^{(i)}) \quad (q = 0,2,4,\dots) \\ M_\delta & \text{if} & t \in [T_{q+1}^{(i)}, T_{q+2}^{(i)}) \end{array} \right.$$

satisfy properties (2.19) and (2.20) on $\mathbb{R}_0^+$.

It is straightforward to see that (2.19) is always satisfied. In order to verify (2.20), we interchange t by s in the first inequality of (2.18), then we have:

$$\left\{ \begin{array}{lcll} \Phi(t,s) & \geq & \alpha(t-s) - \mu t - \ln(K) & t \geq s, \\ \Psi(t,s) & \leq & -\alpha(t-s) + \mu s + \ln(K) & t \geq s. \end{array} \right. \tag{2.21}$$

By using induction, we will verify that there exists a sequence $\left\{T_l^{(i)}\right\}_{l=0}^{+\infty}$ satisfying (2.20). First, by the Proposition 2.1 combined with the fact that

$$\exp\left(\int_s^t d_{rr}^{(i)}(\tau)d\tau\right) \leq \|\Phi_{D_i}(t,s)\|$$

where $\Phi_{D_i}(t,s)$ is the evolution operator of the system (2.15) then there exist constants $\bar{\alpha} \geq 0$, $\bar{\mu} \geq 0$ and $\bar{K} \geq 1$ such that satisfies the following

$$\int_s^t d_{rr}^{(i)}(\tau)d\tau \leq \bar{\alpha}|t-s| + \bar{\mu}s + \ln(\bar{K}). \tag{2.22}$$

Then, using the equation (2.22) we have

$$\Phi(t,s) \leq \bar{\alpha}(t-s) + \bar{\mu}s + \ln(\bar{K}) + |a_i|(t-s) + M_\delta(t-s). \tag{2.23}$$

On the other hand, by (2.21) we obtain

$$\int_s^t d_{rr}^{(i)}(\tau)d\tau \geq \alpha(t-s) - \mu t + a_i(t-s) - M_\delta(t-s) - \ln(K), \tag{2.24}$$

and using the last expression we deduce

$$\Psi(t,s) \geq (-\alpha - (b_i - a_i) - 2M_\delta)(t-s) - \mu t - \ln(K). \tag{2.25}$$

By the equations (2.21), (2.23) and (2.25) we have the following

$$\left\{\begin{array}{lcll} \Phi(t,s) & \geq & (\alpha-\mu)(t-s) - \mu s - \ln(K) & t \geq s, \\ \Phi(t,s) & \leq & (\bar{\alpha} + |a_i| + M_\delta + \bar{\mu})(t-s) + \bar{\mu}s + \ln(\bar{K}) & t \geq s. \end{array}\right. \tag{2.26}$$

$$\left\{\begin{array}{lcll} \Psi(t,s) & \leq & -(\alpha-\mu)(t-s) + \mu s + \ln(K) & t \geq s, \\ \Psi(t,s) & \geq & (-\mu - (b_i - a_i) - 2M_\delta)(t-s) - \mu s - \ln(K) & t \geq s. \end{array}\right. \tag{2.27}$$

Now we will introduce constants and conditions that allow us to obtain the desired result (this conditions are inherent in the nonuniform framework).

Let $N, \xi, p, \bar{\xi}, \bar{p} \in \mathbb{R}$ constants that satisfy:

(C1) $0 < N < \min\{\alpha - \mu, \bar{\alpha} + |a_i| + M_\delta + \bar{\mu}\}$.

(C2) $\max\left\{\ln(\bar{K}), \ln(K)\right\} < p = -\bar{p}$.

(C3) $0 \leq \max\{\bar{\mu}, \mu\} \leq -\bar{\xi} \leq \xi$.

If $s = 0$ in the first inequality of (2.26) we obtain

$$\Phi(t,0) \geq (\alpha - \mu)t - \ln(K), \quad t \geq 0,$$

which implies that $\Phi(t,0)$ is unbounded in $\mathbb{R}_0^+$, since $\alpha > \mu$. In consequence, given $N, \xi, p \in \mathbb{R}$, there exists $T_1^{(i)} > 0$ such that

$$\begin{cases} \Phi(T_1^{(i)},0) &=& N(T_1^{(i)}-0)+\xi 0+p, \\ \Phi(t,0) &<& N(t-0)+\xi 0+p \qquad (0 \leq t < T_1^{(i)}). \end{cases} \tag{2.28}$$

Then we consider the value $\bar{\xi}T_1^{(i)} + \bar{p}$ and

$$T_2^{(i)} = \min\left\{\omega \in \mathbb{R}_0^+ : \Psi(\omega, T_1^{(i)}) = -N(T_1^{(i)}-0) - \xi 0 - p\right\},$$

with $T_2^{(i)} > T_1^{(i)}$. Now we will calculate the slope of the line that joins the points $\bar{\xi}T_{r_1}^{(i)} + \bar{p}$ and $-N(T_{r_1}^{(i)}-0) - \xi 0 - p$, which we will denote by $\bar{N}$. Moreover, $\bar{N}$ satisfies the following technical condition

(C4) $\max\{-(\alpha-\mu), -(\mu+(b_i-a_i)+2M_\delta)\} < \bar{N}$.

Then we have

$$\bar{N} = \frac{(-N(T_1^{(i)}-0)-\xi 0-p)-(\bar{\xi}T_1^{(i)}+\bar{p})}{T_2^{(i)}-T_1^{(i)}},$$
$$\bar{N} = \frac{-N(T_1^{(i)}-0)-\xi 0-\bar{\xi}T_1^{(i)}}{T_2^{(i)}-T_1^{(i)}}.$$

Due to the conditions **(C1)** and **(C3)**, we have that $\bar{N} \leq 0$. In this way, we consider the straight $\bar{N}(t-T_1^{(i)}) + \bar{\xi}T_1 + \bar{p}$.

Based on the above and the equation (2.27), if $s = T_1^{(i)}$ then there exists $T_2^{(i)} > T_1^{(i)}$ such that

$$\begin{cases} \Psi(T_2^{(i)},T_1^{(i)}) &=& \bar{N}(T_2^{(i)}-T_1^{(i)})+\bar{\xi}T_1+\bar{p}, \\ \Psi(t,T_1^{(i)}) &>& \bar{N}(t-T_1^{(i)})+\bar{\xi}T_1^{(i)}+\bar{p}, \qquad T_1^{(i)} \leq t < T_2^{(i)}. \end{cases} \tag{2.29}$$

By (2.26) and (2.28) we obtain that for $t \in [0, T_1^{(i)})$

$$\begin{aligned} -\mu t - \ln(K) - (-\bar{N}t - \bar{\xi}t - \bar{p}) &\leq \int_0^t (d_{rr}^{(i)}(\tau) - (c_r^{(i)}(\tau) + \lambda_r^{(i)}(\tau)))d\tau, \\ &\leq Nt + \xi t + p. \end{aligned}$$

In fact, we have

$$-\mu t - \ln(K) - (-\bar{N}t - \bar{\xi}t - \bar{p}) \leq -\mu t - \ln(K),$$

by the equation (2.26)

$$-\mu t - \ln(K) \leq \int_0^t (d_{rr}^{(i)}(\tau) - (a_i - M_\delta))d\tau,$$

then by the equation (2.28)

$$\int_0^t (d_{rr}^{(i)}(\tau) - (a_i - M_\delta))d\tau < N(t-0) + \xi 0 + p$$

and finally,

$$N(t-0)+\xi 0+p \leq Nt+\xi t+p.$$

On the other hand, from the equations (2.27) and (2.29), we have for $t \in [T_1^{(i)}, T_2^{(i)})$

$$\begin{aligned} \mu t+\ln(K)+Nt+\xi t+p &\geq \int_{T_1^{(i)}}^{t} (d_{rr}^{(i)}(\tau)-(c_r^{(i)}(\tau)+\lambda_r^{(i)}(\tau)))d\tau, \\ &\geq -(-\bar{N}t-\bar{\xi}t-\bar{p}). \end{aligned}$$

Similar to the previous, we have that

$$\mu t+\ln(K)+Nt+\xi t+p \geq \mu t+\ln(K),$$

by the equation (2.27) we have

$$\mu t+\ln(K) \geq \int_{T_1^{(i)}}^{t} (d_{rr}^{(i)}(\tau)-(b_i+M_\delta))d\tau,$$

and then by the equation (2.29)

$$\begin{aligned} \int_{T_1^{(i)}}^{t} (d_{rr}^{(i)}(\tau)-(b_i+M_\delta))d\tau &> \bar{N}(t-T_1^{(i)})+\bar{\xi}T_1^{(i)}+\bar{p}, \\ &\geq -(-\bar{N}t-\bar{\xi}t-\bar{p}). \end{aligned}$$

Thus for $t \in [0, T_2^{(i)})$

$$\left| \int_0^t (d_{rr}^{(i)}(\tau)-(c_r^{(i)}(\tau)+\lambda_r^{(i)}(\tau)))d\tau \right| \leq 2\mu t+2\ln(K)+2J(t)$$

with $J(t)=\max\left\{Nt+\xi t+p, -\bar{N}t-\bar{\xi}t-\bar{p}\right\}$.

As inductive hypothesis, we will assume that there exists $2m+1$ numbers

$$0=T_0^{(i)}<T_1^{(i)}<T_2^{(i)}<\cdots<T_{2m-1}^{(i)}<T_{2m}^{(i)}$$

such that (2.19) is satisfies and for $t \in [0, T_{2m}^{(i)})$

$$\left| \int_0^t (d_{rr}^{(i)}(\tau)-(c_r^{(i)}(\tau)+\lambda_r^{(i)}(\tau)))d\tau \right| \leq 2\mu t+2\ln(K)+2J(t)$$

with $J(t)=\max\left\{Nt+\xi t+p, -\bar{N}t-\bar{\xi}t-\bar{p}\right\}$. By using the first inequality of (2.26) and considering $s=T_{2m}^{(i)}$, we have that

$$\Phi(t, T_{2m}^{(i)}) \geq (\alpha-\mu)(t-T_{2m}^{(i)})-\mu T_{2m}^{(i)}-\ln(K)$$

is unbounded for any $t>T_{2m}^{(i)}$. Then, there exists $T_{2m+1}^{(i)}>T_{2m}^{(i)}$ such that

$$\left\{ \begin{array}{rcl} \Phi(T_{2m+1}^{(i)}, T_{2m}^{(i)}) &=& N(T_{2m+1}^{(i)}-T_{2m}^{(i)})+\xi T_{2m}^{(i)}+p, \\ \Phi(t, T_{2m}^{(i)}) &<& N(t-T_{2m}^{(i)})+\xi T_{2m}^{(i)}+p \quad (T_{2m}^{(i)} \leq t < T_{2m+1}^{(i)}). \end{array} \right. \tag{2.30}$$

Now we consider the value $\bar{\xi}T^{(i)}_{2m+1} + \bar{p}$ and

$$T^{(i)}_{2m+2} = \min\left\{\omega \in \mathbb{R}^+_0 : \Psi(\omega, T^{(i)}_{2m+1}) = -N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) - \xi T^{(i)}_{2m} - p\right\},$$

with $T^{(i)}_{2m+2} > T^{(i)}_{2m+1}$.

As before, let $\bar{N}$ be the slope of the line joining the points

$$\bar{\xi}T^{(i)}_{2m+1} + \bar{p} \text{ and } -N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) - \xi T^{(i)}_{2m+1} - p.$$

By the conditions **(C1)**, **(C2)** and **(C3)** we have $\bar{N} \leq 0$. In this way, we consider the straight

$$\bar{N}(t - T^{(i)}_{2m+1}) + \bar{\xi}T^{(i)}_{2m+1} + \bar{p}.$$

Combining the above straight and the equation (2.27), if $s = T^{(i)}_{2m+1}$ then there exists $T^{(i)}_{2m+2} > T^{(i)}_{2m+1}$ such that

$$\left\{\begin{array}{rcl} \Psi(T^{(i)}_{2m+2}, T^{(i)}_{2m+1}) & = & \bar{N}(T^{(i)}_{2m+2} - T^{(i)}_{2m+1}) + \bar{\xi}T_{2m+1} + \bar{p}, \\ \Psi(t, T^{(i)}_{2m+1}) & > & \bar{N}(t - T^{(i)}_{2m+1}) + \bar{\xi}T^{(i)}_{2m+1} + \bar{p}, \end{array}\right. \tag{2.31}$$

for $T^{(i)}_{2m+1} \leq t < T^{(i)}_{2m+2}$.

Now we will prove that for $t \in [0, T^{(i)}_{2m+2})$ we obtain

$$\left|\int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau\right| \leq 2\mu t + 2\ln(K) + 2J(t).$$

By inductive hypothesis, we have proved the case in which $t \in [0, T^{(i)}_{2m})$. If $t \in [T^{(i)}_{2m}, T^{(i)}_{2m+1})$ we have

$$\begin{aligned} \left|\int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau\right| &= \left|\int_{T^{(i)}_{2m}}^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau\right| \\ &= \left|\int_{T^{(i)}_{2m}}^t (d^{(i)}_{rr}(\tau) - (a_i - M_\delta))d\tau\right|. \end{aligned}$$

By the equations (2.26) and (2.30), as before we obtain that for $t \in [T^{(i)}_{2m}, T^{(i)}_{2m+1})$

$$-\mu t - \ln(K) - (-\bar{N}t - \bar{\xi}t - \bar{p}) \leq \int_{T^{(i)}_{T_{2m}}}^t (d^{(i)}_{rr}(\tau) - (a_i - M_\delta))d\tau \leq Nt + \xi t + p.$$

In the case $t \in [T^{(i)}_{2m+1}, T^{(i)}_{2m+2})$, we have

$$\begin{aligned} \left|\int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau\right| \quad = \quad & \Big|N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \xi T^{(i)}_{2m} + p \\ & + \int_{T^{(i)}_{2m+1}}^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau\Big|. \end{aligned}$$

Then, by the equations (2.27) and (2.31), we have that for $t \in [T^{(i)}_{2m+1}, T^{(i)}_{2m+2})$ is satisfied that

$$\begin{aligned}\mu t + \ln(K) + Nt + \xi t + p \quad &\geq \quad N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \xi T^{(i)}_{2m} + p \\ &\quad + \int_{T^{(i)}_{2m+1}}^{t} (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau \\ &\geq -(-\bar{N}t - \bar{\xi}t - \bar{p}).\end{aligned}$$

In fact, (2.27) and (2.31) implies that

$$\mu t + \ln(K) \quad \geq \int_{T^{(i)}_{2m+1}}^{t} (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau, \tag{2.32}$$

Now considering the two previous inequalities separately in (2.32), we obtain

$$\mu t + \ln(K) + N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \xi T^{(i)}_{2m} + p \geq \int_0^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau$$

and

$$\int_0^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau \geq \bar{N}(t - T^{(i)}_{2m+1}) + \bar{\xi} T^{(i)}_{2m+1} + \bar{p} + N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \xi T^{(i)}_{2m} + p.$$

Then for the first inequality we have

$$\mu t + \ln(K) + Nt + \xi t + p \geq \mu t + \ln(K) + N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \xi T^{(i)}_{2m} + p$$

and on the other hand for the second inequality we have

$$\bar{N}(t - T^{(i)}_{2m+1}) + \bar{\xi} T^{(i)}_{2m+1} + \bar{p} + N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \xi T^{(i)}_{2m} + p \geq -(-\bar{N}t - \bar{\xi}t - \bar{p}).$$

Therefore, for $t \in [0, T^{(i)}_{2m+2})$

$$\left| \int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau \right| \leq 2\mu t + 2\ln(K) + 2J(t).$$

Finally, we will prove that $T^{(i)}_m \to +\infty$ as $m \to +\infty$. For that, first of all we have by the equations (2.26) and (2.30):

$$\begin{aligned}N(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \xi T^{(i)}_{2m} + p &= \int_{T^{(i)}_{2m}}^{T^{(i)}_{2m+1}} (d^{(i)}_{rr}(\tau) - (a_i - M_\delta))d\tau \\ &\leq (\bar{\alpha} + |a_i| + M_\delta + \bar{\mu})(T^{(i)}_{2m+1} - T^{(i)}_{2m}) + \bar{\mu} T^{(i)}_{2m} + \ln(\bar{K}),\end{aligned}$$

which implies

$$(\xi - \bar{\mu})T^{(i)}_{2m} + p - \ln(\bar{K}) \leq (\bar{\alpha} + |a_i| + M_\delta + \bar{\mu} - N)(T^{(i)}_{2m+1} - T^{(i)}_{2m}).$$

By the conditions **(C1)** and **(C2)**, we have the following

$$0 < \frac{p - \ln(\bar{K})}{\bar{\alpha} + |a_i| + M_\delta + \bar{\mu} - N} \leq T^{(i)}_{2m+1} - T^{(i)}_{2m}.$$

On the other hand, in view of the equations (2.27) and (2.31) we have

$$\begin{aligned}\bar{N}(T^{(i)}_{2m+2} - T^{(i)}_{2m+1}) + \bar{\xi}T^{(i)}_{2m+1} + \bar{p} &= \int_{T^{(i)}_{2m+1}}^{T^{(i)}_{2m+2}} (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau \\ &\geq -\mu T^{(i)}_{2m+1} - \ln(K) \\ &\quad (-\mu - (b_i - a_i) - 2M_\delta)(T^{(i)}_{2m+2} - T^{(i)}_{2m+1}),\end{aligned}$$

which implies

$$(\bar{\xi} + \mu)T^{(i)}_{2m+1} + \bar{p} + \ln(K) \geq -(\mu + (b_i - a_i) + 2M_\delta + \bar{N})(T^{(i)}_{2m+2} - T^{(i)}_{2m+1}).$$

Similarly, the conditions **(C2)** and **(C4)** allow us to ensure that

$$0 < \frac{-\bar{p} - \ln(K)}{\mu + (b_i - a_i) + 2M_\delta + \bar{N}} \leq T^{(i)}_{2m+2} - T^{(i)}_{2m+1}.$$

So the above allows us to obtain the existence of $c^{(i)}_r(t)$, $\lambda^{(i)}_r(t)$ defined on $\mathbb{R}^+_0$ verifying (2.19) and finally:

$$\left| \int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau \right| \leq \Delta + \upsilon t \quad (t \in \mathbb{R}^+_0),$$

with $\Delta = 2(\ln(K) + p) \geq 0$ and $\upsilon = \upsilon_\mu$, defined by

$$\upsilon = \max\left\{2(\mu + N + \xi), 2(\mu - \bar{N} - \bar{\xi})\right\}. \tag{2.33}$$

From our definition of $c^{(i)}_r(t)$ and $\lambda^{(i)}_r(t)$ we know that are piecewise continuous. Therefore, there exists continuous functions $\bar{c}^{(i)}_r(t)$, $\bar{\lambda}^{(i)}_r(t)$ satisfying

$$a_i \leq \bar{c}^{(i)}_r(t) \leq b_i \quad \text{and} \quad |\bar{\lambda}^{(i)}_r(t)| \leq M_\delta \quad \text{for any } t \in \mathbb{R}^+_0 \tag{2.34}$$

and

$$\int_0^t \left| (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)) - (\bar{c}^{(i)}_r(\tau) + \bar{\lambda}^{(i)}_r(\tau))d\tau \right| \leq 1,$$

thus

$$\left| \int_0^t [d^{(i)}_{rr}(\tau) - (\bar{c}^{(i)}_r(\tau) + \bar{\lambda}^{(i)}_r(\tau))]d\tau \right| \leq \bar{\Delta} + \upsilon t$$

with $\bar{\Delta} = \Delta + 1$.

As a consequence of this result, we construct the $n_i \times n_i$ matrix:

$$L_i(t) = Diag(\mu_1(t), \dots, \mu_{n_i}(t)),$$

where for any $r \in \{1, \dots, n_i\}$, μ_r are defined by

$$\mu_r(t) = \exp\left(\int_0^t (d_{rr}^{(i)}(\tau) - (\bar{c}_r^{(i)}(\tau) + \bar{\lambda}_r^{(i)}(\tau))d\tau\right),$$

and we conclude that

$$\|L_i(t)\| \leq \Omega \exp(\upsilon t) \quad and \quad \left\|L_i^{-1}(t)\right\| \leq \Omega \exp(\upsilon t) \quad \text{for any} \quad t \in \mathbb{R}_0^+,$$

with $\Omega = \exp(\bar{\Delta})$.

Step 4): The systems (2.14) can be nonuniformly contracted to $[a_i, b_i]$, for any $i \in \{1, \dots, m\}$: The system (2.15) is nonuniform kinematically similar to

$$\dot{z}_i = \Lambda_i(t)z_i, \tag{2.35}$$

with $y_i(t) = L_i(t)z_i(t)$, where $\Lambda_i(t) = L_i^{-1}(t)D_i(t)L_i(t) - L_i^{-1}(t)\dot{L}_i(t)$ is a $n_i \times n_i$ matrix whose rs-coefficient is defined by

$$\{\Lambda_i(t)\}_{rs} = \begin{cases} \bar{c}_r^{(i)}(t) + \bar{\lambda}_r^{(i)}(t) & \text{if} \quad r = s, \\ \frac{\mu_s(t)}{\mu_r(t)} d_{rs}^{(i)}(t) & \text{if} \quad 1 \leq r < s \leq n_i, \\ 0 & \text{if} \quad 1 \leq s < r \leq n_i. \end{cases}$$

We observe that $|d_{rs}^{(i)}(t)| \leq \mathcal{K}_1 \exp(\kappa_1 t)$ with $\mathcal{K}_1 > 0$, for $1 \leq r < s \leq n_i$ and by the equation (2.33), we have $\frac{\mu_s(t)}{\mu_r(t)} \leq \mathcal{K}_2 \exp(\kappa_2 t)$ with $\mathcal{K}_2 > 1$ and $\kappa_2 = 2\upsilon$ with υ as in (2.33), then

$$|\{\Lambda_i(t)\}_{rs}| \leq \mathcal{K}_2\mathcal{K}_1 \exp(\kappa t), \tag{2.36}$$

where $\kappa = \kappa_\varepsilon = \kappa_1 + \kappa_2$.

Let us define the transformation

$$z_i(t) = R_i(t)w_i(t),$$

with

$$R_i(t) = Diag(\exp(-\kappa M_\delta K_\delta t), \eta \exp(-2\kappa M_\delta K_\delta t), \dots, \eta^{n_i - 1} \exp(-n_i \kappa M_\delta K_\delta t)),$$

and we also define K_δ such that $K_\delta M_\delta \geq 1$ and

$$0 < \eta < \frac{M_\delta}{M_\delta + \mathcal{K}_1\mathcal{K}_2}. \tag{2.37}$$

Now, we can see that (2.35) and (2.15) are δ-nonuniform kinematically similar to

$$\dot{w}_i = \Gamma_i(t)w_i,$$

where the rs-coefficient of $\Gamma_i(t)$ is

$$\{\Gamma_i(t)\}_{rs} = \begin{cases} \{\Lambda_i(t)\}_{rs} + r\kappa M_\delta K_\delta & \text{if} \quad r = s, \\ \eta^{s-r}\{\Lambda_i(t)\}_{rs}\exp(-(s-r)\kappa M_\delta K_\delta t) & \text{if} \quad 1 \leq r < s \leq n_i, \\ 0 & \text{if} \quad 1 \leq s < r \leq n_i. \end{cases}$$

Let us observe that $\Gamma_i(t)$ can be written as follows:

$$\Gamma_i(t) = \bar{C}_i(t)I + \bar{B}_i(t),$$

where $\bar{C}_i(t) = Diag(\bar{c}_1(t), \ldots, \bar{c}_{n_i}(t))$ and the rs-coefficient of $\bar{B}_i(t)$ is defined by

$$\left\{\bar{B}_i(t)\right\}_{rs} = \begin{cases} \left\{\bar{\lambda}_r(t)\right\}_{rs} + r\kappa M_\delta K_\delta & \text{if} \quad r = s, \\ \eta^{s-r}\frac{\mu_s(t)}{\mu_r(t)}d_{rs}^{(i)}(t)\exp(-(s-r)\kappa M_\delta K_\delta t) & \text{if} \quad 1 \leq r < s \leq n_i, \\ 0 & \text{if} \quad 1 \leq s < r \leq n_i. \end{cases}$$

By (2.34) and (2.36), we can verify that

$$\begin{aligned} \left\|\bar{B}_i(t)\right\| & \leq M_\delta[1 + r\kappa K_\delta] + \mathcal{K}_1\mathcal{K}_2[\eta + \eta^2 + \cdots + \eta^{n_i}], \\ & \leq M_\delta[1 + n_i\kappa K_\delta] + \mathcal{K}_1\mathcal{K}_2\tfrac{\eta}{1-\eta}. \end{aligned}$$

Recall that $M_\delta = \frac{\delta}{m}$ and by using (2.37) it follows that $\left\|\bar{B}_i(t)\right\| \leq \frac{\delta}{m}K_{\delta,\mu}$, where $K_{\delta,\mu} = 2 + n_i\kappa K_\delta$.

Thus, for any $i \in \{1, \ldots, m\}$ the system (2.15) is $\bar{\delta}$-nonuniform kinematically similar (with $\bar{\delta} = \frac{\delta}{m}$) to

$$\dot{w}_i = [\bar{C}_i(t) + \bar{B}_i(t)]w_i,$$

where

$$\bar{c}_j(t) \in [a_i, b_i] = \Sigma(B_i), \quad j \in \{1, \ldots, n_i\} \quad \text{and} \quad \left\|\bar{B}_i(t)\right\| \leq \frac{\delta}{m}K_{\delta,\mu}.$$

Finally, (2.15) is nonuniformly contracted to $\Sigma(B_i)$.

Step 5): The system (1.1) *can be nonuniformly contracted to* $\Sigma(A)$*:* By using the previous result, we can see that (1.1) is δ-nonuniform kinematically similar to

$$\dot{w} = [C(t) + B(t)]w,$$

where

$$C(t) = Diag(\bar{C}_1(t), \ldots, \bar{C}_m(t)) \quad \text{and} \quad B(t) = Diag(\bar{B}_1(t), \ldots, \bar{B}_m(t)).$$

In consequence, note that

$$C(t) \subset \bigcup_{i=1}^{m} [a_i, b_i] = \Sigma(A) \quad \text{and} \quad \|B(t)\| \leq \delta K_{\delta,\mu}.$$

Finally, the system (1.1) is nonuniformly contracted to $\Sigma(A)$.

□

Remark 2.2. *The inequalities* (2.26) *and* (2.27) *show that the functions* $\Phi(t,s)$ *and* $\Psi(t,s)$ *are bounded. This bounds not necessarily cross to graph this functions, thus the conditions* **(C1)–(C4)** *allows us to find straights that cross it to least once. Moreover this procedure enable us to construct* $\left\{T_l^{(i)}\right\}_{l=0}^{+\infty}$ *which is the sequences of first crossing time of the graph of this function* $\Phi(t,s)$ *and* $\Psi(t,s)$ *with that straights. We have proved that this sequence of crossing time has not accumulations points.*

2.5 Application of the main result

In this section, we present two examples of scalar systems, with their respective spectra and, in addition, we will prove that there exists a Lyapunov transformation that allows each system to contract its spectrum. Finally, we will present a diagonal planar example considering the previous scalar systems

Example 2.1. *Let us consider the scalar differential equation studied in* [17, p.547] :

$$\dot{x} = L_1(t)x, \tag{2.38}$$

with $L_1(t) = \lambda_0 + at\sin t$, $\lambda_0 < a < 0$.

First, we will prove that the nonuniform spectrum of (2.38) *is*

$$\Sigma(L_1) = [\lambda_0 - |a|, \lambda_0 + |a|].$$

In fact, the evolution operator of (2.38) *is given by*

$$\Phi(t,s) = \exp(\lambda_0(t-s) - a\cos t(t-s) - as(\cos t - \cos s) + a(\sin t - \sin s)).$$

For any $\gamma \in \mathbb{R}$, *the evolution operator of the system* $\dot{x} = (L_1(t) - \gamma)x$ *is given by*

$$\Phi_\gamma(t,s) = \exp(-\gamma(t-s))\Phi(t,s). \tag{2.39}$$

For any $\gamma \in (\lambda_0 + |a|, +\infty)$, *it follows from* (2.39) *that*

$$\|\Phi_\gamma(t,s)\| \leq \exp(2|a|)\exp(-(\gamma - \lambda_0 - |a|)(t-s) + 2|a|s), \quad t \geq s,$$

which implies that the system $\dot{x} = (L_1(t) - \gamma)x$ *admits a nonuniform exponential dichotomy with* $P(t) = 1$, $K = \exp(2|a|)$, $\alpha = \gamma - \lambda_0 - |a| > 0$, $\mu = 2|a| > 0$. *Thus*

$$(\lambda_0 + |a|, +\infty) \subset \rho(L_1).$$

For any $\tilde{\gamma} \in (-\infty, \lambda_0 - |a|)$, *it follows from* (2.39) *that*

$$\|\Phi_{\tilde{\gamma}}(t,s)\| \leq \exp(2|a|)\exp((-\tilde{\gamma} + \lambda_0 - |a|)(t-s) + 2|a|s), \quad t \leq s,$$

which implies that the system $\dot{x} = (L_1(t) - \tilde{\gamma})x$ *admits a nonuniform exponential dichotomy with* $P(t) = 0$, $K = \exp(2|a|)$, $\tilde{\alpha} = -\tilde{\gamma} + \lambda_0 - |a| > 0$, $\mu = 2|a| > 0$. *Thus*

$$(-\infty, \lambda_0 - |a|) \subset \rho(L_1).$$

From the above, we have

$$(-\infty, \lambda_0 - |a|) \cup (\lambda_0 + |a|, +\infty) \subset \rho(L_1),$$

which implies that $\Sigma(L_1) \subset [\lambda_0 - |a|, \lambda_0 + |a|]$.

Now we show that $[\lambda_0 - |a|, \lambda_0 + |a|] \subset \Sigma(L_1)$. *To show this, we first prove that* $\lambda_0 + |a| \in \Sigma(L_1)$. *On the contrary, assume that* $\gamma_1 = \lambda_0 + |a|$ *such that* $\dot{x} = (L_1(t) - \gamma_1)x$ *admits a nonuniform exponential dichotomy. We know that either the projector* $P(t) = 0$ *or* $P(t) = 1$. *If* $P(t) = 1$ *then there exist constants* $K \geq 1, \alpha > 0$ *and* $\mu \geq 0$ *such that the following estimate holds*

$$\begin{aligned} \|\Phi_{\gamma_1}(t,s)\| &= \exp(-\gamma_1(t-s))\,\|\Phi(t,s)\|\,, \\ &\leq K\exp(-\alpha(t-s) + \mu s), \quad t \geq s \end{aligned}$$

Substituting $\gamma_1 = \lambda_0 + |a|$, *we have for* $t \geq s$

$$\exp(-|a|(1-\cos t)(t-s) - as(\cos t - \cos s) + a(\sin t - \sin s)) \leq K\exp(-\alpha(t-s) + \mu s),$$

which yields a contradiction for $s = (2k+1)\pi$ *and* $t = 4k\pi$ *since we have*

$$\exp(2|a|\pi) \leq K\exp(-(\alpha - \mu)2k\pi + (\alpha + \mu)\pi)$$

where the left side of the inequality is constant and the right side converge to 0 when $k \to +\infty$ *if* $\alpha > \mu$. *If* $P(t) = 0$ *and* $t \leq s$, *the dichotomy estimate is*

$$\exp(-|a|(1-\cos t)(t-s) - as(\cos t - \cos s) + a(\sin t - \sin s)) \leq K\exp(\alpha(t-s) + \mu s),$$

which also yields a contradiction for $t = 0$ *and* $s = (2k-1)\pi$ *since we have*

$$\exp(2|a|(2k-1)\pi) \leq K\exp(-(\alpha - \mu)(2k-1)\pi)$$

where the left side of last inequality diverge and the right side converge to 0 when $k \to +\infty$ *if* $\alpha > \mu$. *Therefore* $\lambda_0 + |a| \in \Sigma(L_1)$.

Analogously, we can prove that $\lambda_0 - |a| \in \Sigma(L_1)$, *in fact, assume that* $\gamma_1 = \lambda_0 - |a|$ *such that* $\dot{x} = (L_1(t) - \gamma_1)x$ *admits a nonuniform exponential dichotomy. We know that either the projector* $P(t) = 0$ *or* $P(t) = 1$. *If* $P(t) = 1$ *then there exist constants* $K \geq 1, \alpha > 0$ *and* $\mu \geq 0$ *such that the following estimate holds*

$$\begin{aligned} \|\Phi_{\gamma_1}(t,s)\| &= \exp(-\gamma_1(t-s))\,\|\Phi(t,s)\|\,, \\ &\leq K\exp(-\alpha(t-s) + \mu s), \quad t \geq s \end{aligned}$$

Substituting $\gamma_1 = \lambda_0 - |a|$*, we have for* $t \geq s$

$$\exp(|a|(1+\cos t)(t-s) - as(\cos t - \cos s) + a(\sin t - \sin s)) \leq K\exp(-\alpha(t-s) + \mu s),$$

which yields a contradiction for $s = 0$ *and* $t \to +\infty$*. If* $P(t) = 0$ *and* $t \leq s$*, the dichotomy estimate is*

$$\exp(|a|(1+\cos t)(t-s) - as(\cos t - \cos s) + a(\sin t - \sin s)) \leq K\exp(\alpha(t-s) + \mu s),$$

which also yields a contradiction for $t = 0$ *and* $s = (2k+1)\pi$ *since we have*

$$1 \leq K\exp(-(\alpha-\mu)(2k+1)\pi)$$

where the left side of last inequality is constant and the right side converge to 0 when $k \to +\infty$ *if* $\alpha > \mu$*. Therefore* $\lambda_0 - |a| \in \Sigma(L_1)$*.*

By Theorem 2.1, we know that $\Sigma(L_1)$ *is an interval. Thus, for any* $\gamma \in [\lambda_0 - |a|, \lambda_0 + |a|]$*, it follows from the connectedness that* $\gamma \in \Sigma(L_1)$*. Consequently,* $[\lambda_0 - |a|, \lambda_0 + |a|] \in \Sigma(L_1)$*.*

Therefore, $\Sigma(L_1) = [\lambda_0 - |a|, \lambda_0 + |a|]$*.*

Now we claim that (2.38) is nonuniformly contracted to $\Sigma(L_1)$*. Indeed, given a fixed* $\delta > 0$ *and* $\varepsilon_1 = 2|a|$*, we consider the matrix function* $t \to S_1(t) \in M_1(\mathbb{R})$ *defined by*

$$S_1(t) = \exp\left(\frac{\varepsilon_1}{2} t\cos t - \delta\sin t\right),$$

and we can verify that (2.38) is δ*-nonuniformly kinematically similar to*

$$\dot{y} = (C(t) + B(t))y, \quad \text{with} \quad C(t) = \lambda_0 \quad \text{and} \quad B(t) = -\delta\cos(t)\left(1 + \tfrac{\varepsilon_1}{2\delta}\right).$$

The claim follows since $C(t) \in [\lambda_0 - |a|, \lambda_0 + |a|]$ *and* $\|B(t)\| \leq \delta K_{\delta,\varepsilon_1}$*, where* $K_{\delta,\varepsilon_1} = 1 + \frac{\varepsilon_1}{2\delta}$*.*

Example 2.2. *Consider the scalar system shown in [48] defined in* $\mathbb{R}_0^+$

$$\dot{x} = L_2(t)x, \tag{2.40}$$

with $L_2(t) = \lambda_1(\sin(\ln(t+1)) + \cos(\ln(t+1)))$ *and* $\lambda_1 \neq 0$*.*

First, we will prove that the nonuniform spectrum of (2.40) is

$$\Sigma(L_2) = [-|\lambda_1|, |\lambda_1|].$$

In fact, the evolution operator of (2.40) is given by

$$\Phi(t,s) = \exp(\lambda_1(t+1)\sin(\ln(t+1)) - \lambda_1(s+1)\sin(\ln(s+1))).$$

For any $\gamma \in \mathbb{R}$*, the evolution operator of system* $\dot{x} = (L_2(t) - \gamma)x$ *is given by*

$$\Phi_\gamma(t,s) = \exp(-\gamma(t-s))\Phi(t,s). \tag{2.41}$$

For any $\gamma \in (|\lambda_1|, +\infty)$, *it follows from* (2.41) *that*

$$\begin{aligned} \|\Phi_\gamma(t,s)\| &\leq \exp(-\gamma(t-s) + |\lambda_1|(t+s+2)), \\ &\leq \exp(2|\lambda_1|)\exp(-(\gamma-|\lambda_1|)(t-s) + 2|\lambda_1|s), \quad t \geq s, \end{aligned}$$

which implies that the system $\dot{x} = (L_2(t) - \bar{\gamma})x$ *admits a nonuniform exponential dichotomy with* $P(t) = 1$, $K = \exp(2|\lambda_1|)$, $\alpha = \gamma - |\lambda_1| > 0$ *and* $\mu = 2|\lambda_1| > 0$. *Thus*

$$(|\lambda_1|, +\infty) \subset \rho(L_2).$$

For any $\bar{\gamma} \in (-\infty, -|\lambda_1|)$, *it follows from* (2.41) *that*

$$\begin{aligned} \|\Phi_{\bar{\gamma}}(t,s)\| &\leq \exp(-\bar{\gamma}(t-s) + |\lambda_1|(t+s+2)), \\ &\leq \exp(2|\lambda_1|)\exp((-\bar{\gamma}+|\lambda_1|)(t-s) + 2|\lambda_1|s), \quad t \leq s, \end{aligned}$$

which implies that the system $\dot{x} = (L_2(t) - \bar{\gamma})x$ *admits a nonuniform exponential dichotomy with* $P(t) = 0$, $K = \exp(2|\lambda_1|)$, $\bar{\alpha} = -\bar{\gamma} + |\lambda_1| > 0$ *and* $\bar{\mu} = 2|\lambda_1| > 0$. *Thus*

$$(-\infty, -|\lambda_1|) \subset \rho(L_2).$$

From the above, we have

$$(-\infty, -|\lambda_1|) \cup (|\lambda_1|, +\infty) \subset \rho(L_2),$$

which implies that $\Sigma(L_2) \subset [-|\lambda_1|, |\lambda_1|]$.

Now we show that $[-|\lambda_1|, |\lambda_1|] \subset \Sigma(L_2)$. *To show this, we first prove that* $-|\lambda_1| \in \Sigma(L_2)$. *On the contrary, assume that* $\gamma_2 = -|\lambda_1|$ *such that* $\dot{x} = (L_2(t) - \gamma_2)x$ *admits a nonuniform exponential dichotomy. We know that either the projector* $P(t) = 0$ *or* $P(t) = 1$. *If* $P(t) = 1$ *then there exist constant* $K \geq 1$, $\alpha > 0$ *and* $\mu \geq 0$ *such that the following estimate holds*

$$\begin{aligned} \|\Phi_{\gamma_2}(t,s)\| &= \exp(-\gamma_2(t-s))\,\|\Phi(t,s)\|, \\ &\leq K\exp(-\alpha(t-s) + \mu s), \quad t \geq s \end{aligned}$$

Substituting $\gamma_2 = -|\lambda_1|$, *we have for* $t \geq s$

$$\exp(|\lambda_1|(t-s) + \lambda_1(t+1)\sin(\ln(t+1)) - \lambda_1(s+1)\sin(\ln(s+1))) \leq K\exp(-\alpha(t-s) + \mu s),$$

which yields a contradiction for $s = 0$ *and* $t \to +\infty$. *If* $P(t) = 0$ *and* $t \leq s$, *the dichotomy estimate is*

$$\exp(|\lambda_1|(t-s) + \lambda_1(t+1)\sin(\ln(t+1)) - \lambda_1(s+1)\sin(\ln(s+1))) \leq K\exp(\alpha(t-s) + \mu s),$$

which also yields a contradiction for $t = 0$ *and* $s = \exp\left(\frac{3\pi}{2} + 2k\pi\right) - 1$ *since we have*

$$\exp\left(|\lambda_1|\left(\exp\left(\frac{3\pi}{2} + 2k\pi\right) - 1\right) + \lambda_1\exp\left(\frac{3\pi}{2} + 2k\pi\right)\right) \leq K\exp(-(\alpha-\mu)\exp\left(\frac{3\pi}{2} + 2k\pi\right) - 1),$$

where the left side if the last inequality is constant if $\lambda_1 < 0$, *diverge if* $\lambda_1 > 0$ *and the right side converge to 0 where* $k \to +\infty$ *if* $\alpha > \mu$.*Therefore* $-|\lambda_1| \in \Sigma(L_2)$.

Analogously, we can prove that $|\lambda_1| \in \Sigma(L_1)$, *in fact, assume that* $\gamma_2 = |\lambda_1|$ *such that* $\dot{x} =$

$(L_2(t) - \gamma_2)x$ admits a nonuniform exponential dichotomy. We know that either the projector $P(t) = 0$ or $P(t) = 1$. If $P(t) = 1$ then there exist constants $K \geq 1, \alpha > 0$ and $\mu \geq 0$ such that the following estimate holds

$$\begin{aligned} \|\Phi_{\gamma_2}(t,s)\| &= \exp(-\gamma_2(t-s))\,\|\Phi(t,s)\|\,, \\ &\leq K\exp(-\alpha(t-s)+\mu s), \quad t \geq s \end{aligned}$$

Substituting $\gamma_2 = |\lambda_1|$, we have for $t \geq s$

$$\exp(-|\lambda_1|(t-s) + \lambda_1(t+1)\sin(\ln(t+1)) - \lambda_1(s+1)\sin(\ln(s+1))) \leq K\exp(-\alpha(t-s)+\mu s),$$

which yields a contradiction for $s =$ and $t =$. If $P(t) = 0$ and $t \leq s$, the dichotomy estimate is

$$\exp(-|\lambda_1|(t-s) + \lambda_1(t+1)\sin(\ln(t+1)) - \lambda_1(s+1)\sin(\ln(s+1))) \leq K\exp(\alpha(t-s)+\mu s),$$

which also yields a contradiction for $t = 0$, $s = \exp(2k\pi - \frac{\pi}{2}) - 1$ since we have

$$\exp\left((|\lambda_1| + \lambda_1)\left(\exp\left(2k\pi - \frac{\pi}{2}\right) - 1\right) \leq K\exp\left(-\left(\exp\left(2k\pi - \frac{\pi}{2}\right) - 1\right)(\alpha - \mu)\right)\right.$$

where the left side of last inequality is constant if $\lambda_1 < 0$, diverge if $\lambda_1 > 0$ and the right side converge to 0 when $k \to +\infty$ if $\alpha > \mu$. Therefore $|\lambda_1| \in \Sigma(L_2)$. By Theorem 2.1, we know that $\Sigma(L_2)$ is an interval. Thus, for any $\gamma \in [-|\lambda_1|, |\lambda_1|]$, it follows from the connectedness that $\gamma \in \Sigma(L_2)$. Consequently, $[-|\lambda_1|, |\lambda_1|] \in \Sigma(L_2)$.

Therefore, $\Sigma(L_2) = [-|\lambda_1|, |\lambda_1|]$.

Now we claim that (2.40) is nonuniformly contracted to $\Sigma(L_2)$. Indeed, given a fixed $\delta > 0$ and $\varepsilon_2 = 2\lambda_1$, we consider the matrix function $t \to S_2(t) \in M_1(\mathbb{R})$ defined by $S_2(t) = \exp(\frac{1}{2}\delta[(t+1)\sin(\ln(t+1)) + (t+1)\cos(\ln(t+1))])$, and is simple verify that (2.40) is δ-nonuniformly kinematically similar to

$$\dot{y} = (C(t) + B(t))y,$$

with $C(t) = \lambda_1\sin(\ln(t+1))$ and $B(t) = \lambda_1\cos(\ln(t+1)) + \delta\cos(\ln(t+1))$.

The claim follows since $C(t) \in [-|\lambda_1|, |\lambda_1|]$ and $\|B(t)\| \leq \delta K_{\delta,\varepsilon_2}$, where $K_{\delta,\varepsilon_2} = 1 + \frac{\varepsilon_2}{2\delta}$.

Example 2.3. *Consider the planar system defined in $\mathbb{R}_0^+$*

$$\dot{x} = L(t)x, \tag{2.42}$$

where

$$L(t) = \begin{pmatrix} \lambda_0 + at\sin t & 0 \\ 0 & \lambda_1(\sin(\ln(t+1)) + \cos(\ln(t+1))) \end{pmatrix}$$

with the same conditions as in the previous examples and also $\lambda_0 - a < -|\lambda_1|$. We can see that in view of the Lemma 1.7, the nonuniform spectrum of (2.42) is $\Sigma(L) = [\lambda_0 + a, \lambda_0 - a] \cup [-|\lambda_1|, |\lambda_1|]$.

Now we claim that (2.42) is nonuniformly contracted to $\Sigma(L)$. Indeed, given a fixed $\delta > 0$ and

$\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, *we consider the matrix function* $t \to S(t) \in M_2(\mathbb{R})$ *defined by*

$$S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix},$$

where $S_1(t)$ *and* $S_2(t)$ *are as in the previous examples. It is verified that* (2.42) *is* δ*-nonuniformly kinematically similar to*

$$\dot{y} = (C(t) + B(t))y,$$

with

$$C(t) = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \sin(\ln(t+1)) \end{pmatrix}$$

and

$$B(t) = \begin{pmatrix} -\delta\cos(t)\left(1 + \frac{\varepsilon_1}{2\delta}\right) & 0 \\ 0 & \lambda_1\cos(\ln(t+1)) + \delta\cos(\ln(t+1)) \end{pmatrix}.$$

The claim follows since $C(t) \in \Sigma(L)$ *and* $\|B(t)\| \leq \delta K_{\delta,\varepsilon}$, *where* $K_{\delta,\varepsilon} = \max\{K_{\delta,\varepsilon_1}, K_{\delta,\varepsilon_2}\}$.

Chapter 3

Linearization of a nonautonomous unbounded system with nonuniform contraction: A Spectral Approach

In the first chapter we defined the nonuniform spectrum for a nonautonomous linear system and in the second chapter we obtained an important result for this spectrum, which allows us to describe system (1.1) as the system

$$\dot{y} = (C(t) + B(t)) \tag{3.1}$$

with $C(t)$ is a diagonal matrix, $C(t) \in \Sigma(A)$ and $\|B(t)\| \leq \delta K_{\delta,\varepsilon}$.

As presented in the first chapter, in the autonomous context, Hartman–Grobman's theorem allows to establish an equivalence between the flows of system (4) and system (3). For the nonautonomous context and generalizing the Hartman–Grobman theorem, K. J. Palmer [35] worked the topological equivalence between the solutions of system (1.1) and system

$$\dot{x} = A(t)x + f(t,x) \tag{3.2}$$

using tools of the Green's function, assuming that the linear system has an exponential dichotomy, f it is bounded and its Lipschitz constant is small. This same work was done by Á. Castañeda and G. Robledo [15] but in the discrete context and the difference equations.

Following the line of the exponential dichotomy, F. Lin [30] established a topological equivalence between systems (1.1) and (3.2), when system (1.1) has asymptotic stability (which can be related to the identity projector), from a spectral point of view, namely, considering the Lipschitz constant of f related to the spectrum and bounded at the origin. Á. Castañeda and G. Robledo [16] obtained this same result for equations in differences.

It should be noted that L. Barreira and C. Valls have a result of the Hartman-Grobman type [6] if system (1.1) has a nonuniform exponential dichotomy, but the approach presented in that article is totally different from the one presented in this chapter.

The objective of this chapter is to obtain a result such as the Hartman-Grobman Theorem, using the result of nonuniform almost reducibility and the Lyapunov functions and quadratic forms theory.

As we consider (3.2) as a perturbation of system (1.1), for the system (3.1) we consider

$$\dot{y} = (C(t) + B(t))y + g(t, y). \tag{3.3}$$

3.1 Preliminaries.

3.1.1 Properties

In this article we consider the following couple of the systems

$$\begin{cases} \dot{x} = A(t)x & (3.4a) \\ \dot{x} = A(t)x + f(t, x) & (3.4b) \end{cases}$$

where $A\colon \mathbb{R}_0^+ \to M(n, \mathbb{R})$ and $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on (t, x) and

$$\begin{cases} \dot{y} = [C(t) + B(t)]y & (3.5a) \\ \dot{y} = [C(t) + B(t)]y + g(t, y) & (3.5b) \end{cases}$$

where $B, C\colon \mathbb{R}_0^+ \to M(n, \mathbb{R})$ and $g : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on (t, x). Moreover, the following properties are verified:

(P1) For $\nu, \mathcal{M} > 0$, $\|A(t)\| \leq \mathcal{M} \exp(\nu t)$ for any $t \in \mathbb{R}_0^+$.

(P2) The evolution operator $\Phi(t, s)$ of (3.4a) has a nonuniformly bounded growth [47], namely, there exist constants $K_0 \geq 1$, $a \geq 0$ and $\bar{\varepsilon} \geq 0$ such that

$$\|\Phi(t, s)\| \leq K_0 \exp(a\,|t - s| + \bar{\varepsilon}s), \quad t,\ s \in \mathbb{R}_0^+.$$

(P3) The system (3.4a) is nonuniform contractible if there exist $K > 0$, $\alpha > 0$ and $\mu \geq 0$ such that

$$\|\Phi(t, s)\| \leq K \exp(-\alpha(t - s) + \mu s) \quad \text{for any } t \geq s \geq 0. \tag{3.6}$$

(P4) The function f is continuous on (t, x) and is an element of one of the following families of functions:

$$\mathcal{A}_1 = \left\{ \begin{array}{c} f : \sup_{t \in \mathbb{R}_0^+} \|f(t, 0)\| < +\infty \text{ and } \exists\ L_f,\ \beta \geq 0 \text{ s.t.} \\ \|f(t, u) - f(t, v)\| \leq L_f \exp(-2\beta t)\, \|u - v\| \ \ \forall t \in \mathbb{R}_0^+ \end{array} \right\},$$

$$\mathcal{A}_2 = \left\{ f : f \in \mathcal{A}_1 \text{ and } f(t, 0) = 0 \text{ for all } t \in \mathbb{R}_0^+ \right\}.$$

3.1.2 Main Tools

The fundamental tools in our work are the concepts of topological equivalence, introduced by K.J. Palmer in [35], the nonuniform exponential dichotomy which was introduced by L. Barreira and C. Valls in [5], and the δ-nonuniform kinematical similarity.

Here we present the definition with which we will work throughout this chapter.

Definition 3.1. *The systems* (3.4a) *and* (3.4b) *will be called topologically equivalent if there exists a map* $H : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ *with the properties*

(i) *For each fixed* $t \in \mathbb{R}_0^+$*, the map* $\xi \mapsto H(t,\xi)$ *is a bijection.*

(ii) *For any fixed* $t \in \mathbb{R}_0^+$*, the maps* $\xi \mapsto H(t,\xi)$ *and* $\xi \mapsto H^{-1}(t,\xi) = G(t,\xi)$ *are continuous.*

(iii) *If* $\|\xi\| \to +\infty$*, then* $\|H(t,\xi)\| \to +\infty$*.*

(iv) *If* $x(t)$ *is a solution of* (3.4a)*, then* $H(t,x(t))$ *is a solution of* (3.5a)*. Similarly, if* $y(t)$ *is a solution of* (3.4b)*, then* $G(t,y(t))$ *is a solution of* (3.4a)*.*

Remark 3.1. *We can observe that the topological equivalence is an equivalent relation. Moreover, it is easy to verify that* δ*-nonuniform kinematical similarity is a particular case of topological equivalence. Indeed, the properties of Definition 3.1 are verified with* $H(t,\xi) = S^{-1}(\delta,t)\xi$*.*

3.2 Mathematical preliminaries.

The following is the proof of the claim in the Remark 3.1.

Proposition 3.1. *If*

$$\dot{x} = F(t,x(t)) \tag{3.7}$$

and

$$\dot{y} = G(t,y(t)), \tag{3.8}$$

are nonuniformly kinematically similar, then they are topologically equivalent.

Proof. Let $S(t)$ be the nonuniform kinematically similarity between (3.7) and (3.8). Now we define the function $H(t,x) = S^{-1}(t)x$, and if we denote $H_t(x) = H(t,x)$ for any $t \in \mathbb{R}_0^+$ fixed, then $H_t(x)$ is continuous, has inverse $G_t(y) = S(t)y$ for any $t \in \mathbb{R}_0^+$ fixed, which also is continuous. If $X(t)$ is solution of (3.7), then $H(t,X(t)) = S^{-1}(t)X(t) = Y(t)$ is solution of (3.8).

On the other hand, for any fixed $t \in \mathbb{R}_0^+$, we have the following estimate

$$\|G(t,H(t,\xi))\| = \|\xi\| \leq M\exp(\upsilon t)\,\|H(t,\xi)\|\,,$$

and if $\|\xi\| \to +\infty$, then $\|H(t,\xi)\| \to +\infty$. □

Now we will rewrite the definition of nonuniformly kinematically similar and nonuniformly almost reducible, concepts that we already defined in the previous chapters and that will be important in the development of this chapter.

Definition 3.2. ([47]) *Given a $\delta > 0$, the linear system (3.4a) is δ-nonuniformly kinematically similar to*

$$\dot{y} = U(t)y, \tag{3.9}$$

if there exist a Lyapunov's transformation $S(\delta, t)$ and $\upsilon \geq 0$, with

$$\|S(\delta,t)\| \leq M_{\upsilon,\delta}\exp(\upsilon t) \quad \text{and} \quad \left\|S^{-1}(\delta,t)\right\| \leq M_{\upsilon,\delta}\exp(\upsilon t),$$

such that the change of coordinates $y(t) = S^{-1}(\delta,t)x(t)$ transforms the system (3.4a) into (3.9).

Remark 3.2. *The nonuniform kinematical similarity preserves the nonuniform contraction (see more details in chapter 2, Lemma 1.2). Thus, as the systems (3.4a) and (3.5a) are δ–nonuniformly kinematically similar (see Theorem 2.2), and as the system (3.4a) satisfies the condition* **(P2)** *with $K \geq 1$, $\alpha > 0$, $\mu \geq 0$ and if $\alpha > \mu$, then the system (3.5a) admits a nonuniform contraction, i.e., there exist $K_1 \geq 1$, $\alpha_1 > 0$ and $\mu_1 \geq 0$ satisfying*

$$\|\Psi(t,s)\| \leq K_1\exp(-\alpha_1(t-s)+\mu_1 s), \quad t \geq s \geq 0, \tag{3.10}$$

where $\Psi(t,s)$ is the evolution operator of (3.5a).

Definition 3.3. ([9]) *The system (3.4a) is nonuniformly almost reducible to*

$$\dot{y} = C(t)y,$$

if for any $\delta > 0$ and $\varepsilon \geq 0$, there exists a constant $K_{\delta,\varepsilon} \geq 1$ such that (3.4a) is $\delta-$nonuniformly kinematically similar to

$$\dot{y} = [C(t)+B(t)]y, \quad \text{with} \quad \|B(t)\| \leq \delta K_{\delta,\varepsilon}$$

for any $t \in \mathbb{R}_0^+$.

Some comments that we can provide in relation to the nonuniform spectrum and the nonuniform almost reducibility, for the context of this chapter, are the following:

Remark 3.3. *From the nonuniform exponential dichotomy spectrum, assumptions* **(P1)**, **(P2)** *and* **(P3)** *can be better understood. Indeed:* **(P2)** *implies that $\Sigma(A)$ is a finite union of at most $m \leq n$ compact intervals*

$$\Sigma(A) = \bigcup_{i=1}^{m}[a_i, b_i],$$

with $-\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty$.

Therefore, in the charpter 2 we proved that diagonals terms of $C(t)$ are contained in $\Sigma(A)$.

(P3) *implies that* $\Sigma(A) \subset (-\infty, 0)$. *In fact, for* $\lambda \in \mathbb{R}$ *we have that*

$$\begin{aligned} \|\Phi_\lambda(t,s)\| &= \|\exp(-\lambda(t-s))\Phi(t,s)\|, \\ &\leq K\exp(-(\alpha+\lambda)(t-s)+\mu s), \end{aligned}$$

with $\alpha > \mu$ *and* $\Phi_\lambda(t,s)$ *is the evolution operator of system* $\dot{x} = (A(t) - \lambda I)x$. *If* $\alpha + \lambda > \mu$, *so* $\lambda > -(\alpha - \mu)$, *then* $\lambda \in \rho(A)$. *Thus*

$$\Sigma(A) \subset (-\infty, -(\alpha-\mu)) \subset (-\infty, 0).$$

In Theorem 2.2 from chapter 2 it was proved that if **(P1)** *and* **(P2)** *are satisfied, the system* (1.1) *is* δ*-nonuniformly kinematically similar via* $S^{-1}(\delta,t)$ *to* (3.1), *where* $C(t) = \mathrm{Diag}(C_1(t), \ldots, C_n(t))$ *with* $C_i(t) \in \Sigma(A)$ *and* $\|B(t)\| \leq \delta K_{\delta,\varepsilon}$.

In addition, under the same transformation, the system (3.2) is transformed in

$$\dot{y} = (C(t)+B(t))y + S^{-1}(\delta,t)f(t,S(\delta,t)y). \tag{3.11}$$

3.3 Lyapunov function and quadratic forms.

In this section, for the system (3.4a), we obtain a complete characterization of nonuniform contraction in terms of a Lyapunov function which will allow us to construct a topological equivalence between systems (3.4a)–(3.4b) and (3.5a)–(3.5b). For this purpose, we recall the definition of *strict Lypaunov function* and the main results from [29].

Definition 3.4. *Given* $\mathcal{K} \geq 1$ *and* $\upsilon \geq 0$. *We say that a continuous function* $V : [0,+\infty)\times X \to \mathbb{R}_0^+$, *where* X *is a Banach space, is a strict Lyapunov function for* (3.4a) *if*

(V1) $\|x\|^2 \leq V(t,x) \leq \mathcal{K}^2 \exp(2\upsilon t)\|x\|^2$, *for any* $t \geq 0$ *and* $x \in X$,

(V2) $V(t,\Phi(t,s)x) \leq V(s,x)$, *for any* $t \geq s \geq 0$ *and* $x \in X$,

(V3) *Exists* $\gamma > 0$ *such that* $V(t,\Phi(t,s)x) \leq \exp(-2\gamma(t-s))V(s,x)$, $\forall t \geq s \geq 0$ *and* $x \in X$.

The last definition has subtle differences with respect to Liao *et al.* [29]. In fact, we have tailored it in order to relate it with the nonuniform exponential dichotomy. Indeed, we have the following result.

Theorem 3.1. *The system* (3.4a) *has nonuniform contraction if and only if it admits a strict Lyapunov function.*

Proof. Suppose that there exists a strict Lyapunov function for (3.4a). From the conditions **(V1)** and **(V3)** we have

$$\begin{aligned}\|\Phi(t,s)x\|^2 \leq & \quad V(t,\Phi(t,s)x) \leq \exp(-2\gamma(t-s))V(s,x), \\ \leq & \quad \exp(-2\gamma(t-s))\mathcal{K}^2 \exp(2\upsilon s)\,\|x\|^2\end{aligned}$$

which implies that

$$\|\Phi(t,s)x\| \leq \mathcal{K}\exp(-\gamma(t-s)+\upsilon s)\,\|x\|\,.$$

Therefore, (3.4a) admits a nonuniform contraction with $\gamma = \alpha$ and $\upsilon = \mu$.

On the other hand, for $t \geq 0$ and $x \in X$ we define

$$V(t,x) = \sup_{\tau \geq t}\left\{\|\Phi(\tau,s)x\|^2 \exp(2\alpha(\tau-t))\right\}.$$

As (3.4a) admits nonuniform contraction, we have that $V(t,x) \leq K^2\exp(2\mu t)\,\|x\|^2$. If we consider $\tau = t$, then $\|x\|^2 \leq V(t,x)$. Now, for $t \geq s \geq 0$

$$\begin{aligned}V(t,\Phi(t,s)x) &= \sup\nolimits_{\tau\geq t}\left\{\|\Phi(\tau,t)\Phi(t,s)x\|^2\exp(2\alpha(\tau-t))\right\}, \\ &= \exp(2\alpha(s-t))\sup\nolimits_{\tau\geq t}\left\{\|\Phi(\tau,s)x\|^2\exp(2\alpha(\tau-s))\right\}, \\ &\leq \exp(2\alpha(s-t))\sup\nolimits_{\tau\geq s}\left\{\|\Phi(\tau,s)x\|^2\exp(2\alpha(\tau-s))\right\}, \\ &= \exp(-2\alpha(t-s))V(s,x).\end{aligned}$$

Therefore, V is a strict Lyapunov function for (3.4a).

□

Now we will focus in Lyapunov functions that are defined in terms of quadratic forms. Let $\mathcal{S}(t) \in \mathcal{B}(X)$ be a symmetric positive-definite operator for $t \geq 0$, where $\mathcal{B}(X)$ the space of bounded linear operators in a Banach space X. A *quadratic Lyapunov function* V is given as

$$V(t,x) = \langle \mathcal{S}(t)x, x\rangle\,. \tag{3.12}$$

Remark 3.4. *Given two linear operators M, N, we write $M \leq N$ if they verify $\langle Mx, x\rangle \leq \langle Nx, x\rangle$ for $x \in X$.*

The following result (see [29, Theorem 2.2] with $\mu(t) = e^t$) establishes a characterization of nonuniform contraction in terms of the existence of quadratic Lypaunov function.

Proposition 3.2. *Assume that there exist constants $c > 0$ and $d \geq 1$ such that*

$$\|\Phi(t,s)\| \leq c, \quad \textit{whenever} \quad t-s \leq \ln(d) \tag{3.13}$$

Then (3.4a) admits a nouniform contraction if and only if there exist symmetric positive definite *operators* $\mathcal{S}(t)$ *and constant* $\mathcal{C}$,$\mathcal{K}_1 > 0$ *such that* $S(t)$ *is of class* C^1 *in* $t \geq 0$ *and*

$$\|\mathcal{S}(t)\| \leq \mathcal{C}\mathcal{K}_1 \exp(2\mu t), \tag{3.14}$$

$$\mathcal{S}'(t) + A^*(t)\mathcal{S}(t) + \mathcal{S}(t)A(t) \leq (-Id + \mathcal{K}_1\mathcal{S}(t)). \tag{3.15}$$

Proof. First we consider the linear operator

$$\mathcal{S}(t) = \int_t^{+\infty} \Phi(\tau,t)^*\Phi(\tau,t)\exp(2(\alpha-\varrho)(\tau-t))d\tau,$$

for some constant $\varrho \in (0,\alpha)$ and $\Phi(\tau,t)^*$ represents the adjoint operator of $\Phi(\tau,t)$. Clearly , $\mathcal{S}(t)$ symmetric for each $t \geq 0$. Moreover, by (3.12), we note that

$$\begin{aligned}
\|V(t,x)\| &= \int_t^{+\infty} \|\Phi(\tau,t)x\|^2 \exp(2(\alpha-\varrho)(\tau-t))d\tau,\\
&\leq K^2\exp(2\mu t)\,\|x\|^2 \int_t^{+\infty} \exp(-2\varrho(\tau-t))d\tau,\\
&= \frac{K^2\exp(2\mu t)}{2\varrho}\,\|x\|^2 .
\end{aligned}$$

Since the operator $\mathcal{S}(t)$ is symmetric for any $t \geq 0$, then we have that

$$\|\mathcal{S}(t)\| = \sup_{\|x\|=1} V(t,x) \leq \frac{K^2\exp(2\mu t)}{2\varrho},$$

and therefore (3.14) holds. Since

$$\frac{\partial\Phi(\tau,t)}{\partial t} = -\Phi(\tau,t)A(t), \qquad \frac{\partial\Phi(\tau,t)^*}{\partial t} = -A(t)^*\Phi(\tau,t)^*,$$

we find that $\mathcal{S}(t)$ is of class C^1 in t with derivative

$$\begin{aligned}
\mathcal{S}'(t) = -Id - &\int_t^{+\infty} A(t)^*\Phi(\tau,t)^*\Phi(\tau,t)\exp(2(\alpha-\varrho)(\tau-t))d\tau\\
-&\int_t^{+\infty} \Phi(\tau,t)^*\Phi(\tau,t)A(t)\exp(2(\alpha-\varrho)(\tau-t))d\tau\\
-2(\alpha-\varrho)&\int_t^{+\infty} \Phi(\tau,t)^*\Phi(\tau,t)\exp(2(\alpha-\varrho)(\tau-t))d\tau,
\end{aligned}$$

which implies that

$$\mathcal{S}'(t) = -Id - A(t)^*\mathcal{S}(t) - \mathcal{S}(t)A(t) - 2(\alpha-\varrho)\mathcal{S}(t).$$

Therefore

$$\mathcal{S}'(t) + A(t)^*\mathcal{S}(t) + \mathcal{S}(t)A(t) = -(Id + 2(\alpha-\varrho)\mathcal{S}(t)), \tag{3.16}$$

which establishes (3.15) with

$$\mathcal{K} = 2(\alpha-\varrho). \tag{3.17}$$

On the other hand, set $x(t) = \Phi(t,\tau)x(\tau)$. By (3.14), we have

$$V(t,x(t)) \leq \|\mathcal{S}(t)\|\,\|x(t)\|^2 \leq \mathcal{C}K^2\exp(2\mu t)\,\|x\|^2 . \tag{3.18}$$

The following results allow us conclude this implicance

Lemma 3.1. *There exists a constant $\eta > 0$ such that*

$$V(t, x(t)) \geq \eta \|x\|^2 . \tag{3.19}$$

Proof. Note that

$$\begin{aligned} \frac{d}{dt}V(t,x(t)) &= \big\langle \mathcal{S}'(t)x(t), x(t)\big\rangle + \langle \mathcal{S}(t)A(t)x(t), x(t)\rangle + \langle \mathcal{S}(t)x(t), A(t)x(t)\rangle , \\ &= \big\langle (\mathcal{S}'(t) + \mathcal{S}(t)A(t) + A(t)^*\mathcal{S}(t))x(t), x(t)\big\rangle . \end{aligned} \tag{3.20}$$

Hence, by condition (3.15) and the fact that $\mathcal{K} > 0$ we obtain

$$\frac{d}{dt}V(t,x(t)) \leq -\|x\|^2 .$$

Now given $\tau > 0$, take $t > \tau$ such that $\exp(t) = d\exp(\tau)$ with d as in (3.13). Then

$$\begin{aligned} V(t,x(t)) - V(\tau, x(\tau)) &= \int_\tau^t \frac{d}{dv}V(v,x(v))dv, \\ &\leq -\int_\tau^t \|x\|^2 dv, \\ &= -\int_\tau^t \|\Phi(v,\tau)x(\tau)\|^2 dv, \\ &\leq -\|x\|^2 \int_\tau^t \frac{1}{\|\Phi(\tau,v)\|^2}dv. \end{aligned}$$

It follows from (3.13) that

$$\begin{aligned} V(t,x(t)) - V(\tau, x(\tau)) &\leq -\frac{1}{c^2}\|x(\tau)\|^2 \int_\tau^t 1dv, \\ &= -\frac{\log(d)}{c^2}\|x(\tau)\|^2 . \end{aligned}$$

Since $V(t,x(t)) \geq 0$, we have

$$V(\tau, x(\tau)) \geq V(\tau,x(\tau)) - V(t,x(t)) \geq \frac{\log(d)}{c^2}\|x\|^2 .$$

which yields (3.19) with $\eta = \frac{\log(d)}{c^2}$. □

Lemma 3.2. *For $t \geq \tau$ we have*

$$V(t,x(t)) \leq \exp(-\mathcal{K}(t-\tau))V(\tau,x(\tau)).$$

Proof. By (3.15) and (3.20) we have that

$$\frac{d}{dt}V(t,x(t)) \leq -\mathcal{K}V(t,x(t)).$$

Therefore

$$V(t,x(t)) - V(\tau,x(\tau)) = \int_\tau^t \frac{d}{dv}V(v,x(v))dv \le -\mathcal{K}\int_\tau^t V(v,x(v))dv.$$

It follows from Gronwall lemma that

$$V(t,x(t)) \le \exp(-\mathcal{K}(t-\tau))V(\tau,x(\tau)),$$

which yields the desired result. □

By lemmas 3.1 and 3.2 together with (3.18), we obtain

$$\begin{aligned}
\|\Phi(t,\tau)x(\tau)\|^2 &= \|x(t)\|^2, \\
&\le \eta^{-1}V(t,x(t)), \\
&\le \eta^{-1}\exp(-\mathcal{K}(t-\tau))V(\tau,x(\tau)), \\
&\le \eta^{-1}CK^2\exp(2\mu\tau)\exp(-\mathcal{K}(t-\tau))\|x(\tau)\|^2,
\end{aligned}$$

and therefore

$$\|\Phi(t,\tau)\| \le (\eta^{-1}CK^2)^{\frac{1}{2}}\exp(\mu\tau)\exp\left(-\frac{\mathcal{K}}{2}(t-\tau)\right).$$

□

Remark 3.5. *If the system (3.4a) satisfies the properties **(P3)**, this implies the condition (3.13).*

3.4 Main results

The principal results of this chapter are the following:

Theorem 3.2. *Consider the couple of system (3.5a)–(3.5b) such that $C_i(t) \in \Sigma(A)$ for $i = 1,\dots,n$ and $\|B(t)\| \le \delta K_{\delta,\epsilon}$. If* **(P1)–(P3)** *are satisfied, then*

(1) *If $y(t)$ is solution of (3.5b) and $\alpha_1 > \mu_1$, then for $L_g < \alpha_1 - \mu_1$, we have*

$$\frac{dV(t,y(t))}{dt} \le -2[\alpha_1 - \mu_1 - L_g]V(t,y(t)), \tag{3.21}$$

where $V(t,x)$ is a Lyapunov function associated to (3.5a).

(2) *The systems (3.5a)–(3.5b) are topologically equivalent.*

Theorem 3.3. *If the properties* **(P1)-(P4)** *are verified with $0 < \delta < \alpha - \mu$ and $f \in \mathcal{A}_2$ such that*

$$L_f \le \frac{\delta}{M_1^2}, \tag{3.22}$$

with $\|S(\delta,t)\| \leq M_1 \exp(\beta t)$ *and* $\left\|S^{-1}(\delta,t)\right\| \leq M_1 \exp(\beta t)$*, for some* $\beta > 0$*, then the systems* (3.4a) *and* (3.4b) *are topologically equivalent.*

Theorem 3.4. *If the properties* **(P1)-(P4)** *are verified with* $0 < \delta < \alpha - \mu$ *and* $f \in \mathcal{A}_1$ *such that*

$$L_f \leq \min\left\{\frac{\delta}{M_1^2}, \frac{\alpha}{K}\right\}, \tag{3.23}$$

then the systems(3.4a) *and* (3.4b) *are topologically equivalent.*

3.5 Some basic results

The following proposition is a classical result of local continuity with respect to the initial conditions for differential equations.

Proposition 3.3. *Let us consider the differential equation*

$$\dot{x} = F(t,x) \tag{3.24}$$

where $F \in \mathcal{A}_2$*, then for the solution* $X(t,s,u)$ *of* (3.24) *with* $X(s,s,u) = u$*, we have that*

$$\|u-v\| \exp(-L_F|t-s|) \leq \|X(t,s,u) - X(t,s,v)\| \leq \|u-v\| \exp(L_F|t-s|).$$

Proof. For any $u \in \mathbb{R}^n$, we have

$$X(t,s,u) = u + \int_s^t F(X(r,s,u),r)dr,$$

so

$$\|X(t,s,u) - X(t,s,v)\| \leq \|u-v\| + \left|\int_s^t L_F \|X(r,s,u) - X(r,s,v)\| \, dr\right|.$$

By Gronwall's Lemma,

$$\|X(t,s,u) - X(t,s,v)\| \leq \|u-v\| \exp(L_F|t-s|).$$

Replacing u and v with $X(s,t,u)$ and $X(s,t,v)$ respectively, it follows that

$$\|X(s,t,u) - X(s,t,v)\| \geq \|u-v\| \exp(-L_F|t-s|).$$

□

The following result is an extension to the nonuniform context of [31, Proposition 5].

Proposition 3.4. *Assume that the system* (3.4a) *has a nonuniform exponential dichotomy on* $\mathbb{R}_0^+$ *with* $K \geq 1$*, constants* $\alpha > 0$*,* $\mu \geq 0$ *and* $P(t) = I$ *for any* $t \in \mathbb{R}_0^+$*. Let us consider the nonlinear perturbation*

$$\dot{x} = A(t)x + \mathcal{F}(t, x(t), \kappa) \tag{3.25}$$

where $\mathcal{F} : \mathbb{R}_0^+ \times \mathbb{R}^n \times \boldsymbol{B} \to \mathbb{R}^n$ *and* $\boldsymbol{B}$ *is a Banach space. Moreover,* $\mathcal{F}$ *satisfies the following conditions:*

(i) $\mathcal{F}(t, x, \kappa)$ *is bounded with respect a* t*, for all* $x \in \mathbb{R}^n$ *and* $\kappa \in \boldsymbol{B}$ *fixed with the norm*

$$\|\mathcal{F}(x, \kappa)\|_A = \sup_{t \in \mathbb{R}_0^+} \exp(-\mu t) \|\mathcal{F}(t, x, \kappa)\|.$$

(ii) *There exist* $L_{\mathcal{F}} > 0$ *such that*

$$\|\mathcal{F}(t, x_1, \kappa) - \mathcal{F}(t, x_2, \kappa)\| \leq L_{\mathcal{F}} \exp(-2\mu t) \|x_1 - x_2\|$$

for any $t \in \mathbb{R}_0^+$ *and* $\kappa \in \boldsymbol{B}$.

(iii) $K_0 = \sup_{t \in \mathbb{R}_0^+, \kappa \in \boldsymbol{B}} \|\mathcal{F}(t, 0, \kappa)\| < +\infty$

If $KL_{\mathcal{F}} < \alpha$ *then for any fixed* $\kappa \in \boldsymbol{B}$ *the system* (3.25) *has a unique bounded solution* $Z(t, \kappa)$*, with the norm* $\|\cdot\|_A$*, described by*

$$Z(t, \kappa) = \int_0^t \Phi(t, \tau) \mathcal{F}(\tau, Z(\tau, \kappa), \kappa) d\tau. \tag{3.26}$$

such that $\sup_{t \in \mathbb{R}_0^+, \kappa \in \boldsymbol{B}} \|Z(t, \kappa)\| < +\infty$.

Proof. Let us consider a fixed $\kappa \in \boldsymbol{B}$ and construct the sequence $\{\varphi_j\}_j$ recursively defined by

$$\varphi_{j+1}(t, \kappa) = \int_0^t \Phi(t, \tau) \mathcal{F}(\tau, \varphi_j(\tau, \kappa), \kappa) d\tau$$

and

$$\varphi_0(t, \kappa) = \int_0^t \Phi(t, \tau) \mathcal{F}(\tau, 0, \kappa) d\tau,$$

where $\varphi_0(t, \kappa) \in \boldsymbol{C}$, where $\boldsymbol{C}$ is defined by

$$\boldsymbol{C} = \left\{ \begin{array}{c} U : \mathbb{R}_0^+ \times \boldsymbol{B} \to \mathbb{R}^n \text{: for any } \kappa \in \boldsymbol{B} \text{ fixed,} \\ \|U(\kappa)\|_A < +\infty \text{ and } U \text{ is continuous in } (t, \kappa) \end{array} \right\}$$

with $\|U(\kappa)\|_A = \sup_{t \in \mathbb{R}_0^+} \exp(-\mu t) \|U(t, \kappa)\|$.

In the first place we will proof that $(\boldsymbol{C}, \|\cdot\|_A)$ is a Banach space. Indeed, let $\{U_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\boldsymbol{C}$, then for any $\varepsilon > 0$ and for $\tau \in \mathbb{R}_0^+$ fixed, there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$

$$\|U_n(\kappa) - U_m(\kappa)\|_A = \sup_{t \in \mathbb{R}_0^+} \exp(-\mu t) \|U_n(t, \kappa) - U_m(t, \kappa)\| < \varepsilon,$$

but the expression $\exp(-\mu\tau)\left\|U_n(\tau,\kappa)-U_m(\tau,\kappa)\right\| \leq \left\|U_n-U_m\right\|_A$ implies that

$$\left\|U_n(\tau,\kappa)-U_m(\tau,\kappa)\right\| \leq \exp(\mu\tau)\varepsilon,$$

then $\{U_n(\tau,\kappa)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^n$, so we obtain a well-defined function $U:\mathbb{R}_0^+\times\boldsymbol{B}\to\mathbb{R}^n$ that satisfies $U(\tau,\kappa)=\lim_{n\to+\infty}U_n(\tau,\kappa)$ for τ,κ fixed. Therefore we have

$$\left\|U(\tau,\kappa)-U_n(\tau,\kappa)\right\| = \lim_{m\to+\infty}\left\|U_m(\tau,\kappa)-U_n(\tau,\kappa)\right\| \leq \lim_{m\to+\infty}\exp(\mu\tau)\varepsilon = \exp(\mu\tau)\varepsilon,$$

then

$$\exp(-\mu\tau)\left\|U(\tau,\kappa)-U_n(\tau,\kappa)\right\| < \varepsilon,$$

so

$$\sup_{\tau\in\mathbb{R}_0^+}\exp(-\mu\tau)\left\|U(\tau,\kappa)-U_n(\tau,\kappa)\right\| \leq \varepsilon.$$

Thus, $\left\|U(\kappa)\right\|_A \leq \left\|U(\kappa)-U_n(\kappa)\right\|_A+\left\|U_n(\kappa)\right\|_A<\infty$ for big $n\in\mathbb{N}$ and U is continuous due to the continuity of U_k, then $U\in\boldsymbol{C}$, so $(\boldsymbol{C},\left\|\cdot\right\|_A)$ is a Banach space.

Now we will prove by induction that $\varphi_j\in\boldsymbol{C}$ for any $j\in\mathbb{N}\cup\{0\}$. Indeed, if $\varphi_j\in\boldsymbol{C}$, we estimate $\left\|\varphi_{j+1}(\kappa)\right\|_A$

$$\left\|\varphi_{j+1}(t,\kappa)\right\| \leq \int_0^t K\exp(-\alpha(t-\tau)+\mu\tau)(L_{\mathcal{F}}\exp(-2\mu\tau)\left\|\varphi_j(\tau,\kappa)\right\|+K_0),$$

from which it follows that if $K_j=\left\|\varphi_j(\kappa)\right\|_A$, then

$$\begin{aligned}\exp(-\mu t)\left\|\varphi_{j+1}(t,\kappa)\right\| &\leq \int_0^t K\exp(-\alpha(t-\tau))(L_{\mathcal{F}}K_j+\exp(\mu\tau)K_0)d\tau,\\ &< \frac{KK_jL_{\mathcal{F}}}{\alpha}+\frac{KK_0}{\mu}<+\infty\end{aligned}$$

and we obtain

$$\left\|\varphi_{j+1}(\kappa)\right\|_A=\sup_{t\in\mathbb{R}_0^+}\exp(-\mu t)\left\|\varphi_{j+1}(t,\kappa)\right\| \leq \frac{KK_jL_{\mathcal{F}}}{\alpha}+\frac{KK_0}{\mu}<+\infty.$$

From the above, we can consider a map $T:\boldsymbol{C}\to\boldsymbol{C}$ given by

$$T(Z(t,\kappa))=\int_0^t\Phi(t,\tau)\mathcal{F}(\tau,Z(\tau,\kappa),\kappa)d\tau,$$

which is well defined. Since we have that $KL_{\mathcal{F}}<\alpha$, we have that T is a contraction, indeed

$$\begin{aligned}\left\|T(Z_1(t,\kappa))-T(Z_2(t,\kappa))\right\| &\leq \int_0^t KL_{\mathcal{F}}\exp(-\alpha(t-\tau)+\mu\tau-2\mu\tau)\left\|Z_1(\tau,\kappa)-Z_2(\tau,\kappa)\right\|d\tau,\\ \left\|T(Z_1(\kappa))-T(Z_2(\kappa))\right\|_A &\leq \frac{KL_{\mathcal{F}}}{\alpha}\left\|Z_1(\kappa)-Z_2(\kappa)\right\|_A,\end{aligned}$$

which implies that $\{\varphi_j\}$ is the unique sequences in $\boldsymbol{C}$ satisfying the recursivity stated above.

Now we will prove that $\{\varphi_j\}$ is a Cauchy sequence in the Banach space $(\boldsymbol{C},\left\|\cdot\right\|_A)$. We proceed inductively. We observe that, firstly

$$\begin{aligned} \|\varphi_1(t,\kappa)-\varphi_0(t,\kappa)\| \leq & \int_0^t K\exp(-\alpha(t-\tau)+\mu\tau)L_{\mathcal{F}}\exp(-2\mu\tau)\,\|\varphi_0(\tau,\kappa)\|\,d\tau, \\ \leq & \; KL_{\mathcal{F}}\frac{KK_0}{\alpha}\int_0^t \exp(-\alpha(t-\tau))d\tau \leq \bar{K}\frac{KL_{\mathcal{F}}}{\alpha}, \end{aligned}$$

which implies that

$$\|\varphi_1(\kappa)-\varphi_0(\kappa)\|_A \leq \bar{K}\frac{KL_{\mathcal{F}}}{\alpha}$$

with $\bar{K}=\dfrac{KK_0}{\alpha}$.

As inductive hypothesis, we have that $\|\varphi_j(\kappa)-\varphi_{j-1}(\kappa)\|_A \leq \bar{K}\left(\dfrac{KL_{\mathcal{F}}}{\alpha}\right)^j$, and therefore

$$\begin{aligned} \|\varphi_{j+1}(t,\kappa)-\varphi_j(t,\kappa)\| \leq & \int_0^t K\exp(-\alpha(t-\tau)+\mu\tau)L_{\mathcal{F}}\exp(-2\mu\tau)\,\|\varphi_j(\tau,\kappa)-\varphi_{j-1}(\tau,\kappa)\|\,d\tau, \\ \leq & \; \bar{K}KL_F\left(\frac{KL_{\mathcal{F}}}{\alpha}\right)^j\int_0^t \exp(-\alpha(t-\tau))d\tau \leq \bar{K}\left(\frac{KL_{\mathcal{F}}}{\alpha}\right)^{j+1}, \\ \|\varphi_{j+1}(\kappa)-\varphi_j(\kappa)\|_A \leq & \; \bar{K}\left(\frac{KL_{\mathcal{F}}}{\alpha}\right). \end{aligned}$$

Finally, for all $\varepsilon>0$ there exists $N(\varepsilon)\in\mathbb{N}$ such that for any $n,m\geq N$ we have

$$\begin{aligned} \|\varphi_n(\kappa)-\varphi_m(\kappa)\|_A \leq & \; \bar{K}\left(\tfrac{KL_{\mathcal{F}}}{\alpha}\right)^{m+1}\left(1+\tfrac{KL_{\mathcal{F}}}{\alpha}+\cdots+\left(\tfrac{KL_{\mathcal{F}}}{\alpha}\right)^{n-(m-1)}\right) \\ & \leq \bar{K}\left(\frac{KL_{\mathcal{F}}}{\alpha}\right)^N\left(\frac{1-\left(\frac{KL_{\mathcal{F}}}{\alpha}\right)^{n-m}}{1}-\frac{KL_{\mathcal{F}}}{\alpha}\right), \\ & \leq \bar{K}\left(\frac{KL_F}{\alpha}\right)^N\left(\frac{1}{1-\frac{KL_F}{\alpha}}\right)<\varepsilon. \end{aligned}$$

which proves that $\{\varphi_j\}$ is a Cauchy sequence in the Banach space $\boldsymbol{C}$ convergent to the fixed point $Z(t,\kappa)$ defined by (3.26).

Considering a fixed $\kappa\in\boldsymbol{B}$ we have that $\|Z(\kappa)\|_A < C(\kappa)$. That is, $Z(\cdot,\kappa)\in\boldsymbol{C}$ but its bound $C(\kappa)$ could be dependent of κ. However, we will prove that $C(\kappa)$ has an upper bounded independent of κ. Indeed, combining the properties (ii), (iii) with the nonuniform exponential dichotomy of (3.4a), we have that

$$\begin{aligned}\|Z(t,\kappa)\| \leq\ & KL_{\mathcal{F}}\int_0^t \exp(-\alpha(t-\tau)+\mu\tau)\exp(-2\mu\tau)\,\|Z(\tau,\kappa)\|\,d\tau \\ & +KK_0\int_0^t \exp(-\alpha(t-\tau)+\mu\tau)d\tau,\end{aligned}$$

which implies that

$$\begin{aligned}\exp(-\mu t)\,\|Z(t,\kappa)\| \leq & KL_{\mathcal{F}}\int_0^t \exp(-\alpha(t-\tau))\exp(-\mu\tau)\,\|Z(\kappa)\|_A\,d\tau \\ & +KK_0\int_0^t \exp(-\alpha(t-\tau))d\tau.\end{aligned}$$

Thus,

$$\exp(-\mu t)\,\|Z(t,\kappa)\| \leq \frac{KL_{\mathcal{F}}C(\kappa)}{\alpha}+\frac{KK_0}{\alpha},$$

and taking supremum over $t\in\mathbb{R}_0^+$, we obtain

$$C(\kappa) \leq \frac{KK_0}{\alpha}\left(1-\frac{KL_{\mathcal{F}}}{\alpha}\right)^{-1}$$

□

3.6 Proof of Theorem 3.2

We will follow the lines of proof of the Lemma of the Palmer's article [35, p. 11] in order to obtain (1) and (2) of our Theorem. We point out that in the calculations of the derivative of V with respect t evaluated at the origin, we are considering only the right side derivative.

Let $x(t)=X(t,\tau,\xi)$ be the solution of (3.5a) such that $x(\tau)=\xi\neq 0$ and $y(t)=Y(t,s,\omega)$ be the solution of (3.5b) such that $y(s)=\omega\neq 0$.

As the system (3.5a) has nonuniform contraction (by Remark 3.2), we have that its evolution operator satisfies (3.13), we can use Proposition 3.2 to obtain a symmetric positive definite operator $\mathcal{S}(t)$ which define a strict Lyapunov function $V(t)$ associated to the system (3.5a). Thus, by using the construction of $V(t)$, (3.14), Remark 3.2 and the Lipschitz constant L_g of function g, we obtain that:

$$
\begin{aligned}
\frac{dV(t,y(t))}{dt} =& \left\langle \mathcal{S}'(t)y(t), y(t) \right\rangle + \langle \mathcal{S}(t)[A(t)y(t) + g(t,y(t))], y(t)\rangle \\
&+ \langle \mathcal{S}(t)y(t), A(t)y(t) + g(t,y(t))\rangle , \\
\\
=& \langle \mathcal{S}(t)y(t) + \mathcal{S}(t)A(t)y(t) + A^*(t)\mathcal{S}(t)y(t), y(t)\rangle \\
&+ \langle \mathcal{S}(t)g(t,y(t)), y(t)\rangle + \langle \mathcal{S}(t)y(t), g(t,y(t))\rangle , \\
\\
\leq& \langle -[Id + 2(\alpha_1 - \mu_1)\mathcal{S}(t)y(t), y(t)]\rangle + 2\,\langle \mathcal{S}(t)g(t,y(t)), y(t)\rangle , \\
\\
\leq& \langle -2(\alpha_1 - \mu_1)\mathcal{S}(t)y(t), y(t)\rangle + 2\,\langle \mathcal{S}(t)L_g y(t), y(t)\rangle , \\
\\
=& -2(\alpha_1 - \mu_1)V(t,y(t)) + 2L_g V(t,y(t)), \\
\\
\leq& -2[\alpha_1 - \mu_1 - L_g]V(t,y(t)),
\end{aligned}
$$

and the part (1) of our result follows.

Notice that if we consider $x(t)$, then in the previous inequality we have

$$\frac{dV(t,x(t))}{dt} \leq -2[\alpha_1 - \mu_1]V(t,x(t)) \leq -2[\alpha_1 - \mu_1 - L_g]V(t,x(t)). \tag{3.27}$$

Now we will prove that second statement our result. From Lemma 3.2 and considering $\gamma = \alpha_1 - \mu_1 - L_g > 0$, we have that

$$V(t,x(t)) \leq V(s,x(s))\exp(-\bar{\gamma}(t-s)), \quad t \geq s$$

with $\bar{\gamma} = 2\gamma$, then $V(t,x(t))$ is strictly decreasing and converges to 0 as t tends to infinity. Now given $\varepsilon > 0$, let $\ell = \ell(\varepsilon) > 0$ such that there exists a unique $T = T(\tau,\xi)$ that satisfies

$$V(T,x(T)) = \frac{\ell}{2}.$$

It is easy to see that $T(\tau,\varepsilon)$ is a continuous function of (τ,ξ) for $\xi \neq 0$. Now we define

$$H(\tau,\xi) = \begin{cases} Y(\tau, T(\tau,\xi), X(T(\tau,\xi),\tau,\xi)) & if \quad \xi \neq 0, \\ \\ 0 & if \quad \xi = 0. \end{cases} \tag{3.28}$$

Clearly, $H(\tau,\xi)$ is continuous for $\xi \neq 0$. With the purpose to discuss its continuity at $\xi = 0$, we analyze the behaviour of $|T(\tau,\xi)-\tau|$ as ξ tends to 0. By **(V1)** and Proposition 3.3 we have

$$\begin{aligned}\frac{\ell}{2} &= V(T(\tau,\xi),X(T(\tau,\xi),\tau,\xi)),\\ &\leq \mathcal{K}^2\exp(2\upsilon T(\tau,\xi))\,\|X(T(\tau,\xi),\tau,\xi)\|^2,\\ &= \mathcal{K}^2\exp(2\upsilon T(\tau,\xi))\exp(2L_F|T(\tau,\xi)-\tau|)\,\|\xi\|^2.\end{aligned}$$

Then

$$\exp(-|T(\tau,\xi)-\tau|)\leq\left(\frac{2\mathcal{K}\exp(2\upsilon T(\tau,\xi))\,\|\xi\|^2}{\ell}\right)^{\frac{1}{2L_F}},\tag{3.29}$$

where $L_F=|\bar{a}_1|+\delta K_{\delta,\varepsilon}$. Notice that the system (3.5a) has nonuniform contraction with its evolution operator with nonuniformly bounded growth (see Remark 2.1) which imply that the spectrum $\Sigma(C(t)+B(t))=\bigcup_{i=1}^{m}[\bar{a}_i,\bar{b}_i]\subset(-\infty,0)$.

Now by Lemma 3.1, there exists $\eta>0$ such that

$$\begin{aligned}\eta\,\|H(\tau,\xi)\|^2 &\leq V(\tau,H(\tau,\xi)),\\ &\leq V(\tau,Y(\tau,T,X(T,\tau,\xi))).\end{aligned}$$

However, we have

$$\|\xi\|\leq\left(\frac{\ell\exp(-2\upsilon\tau)}{2\mathcal{K}^2}\right)^{\frac{1}{2}}\Rightarrow V(\tau,\xi)\leq\mathcal{K}^2\exp(2\upsilon\tau)\,\|\xi\|^2\leq\frac{\ell}{2}\Rightarrow T(\tau,\xi)\leq\tau.$$

It follows by Lemma 3.2 and (3.29) that

$$\begin{aligned}\|H(\tau,\xi)\|^2 &\leq \eta^{-1}\exp(-\bar{\gamma}(\tau-T))V(T,Y(T,T,X(T,\tau,\xi))),\\ &= \frac{\eta^{-1}\ell\exp(-\bar{\gamma}(\tau-T))}{2},\\ &\leq \left(\frac{\eta^{-1}\ell}{2}\right)\left(\frac{2\mathcal{K}\exp(2\upsilon T(\tau,\xi))\,\|\xi\|^2}{\ell}\right)^{\frac{\bar{\gamma}}{2L_F}}.\end{aligned}$$

Hence, if $\|\xi\|\leq\left(\frac{\ell\exp(-2\upsilon\tau)}{2\mathcal{K}^2}\right)^{\frac{1}{2}}$ we obtain

$$\|H(\tau,\xi)\|\leq\left(\frac{\eta^{-1}\ell}{2}\right)^{\frac{1}{2}}\left(\frac{2\mathcal{K}\exp(2\upsilon T(\tau,\xi))\,\|\xi\|^2}{\ell}\right)^{\frac{\bar{\gamma}}{4L_F}}.$$

On the other hand, by Lemma 3.1 and Proposition 3.3 we have that

$$\begin{aligned}\frac{\ell}{2} &= V(T(\tau,\xi), X(T(\tau,\xi),\tau,\xi)),\\ &\geq \eta \left\|X(T(\tau,\xi),\tau,\xi)\right\|^2,\\ &\geq \eta \left\|\xi\right\|^2 \exp(-2L_F|T(\tau,\xi)-\tau|).\end{aligned}$$

Thus,

$$\exp(|T(\tau,\xi)-\tau|) \geq \left(\frac{2\eta\left\|\xi\right\|^2}{\ell}\right)^{\frac{1}{2L_F}}. \tag{3.30}$$

Notice that

$$\|\xi\| \geq \left(\frac{\ell}{2\eta}\right)^{\frac{1}{2}} \Rightarrow \frac{\ell}{2} \leq \eta\left\|\xi\right\|^2 \leq V(\tau,\xi) \Rightarrow T(\tau,\xi) \geq \tau.$$

Then if $\|\xi\| \geq \left(\frac{\ell}{2\eta}\right)^{\frac{1}{2}}$, by (3.30) and **(V3)**, we have

$$\begin{aligned}\|H(\tau,\xi)\|^2 &\geq \frac{\exp(-2\upsilon\tau)}{\mathcal{K}^2} V(\tau, H(\tau,\xi)),\\ &= \frac{\exp(-2\upsilon\tau)}{\mathcal{K}^2} V(\tau, Y(\tau, T, X(T,\tau,\xi))),\\ &\geq \frac{\exp(-2\upsilon\tau + \bar{\gamma}(T-\tau))}{\mathcal{K}^2} V(T, Y(T,T,X(T,\tau,\xi))),\\ &= \frac{\ell\exp(-2\upsilon\tau)}{2\mathcal{K}^2}\exp(\bar{\gamma}(T-\tau)),\\ &\geq \frac{\ell\exp(-2\upsilon\tau)}{2\mathcal{K}^2}\left(\frac{2\eta\left\|\xi\right\|^2}{\ell}\right)^{\frac{\bar{\gamma}}{2L_F}}\end{aligned}$$

Therefore

$$\|H(\tau,\xi)\| \geq \left(\frac{\ell\exp(-2\upsilon\tau)}{2\mathcal{K}^2}\right)^{\frac{1}{2}}\left(\frac{2\eta\left\|\xi\right\|^2}{\ell}\right)^{\frac{\bar{\gamma}}{4L_F}}.$$

Now we proof that if $x(t)$ is a solution of (3.5a), $H(t,x(t))$ is a solution of (3.5b).

If $\xi = 0$,

$$H(t, X(t,\tau,\xi)) = H(t,0) = 0.$$

In the case when $\xi \neq 0$, we have that

$$\begin{aligned}H(t,X(t,\tau,\xi)) =& Y(t, T(t,X(t,\tau,\xi)), X(T(t,X(t,\tau,\xi)),t,X(t,\tau,\xi))),\\ =& Y(t,T(t,X(t,\tau,\xi)), X(T(t,X(t,\tau,\xi)),\tau,\xi)).\end{aligned}$$

On the one hand we have

$$\frac{\ell}{2} = H(T(\tau,\xi), X(T(\tau,\xi),\tau,\xi)) = H(T(\tau,\xi), X(T(\tau,\xi),t,X(t,\tau,\xi))), \tag{3.31}$$

and on the other hand

$$\frac{\ell}{2} = H(T(t, X(t, \tau, \xi)), X(T(t, X(t, \tau, \xi)), t, X(t, \tau, \xi))), \tag{3.32}$$

and by the equations (3.31) and (3.32), we deduce that $T(t, X(t, \tau, \xi)) = T(\tau, \xi)$. Hence, for all $t, \tau \geq 0$, and $\xi \neq 0$,

$$H(t, X(t, \tau, \xi)) = Y(t, T(\tau, \xi), X(T(\tau, \xi), \tau, \xi)), \tag{3.33}$$

which is a solution of (3.5b).

Similarly, we define a mapping

$$G(\tau, \xi) = \begin{cases} X(\tau, S(\tau, \xi), Y(S(\tau, \xi), \tau, \xi)) & if \quad \xi \neq 0, \\ \\ 0 & if \quad \xi = 0, \end{cases} \tag{3.34}$$

where $S = S(\tau, \xi)$ is the unique time s such that

$$V(S, y(S)) = \frac{\ell}{2}.$$

We can deduce similar properties to those of the function H for G and, moreover we have

$$G(t, Y(t, \tau, \xi)) = X(t, S(\tau, \xi), Y(S(\tau, \xi), \tau, \xi)),\ \xi \neq 0,$$

which is obtained in a similar way to (3.33).

To prove that $H(\tau, G(\tau, \xi)) = \xi$, if $S = S(\tau, y)$ we note that $\frac{\ell}{2}$ can be written as

$$\frac{\ell}{2} = V(T(S, Y(S, \tau, y)), X(T(S, Y(S, \tau, y)), S, Y(S, \tau, y))), \tag{3.35}$$

and as

$$\frac{\ell}{2} = V(S, Y(S, \tau, y)) = V(S, X(S, S, Y(S, \tau, y))). \tag{3.36}$$

From the equations (3.35) and (3.36) we can assure that

$$T(S(\tau, y), Y(S(\tau, y), \tau, y)) = S(\tau, y). \tag{3.37}$$

Therefore we have

$$\begin{aligned} H(\tau, G(\tau, \xi)) =& H(\tau, X(\tau, S(\tau, \xi), Y(S(\tau, \xi), \tau, \xi))), \\ =& Y(\tau, T(S, Y(S, \tau, \xi)), X(T(S, Y(S, \tau, \xi)), S, Y(S, \tau, \xi))), \end{aligned}$$

and from (3.37), then we obtain

$$\begin{aligned} H(\tau, G(\tau, \xi)) =& Y(\tau, S, X(S, S, Y(S, \tau, \xi))), \\ =& Y(\tau, S, Y(S, \tau, \xi)) = \xi. \end{aligned}$$

In a similar way, we can obtain that

$$G(\tau, H(\tau, \xi)) = \xi,$$

for all $\tau \in \mathbb{R}_0^+$, $\xi \in \mathbb{R}^n$.

3.7 Proof Theorem 3.3

This result is a consequence of the Theorem 3.2. Indeed, we have that (3.4a) and (3.5a) are topologically equivalent through of the matrix $S(\delta, t)$. Then the systems (3.5a) and (3.11) are topologically equivalent through of the matrix $S(\delta, t)$ also. If we denote $g(t, y) = S^{-1}(\delta, t) f(t, S(\delta, t)y)$, then $g \in \mathcal{A}_2$ with $L_g = M_1^2 L_f$. In fact,

$$\begin{aligned}
\|g(t, y_1) - g(t, y_2)\| &= \left\|S^{-1}(\delta, t) f(t, S(\delta, t)y_1) - S^{-1}(\delta, t) f(t, S(\delta, t)y_2)\right\|, \\
&\leq M_1 \exp(\beta t) \left\|f(t, S(\delta, t)y_1) - f(t, S(\delta, t)y_2)\right\|, \\
&\leq M_1 L_f \exp(\beta t) \exp(-2\beta t) \left\|S(\delta, t)y_1 - S(\delta, t)y_2\right\|, \\
&\leq M_1^2 L_f \left\|y_1 - y_2\right\|.
\end{aligned}$$

Since $L_g \leq \delta < \alpha - \mu$, by combining Theorem 3.2 and the fact that topological equivalence is a equivalence relation, the systems (3.4a) and (3.4b) are topologically equivalent.

3.8 Proof of Theorem 3.4.

We take the function $f_0(t, x) = f(t, x) - f(t, 0)$, then $f \in \mathcal{A}_1$ implies $f_0 \in \mathcal{A}_2$. Indeed, $f_0(t, 0) = 0$ and

$$\|f_0(t, x_1) - f_0(t, x_2)\| = \|f(t, x_1) - f(t, x_2)\| \leq L_f \exp(-2\beta t) \|x_1 - x_2\|),$$

for any $t \in \mathbb{R}_0^+, x_1, x_2 \in \mathbb{R}^n$ and some $\beta \geq 0$. As f and f_0 have the same Lipschitz constant, by Theorem 3.3 and inequality (3.23) it is sufficient to prove that the systems (3.4b) and

$$\dot{x} = A(t)x + f_0(t, x) \tag{3.38}$$

are topologically equivalent. By the condition **(P3)** there exist constants $K \geq 1$, $\alpha > 0$ and $\mu \geq 0$ satisfying (3.6). For the unique solution $X(t, \tau, \xi)$ of (3.38) passing through ξ at $t = \tau$, we define the function $F : \mathbb{R}_0^+ \times \mathbb{R}^n \times \boldsymbol{B} \to \mathbb{R}^n$, with $\boldsymbol{B} = \mathbb{R}_0^+ \times \mathbb{R}^n$, as

$$\begin{aligned}
F(t, y, (\tau, \xi)) &= f(t, y + X(t, \tau, \xi)) - f_0(t, X(t, \tau, \xi)), \\
&= f(t, y + X(t, \tau, \xi)) - f(t, X(t, \tau, \xi)) + f(t, 0).
\end{aligned}$$

If $K_0 = \sup\left\{t \in \mathbb{R}_0^+ : \; \|f(t,0)\|\right\}$ then

$$\begin{cases} \|F(t,y,(\tau,\xi))\| \leq L_f \exp(-2\beta t)\,\|y\| + K_0, \\ \\ \|F(t,y_1,(\tau,\xi)) - F(t,y_2,(\tau,\xi))\| \leq L_f \exp(-2\beta t)\,\|y_1 - y_2\|. \end{cases}$$

We note that F verifies the hypothesis of Proposition 3.4, which implies that the system

$$\dot{z} = A(t)z + F(t,z,(\tau,\xi)) \tag{3.39}$$

has a unique bounded and continues solution $Z(t,(\tau,\xi))$ defined by

$$Z(t,(\tau,\xi)) = \int_0^t \Phi(t,s)[f(s,Z(s,(\tau,\xi)) + X(s,\tau,\xi)) - f_0(s,X(s,\tau,\xi))]ds$$

with the norm

$$\|Z\| = \sup_{t\in\mathbb{R}_0^+,(\tau,\xi)\in\mathbb{R}_0^+\times\mathbb{R}^n} \exp(-\beta t)\,\|Z(t,(\tau,\xi))\| = M_0 < +\infty.$$

Now, let us construct the map $H : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$H(\tau,\xi) = \xi + Z(\tau,(\tau,\xi)). \tag{3.40}$$

Lemma 3.3. *For any $(r,t) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and $(\tau,\xi) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ we have that*

$$Z(r,(t,X(t,\tau,\xi))) = Z(r,(\tau,\xi)). \tag{3.41}$$

Proof. Firstly, we note that

$$\begin{aligned} Z(r,(t,X(t,\tau,\xi)) &= \int_0^r \Phi(r,s)[f(s,Z(s,(t,X(t,\tau,\xi))) + X(s,t,X(t,\tau,\xi))) \\ &\quad - f_0(s,X(s,t,X(t,\tau,\xi)))]ds, \\ &= \int_0^r \Phi(r,s)[f(s,Z(s,(t,X(t,\tau,\xi))) + X(s,\tau,\xi))] \\ &\quad - f_0(s,X(s,\tau,\xi))]ds \end{aligned}$$

and

$$Z(r,(\tau,\xi)) = \int_0^r \Phi(r,s)[f(s,Z(s,(\tau,\xi)) + X(s,\tau,\xi)) - f_0(s,X(t,\tau,\xi))]ds.$$

Secondly, we have the following estimate

$$\|Z(r,(t,X(t,\tau,\xi)))-Z(r,(\tau,\xi))\| =$$

$$\left\|\int_0^r \Phi(r,s)[f(s,Z(s,(t,X(t,\tau,\xi)))+X(s,\tau,\xi))-f(s,Z(s,(\tau,\xi))+X(s,\tau,\xi))]ds\right\|$$

$$\leq \int_0^r KL_f\exp(-\alpha(r-s)+\mu s)\exp(-2\mu s)\left\|Z(s,(t,X(t,\tau,\xi)))-Z(s,(\tau,\xi))\right\|ds,$$

$$\leq \frac{KL_f}{\alpha}\sup_{r\in\mathbb{R}_0^+}\exp(-\mu r)\left\|Z(r,(t,X(t,\tau,\xi)))-Z(r,(\tau,\xi))\right\|$$

and the Lemma follows. □

Lemma 3.4. *If $t\mapsto X(t,\tau,\xi)$ is solution of* (3.38) *such that $X(\tau,\tau,\xi)=\xi$, then $t\mapsto H(t,X(t,\tau,\xi))$ is solution of* (3.4b).

Proof. Combining the equations (3.40) and (3.41), we have that

$$H(t,X(t,\tau,\xi)) = X(t,\tau,\xi)+Z(t,(\tau,\xi)),$$

and a simple computation allows us to verify the statement. □

Lemma 3.5. *The map $\xi\mapsto H(\tau,\xi)$ is continuous for any fixed $\tau\in\mathbb{R}_0^+$.*

Proof. By (3.40), the only thing that we should prove is that the map $\xi\mapsto Z(\tau,(\tau_0,\xi))$ is continuous for any fixed τ. Indeed, let us recall $\tau\mapsto Z(\tau,(\tau,\xi))$ is the unique bounded solution in $\boldsymbol{C}$ of (3.39), which was constructed by successive approximations in Proposition 3.4. That is

$$\lim_{j\to+\infty} Z_j(\tau,(\tau_0,\xi)) = Z(\tau,(\tau_0,\xi)),$$

where

$$Z_{j+1}(\tau,(\tau_0,\xi)) = \int_0^\tau \Phi(\tau,s)F(s,Z_j(s,(\tau_0,\xi)),(\tau_0,\xi))ds.$$

Moreover we know that for any $\varepsilon>0$, there exists $J=J(\varepsilon)>0$ such that for any $j>J$ it follows that

$$\begin{aligned}\left\|Z(\tau,(\tau_0,\xi))-Z(\tau,(\tau_0,\xi'))\right\| &\leq \left\|Z(\tau,(\tau_0,\xi))-Z_j(\tau,(\tau_0,\xi))\right\| \\ &\quad+\left\|Z_j(\tau,(\tau_0,\xi))-Z_j(\tau,(\tau_0,\xi'))\right\| \\ &\quad+\left\|Z_j(\tau,(\tau_0,\xi'))-Z(\tau,(\tau_0,\xi'))\right\| \\ &< \frac{2}{3}\varepsilon+\left\|Z_j(\tau,(\tau_0,\xi))-Z_j(\tau,(\tau_0,\xi'))\right\|.\end{aligned}$$

We will prove by induction that for any $j \in \mathbb{N}$, there exists $\delta_j > 0$ such that

$$\left\| Z_j(\tau,(\tau_0,\xi)) - Z_j(\tau,(\tau_0,\xi')) \right\| < \frac{\varepsilon}{3} \quad \text{if} \quad \left\| \xi - \xi' \right\| < \delta_j. \tag{3.42}$$

Indeed, we cosider an initial term

$$Z_0(\tau,(\tau,\xi)) = Z_0(\tau,(\tau,\xi')) = \phi_0 \in \boldsymbol{C}$$

and suppose that (3.42) is verified for some j as inductive hypothesis. Now, we have that

$$\left\| Z_{j+1}(\tau,(\tau_0,\xi)) - Z_{j+1}(\tau,(\tau_0,\xi')) \right\| \leq \Delta$$

where

$$\Delta = \left\| \int_0^\tau \Phi(\tau,s)[F(s,Z_j(s,(\tau_0,\xi)),(\tau_0,\xi)) - F(s,Z_j(s,(\tau_0,\xi')),(\tau_0,\xi'))]ds \right\|$$

From the definition and properties of F, by Gronwall's Lemma and inductive hypothesis, we have that

$$\begin{aligned}
\Delta \leq & \left\| \int_0^\tau \Phi(\tau,s)[f(s,Z_j(s,(\tau_0,\xi)) + X(s,\tau_0,\xi)) - f(s,Z_j(s,(\tau_0,\xi')) + X(s,\tau_0,\xi'))]ds \right\| \\
& + \left\| \int_0^\tau \Phi(\tau,s)[f_0(s,X(s,\tau_0,\xi)) - f_0(s,X(s,\tau_0,\xi'))]ds \right\|, \\
& \leq \int_0^\tau KL_f \exp(-\alpha(\tau-s) - \mu s)[\left\| Z_j(s,(\tau_0,\xi)) - Z_j(s,(\tau_0,\xi')) \right\| + \left\| X(s,\tau_0,\xi) - X(s,\tau_0,\xi') \right\|]ds \\
& + \int_0^\tau KL_{f_0} \exp(-\alpha(\tau-s) - \mu s)[\left\| X(s,\tau_0,\xi) - X(s,\tau_0,\xi') \right\|]ds, \\
& \leq \frac{\varepsilon}{3} KL_f \int_0^\tau \exp(-\alpha(\tau-s) - \mu s + \mu s)ds \\
& + K(2L_f) \int_0^\tau \exp(-\alpha(\tau-s)) \left\| \xi - \xi' \right\| \exp(L_F(\tau-s))ds, \\
& \leq \frac{\varepsilon}{3} \frac{KL_f}{\alpha} + \frac{K(2L_f)}{\alpha} \exp(L_f\tau) \left\| \xi - \xi' \right\|
\end{aligned}$$

and (3.42) is satisfied for $j+1$ when we choose

$$\delta_{j+1} = \min \left\{ \delta_j, \left(1 - \frac{KL_f}{\alpha}\right) \exp(-L_f\tau) \frac{\alpha}{K(2L_f)} \frac{\varepsilon}{3} \right\}$$

and we can prove the continuity of $\xi \mapsto Z(\tau,(\tau_0,\xi))$. All of the above allows us to conclude that H is continuous for any fixed τ.

□

Remark 3.6. *We note that if $Y(t,\tau,\xi)$ is the unique solution of (3.4b) passing through ξ at $t=\tau$, we can define the function $\tilde{F}:\mathbb{R}_0^+\times\mathbb{R}^n\times\boldsymbol{B}\to\mathbb{R}^n$ as*

$$\begin{aligned}\tilde{F}(t,\tilde{y},(\tau,\xi)) &= f_0(t,\tilde{y}+Y(t,\tau,\xi))-f(t,Y(t,\tau,\xi)),\\ &= f(t,\tilde{y}+Y(t,\tau,\xi))-f(t,0)-f(t,Y(t,\tau,\xi)).\end{aligned}$$

and obtain

$$\begin{cases}\left\|\tilde{F}(t,\tilde{y},(\tau,\xi))\right\| \leq L_f\exp(-2\beta t)\left\|\tilde{y}\right\|+K_0,\\[2ex] \left\|\tilde{F}(t,\tilde{y}_1,(\tau,\xi))-\tilde{F}(t,\tilde{y}_2,(\tau,\xi))\right\| \leq L_f\exp(-2\beta t)\left\|\tilde{y}_1-\tilde{y}_2\right\|.\end{cases}$$

In the same way $\tilde{F}$ satisfies the hypothesis of Proposition 3.4, which implies that the system

$$\dot{z}=A(t)z+\tilde{F}(t,z,(\tau,\xi))$$

has a unique bounded solution $\tilde{Z}(t,(\tau,\xi))$ defined by

$$\tilde{Z}(t,(\tau,\xi))=\int_0^t\Phi(t,s)[f_0(s,\tilde{Z}(s,(\tau,\xi))+Y(s,\tau,\xi))-f(s,Y(s,\tau,\xi))]ds.$$

As a consequence of the previous remark, we can construct the map $G:\mathbb{R}_0^+\times\mathbb{R}^n\to\mathbb{R}^n$ as

$$G(\tau,\xi)=\xi+\tilde{Z}(\tau,(\tau,\xi)).$$

and we prove the following results that are similar to the previous one.

Lemma 3.6. *For any $(r,t)\in\mathbb{R}_0^+\times\mathbb{R}_0^+$ and $(\tau,\xi)\in\mathbb{R}_0^+\times\mathbb{R}^n$ we have that*

$$\tilde{Z}(r,(t,Y(t,\tau,\xi)))=\tilde{Z}(r,(\tau,\xi)).$$

Lemma 3.7. *If $t\mapsto Y(t,\tau,\xi)$ is solution of (3.4a) such that $Y(\tau,\tau,\xi)=\xi$, then $t\mapsto G(t,Y(t,\tau,\xi))$ is solution of (3.38).*

Lemma 3.8. *The map $\xi\mapsto G(\tau,\xi)$ is continuous for any fixed $\tau\in\mathbb{R}_0^+$.*

Finally, from all these Lemmas, we can conclude that the systems (3.4b) and (3.38) are topologically equivalent, which is enough to prove the result. □

It is important to mention that although a homeomorphism was constructed between (3.4a) and (3.4b), one of the objectives of this thesis was to achieve some kind of regularity for this homeomorphism. In that line, in the article [20] is used the spectral theory to prove that the linearization is simultaneously differentiable in the origin and Hölder continuous in an neighborhood of the origin by using a different approach to this thesis .

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