Universidad de Santiago de Chile

Facultad de Ciencia

Doctorado en Ciencia mención en Matemática

New challenges in Volterra equations and the discrete fractional Laplacian

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Trabajo para optar al grado de Doctor en Ciencia mención en Matemática

Santiago - Agosto de 2020

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Trabajo de Titulación presentado a la Facultad de Ciencia, en cumplimiento parcial de los requisitos exigidos para optar al grado de **Doctor en Ciencia mención en Matemática**. Este trabajo de graduación fue presentado bajo la supervisión del profesor guía Dr. Carlos Lizama Yáñez, y los profesores supervisores del trabajo de tesis Dr. Valentin Keyantuo de la Universidad de Puerto Rico, Dr. Rodrigo Ponce de la Universidad de Talca, Dr. Mahamadi Warma de la George Mason University, y Dr. Sebastián Zamorano de la Universidad de Santiago de Chile.

Dr. Carlos Lizama	Dr. Mahamadi Warma
Dr. Rodrigo Ponce	Dr. Valentin Keyantuo
Dr. Sebastián Zamorano	Director: Rafael Labarca

Dedicado a

mi familia

Acknowledgment

The present thesis was supported by CONICYT under grant Doctorado Nacional Chile 2016 NO:21160120.

Contents

Chapter 1

Introduction

Evolution equations of fractional order in time have become an active area of study due to the fact that many phenomena have been found to be modeled using such equations. The idea of fractional derivative is not new, but the realization that they provide better models for a large array of problems in science and engineering is rather recent. Phenomena with memory effects, anomalous diffusion, polymer science, rheology, material science are but some of the areas where modeling with fractional derivatives has proven successful in uncovering properties that were not detectable with the usual differential equations. The references [26, 27, 40, 43, 47, 48, 50, 51, 52, 59, 61, 71, 73, 74], cover several of these phenomena and demonstrate the importance of the fractional model.

The basic functions relevant to these studies are the Mittag-Leffler function, the Wright functions and their generalizations. Fourier and Laplace transforms constitute the fundamental mathematical methods for the treatment of these equations, combined with some facts from operator theory such as semigroups, cosine functions and more specifically the so-called resolvent families. The thesis [12] is an early reference on the topic with a functional analytic perspective. Several monographs have appeared on fractional calculus and its applications, e.g. [48, 50, 55, 61].

In this thesis we study continuous and discrete fractional models. Topics such as existence, uniqueness, and other properties for some classes of abstract evolution equations on diverse Banach spaces are the main focus of this work.

Firstly, we observe that fractional derivatives in continuous sense of order $\alpha > 0$ are defined through the finite convolution between a given function and the special kernel

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0.$$

We can ask if it is possible to study fractional differential equations involving a more general kernel $b(t)$. This observation has been taken by some authors, in order to relax the properties of the memory kernel g_α, which is the main responsible of the behavior and qualitative properties of fractional models. This leads to the analysis of equations in the form

$$\partial_t(b \star [x - x_0])(t) + Ax(t) = f(t, x(t)), \tag{1.0.1}$$

with initial datum $x(0) = x_0$, where A denotes a closed linear operator defined on a Banach space X and $\star$ denotes the finite convolution on $\mathbb{R}^+$. In this way, the non-local time term on the left hand side includes such classical cases as the Riemann-Liouville and the Caputo fractional derivatives, respectively, for convenient choices of b. We remark that, in general, there is a closed relationship between fractional equations and Volterra integral equations. For instance, Kemppainen et. al. [33] and Vergara and Zacher [64] studied decay estimates for non-local in time subdiffusion equations in the form (1.0.1) when $A = -\Delta$, the Laplace operator on $\mathbb{R}^N$ by means of this equivalence with Volterra equations. This is due to the fact that, as observed in the references [33] and [64] the kernel b satisfies the following remarkable property:

($\mathcal{PC}$) $b \in L^1_{loc}(\mathbb{R}^+)$ is non-negative, and non-increasing and there exists a kernel $a \in L^1_{loc}(\mathbb{R}^+)$ such that $a \star b \equiv 1$ in $(0, \infty)$.

In particular, if b satisfies condition ($\mathcal{PC}$) then a is completely positive cf. [17, Theorem 2.2] or [58, Remarks on p.326]. For example, for $b(t) = g_{1-\alpha}(t)$ we have $a(t) = g_\alpha(t)$ where $0 < \alpha < 1$. Another very interesting example is given by

$$b(t) = \int_0^1 g_\beta(t)d\beta \qquad \text{and} \qquad a(t) = \int_0^\infty \frac{e^{-st}}{1+s}ds. \tag{1.0.2}$$

In this case the operator $\partial_t(b \star \cdot)$ is a so-called operator of distributed order, see e.g. [39]. More examples are discussed in [64, Section 6]. In this thesis, we study the following model

$$\partial_t(b \star [x - h(\cdot, x(\cdot))])(t) + A(x(t) - h(t, x(t))) = f(t, x(t)), \quad t \geq 0. \tag{1.0.3}$$

Our initial observation is that under the hypothesis of the condition $(\mathcal{PC})$, the equation (1.0.3) is equivalent to the following class of abstract integral Volterra equations

$$\begin{cases} x(t) - h(t, x(t)) & + \displaystyle\int_0^t a(t-s)A[x(s) - h(s, x(s))]ds \\ & = \displaystyle\int_0^t a(t-s)f(s, x(s))ds + x_0 - h(0, x_0), \\ x(0) = x_0. & \end{cases} \tag{1.0.4}$$

In the paper [71] by Zhang and Liu, the existence of mild solutions in a particular case of (1.0.4) was proved, based on the Hausdorff measure of non-compactness. Alvarez and Lizama [4] is another reference on the application of measure of non-compactness to the existence of mild solutions for evolution equations.

On the other hand, the existence of globally attractive solutions for mathematical models is a very challenging topic that is drawing the attention of many researchers in the last decade. For instance, Alzabut and Abdeljawad in [5] studied the existence of a globally attractive periodic solutions of an impulsive delay logarithmic population model. Bartuccelli, Deane and Gentile [9] analyzed globally and locally attractive solutions for quasi-periodically forced systems, and Li and Cheng [42] established conditions for the existence of globally attractive periodic solutions of a perturbed functional differential equation. In general, the attracting character of the solutions can be deduced by different methods. For instance, using a result due to Tang [62], or via the measure of non-compactness due to Banás [8]. In this thesis, we will take this last approach.

Motivated by the initial observation, we ask ourselves: Under which conditions on a general kernel $a(t)$ there exists globally attractive mild solutions for the abstract model (1.0.4)?

One of our purposes in this thesis is to provide an answer to this question. Roughly speaking, we find the following class of kernels:

($\mathcal{K}$) $a \in C((0,\infty)) \cap L^1_{loc}(\mathbb{R}^+)$ is completely monotonic.

We notice that both kernels $a(t) = g_q(t)$, $0 < q < 1$ and $a(t)$ given in (1.0.2) satisfy the property ($\mathcal{K}$).

Parallel to the above description of continuous cases, discrete models have been attracting attention due to their applications to modeling on networks and to their use in approximation theory. A seminal paper is [10], which study several concrete difference operators. More recent references are [1, 14, 21, 36, 46].

In the second part of this thesis, we first study the fundamental solution of a semidiscrete model: continuous in time ($\mathbb{R}^+$) and discrete in space ($\mathbb{Z}$). We first consider the following model

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = -(-\Delta_d)^{\alpha} u(n,t) + g(n,t), \quad t > 0,\, n \in \mathbb{Z}, \\ u(n,0) = \varphi(n),\, u_t(n,0) = \psi(n). \end{cases} \tag{1.0.5}$$

Here, $0 < \alpha < 1$ and $-(-\Delta_d)^{\alpha}$ is the discrete fractional Laplacian. We assume that $1 < \beta \leq 2$ (in the case $0 < \beta \leq 1$ we have to assume only one initial condition). This case was previously studied in [34] from a slightly different perspective. We recall that Δ_d is the discrete Laplacian:

$$\Delta_d f(n) = f(n+1) - 2f(n) + f(n-1), \quad n \in \mathbb{Z}, \quad f \in l^p(\mathbb{Z}),\ 1 \leq p \leq \infty,$$

whereas the discrete fractional Laplacian is defined by:

$$(-\Delta_d)^{\alpha} f(n) = \sum_{k \in \mathbb{Z}} K^{\alpha}(n-k) f(k),\, n \in \mathbb{Z},\, f \in l^p(\mathbb{Z}),\ 1 \leq p \leq \infty,$$

where the coefficients K^{α} are given by:

$$K^{\alpha}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (4\sin^2(\theta/2))^{\alpha} e^{-in\theta} d\theta = \frac{(-1)^n \Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)},\, n \in \mathbb{Z}.$$

Interest in this particular semi discrete model comes partially from an equation studied by Mainardi, Luchko and Pagnini [49]. We observe that, recently, several authors have studied equations involving the discrete fractional Laplacian, for example, we cite [14, 15, 31, 32].

We remark that the model (1.0.5) includes many well-known semi-discrete equations as particular case:

- For $\alpha = 1, \beta = 2$ we get the *semi-discrete wave equation* (see [46]).
- For $\alpha = \beta = 1$ we get the *semi-discrete heat equation* (see [14]).
- For $\alpha = \frac{1}{2}, \beta = 1$ we get the *semi-discrete Poisson equation* (see [14]).
- For $\alpha = 1, \beta = \frac{1}{3}$ we get the *semi-discrete Airy equation* (see [63]).

Other references on the study of fundamental solutions of this models are [15] and [46] . Particularly, in [46], the authors combine operator theory techniques with the properties of the Bessel functions to develop a theory of analytic semigroups and cosine operators generated by Δ_d and $-(-\Delta_d)^\alpha$.

On the other hand, we observe that (1.0.5) is a particular case of the more general model

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -A^\alpha u(n,t) + f(n,t) & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases} \tag{1.0.6}$$

where A is a bounded linear operator. In the particular case that $A = \nabla$ is the *backward Euler operator* or $-A = \Delta$ is the *forward Euler operator*, this model was studied in [1], where maximum and comparison principles in the context of harmonic analysis are proved.

Our purpose in the last part of this thesis is to investigate in an unified way the fundamental solutions of the fractional semi discrete models (1.0.5) and (1.0.6).

In order to provide simultaneously in our analysis the sub diffusive and super difussive cases associated to this models, in this thesis we include the representation of the fundamental solutions for the following semi-discrete equations:

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases}$$

in case $0 < \beta \le 1$ and

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n),\ \ u_t(n,0) = \phi(n), & n \in \mathbb{Z}, \end{cases}$$

in case $1 < \beta \leq 2$. In both cases, B will be the convolution operator $Bf(n) := (b * f)(n)$ defined on $\ell^p(\mathbb{Z})$, $p \in [1, \infty]$, $b \in \ell^1(\mathbb{Z})$ and $\beta \in (0, 2]$ is a real number.

Our key observation is that discrete fractional operators can be obtained by allowing fractional powers of b as element of the Banach algebra $\ell^1(\mathbb{Z})$. This original approach, that we provide in this thesis, allow us to obtain new insights by introducing a completely new method to analyze both qualitative behavior and fundamental solutions of the above described models in an unified way.

At the end, we present relevant semi-discrete models as particular case of the general theory, for instance, the discrete Nagumo equation, r-transport equations and a new and interesting second order discrete equation (involving a new discrete operator, namely Δ_{dd}) that we have named the De Juhasz equation, which appears in the Bateman's [10] in connection with surges in springs and connected systems of springs.

This thesis is organized in five chapters. In the second chapter, we give the concepts and main results that are used in the development of this work. In the third chapter, we prove the existence of globally attractive mild solution for (1.0.3), where $t \in \mathbb{R}^+$. In the fourth chapter, we give the explicit solution for the discrete model equation involving the discrete fractional Laplacian and we prove that, for each $t \in \mathbb{R}^+$, the fundamental solution lies in $\ell^\infty(\mathbb{Z})$. Also, we consider a perturbation on the discrete Laplacian, and we get similar results. In the fifth chapter, we study the fundamental solution for the general linear model involving a convolution operator with symbol in $\ell^1(\mathbb{Z})$, via Banach algebras. Finally, we apply this result for several difference operators and their fractional powers.

Chapter 2

Preliminaries

In this chapter we present a summary of definitions and main results of the literature that will be used throughout this thesis. Mainly, we give the basic concepts related to semigroup, cosine and resolvent families of operators, almost sectorial operators, special functions and subordination principle on Wright function. Also we give information about of fractional derivative in Caputo sense, measure of non-compactness, as well as other concepts that will be useful in this thesis.

2.1 Semigroup and cosine families of operators

The notions of semigroups and cosine operators are a powerful tool in order to study the abstract Cauchy problem of order 1 and 2, respectively. In what follows, we denote $\mathcal{B}(X)$ as the set of all bounded linear operators from X into X.

Definition 2.1.1. Let X be a Banach space. A one parameter family $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is a *semigroup of bounded linear operators* on X if

1. $T(0) = I$, the identity operator on X.

2. $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$. This is called the semigroup property or functional equation for a semigroup.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists}\} \tag{2.1.1}$$

and

$$Ax = \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}, \ x \in D(A) \tag{2.1.2}$$

is the *infinitesimal generator* of the semigroup $T(t)$, $D(A)$ is the domain of A. The semigroup of a linear operator $\{T(t)\}_{t\geq 0}$ is *uniformly continuous* if

$$\lim_{t\to 0^+} ||T(t) - I|| = 0. \tag{2.1.3}$$

In this case, we have an explicit representation for this semigroup, given by the expression

$$T(t)x = \sum_{j=0}^{\infty} \frac{(At)^j}{j!}, \quad x \in X.$$

A semigroup of linear operators $\{T(t)\}_{t\geq 0}$ is *strongly continuous* or C_0-*semigroup* if

$$\lim_{t\to 0^+} ||T(t)x - x|| = 0, \quad \forall x \in X. \tag{2.1.4}$$

The semigroup theory arises in the study of the solution for the *abstract Cauchy problem of first order*

$$\begin{cases} \frac{d}{dt}u(t) = Au(t), & t \in \mathbb{R}^+, \\ u(0) = x, & x \in D(A), \end{cases} \tag{2.1.5}$$

where $u : \mathbb{R}^+ \to X$.

If A is the generator of a strongly continuous semigroup, then the function defined by $u(t) := T(t)x$ is the unique solution for the abstract Cauchy problem (2.1.5). For more information, see [7, 18, 53].

Definition 2.1.2. Let X be a Banach space. A one parameter family $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{B}(X)$ is a *Cosine family of bounded linear operators* if

i) $C(0) = I$.

ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0^+} 2\frac{C(t)x - x}{t^2} \text{ exists}\} \tag{2.1.6}$$

and

$$Ax = 2\frac{C(t)x - x}{t^2}, \; x \in D(A) \tag{2.1.7}$$

is the *infinitesimal generator* of the Cosine family $\{C(t)\}_{t\in\mathbb{R}}$ and $D(A)$ is the domain of A.

When the operator A is bounded on X, we have a explicit representation for the cosine operator given by the expression

$$C(t)x = \sum_{j=0}^{\infty} \frac{(-1)^j (At)^{2j}}{(2j)!} x, \quad x \in X.$$

A Cosine family of operators $\{C(t)\}_{t\geq 0}$ is *strongly continuous* if for each $x \in X$, the application $t \to C(t)x$ is continuous.

We consider the second order differential equation

$$\begin{cases} \frac{d^2}{dt^2}u(t) = Au(t), & t \in \mathbb{R}, \\ u(0) = x, & x \in D(A), \\ u'(0) = 0, & \end{cases} \tag{2.1.8}$$

where $u : \mathbb{R} \to X$. If A is the generator of a strongly continuous cosine family, then the function defined by $u(t) := C(t)x$ is the unique solution for the abstract Cauchy problem (2.1.5). For more information, see [7, 19].

2.2 Resolvent and integral resolvent families of operators.

In this section, we give the definition of *resolvent* and *integral resolvent families of operators*, which are an important tool for the formulation of mild solution for integral equations.

Definition 2.2.1. Let B be a closed linear operator with domain $D(B) \subset X$ and $a \in L^1_{loc}(\mathbb{R}^+)$. A family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called a *resolvent* with generator B if the following conditions are satisfied:

(1) $S(t)$ is strongly continuous on $\mathbb{R}^+$ and $S(0) = I$;

(2) $S(t)$ commutes with B, which means that $S(t)D(B) \subset D(B)$ and $BS(t)x = S(t)Bx$, for all $x \in D(B)$ and $t \geq 0$;

(3) The resolvent equation holds:

$$S(t)x = x + \int_0^t a(t-s)BS(s)xds, \quad \text{for all } x \in D(B),\ t \geq 0. \tag{2.2.1}$$

Definition 2.2.2. Let B be a closed linear operator with domain $D(B) \subset X$ and $a \in C(\mathbb{R}^+)$. A strongly continuous family $\{P(t)\}_{t\geq 0}$ contained in $\mathcal{B}(X)$ is called an *integral resolvent* with generator B if the following conditions are satisfied:

(1) $P(0) = a(0)I$;

(2) $P(t)$ commutes with B;

(3) The integral resolvent equation holds:

$$P(t)x = a(t)x + \int_0^t a(t-s)BP(s)xds \quad \text{for all } x \in D(B),\ t \geq 0. \tag{2.2.2}$$

Definition 2.2.3. A resolvent $S(t)$ (resp. an integral resolvent $P(t)$) is called analytic if the function $S : \mathbb{R}^+ \to B(X)$ (resp. $P : \mathbb{R}^+ \to B(X)$) admits a analytic extension to a sector $\Sigma(0,\theta) := \{z \in \mathbb{C} : |arg(z)| < \theta, \quad \theta \in (0, \frac{\pi}{2})\}$.

An analytic resolvent $S(t)$ (resp. an integral resolvent $P(t)$) is said to be of analyticity type (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is $M = M(\omega, \theta)$ such that:

$$\|S(z)\| \leq Me^{\omega Rez} \quad (\text{resp.} \|P(z)\| \leq Me^{\omega Rez}).$$

Let f be exponentially bounded of order ω. We denote the Laplace transform of f by

$$\mathcal{L}(f)(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \text{Re}(\lambda) > \omega.$$

We define the *finite convolution* on $\mathbb{R}^+$ by

$$(f \star g)(t) := \int_0^t f(t-s)g(s)\, ds.$$

Directly from [57, Theorem 0.1, p.5] we have the following result.

Proposition 2.2.4. *Let $\{P(t)\}_{t\geq 0}$ be an analytic integral resolvent of type (ω_0, θ_0) and $a(t)$ of exponential growth. Let $\widehat{P}(t)$ be denote the Laplace transform of $\{P(t)\}_{t\geq 0}$. Then, for each $\omega > \omega_0$ and $\theta < \theta_0$:*

$$\|\widehat{P}(\lambda)\| \leq \frac{C}{|\omega - \lambda|}, \qquad \lambda \in \Sigma\left(\omega, \theta + \frac{\pi}{2}\right), \quad \textit{for some} \;\; C = C(\omega, \theta) > 0. \tag{2.2.3}$$

The relation between resolvent and integral resolvent families is given in the following proposition.

Proposition 2.2.5. *Suppose that $a \in C^1(\mathbb{R}^+)$, then*

$$\int_0^t P(s)xds = \int_0^t a(t-s)S(s)xds, \quad t \geq 0, \quad x \in X. \tag{2.2.4}$$

In particular, if B is the generator of a resolvent family, then B is also the generator of an integral resolvent family, given by the formula

$$P(t)x = a(0)S(t)x + \int_0^t a'(t-s)S(s)xds, \quad t \geq 0, \quad x \in X. \tag{2.2.5}$$

Proof. Using the identities (2.2.1) and (2.2.2), we have, in view of the commutativity of the convolution

$$\begin{aligned}(S\star P)(t) &= (I\star P)(t)+B(a\star S\star P)(t)=(I\star P)(t)+S\star(B(a\star P))(t)\\ &= (I\star P)(t)+(S\star[P-a])(t)=(I\star P)(t)+(S\star P)(t)-(S\star a)(t).\end{aligned}$$

Therefore, we obtain (2.2.4). Differentiating (2.2.4) with respect to t, we obtain (2.2.5). □

2.3 Almost sectorial operators

In this section, we recall the definition and properties of *almost sectorial operators*. Let S^0_μ with $0<\mu<\pi$ be the open sector:

$$S^0_\mu=\{z\in\mathbb{C}\setminus\{0\}:|\arg(z)|<\mu\}$$

and S_μ be its closure, that is:

$$S_\mu=\{z\in\mathbb{C}:|\arg(z)|\leq\mu\}. \tag{2.3.1}$$

We denote by $\rho(A)$ the *resolvent set of the linear operator* A and $\sigma(A):=\mathbb{C}\setminus\rho(A)$ the *spectrum of the linear operator* A . For $z\in\rho(A)$ we denote the resolvent operator

$$R(z;A):=(zI-A)^{-1}.$$

Definition 2.3.1. [54] Let $-1<\gamma<0$ and $0\leq\omega<\pi$ be given. By $\Theta^\gamma_\omega(X)$ we denote the family of all linear and closed operators $A:D(A)\subseteq X\to X$, which satisfy:

(1) $\sigma(A)\subseteq S_\omega$, and

(2) for every $\omega<\mu<\pi$ there exists a constant C_μ such that:

$$||R(z;A)||_{\mathcal{B}(X)}\leq C_\mu|z|^\gamma,\quad\text{for all}\quad z\in\mathbb{C}\setminus S_\mu.$$

Definition 2.3.2. A linear operator A is called an *almost sectorial operator* on X if $A\in\Theta^\gamma_\omega(X)$.

Given $A \in \Theta^{\gamma}_{\omega}(X)$, we denote $X_\alpha := D(A^\alpha)$, and $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$. The existence of the complex power A^α is a consequence of the functional calculus developed in [54, Section 2]. See also [54, Theorem 3.2 and Proposition 3.3] for their main properties. In contrast to the case of sectorial operators, having $0 \in \rho(A)$ does not imply that the complex powers $A^{-\alpha}$ with $\operatorname{Re}\alpha > 0$ are bounded. However, the operator $A^{-\alpha}$ belongs to $\mathcal{B}(X)$ whenever $\operatorname{Re}\alpha > 1+\gamma$. See [54, Proposition 3.4].

Consider $A \in \Theta^{\gamma}_{\omega}(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. We denote, for $t \in S^0_{\frac{\pi}{2}-\omega}$,

$$T(t) := \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-tz}R(z;A)dz, \tag{2.3.2}$$

where $\omega < \theta < \mu < \frac{\pi}{2} - |\arg\ t|$ and Γ_θ denotes the path

$$\{re^{-i\theta}\ : r > 0\,\} \cup \{re^{i\theta}\ :\ r > 0\},$$

oriented such that S^0_θ lies to the left of Γ_θ. We have the following properties on $\{T(t)\}_{t\geq 0}$. For a proof, see [54, Theorem 3.9].

(i) $\{T(t)\}_{t\geq 0}$ is an analytic semigroup in $S^0_{\frac{\pi}{2}-\omega}$ and

$$\frac{d^n}{dt^n}T(t) = (-A)^n T(t), \quad \text{for all} \ \ t \in S^0_{\frac{\pi}{2}-\omega}.$$

(ii) There exists a constant $C_0 = C_0(\gamma) > 0$ such that

$$||T(t)||_{\mathcal{B}(X)} \leq C_0 t^{-\gamma-1}, \quad \text{for all} \ \ t > 0.$$

(iii) The range $R(T(t))$ of $\{T(t)\}_{t\geq 0}$ is contained in $D(A^\infty)$, for each $t \in S^0_{\frac{\pi}{2}-\omega}$. In particular for all $\beta \in \mathbb{C}$ with $Re\beta > 0$, $R(T(t)) \subset D(A^\beta)$ and

$$A^\beta T(t)x = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz}R(z;A)x dz, \quad \text{for all} \ \ x \in X,$$

and hence there exists a constant $C' = C'(\gamma,\beta) > 0$ such that:

$$||A^\beta T(t)||_{\mathcal{B}(X)} \leq C' t^{-\gamma-Re\beta-1}, \quad \text{for all} \ \ t > 0.$$

(iv) If $\beta > 1+\gamma$, then $D(A^\beta) \subset \Sigma_T$, where Σ_T is the continuity set of the semigroup $\{T(t)\}_{t\geq 0}$. That is:

$$\Sigma_T = \{x \in X : \lim_{t\to 0} T(t)x = x\}.$$

Recall that $R(z;A)$ denotes the resolvent operator of A. The relation between the resolvent operator of A and the semigroup $\{T(t)\}_{t\geq 0}$ is characterized by the following Lemma:

Lemma 2.3.3. *[54, Theorem 3.13] Let $A \in \Theta^{\gamma}_{\omega}(X)$, with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$ be given. Then, for every $\lambda \in \mathbb{C}$, with $Re\lambda > 0$, we have:*

$$R(\lambda; -A) = \int_0^{\infty} e^{-\lambda t} T(t) dt. \tag{2.3.3}$$

In other words, if $A \in \Theta^{\gamma}_{\omega}(X)$ then $-A$ is the generator of an analytic semigroup of growth order $\gamma + 1$.

2.4 Special functions

Special functions play a crucial role in the study of differential equations of fractional order. In this section, we give the basic theory of the Mittag-Leffler function, the Wright functions of one and two parameters, the generalized Wright function, the Lévy stable distribution, Bessel function of first kind and the modified Bessel function of first order.

2.4.1 Mittag-Leffler function

The well-known *Mittag-Leffler* function (see e.g. [25, 26, 55, 61]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{Ha} e^{\mu} \frac{\mu^{\alpha-\beta}}{\mu^{\alpha} - z} d\mu, \quad \alpha > 0,\ \beta \in \mathbb{C}, z \in \mathbb{C}, \tag{2.4.1}$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ counterclockwise. The Laplace transform of the Mittag-Leffler function is given by (see [35])

$$\int_0^{\infty} e^{-\lambda t} t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\pm \omega t^{\alpha})\ dt = \frac{k! \lambda^{\alpha-\beta}}{(\lambda^{\alpha} \mp \omega)^{k+1}}, \quad \alpha > 0,\ \beta \in \mathbb{C}, \quad \mathrm{Re}(\lambda) > |\omega|^{\frac{1}{\alpha}}. \tag{2.4.2}$$

Note that Mittag-Leffler function generalizes some well known functions:

- $E_{0,1}(z) = \frac{1}{1-z}$.
- $E_{1,1}(z) = e^z$.
- $E_{2,1}(z) = \cosh(\sqrt{z})$.
- $\displaystyle\int_0^z E_{0,1}(s^2)\ ds = \arctan(z)$.
- $\displaystyle\int_0^z E_{2,1}(s^2)\ ds = \sin(z)$.

For $z \in \mathbb{C}$, and $0 < \alpha < 1$, we also have

$$\begin{aligned} E_{2\alpha,1}(-z^2) &= \int_0^\infty \cos(zt)\Phi_\alpha(t)dt, \\ zE_{2\alpha,1+\alpha}(-z^2) &= \int_0^\infty \sin(zt)\Phi_\alpha(t)dt, \end{aligned} \tag{2.4.3}$$

see [28, Formula 2.29]. For more details on the Wright functions Φ_α, we refer to [25, 26, 28, 55, 61, 69] and the references therein. Some properties of the Mittag-Leffler function are the following.

Proposition 2.4.1. *Let $\alpha, \gamma > 0$ and $m \in \mathbb{N} \cup \{0\}$ be given.*

i) $E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}$.

ii) $E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z)$.

iii) $\left(\frac{d}{dz}\right)^m [z^{\gamma-1}E_{\alpha,\gamma}(z^\alpha)] = z^{\gamma-m-1}E_{\alpha,\gamma-m}(z^\alpha)$.

A proof of the above proposition can be found in [55, p. 22, Formula (1.83)].

2.4.2 Wright functions

In this subsection, we present the Wright function of one parameter Φ_β, the Wright function of two parameters $\Phi_{\alpha,\beta}$, the scaled Wright function $\psi_{\alpha,\beta}$ and the generalized Wright function ${}_p\Psi_q$. We give the principal properties and the relationship between these functions.

Wright function of one parameter

The well-known *Wright type function with one parameter* on $\mathbb{C}$ [25, Formula (28)] (see also [55, 61, 69]) is defined as

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{2\pi i} \int_\gamma \mu^{\alpha-1} e^{\mu + z\mu^\alpha} d\mu, \quad 0 < \alpha < 1, \tag{2.4.4}$$

where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise.

An interesting fact is the following relationship between the Wright and the Mittag-Leffler functions.

Proposition 2.4.2. *Let $z \in \mathbb{C}$, $t > 0$ and $0 < \alpha, \gamma < 1$. Then the following properties hold:*

(i) $E_{\gamma,1}(z) = \displaystyle\int_0^\infty \Phi_\gamma(t) e^{zt} dt.$

(ii) $\Phi_\alpha(t) \geq 0.$

(iii) $\displaystyle\int_0^\infty \Phi_\alpha(t) dt = 1.$

For a proof, see [2, Section 2]. It follows from (ii) and (iii) that Φ_α is a probability density function on $\mathbb{R}^+$. The following formula, on the moments, holds

$$\int_0^\infty x^p \Phi_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(\alpha p + 1)}, \quad p > -1, \quad 0 < \alpha < 1. \tag{2.4.5}$$

The formula (2.4.5) is derived from the representation (2.4.4) and can be found in [25, formula after (38)]. Note that in this reference the notation $M(x,\alpha) := \Phi_\alpha(x)$ is used. For more details on the Wright type functions, we refer to [12, 25, 47, 69] and the references therein.

Wright function of two parameters

The *Wright function of two parameters* is defined by

$$\Phi_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n+\beta)} = \frac{1}{2\pi i}\int_\gamma \mu^{-\beta} e^{\mu + z\mu^\alpha} d\mu, \quad 0<\alpha<1,\ \beta\in\mathbb{C}. \tag{2.4.6}$$

Note from the above definition that

$$\Phi_\alpha(z) = \Phi_{-\alpha,1-\alpha}(z). \tag{2.4.7}$$

Scaled Wright function of two parameters

Now, we define the *scaled Wright function*, as follows

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1}\Phi_{-\alpha,\beta}(-st^{-\alpha}), \quad t>0,\ s\in\mathbb{C}. \tag{2.4.8}$$

Note that using the change of variable $z=\frac{\mu}{t}$, we get the integral representation

$$\psi_{\alpha,\beta}(t,s) = \int_\gamma z^{-\beta} e^{tz - sz^\alpha} d\mu, \quad t,s>0. \tag{2.4.9}$$

We remark that $\psi_{\alpha,\beta}(t,s) > 0$, for all $t,s>0$. For more details and properties, see [2, Theorem 3.2]. Using (2.4.7) and (2.4.8) we obtain the following relation between the scaled Wright function and Wright function of one parameter

$$\psi_{\alpha,1-\alpha}(s,t) = t^{-\alpha}\Phi_\alpha(st^\alpha), \quad 0<\alpha<1, t>0,\ s\in\mathbb{C}. \tag{2.4.10}$$

Generalized Wright function

Now, we present the generalized Wright function (see [37, 38, 67, 68]) that we have briefly mentioned in the introduction. It is defined by

$$F(z) \equiv {}_p\Psi_q \left[\begin{array}{c|c} (a_i, \alpha_i)_{1,p} & \\ & z \\ (b_j, \beta_j)_{1,q} & \end{array} \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \tag{2.4.11}$$

where $z, a_i, b_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, ..., p$ and $j = 1, ..., q$. Results on the radius of convergence of the series (2.4.11) is presented in the following theorem. We refer to [38, Theorem 1.5 p.56.].

Theorem 2.4.3. *Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}, (i = 1, ..., p; j = 1, ..., q)$, and let:*

$$\begin{aligned} \kappa &= \textstyle\sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l, \\ \delta &= \prod_{l=1}^{p} |\alpha_l|^{-\alpha_l} \prod_{j=1}^{q} |\beta_j|^{\beta_j}, \\ \mu &= \sum_{j=1}^{q} b_j - \sum_{l=1}^{p} a_l - \frac{p-q}{2}. \end{aligned}$$

(i) If $\kappa > -1$, then the series in (2.4.11) is absolutely convergent for all $z \in \mathbb{C}$.

(ii) If $\kappa = -1$, then the series in (2.4.11) is absolutely convergent for $|z| < \delta$ and for $|z| = \delta$ and $\mathrm{Re}(\mu) > \frac{1}{2}$.

Furthermore, if $c_k = \dfrac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \dfrac{1}{k!}$, then, using Stirling's formula (2.5.2), for k sufficiently large, we have that

$$|c_k| \sim A \left(\frac{k}{e} \right)^{-(\kappa+1)k} \delta^{-k} k^{-(\mathrm{Re}(\mu)+\frac{1}{2})}, \tag{2.4.12}$$

where

$$A = (2\pi)^{\frac{p-q-1}{2}} \frac{\prod_{l=1}^{p} |\alpha_l|^{\mathrm{Re(a_l)}-\frac{1}{2}}}{\prod_{j=1}^{q} |\beta_j|^{\mathrm{Re(b_j)}-\frac{1}{2}}}.$$

Remark 2.4.4. The Mittag-Leffler and Wright functions can be recovered as special cases of the generalized Wright function (with an obvious extension of notation). Indeed,

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = {}_0\Psi_1 \left[\begin{array}{c|c} -- & \\ & z \\ (\beta, \alpha) & \end{array} \right],$$

and

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = {}_0\Psi_1 \left[\begin{array}{c|c} -- & \\ & z \\ (1-\alpha, -\alpha) & \end{array} \right].$$

The Wright function with two-parameters $W_{\lambda,\mu}(z)$ has the representation:

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)} = {}_0\Psi_1 \left[\begin{array}{c|c} -- & \\ & z \\ (\mu, \lambda) & \end{array} \right].$$

2.4.3 Stable Lévy distribution

We present the following function, called stable *Lévy distribution*, defined for $0 < \alpha < 1$ by

$$f_{t,\alpha}(\lambda) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - tz^\alpha} dz, & \sigma > 0, \quad t > 0, \quad \lambda \geq 0, \\ 0 & \lambda < 0, \end{cases} \tag{2.4.13}$$

where the branch of z^α is taken so that $\mathrm{Re}(z^\alpha) > 0$ for $\mathrm{Re}(z) > 0$. This branch is single-valued in the z-plane cut along the negative real axis. These functions were introduced by S. Bochner [13] in the study of certain stochastic processes. K. Yosida [70] used them systematically in the study of semigroups generated by fractional powers of uniformly bounded semigroups of linear operators. The Lévy distribution are the density functions associated with the stable Lévy processes in the rotational invariant case, and are related to the fractional Brownian motion.

Remark 2.4.5. The following properties hold:

(i) $\displaystyle\int_0^\infty e^{-\lambda a} f_{t,\alpha}(\lambda) d\lambda = e^{-ta^\alpha}, \quad t > 0, \quad a > 0, \quad 0 < \alpha < 1.$

(ii) $f_{t,\alpha}(\lambda) \geq 0, \quad \lambda > 0, \; t > 0, \quad 0 < \alpha < 1.$

(iii) $\displaystyle\int_0^\infty f_{t,\alpha}(\lambda) d\lambda = 1, \quad t > 0, \quad 0 < \alpha < 1.$

(iv) $\displaystyle f_{t+s,\alpha}(\lambda) = \int_0^\lambda f_{t,\alpha}(\lambda - \mu) f_{s,\alpha}(\mu) d\mu, \quad \lambda > 0, \quad t, s > 0, \quad 0 < \alpha < 1.$

(v) $\int_0^\infty e^{\lambda z} f_{\lambda,\alpha}(t)d\lambda = t^{\alpha-1}E_{\alpha,\alpha}(zt^\alpha), \quad z \in \mathbb{C}, \quad t > 0, \quad 0 < \alpha < 1.$

For a proof of (i)-(iv), see [70, p.260-262]. Concerning (iv) we have to observe in [70, Proposition 1, p.260] that by definition $f_{t,\alpha}(\lambda) = 0$ for $\lambda < 0$. For the property (v) we refer to the paper [2, Theorem 3.2 (iii)].

Note that the Lévy functions can be defined using the Wright functions of one and two parameters. In the case of one parameter, we have the representation

$$f_{t,\alpha}(\lambda) = \frac{\alpha t}{\lambda^{\alpha+1}}\Phi_\alpha(t\lambda^{-\alpha}), \tag{2.4.14}$$

which follows from [35, (2.11)] and [70, Prop. 1, p.260]. For the case of the Wright function with two parameters, it follows from [2, Formulas (17), (18) and (32)] that

$$f_{t,\alpha}(\lambda) = \frac{1}{\lambda}\Phi_{-\alpha,0}(-t\lambda^{-\alpha}).$$

Furthermore, note that from (2.4.9), we get

$$f_{s,\alpha}(t) = \int_\gamma e^{tz-sz^\alpha}d\mu = \psi_{\alpha,0}(t,s), \quad t, s > 0. \tag{2.4.15}$$

2.4.4 Bessel functions

The Bessel function are the standard solutions to the Bessel's differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0,$$

where $\nu \in \mathbb{R}$ is the *order* of the Bessel function. Since this equation is of second order, it has two linearly independent solutions, namely $J_\nu(x)$ and $Y_\nu(x)$.

Bessel function of first kind $J_\nu(x)$

The $J_\nu(x)$ function is known as *Bessel function of first kind*, and is given by the following expression

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\nu+1)n!} \left(\frac{x}{2}\right)^{2n+\nu}, \quad x \geq 0. \tag{2.4.16}$$

We recall some properties of the Bessel functions.

- Sign of $J_\nu(x)$

$$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x). \tag{2.4.17}$$

- The generating function (see [29, Formula 8.511], [6]):

$$\sum_{n\in\mathbb{Z}} J_n(x) z^n = e^{\frac{x}{2}(z-\frac{1}{z})}, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \geq 0. \tag{2.4.18}$$

- Aditivity of $J_\nu(x)$

$$J_n(x+y) = \sum_{k\in\mathbb{Z}} J_{n-k}(x) J_k(y). \tag{2.4.19}$$

- The integral representation of J_ν (see [6, Chapter 4]) is given by

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \left(\frac{x}{2}\right)^\nu \int_{-1}^{1} e^{ixt}(1-t^2)^{\nu-\frac{1}{2}} dt, \ \mathrm{Re}(\nu) > -\frac{1}{2}. \tag{2.4.20}$$

Remark 2.4.6. From (2.4.20), one easily deduces the Poisson integral representation

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \left(\frac{x}{2}\right)^\nu \int_0^\pi \cos(x\cos\theta) \sin^{2\alpha}(\theta)\, d\theta. \tag{2.4.21}$$

Modified Bessel function of first kind $I_\nu(x)$

For $\nu \in \mathbb{R}$, I_ν denotes the *Modified Bessel functions of the first kind*, defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\nu+1)n!} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{2.4.22}$$

This function satisfies, among others, the following properties:

- Positivity

$$I_n(x) \geq 0, \quad n \in \mathbb{Z},\ x \geq 0. \tag{2.4.23}$$

- Aditivity of $J_\nu(x)$

$$I_n(x+y) = \sum_{k\in\mathbb{Z}} I_{n-k}(x) I_k(y). \tag{2.4.24}$$

- Sign of I_n

$$I_{-n}(x) = I_n(x) = (-1)^n I_n(-x). \tag{2.4.25}$$

- Generating property of I_n

$$\sum_{n\in\mathbb{Z}} I_n(x) z^n = e^{\frac{x}{2}(z+\frac{1}{z})}, \qquad z \in \mathbb{C}\setminus\{0\}, \quad x \geq 0. \tag{2.4.26}$$

Remark 2.4.7. From the definition of $J_\nu(x)$ and $I_\nu(x)$ it is clear that

$$|J_\nu(x)| \leq I_\nu(x), \quad \nu \in \mathbb{R}^+. \tag{2.4.27}$$

Furthermore, we have

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z), \qquad J_{\frac{-1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos(z),$$

see [29, Formula 8.464].

For $\nu > -1$, and $\beta, \alpha > 0$,

$$\int_0^\infty e^{-\alpha t} I_\nu(\beta t)dt = \frac{\beta^{-\nu}\left(\alpha - \sqrt{\alpha^2-\beta^2}\right)^\nu}{\sqrt{\alpha^2-\beta^2}}, \tag{2.4.28}$$

$$\int_0^\infty e^{-\alpha t} J_\nu(\beta t)dt = \frac{\beta^{-\nu}\left(\sqrt{\alpha^2+\beta^2} - \alpha\right)^\nu}{\sqrt{\alpha^2+\beta^2}}, \tag{2.4.29}$$

see, for example, [29, Formula 6.611].

For $\nu > -1$, and $\beta, \alpha > 0$,

$$\int_0^\infty e^{-\alpha t} J_\nu(\beta t) t^{\nu+1} dt = \frac{(2\alpha)(2\beta)^\nu \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}(\alpha^2+\beta^2)^{\nu+\frac{3}{2}}}, \tag{2.4.30}$$

see, for example, [29, Formula 6.623(2)].

2.5 Miscellaneous

In this section we give various definitions and results used in this thesis.

Definition 2.5.1. Let $f \in C^m(\mathbb{R}^+, \mathbb{R}) := \{f : \mathbb{R}^+ \to \mathbb{R} : f^{(m)} \in C(\mathbb{R})\}$ and $\alpha \in \mathbb{R}^+$, with $m - 1 \leq \alpha \leq m$ be given. We define the *Caputo fractional derivative of order* α as follows

$$\mathbb{D}_t^\alpha f(t) := \int_0^t g_{m-\alpha}(t-s) f^{(m)}(s) = (g_{m-\alpha} \star f^{(m)})(t), \tag{2.5.1}$$

where $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is called the convolution kernel. For more information, see ([55, page 78]).

Some useful properties are presented in the following proposition (see [45, Example 3.3] and [55, Section 2.8.3]).

Proposition 2.5.2. *Let $f \in C^m(\mathbb{R}^+)$ and $\alpha, \beta > -1$ be such that $\alpha + \beta > -1$, with $\alpha \in (m-1, m)$, $m \in \mathbb{Z}$. Then*

i) $\mathcal{L}(g_\alpha)(\lambda) = \lambda^{-\alpha}$.

ii) $(g_\alpha \star g_\beta)(t) = g_{\alpha+\beta}(t)$.

iii) $\mathcal{L}(\mathbb{D}_t^\alpha f)(\lambda) = \lambda^\alpha \mathcal{L}(f)(\lambda) - \sum_{j=0}^{m-1} \lambda^j f^{(m-j-1)}(0), \qquad \operatorname{Re} \alpha > 0, \ \operatorname{Re} \lambda > 0.$

In order to define the notion of measure of non-compactness we consider X a complex Banach space with norm $\|\cdot\|$ and $B(x, r)$ denotes the closed ball centered at x with radius r. By $\mathcal{M}_X$ we denote the family of all nonempty and bounded subsets of X. The subfamily consisting of all relatively compact sets is denoted by $\mathcal{N}_X$. As usual, for a linear operator A, we denote by $D(A)$ the domain of A.

Definition 2.5.3. [8] A function $\mu : \mathcal{M}_X \to \mathbb{R}^+$ is said to be a measure of noncompactness in X

if it satisfies the following conditions:

(1) The set $Ker\ \mu := \{\Omega \in \mathcal{M}_X : \mu(\Omega) = 0\}$ is nonempty and $Ker\ \mu \subset \mathcal{N}_X$;

(2) $\Omega \subset \Omega_0$ implies $\mu(\Omega) \leq \mu(\Omega_0)$, for each $\Omega, \Omega_0 \in \mathcal{M}_X$;

(3) $\mu(Conv(\Omega)) = \mu(\Omega)$, where $Conv(\Omega)$ denotes the convex hull of Ω;

(4) $\mu(\overline{\Omega}) = \mu(\Omega)$, where $\overline{\Omega}$ denotes the closure of $\Omega \in \mathcal{M}_X$;

(5) $\mu(\lambda\Omega + (1-\lambda)\Omega_0) \leq \lambda\mu(\Omega) + (1-\lambda)\mu(\Omega_0)$, for $\lambda \in [0,1]$ and $\Omega, \Omega_0 \in \mathcal{M}_X$;

(6) If $\{\Omega_n\}$ is a sequence of sets in $\mathcal{M}_X$ such that $\Omega_{n+1} \subset \Omega_n$, $\overline{\Omega_n} = \Omega_n$, with $n = 1, 2, ...$, and if $\lim\limits_{n\to\infty} \mu(\Omega_n) = 0$, then the intersection $\Omega_\infty = \bigcap\limits_{n=1}^{\infty} \Omega_n$ is nonempty.

The following is a fixed point theorem of Darbo type for measures of noncompactness.

Lemma 2.5.4. *[8] Let $\mathfrak{M}$ be a nonempty, bounded, closed, and convex subset of a Banach space X, and let $H : \mathfrak{M} \to \mathfrak{M}$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$, such that:*

$$\mu(H(\Omega)) \leq k\mu(\Omega),$$

for any nonempty subset Ω of $\mathfrak{M}$. Then H has a fixed point in $\mathfrak{M}$.

The next Lemma is a useful tool in order to estimate the asymptotic behaviour of the Gamma function.

Lemma 2.5.5. *The Stirling's formula is given by*

$$\Gamma(z) \equiv \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z O\left(1 + (\frac{1}{z})\right), \tag{2.5.2}$$

when $|z|$ is sufficiently large, and $|arg(z)| < \pi - \epsilon$, $\epsilon > 0$.

The following identity holds

Lemma 2.5.6. *Assume that Re(a) > 0 and* $b \in \mathbb{R}$. *Then*

$$\int_0^\pi (\sin(t))^{a-1} e^{ibt} dt = \frac{\pi}{2^{a-1}} \frac{e^{i\pi b/2}}{aB((a+b+1)/2,(a-b+1)/2)}, \tag{2.5.3}$$

where $B(\cdot,\cdot)$ *is the usual* Beta function *defined by*

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\ dt, \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0.$$

.

We recall the relationship between Gamma and Beta functions (see [55, Section 1.1.4])

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Chapter 3

Existence of globally attractive mild solutions for nonlocal equations.

In this chapter we present results obtained in the joint paper [22], where the existence of at least one globally attractive mild solution to the equation

$$\partial_t(b \star [x - h(\cdot, x(\cdot))])(t) + A(x(t) - h(t, x(t))) = f(t, x(t)), \quad t \geq 0, \tag{3.0.1}$$

is obtained under the assumption, among other hypothesis, that A is an almost sectorial operator on a Banach space X and the kernel b belongs to a large class, which covers many relevant cases from applications, in particular the important case of time-fractional evolution equations of neutral type. Furthermore, in this chapter, we consider $J = \mathbb{R}^+$ in order to use the definitions and results of resolvent and integral families of operators, presented in the Chapter 2, Section 2.2.

We recall our initial observation that under the hypothesis of the condition $(\mathcal{PC})$: $b \in L^1_{loc}(\mathbb{R}^+)$ is non-negative, and non-increasing and there exists a kernel $a \in L^1_{loc}(\mathbb{R}^+)$ such that $a \star b \equiv 1$ in

$(0, \infty)$, then the above equation is equivalent to the following class of abstract integral Volterra equations

$$\begin{cases} x(t) - h(t, x(t)) & + \displaystyle\int_0^t a(t-s)A[x(s) - h(s, x(s))]ds \\ & = \displaystyle\int_0^t a(t-s)f(s, x(s))ds + x_0 - h(0, x_0), \\ x(0) = x_0. \end{cases} \tag{3.0.2}$$

3.1 The linear case

We first consider the linear problem

$$\begin{cases} x(t) - h(t) = \displaystyle\int_0^t a(t-s)(-A)[x(s) - h(s)]ds + \int_0^t a(t-s)f(s)ds + x_0 - h(0), \\ x(0) = x_0, \end{cases} \tag{3.1.1}$$

where $a \in L^1_{loc}(\mathbb{R}^+)$, A is a closed linear operator defined on a Banach space X, $f : \mathbb{R}^+ \to X$ and $h : \mathbb{R}^+ \to [D(A)]$ are given functions and $x_0 \in X$. By a strong solution of (3.1.1) we mean a function $x \in C(\mathbb{R}^+; [D(A)])$ that satisfies (3.1.1).

Definition 3.1.1. Let $a \in C^1(\mathbb{R}^+)$ be given. Suppose that $-A$ is the generator of a resolvent $\{S(t)\}_{t\geq 0}$. Given functions $f : \mathbb{R}^+ \times X \to X$, $h : \mathbb{R}^+ \times X \to [D(A)]$ and $x_0 \in X$, we call the continuous function $x : \mathbb{R}^+ \to X$ given by

$$x(t) = S(t)(x_0 - h(0)) + h(t) + \int_0^t P(t-s)f(s)ds, \quad t \geq 0, \tag{3.1.2}$$

a mild solution of (3.1.1), where $\{S(t)\}_{t\geq 0}$ and $\{P(t)\}_{t\geq 0}$ are given by (2.2.1) and (2.2.2), respectively.

An important fact of the mild solutions is presented in the following result.

Proposition 3.1.2. *Let $f : \mathbb{R}^+ \times X \to [D(A)]$ be given. If $x_0 \in D(A)$ then each mild solution is* *a strong solution.*

Proof. Define $B = -A$ and $x_1 := x_0 - h(0)$. Since $x_0 \in D(A)$, where A is closed, and $f(t) \in D(A)$ for all $t \geq 0$, we have $S(t)x_1 \in D(A)$ and $(P \star f)(t) \in D(A)$ for all $t \geq 0$, respectively. Therefore $x(t) \in D(A)$ for all $t \geq 0$. Then by (3.2.2), properties of the convolution, Definition 2.2.1 and Definition 2.2.2 we obtain

$$\begin{aligned}(a \star B(x-h))(t) &= (a \star B[Sx_1 + P \star f])(t) = (a \star BS)(t)x_1 + (a \star BP \star f)(t) \\ &= S(t)x_1 - x_1 + ((P-a) \star f)(t) = x(t) - h(t) - x_1 - (a \star f)(t)\end{aligned}$$

which proves the proposition.

$\square$

3.2 The non-linear case

Now, we consider the following nonlinear abstract integral equation:

$$\begin{cases} x(t) - h(t, x(t)) &= \displaystyle\int_0^t a(t-s)(-A)[x(s) - h(s, x(s))]ds \\ &+ \displaystyle\int_0^t a(t-s)f(s, x(s))ds + x_0 - h(0, x_0), \\ x(0) &= x_0, \end{cases} \tag{3.2.1}$$

where $a \in L^1_{loc}(\mathbb{R}^+)$, A is an almost sectorial operator on a complex Banach space X, $f : \mathbb{R}^+ \times X \to X$ and $h : \mathbb{R}^+ \times X \to [D(A)]$ are given functions and $x_0 \in X$. By a strong solution of (3.2.1) we mean a function $x \in C(\mathbb{R}^+; [D(A)])$ that satisfies (3.2.1).

Given an almost sectorial operator A there exists kernels $a \in C^1(\mathbb{R}^+)$ such that $B := -A$ is the generator of simultaneously a resolvent $\{S(t)\}_{t\geq 0}$ and an integral resolvent $\{P(t)\}_{t\geq 0}$. We denote the set of all such kernels by $\mathcal{K}$.

For example, if A is an almost sectorial operator, we know that $-A$ is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$, and is explicitly given by the formula (4.1.4). Then, by [57, Corollary 2.4, p.56] we obtain that the class of kernels $a \in C((0,\infty)) \cap L^1_{loc}(\mathbb{R}^+)$ that are completely monotonic, belongs to $\mathcal{K}$.

Now, we give the definition of mild solution to the equation (3.2.1).

Definition 3.2.1. Let A be an almost sectorial operator and $a \in \mathcal{K}$. Given functions $f : \mathbb{R}^+ \times X \to X$, $h : \mathbb{R}^+ \times X \to [D(A)]$ and $x_0 \in X$, we say that a continuous function $x : \mathbb{R}^+ \to X$ that satisfies the equation

$$x(t) = S(t)(x_0 - h(0,x_0)) + h(t,x(t)) + \int_0^t P(t-s)f(s,x(s))ds, \quad t \geq 0, \tag{3.2.2}$$

is a mild solution of (3.2.1), where $\{S(t)\}_{t\geq 0}$ and $\{P(t)\}_{t\geq 0}$ are given by (2.2.1) and (2.2.2) respectively.

Definition 3.2.2. A mild solution $x(t)$ of (3.0.1) is said to be globally attractive if:

$$\lim_{t\to\infty} (x(t) - y(t)) = 0,$$

for any mild solution $y(t)$ of (3.0.1).

3.3 Existence of globally attractive mild solutions

In this section we present the main result of this chapter, which gives us the sufficient conditions in order to guarantee the existence of mild solution globally attractive for the integral equation (3.2.1). For this, we use a fixed point Theorem and a measure of noncompactness.

3.3.1 Measure of modulus of continuity

For any nonempty and bounded subset Y of the space $BC(\mathbb{R}^+, X)$ consisting of all continuous and bounded functions with domain $\mathbb{R}^+$ and range X, and a positive number T, we denote by $\omega^T(x,\epsilon)$

the modulus of continuity of a function x on the interval $[0,T]$, where $x \in Y$ and $\epsilon \geq 0$. Namely:

$$\omega^T(x,\epsilon) = \sup\{||x(t)-x(s)|| : t,s \in [0,T], |t-s| \leq \epsilon\}.$$

We then write additionally

$$\begin{aligned}
&\omega^T(Y,\epsilon) = \sup\{\omega^T(x,\epsilon) : x \in Y\},\\
&\omega_0^T(Y) = \lim_{\epsilon\to 0} \omega^T(Y,\epsilon),\\
&\omega_0(Y) = \lim_{T\to\infty} \omega_0^T(Y),
\end{aligned}$$

and

$$\operatorname{diam}(Y_t) = \sup\{||x(t)-y(t)|| : x,y \in Y\}.$$

Finally, consider the function μ defined on the family $\mathcal{M}_{BC(\mathbb{R}^+,X)}$ by the formula:

$$\mu(Y) = \omega_0(Y) + \limsup_{t\to\infty} \operatorname{diam}(Y). \tag{3.3.1}$$

3.3.2 Existence of globally attractive mild solution

Let $BC(\mathbb{R}^+, X_\alpha)$ be denote the Banach space consisting of all real functions defined as bounded and continuous from $\mathbb{R}^+$ to X_α with the norm $\|x\|_\infty = \sup_{t\geq 0} \|x(t)\|_\alpha$. Recall that $X_\alpha = D(A^\alpha)$. The main result of this chapter is the following theorem.

Theorem 3.3.1. *Let $A \in \Theta_\omega^\gamma(X)$ be given, with $-1 < \alpha+\gamma < 0,\ 0 < \alpha < 1$ and $0 < \omega < \frac{\pi}{2}$ and $a \in \mathcal{K}$. Assume that:*

(H1) $f : \mathbb{R}^+ \times X_\alpha \to X$ is continuous, and there exists a positive function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$\begin{cases} ||f(t,x)|| \leq \nu(t) \text{ for all } x \in X; \\ \text{the function } \quad s \to ||A^\alpha P(t-s)||\nu(s) \text{ belongs to } L^1([0,t[,\mathbb{R}^+), \text{ and} \end{cases} \tag{3.3.2}$$

$$\lim_{t\to\infty} \eta(t) := \lim_{t\to\infty} \int_0^t ||A^\alpha P(t-s)||\nu(s)ds = 0. \tag{3.3.3}$$

(H2) $h : \mathbb{R}^+ \times X_\alpha \to X_\alpha$ *is bounded, continuous and there exist a constant* $L \in (0,1)$ *such that:*

$$||h(t_1, x(t_1)) - h(t_2, x(t_2))||_\alpha \leq L(|t_1 - t_2| + ||x(t_1) - x(t_2)||_\alpha),$$

for all $x \in BC(\mathbb{R}^+, X_\alpha)$.

(H3) For each nonempty, bounded set $D \subset BC(\mathbb{R}^+, X_\alpha)$, *the family of functions*

$$\{t \to h(t, \varphi(t)) : \varphi \in D\}$$

is equicontinuous.

(H4) The operator $\tau \to A^\alpha P(\tau)$ *is bounded.*

(H5) The resolvent family $\{S(t)\}_{t\geq 0}$ *and* $\{P(t)\}_{t\geq 0}$ *are uniformly continuous for* $t > 0$ *and*

$$\sup_{t\geq 0} ||A^\alpha S(t)x|| < \infty, \quad \text{for all } x \in X.$$

(H6)

$$\lim_{\tau\to t} \int_0^t ||A^\alpha (P(\tau - s) - P(t - s))||\nu(s)ds = 0, \quad \forall t \geq 0.$$

Then:

(1) The problem (3.2.1) *has at least a mild solution on* $BC(\mathbb{R}^+, X_\alpha)$.

(2) Mild solutions of (3.2.1) *are globally attractive.*

Proof. The proof consist in several steps. First, we consider the operator H defined as follows:

$$(Hx)(t) = S(t)(x_0 - h(0, x_0)) + h(t, x(t)) + \int_0^t P(t - s)f(s, x(s))ds, \quad t \geq 0. \tag{3.3.4}$$

Step 1 We prove that there exists a ball

$$B_r = \{x \in BC(\mathbb{R}^+, X_\alpha) : ||x||_\infty \leq r\}$$

with radius r and centered at 0, such that $H(B_r) \subset B_r$. In fact, for any $r > 0$ and $x \in B_r$, in view of (H2),

$$||h(t, x(t))||_\alpha \leq ||h(t, x(t)) - h(t, 0)||_\alpha + ||h(t, 0)||_\alpha \leq Lr + M_1, \tag{3.3.5}$$

where $M_1 := \sup_{t\in\mathbb{R}^+} ||h(t,0)||_\alpha < \infty$ since h is bounded. Moreover, by (3.3.3) in (H2), we get $\sup\limits_{t\in\mathbb{R}^+} \eta(t) \leq K$ for a positive constant K. Let $x \in B_r$ be arbitrary, then by (3.3.5) and (H5)

$$\begin{aligned}
||H(x)(t)||_\alpha &\leq ||S(t)(x_0 - h(0,x_0))||_\alpha + ||h(t,x(t))||_\alpha + \int_0^t ||P(t-s)f(s,x(s))||_\alpha ds \\
&\leq ||A^\alpha S(t)(x_0 - h(0,x_0))|| + Lr + M_1 + \int_0^t ||A^\alpha P(t-s)f(s,x(s))||ds \\
&\leq \sup_{t\geq 0} ||A^\alpha S(t)(x_0 - h(0,x_0)|| + Lr + M_1 + \sup_{t\geq 0}\int_0^t ||A^\alpha P(t-s)\nu(s)||ds.
\end{aligned}$$

Choose r such that:

$$r \geq \frac{\sup\limits_{t\geq 0} ||A^\alpha S(t)(x_0 - h(0,x_0))|| + M_1 + K}{1-L},$$

then:

$$||(Hx)(t)||_\alpha \leq r,$$

that is, $H(B_r) \subset B_r$.

Step 2 We prove that the operator H is continuous on B_r. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in B_r, such that $x_n \to x \in B_r$, as $n \to \infty$. Then:

$$\|f(s,x_n(s)) - f(s,x(s))\| \to 0, \quad n \to \infty, \tag{3.3.6}$$

because the function f is continuous on $\mathbb{R}^+ \times X_\alpha$. Given $T > 0$ and for every $t \in [0,T]$ fixed, using (H2) and (H4) we obtain:

$$\begin{aligned}
||H(x_n)(t) - H(x)(t)||_\alpha &\leq ||h(t,x_n(t)) - h(t,x(t))||_\alpha \\
&\quad + \int_0^t ||P(t-s)(f(s,x_n(s)) - f(s,x(s)))||_\alpha ds \\
&\leq L||x_n - x||_\infty + \int_0^t ||A^\alpha P(t-s)||\,||(f(s,x_n(s)) - f(s,x(s)))||ds.
\end{aligned} \tag{3.3.7}$$

Now, consider $g_n(s) = ||A^\alpha P(t-s)||\,||(f(s,x_n(s)) - f(s,x(s))||$. By (3.3.6), we have:

$$\lim_{n\to\infty} g_n(s) = 0,$$

and by (H1)

$$g_n(s) \leq 2||A^\alpha P(t-s)||\nu(s) \in L^1([0,t[,\mathbb{R}^+)$$

which follows by (3.3.2). By the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\to\infty}\int_0^t ||A^\alpha P(t-s)||\,||(f(s,x_n(s)) - f(s,x(s))||ds = 0,$$

and clearly $\lim_{n\to\infty} L||x_n - x||_\infty = 0$. Therefore, by (3.3.7) we have:

$$\lim_{n\to\infty} ||H(x_n)(t) - H(x)(t)||_\alpha = 0.$$

This proves that H is continuous en B_r.

Step 3 Let Ω be an arbitrary nonempty subset of B_r. We prove that:

$$\mu(H(\Omega)) \leq L\mu(\Omega).$$

Indeed, let us choose $x \in \Omega$ and t_1, t_2 with $|t_1 - t_2| \leq \epsilon$. For $0 < t_1 < t_2 \leq T$, we have:

$$\begin{aligned} ||H(x)(t_2) - H(x)(t_1)||_\alpha &\leq ||(S(t_2) - S(t_1))(x_0 - h(0,x_0))||_\alpha \\ &\quad + ||h(t,x(t_2)) - h(t,x(t_1))||_\alpha \\ &\quad + ||\int_0^{t_1}(P(t_2-s) - P(t_1-s))f(s,x(s))ds||_\alpha \\ &\quad + \int_{t_1}^{t_2} ||A^\alpha P(t_2-s)||\nu(s)ds \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.3.8}$$

As a consequence of the continuity of $\{S(t)\}_{t\geq 0}$ in the operator topology for $t > 0$, we have that

$$I_1 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

By (H3), we see that:

$$I_2 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

For I_3, we have:

$$||\int_0^{t_1}(P(t_2-s) - P(t_1-s))f(s,x(s))ds||_\alpha \leq \int_0^{t_1} ||A^\alpha(P(t_2-s) - P(t_1-s))||\nu(s)ds$$

and by (H6), we have that:

$$I_3 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

Finally, by (H1), (3.3.2), and by continuity of the integral, we have:

$$I_4 \leq \int_{t_1}^{t_2} ||A^\alpha P(t-s)||\nu(s)ds \to 0 \quad \text{as} \quad t_2 \to t_1.$$

Thus, we obtain:

$$\omega_0^T(H(\Omega)) = 0.$$

Consequently, we have:

$$\omega_0(H(\Omega)) = 0. \tag{3.3.9}$$

Now, by our assumptions, for arbitrary fixed $t \in \mathbb{R}^+$ and $x, y \in \Omega$ we deduce that:

$$\begin{aligned}
& ||Hx(t) - Hy(t)||_\alpha \\
\leq \quad & ||h(t, x(t)) - h(t, y(t))||_\alpha + \int_0^t ||P(t-s)(f(s, x(s)) - f(s, y(s))ds||_\alpha ds \\
\leq \quad & L||x(t) - y(t)||_\alpha + 2M_2\eta(t).
\end{aligned}$$

By (3.3.3), we have:

$$\limsup_{t\to\infty} \operatorname{diam}(H(\Omega))(t) \leq L \limsup_{t\to\infty} \operatorname{diam} \Omega(t). \tag{3.3.10}$$

Therefore, using the measure of non-compactness μ defined in (3.3.1) and using (3.3.9) and (3.3.10), we obtain:

$$\mu(H(\Omega)) \leq L\mu(\Omega). \tag{3.3.11}$$

Step 4 We prove that the conclusion (1) is true. In fact, since $0 < L < 1$, in view of (3.3.11) and Lemma 2.2, we deduce that the operator H has a fixed point x in the ball B_r. Hence, equation (3.2.1) has at least one mild solution $x(t)$.

Step 5 We prove that the conclusion (2) is true.

Indeed, for any other mild solution $y(t)$ of equation (3.2.1), we have:

$$\begin{aligned}
||x(t) - y(t)||_\alpha &\leq ||Hx(t) - Hy(t)||_\alpha \\
&\leq L||x(t) - y(t)||_\alpha + 2M_2\eta(t).
\end{aligned}$$

Then, by (3.3.3), we have:

$$\lim_{t\to\infty} ||x(t) - y(t)||_\alpha \leq \frac{2M_2}{1-L} \lim_{t\to\infty} \eta(t) = 0.$$

That is, mild solutions of (3.2.1) are globally attractive.

□

We finish this subsection with the following practical criteria in order to verify condition (H6).

Proposition 3.3.2. *Let $A \in \Theta_\omega^\gamma(X)$, with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$, $\omega > 0$ be fixed, $a \in C^1(\mathbb{R}^+)$ such that $a(0) = 0$ and A the generator of an analytic integral resolvent $\{P(t)\}_{t\geq 0}$. Then:*

$$\lim_{\tau \to t} \int_0^t ||A^\alpha (P(\tau - s) - P(t - s)||\nu(s) ds = 0, \quad \forall t > 0, \quad \alpha \in (0, 1).$$

Proof. Since $A^\alpha = A^{-(1-\alpha)}A$, where $A^{-(1-\alpha)} \in \mathcal{B}(X)$, for $\alpha \in (0,1)$, it is enough to prove that:

$$\lim_{\tau \to t} \int_0^t ||A(P(\tau - s) - P(t - s))||\nu(s) ds = 0, \quad \forall t > 0.$$

We first note the identity

$$A(P(r) - P(s)) = A \int_s^r P'(\tau) d\tau = \int_s^r AP'(\tau) d\tau.$$

By means of Lemma 2.1 in [57] we can obtain the following estimate

$$|\frac{1}{\hat{a}(\lambda)}| \leq e^{c(|\lambda - \omega|^\varsigma)} \tag{3.3.12}$$

on a sector $\Sigma(\omega, \frac{\pi}{2} + \theta)$ where $\omega > \omega_0$, $\theta < \theta_0$ are fixed but arbitrary and where $\varsigma := \frac{\pi}{\pi + 2\theta'}$ with $\theta < \theta' < \theta_0$. See also [57, Formula (2.19) p. 58]. Note that

$$AP'(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} A\widehat{P'}(\lambda) d\lambda = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} A\lambda \widehat{P}(\lambda) d\lambda, \tag{3.3.13}$$

because $a(0) = 0$, and being $\Gamma = \Gamma_1 \cup \Gamma_2$, defined by:

$$\Gamma_1 = re^{i(\frac{\pi}{2} + \theta)} + \omega, \quad \Gamma_2 = Re^{i\varphi} + \omega,$$

with $R = t^{-(1-\varsigma)^{-1}}, r > R$ and $\varphi \in [0, \frac{\pi}{2} + \theta]$. Taking Laplace's transform in both sides of (2.2.2), we obtain:

$$\widehat{P}(\lambda) = \left(\frac{1}{\widehat{a}(\lambda)} + A \right)^{-1}.$$

Using the identity $I = \left(\frac{1}{\widehat{a}(\lambda)} + A \right) \widehat{P}(\lambda)$ we have $A\lambda \widehat{P}(\lambda) = \lambda - \frac{\lambda}{\widehat{a}(\lambda)} \widehat{P}(\lambda)$. Therefore, from (3.3.13) and Cauchy's theorem, we get

$$\begin{aligned} AP'(t) &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} A\lambda \widehat{P}(\lambda) d\lambda = -\frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \left[\frac{\lambda}{\widehat{a}(\lambda)} \widehat{P}(\lambda) - \lambda \right] d\lambda \\ &= -\frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \frac{\lambda}{\widehat{a}(\lambda)} \widehat{P}(\lambda) d\lambda. \end{aligned}$$

Now, by (2.2.3) and (3.3.12) we conclude that

$$\begin{aligned}
\|AP'(t)\| &= \frac{1}{2\pi}\int_\Gamma |e^{\lambda t}|\frac{|\lambda|}{|\widehat{a}(\lambda)|}\|\widehat{P}(\lambda)\||d\lambda| \\
&\leq \frac{C}{2\pi}\int_\Gamma e^{Re\lambda t}\frac{|\lambda|}{|\lambda-\omega|}e^{c|\lambda-\omega|^\varsigma}|d\lambda| \\
&= \frac{C}{2\pi}\int_{\Gamma_1} e^{Re\lambda t}\frac{|\lambda|}{|\lambda-\omega|}e^{c|\lambda-\omega|^\varsigma}|d\lambda| + \frac{C}{2\pi i}\int_{\Gamma_2} e^{Re\lambda t}\frac{|\lambda|}{|\lambda-\omega|}e^{c|\lambda-\omega|^\varsigma}|d\lambda| \\
&= \frac{C}{2\pi}\int_R^\infty e^{\omega t}e^{-rt\sin\theta}\frac{|\omega+r|}{r}e^{cr^\varsigma}dr + \frac{C}{2\pi i}\int_0^{\frac{\pi}{2}+\theta} e^{\omega t}e^{-Rt\cos\varphi}\frac{\omega+R}{R}e^{cR^\varsigma}Rd\varphi \\
&\leq \frac{Ce^{\omega t}}{2\pi}\Bigg[|\omega|\int_R^\infty e^{-rt\sin\theta}\, e^{cr^\varsigma}\frac{dr}{r} + \int_R^\infty e^{-rt\sin\theta}e^{cr^\varsigma}dr \\
&\qquad + R\int_0^{\frac{\pi}{2}+\theta} e^{Rt\cos\varphi}e^{cR^\varsigma}d\varphi + |\omega|\int_0^{\frac{\pi}{2}+\theta} e^{Rt\cos\varphi}e^{cR^\varsigma}d\varphi\Bigg] \\
&=: \frac{Ce^{\omega t}}{2\pi}[(I)+(II)+(III)+(IV)].
\end{aligned}$$

We observe that the term (I) appeared in [57], page 59 and can be estimated by $t^{\frac{\pi}{2\theta'}}$. For (II), we use the change of variable $b = t\sin\theta, s = rb$, and the identity 12 on page 710 of [56], together with the fact that $0 < \varsigma < 1$. We deduce that the integral is finite. Consequently, we can define, for all $t > 0$:

$$\begin{aligned}
\Psi(t) &:= \frac{Ce^{\omega t}}{2\pi i}\Bigg[|\omega|\int_R^\infty e^{-rt\sin\theta}\frac{e^{cr^\varsigma}}{r}dr + \int_R^\infty e^{-rt\sin\theta}e^{cr^\varsigma}dr \\
&\qquad + R\int_0^{\frac{\pi}{2}+\theta} e^{Rt\cos\varphi}e^{cR^\varsigma}d\varphi + |\omega|\int_0^{\frac{\pi}{2}+\theta} e^{Rt\cos\varphi}e^{cR^\varsigma}d\varphi\Bigg].
\end{aligned}$$

Therefore Ψ is a positive continuous function satisfying $\|AP'(t)\| \leq \Psi(t)$ for all $t > 0$. In this way, we conclude that:

$$\begin{aligned}
||A(P(\tau-s)-P(t-s))|| &= ||\int_{t-s}^{\tau-s} AP'(r)dr|| \leq \int_{t-s}^{\tau-s}\|AP'(r)\|dr \\
&\leq \int_{t-s}^{\tau-s}\Psi(r)dr =: \Phi(\tau).
\end{aligned}$$

where Φ is positive, increasing and $\Phi(\tau) \to 0$ as $\tau \to t$. Therefore, by the monotone convergence theorem, we obtain:

$$\begin{aligned}
\lim_{\tau\to t}\int_0^t ||A(P(\tau-s)-P(t-s))||\nu(s)ds &\leq \lim_{\tau\to t}\int_0^t\int_{t-s}^{\tau-s}\Psi(r)\nu(s)drds \\
&= \int_0^t\left[\lim_{\tau\to t}\int_{t-s}^{\tau-s}\Psi(r)\right]\nu(s)drds \\
&= \int_0^t\left[\lim_{\tau\to t}\Phi(\tau)\right]\nu(s)drds = 0.
\end{aligned}$$

□

3.4 An application

For $0 < q < 1$, we consider the problem

$$\begin{cases} \mathbb{D}_t^q[x(t) - h(t, x(t))] = (-A)[x(t) - h(t, x(t))] + f(t, x(t)) \\ x(0) = x_0, \end{cases} \tag{3.4.1}$$

We recall that $\mathbb{D}_t^q$ denotes the fractional derivative of order $q > 0$ in the sense of Caputo defined in (2.5.1) and $A \in \Theta_\omega^\gamma(X)$. Then, we may convolve with g_q both sides in (3.4.1) and obtain the equivalent problem

$$\begin{cases} x(t) - h(t, x(t)) & = \int_0^t g_q(t-s)(-A)[x(s) - h(s, x(s))]ds \\ & \quad + \int_0^t g_q(t-s) f(s, x(s))ds + x_0 - h(0, x_0), \\ x(0) = x_0, \end{cases} \tag{3.4.2}$$

which takes the form of the abstract model (3.2.1) with $a(t) = g_q(t)$. Therefore, our first application retrieve the main result in [43].

We define the following operators.

$$\mathcal{S}_q(t)x = \int_0^\infty \Phi_q(\sigma) T(\sigma t^q) x d\sigma, \quad t \in S^0_{\frac{\pi}{2}-\omega}, \quad x \in X, \tag{3.4.3}$$

$$\mathcal{P}_q(t)x = \int_0^\infty q\sigma \Phi_q(\sigma) T(\sigma t^q) x d\sigma, \quad t \in S^0_{\frac{\pi}{2}-\omega}, \quad x \in X, \tag{3.4.4}$$

and $\Phi_q(\sigma)$ is the Wright function of one parameter given by (2.4.4).

First, we present the following Lemma (see [65]).

Lemma 3.4.1. *For each $t \in S^0_{\frac{\pi}{2}-\omega}$, the operators $\{\mathcal{P}_q(t)\}_{t\geq 0}$ and $\{\mathcal{S}_q(t)\}_{t\geq 0}$ have the following properties:*

(a) For $q > 0$, $-1 < \alpha + \gamma < 0$, C_0 depending of γ, q and C' depending of γ, q, α, we have:

$$\begin{aligned} ||\mathcal{S}_q(t)x|| &\leq \tfrac{C_0 \Gamma(-\gamma)}{\Gamma(1-q(1+\gamma))} t^{-q(1+\gamma)} ||x||, \\ ||\mathcal{P}_q(t)x|| &\leq \tfrac{q C_0 \Gamma(1-\gamma)}{\Gamma(1-q\gamma)} t^{-q(1+\gamma)} ||x||, \\ ||A^\alpha \mathcal{P}_q(t)x|| &\leq \tfrac{q C' \Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))} t^{-q(1+\gamma+\alpha)} ||x||, \end{aligned} \tag{3.4.5}$$

(b) For $t > 0$, $\{\mathcal{P}_q(t)\}_{t\geq 0}$ and $\{\mathcal{S}_q(t)\}_{t\geq 0}$ are continuous in the uniform operator topology.

Corollary 3.4.2. *Let $-1 < \alpha + \gamma < 0$ and $0 < \alpha < \beta < 1$ be given. Assume that:*

(F1) $f : \mathbb{R}^+ \times X_\alpha \to X$ is continuous, and there exists a positive function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$\begin{cases} ||f(t,x)|| \leq \nu(t), \\ \textit{the function} \quad s \to \frac{\nu(s)}{(t-s)^{1+q(\gamma+q)}} \ \textit{belongs to}\ L^1([0,t[,\mathbb{R}^+), \end{cases} \tag{3.4.6}$$

$$\lim_{t\to\infty} \eta(t) := \lim_{t\to\infty} \int_0^t \frac{\nu(s)}{(t-s)^{1+q(\gamma+\alpha)}} ds = 0 \tag{3.4.7}$$

(F2) The function $h : \mathbb{R}^+ \times X_\alpha \to X_\alpha$ is bounded, continuous and there exists a constant $L \in (0,1)$ such that:

$$||h(t_1, x(t_1)) - h(t_2, x(t_2))||_\alpha \leq L(|t_1 - t_2| + ||x(t_1) - x(t_2)||_\alpha).$$

(F3) For each nonempty, bounded set $D \subset BC(\mathbb{R}^+, X_\alpha)$, the family of functions

$$\{t \to h(t, \varphi(t)) : \varphi \in D\}$$

is equicontinuous.

Then:

(1) For every $x_0 \in D(A^{\alpha+\beta})$ with $\beta > 1 + \gamma$, the problem (3.4.1) has at least mild solution on $BC(\mathbb{R}^+, X_\alpha)$.

(2) All solution are globally attractive.

Proof. Since $A \in \Theta^\gamma_\omega(X)$, the operator A generates an analytic semigroup $\{T(t)\}_{t\geq 0}$. Let $\{S(t)\}_{t\geq 0}$ and $\{P(t)\}_{t\geq 0}$ be operators defined by:

$$S(t) := \mathcal{S}_q(t),$$
$$P(t) := t^{q-1}\mathcal{P}_q(t),$$

where $\{\mathcal{S}_q(t)\}_{t\geq 0}$ and $\{\mathcal{P}_q(t)\}_{t\geq 0}$ are defined in (3.4.3) and (3.4.4), respectively.

Define $a(t) = g_q(t)$. It is not difficult to see that $\{S(t)\}_{t\geq 0}$ is a resolvent family and $\{P(t)\}_{t\geq 0}$ is an integral family, both generated by $-A$.

With this, the mild solution of equation (3.4.1) is a solution of the problem:

$$x(t) = \mathcal{S}_q(t)(x_0 - h(0, x_0)) + h(t, x(t)) + \int_0^t (t-s)^{q-1}\mathcal{P}_q(t-s)f(s, x(s))ds, \quad t \geq 0, \tag{3.4.8}$$

We will prove that, with these ingredients, all the hypothesis of Theorem 5.4.1 are satisfied. Indeed:

(i) For (H1), we prove that the function $s \to ||A^\alpha \mathcal{P}_q(t-s)||\nu(s)$ belongs to $L^1([0,t[, \mathbb{R}^+)$. In fact, by (3.4.5), we have:

$$||A^\alpha (t-s)^{q-1}\mathcal{P}_q(t-s)||\nu(s) \leq C\frac{\nu(s)}{(t-s)^{1+q(\gamma+q)}},$$

and in view of (3.4.6), the claim is proved. Also, by (3.4.6) and (3.4.7), we have:

$$\lim_{t\to\infty}\int_0^t ||A^\alpha P(t-s)||\nu(s)ds \leq \lim_{t\to\infty}\int_0^t \frac{\nu(s)}{(t-s)^{1+q(\gamma+\alpha)}}ds = 0.$$

(ii) The hypotheses (H2) and (H3) are the same than (F2) and (F3).

(iii) The hypothesis (H4) is a consequence of the estimates (3.4.5).

(iv) We prove (H5). Indeed, by the property a), we obtain

$$\begin{aligned} ||A^\alpha \mathcal{S}_q(t)x|| &\leq \int_0^\infty \Phi_q(\sigma)||A^\alpha T(\sigma t^q)x||d\sigma \\ &\leq K\int_0^\infty \Phi_q(\sigma)(\sigma t^q)^{-\gamma-\alpha-1}d\sigma \\ &\leq K'\int_0^\infty \Phi_q(\sigma)\sigma^{-\gamma-\alpha-1}d\sigma, \end{aligned}$$

with $r = -\gamma - \alpha - 1 > -1$, by hypothesis. Then, by (2.4.5):

$$\sup_{t\geq 0}||A^\alpha \mathcal{S}_q(t)x|| < \infty.$$

(v) We will prove (H6). In fact, we have

$$
\begin{aligned}
& \int_0^{t_1} ||A^\alpha((t_2-s)^{q-1}\mathcal{P}_q(t_2-s)-(t_1-s)^{q-1}\mathcal{P}_q(t_1-s))||\nu(s)ds \\
= & \int_0^{t_1} ||(t_2-s)^{q-1}\mathcal{P}_q(t_2-s)-(t_1-s)^{q-1}\mathcal{P}_q(t_1-s)||_\alpha \nu(s)ds \\
\leq & \int_0^{t_1} ||\mathcal{P}_q(t_2-s)[(t_2-s)^{q-1}-(t_1-s)^{q-1}]||_\alpha \nu(s)ds \\
+ & \int_0^{t_1} ||(t_1-s)^{q-1}[\mathcal{P}_q(t_2-s)-\mathcal{P}_q(t_1-s)||_\alpha \nu(s)ds.
\end{aligned}
$$

By (3.4.5), we obtain:

$$
\begin{aligned}
& \int_0^{t_1} ||\mathcal{P}_q(t_2-s)[(t_2-s)^{q-1}-(t_1-s)^{q-1}]||_\alpha \nu(s)ds \\
\leq & \; qC' \tfrac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha)} \int_0^{t_1} \frac{|(t_2-s)^{q-1}-(t_1-s)^{q-1}|}{(t_2-s)^{q-1}} \frac{\nu(s)}{(t_2-s)^{1+q(\gamma+\alpha)}} ds.
\end{aligned}
$$

Therefore, by (3.4.6), we get:

$$
\int_0^{t_1} ||\mathcal{P}_q(t_2-s)[(t_2-s)^{q-1}-(t_1-s)^{q-1}]||_\alpha \nu(s)ds \to 0, \quad \text{as} \quad t_2 \to t_1.
$$

Furthermore, for $\epsilon > 0$, small enough, we obtain:

$$
\begin{aligned}
& \int_0^{t_1} ||(t_1-s)^{q-1}[\mathcal{P}_q(t_2-s)-\mathcal{P}_q(t_1-s)||_\alpha \nu(s)ds \\
\leq & \; q\int_0^{t_1}\int_0^\infty \sigma\Phi_q(\sigma)(t_1-s)^{q-1}||T((t_2-s)^q\sigma)-T((t_1-s)^q\sigma||_\alpha \nu(s)d\sigma ds \\
\leq & \; q\int_0^{t_1-2\epsilon}\int_0^\infty \sigma\Phi_q(\sigma)(t_1-s)^{q-1}||T((t_2-s)^q\sigma-\epsilon^q\sigma)-T((t_1-s)^q\sigma-\epsilon^q\sigma)||\times \\
& \qquad\qquad \times ||A^\alpha T(\epsilon^q\sigma)||\nu(s)d\sigma ds \\
& +M_2\int_{t_1-2\epsilon}^{t_1}\left(\frac{(t_1-s)^{q-1}}{(t_1-s)^{q(\alpha+\gamma+1)}}+\frac{(t_1-s)^{q-1}}{(t_2-s)^{q(\alpha+\gamma+1)}}\right)\nu(s)ds \\
\leq & \; \frac{qC'}{\epsilon^{q(\gamma+q+1)}}\int_0^{t_1-2\epsilon}\int_0^\infty \sigma^{-(\gamma+q)}\Phi_q(\sigma)||T((t_2-s)^q\sigma-\epsilon^q\sigma)-T((t_1-s)^q\sigma-\epsilon^q\sigma)||\times \\
& \qquad\qquad \times \tfrac{\nu(s)}{(t_1-s)^{1-q}} d\sigma ds \\
& +M_2\int_{t_1-2\epsilon}^{t_1}\left(\frac{(t_1-s)^{q-1}}{(t_1-s)^{q(\alpha+\gamma+1)}}+\frac{(t_1-s)^{q-1}}{(t_2-s)^{q(\alpha+\gamma+1)}}\right)\nu(s)ds \\
= & \; J_1+J_2.
\end{aligned}
$$

The continuity of the function $t \to ||T(t)||$ for $t \in (0,T)$ implies that:

$$
J_1 \to 0 \quad \text{as} \quad t_2 \to t_1.
$$

Furthermore, it is easy to see that:

$$J_2 \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Therefore:

$$\lim_{t_2 \to t_1} \int_0^{t_1} ||A^\alpha((t_2 - s)^{\alpha-1}\mathcal{P}_\alpha(t_2 - s) - (t_1 - s)^{\alpha-1}\mathcal{P}_q(t_1 - s))||\nu(s)ds = 0.$$

$\square$

Chapter 4

The fractional Cauchy problem and the discrete fractional Laplacian.

In this chapter we present the main results of the joint paper [21], that concerns with the study of the explicit representation of the unique solution for the following time/space fractional evolution equation:

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) + (-\Delta_d)^{\alpha} u(n,t) = g(n,t), & n \in \mathbb{Z},\ t > 0. \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n) & n \in \mathbb{Z}, \end{cases} \tag{4.0.1}$$

in terms of generalized Wright functions. The parameters α and β belong to $(0,1]$ and $(0,2]$ respectively, that will be analyzed separately. We recall that $(-\Delta_d)^{\alpha}$ is the discrete fractional Laplace operator defined below, and $\mathbb{D}_t^{\beta}$ is the Caputo fractional derivative of order β.

We will prove that the unique solution of the system (4.0.1) has the following representation involving the functions $G_{\alpha,\beta}(n,t)$, $H_{\alpha,\beta}(n,t)$ and $L_{\alpha,\beta}(n,t)$, which are given, for $n \in \mathbb{Z}$ and $t \geq 0$:

$$
\begin{aligned}
G_{\alpha,\beta}(n,t) &:= (-1)^n \; {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ \\ (1,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right] \\
&= \sum_{j=0}^{\infty} (-1)^j K^{\alpha j}(n) g_{\beta j+1}(t), \qquad (4.0.2)
\end{aligned}
$$

and

$$
\begin{aligned}
H_{\alpha,\beta}(n,t)) &:= (-1)^n t \; {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ \\ (2,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right] \\
&= \sum_{j=0}^{\infty} (-1)^j K^{\alpha j}(n) g_{\beta j+2}(t), \qquad (4.0.3)
\end{aligned}
$$

and

$$
\begin{aligned}
L_{\alpha,\beta}(n,t)) &:= (-1)^n t^{\beta-1} \; {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ \\ (\beta,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right] \\
&= \sum_{j=0}^{\infty} (-1)^j K^{\alpha j}(n) g_{\beta j+\beta}(t). \qquad (4.0.4)
\end{aligned}
$$

We call $G_{\alpha,\beta}(\cdot,t)$ the *fundamental solution* of the system (4.0.1). Using the classical Stirling's formula (2.5.2) and the Fubini-Tonelli Theorem, we shall show that $G_{\alpha,\beta}(\cdot,t)$, $H_{\alpha,\beta}(\cdot,t)$ and $L_{\alpha,\beta}(\cdot,t)$ belong to $\ell^1(\mathbb{Z})$. This result will allow us to solve the system (4.0.1) for arbitrary initial values $\varphi,\psi \in \ell^\infty(\mathbb{Z})$.

4.1 The discrete fractional Laplacian

In this section, we recall the definition of the *discrete fractional Laplacian*, denoted by$(-\Delta_d)^\alpha$ for $0 < \alpha \le 1$. This operator has been deeply treated in [15, 16, 21, 46] and it representation is given by

$$(-\Delta_d)^\alpha f(n) = \sum_{k\in\mathbb{Z}} K^\alpha(n-k)f(k), \quad n \in \mathbb{Z},\ f \in \ell^p(\mathbb{Z}),\ 1 \le p \le \infty, \tag{4.1.1}$$

where the coefficients K^α are given by

$$K^\alpha(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} (4\sin^2(\theta/2))^\alpha e^{-in\theta} d\theta = \frac{(-1)^n\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)}, \quad n \in \mathbb{Z}. \tag{4.1.2}$$

We recall that for a given sequence $f = (f(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, the discrete Fourier transform is defined by

$$\mathcal{F}_{\mathbb{Z}}(f)(\theta) = \sum_{n\in\mathbb{Z}} f(n)e^{in\theta} = \sum_{n\in\mathbb{Z}} f(n)z^n \equiv \hat{f}(z), \quad \theta \in \mathbb{T}, \quad z = e^{i\theta}, \tag{4.1.3}$$

where $\mathbb{T} \equiv \mathbb{R}/(2\pi\mathbb{Z})$ is the one-dimensional torus, that we identify with the interval $[-\pi, \pi]$.

Given $\varphi \in L^1[-\pi, \pi]$, its inverse discrete Fourier transform is given by the formula

$$\mathcal{F}_{\mathbb{Z}}^{-1}(\varphi)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}.$$

Therefore

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}_{\mathbb{Z}}(f)(\theta)e^{-in\theta}\, d\theta = \frac{1}{2\pi i}\int_{|z|=1} z^{n-1}\hat{f}(z)dz, \quad n \in \mathbb{Z}.$$

The convolution theorem

$$\mathcal{F}_{\mathbb{Z}}(f * g) = \mathcal{F}_{\mathbb{Z}}(f)\mathcal{F}_{\mathbb{Z}}(g),$$

holds, where $*$ denotes the usual discrete convolution in $\ell^1(\mathbb{Z})$ given by

$$(f * g)(n) := \sum_{k\in\mathbb{Z}} f(n-k)g(k).$$

In what follows in this chapter, for simplicity, we denote $\mathcal{F} := \mathcal{F}_{\mathbb{Z}}$ and $\mathcal{F}(f)(z) := \hat{f}(z)$. Also we let

$$J(z) := z + \frac{1}{z} - 2, \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.1.4}$$

We observe that the function $J(z)$ is a translation of the well known Joukowsky transform. It is straightforward to see that for $|z|=1$ the values of $J(z)$ lie in the interval $[-4,0]$.

We conclude this subsection with the following observation.

Remark 4.1.1. The following properties hold.

(i) $-J(z)\geq 0$ for all $z\in\mathbb{C},\ |z|=1$.

(ii) $J(z)=-4\sin^2(\theta/2)$, if $z=e^{i\theta}$.

(iii) $\mathcal{F}(K^\alpha(n))(z)=(-J(z))^\alpha$ for all $z\in\mathbb{C},\ z=e^{i\theta},\ \theta\in\mathbb{R}$, where K^α is given in (4.1.2).

4.2 Explicit solution in the superdiffusive case

In this section, we study the equation (4.0.1), considering the parameters $\alpha\in(0,1]$ and $\beta\in(1,2]$. The main result is presented in the following Theorem.

Theorem 4.2.1. *Let $\varphi,\phi\in\ell^\infty(\mathbb{Z})$ and $g:\mathbb{Z}\times\mathbb{R}^+\to\mathbb{C}$ be such that, for each $t>0$, $g(\cdot,t)\in\ell^\infty(\mathbb{Z})$ and $\sup_{s\in[0,t]}||g(\cdot,s)||_{\ell^\infty(\mathbb{Z})}<\infty$. Then the function*

$$\begin{aligned}u(n,t)=&\sum_{m\in\mathbb{Z}}G_{\alpha,\beta}(n-m,t)\varphi(m)+\sum_{m\in\mathbb{Z}}H_{\alpha,\beta}(n-m,t)\phi(m)\\&+\sum_{m\in\mathbb{Z}}\int_0^t L_{\alpha,\beta}(n-m,t-s)g(m,s)ds,\end{aligned}\tag{4.2.1}$$

is the unique solution of the initial value problem (4.0.1)*, where $G_{\alpha,\beta}(n,t)$, $H_{\alpha,\beta}(n,t)$) and $L_{\alpha,\beta}(n,t)$) are given by* (4.0.2), (4.0.3) *and* (4.0.4)*, respectively. Furthermore, $G_{\alpha,\beta}(\cdot,t),H_{\alpha,\beta}(\cdot,t)$, $L_{\alpha,\beta}(\cdot,t)\in\ell^1(\mathbb{Z})$, for every $t>0$ and $u(\cdot,t)\in\ell^\infty(\mathbb{Z})$, for every $t>0$.*

Proof. We prove the result in several steps.

Step 1. First we show the explicit solution. Taking the discrete Fourier transformation of (4.0.1), we obtain that

$$\begin{cases} \mathbb{D}_t^\beta \hat{u}(z,t) = -(-J(z))^\alpha \hat{u}(z,t) + \hat{g}(z,t) \\ \hat{u}(z,0) = \hat{\varphi}(z),\ \hat{u}_t(z,0) = \hat{\phi}(z). \end{cases} \tag{4.2.2}$$

Now, taking Laplace's transformation to (4.2.2), we get

$$\mathcal{L}(\hat{u})(z,\lambda) = \frac{\lambda^{\beta-1}}{\lambda^\beta + (-J(z))^\alpha}\hat{\varphi}(z) + \frac{\lambda^{\beta-2}}{\lambda^\beta + (-J(z))^\alpha}\hat{\phi}(z)(z) + \frac{1}{\lambda^\beta + (-J(z))^\alpha}\hat{g}(z,\lambda).$$

From (2.4.2), we get the inverse Laplace transform for (4.2), given by

$$\begin{aligned} \hat{u}(z,t) =& E_{\beta,1}\left(-(-J(z))^\alpha t^\beta\right)\hat{\varphi}(z) + tE_{\beta,2}\left(-(-J(z))^\alpha t^\beta\right)\hat{\phi}(z) \\ &+ t^{\beta-1}E_{\beta,\beta}\left(-(-J(z))^\alpha t^\beta\right) \star \hat{g}(z,t). \end{aligned} \tag{4.2.3}$$

Calculating the inverse discrete Fourier transform of (4.2.3) and using (4.1.1) we get that

$$E_{\beta,1}(-(-J(z))^\alpha t^\beta) = E_{\beta,1}\left(-4^\alpha \sin^{2\alpha}(\theta/2)\, t^\beta\right) = \sum_{k=0}^{\infty} \frac{(-4^\alpha \sin^{2\alpha}\left(\frac{\theta}{2}\right) t^\beta)^k}{\Gamma(\beta k+1)}. \tag{4.2.4}$$

Thus

$$\begin{aligned} G_{\alpha,\beta}(n,t) &:= \mathcal{F}^{-1}\left((E_{\beta,1}(-(-J(z))^\alpha t^\beta)\right)(n) \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta n}\sum_{k=0}^{\infty}\frac{(-4^\alpha \sin^{2\alpha}\left(\frac{\theta}{2}\right)t^\beta)^k}{\Gamma(\beta k+1)}d\theta \\ &= \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k 4^{\alpha k} t^{\beta k}}{\Gamma(\beta k+1)}\int_0^{2\pi} e^{-i\theta n}\sin^{2\alpha k}(\theta/2)\,d\theta. \end{aligned} \tag{4.2.5}$$

Using the change of variable $\frac{\theta}{2} = \rho$, we get that

$$\int_0^{2\pi} e^{-i\theta n}\sin^{2\alpha k}(\theta/2)\,d\theta = 2\int_0^{\pi} e^{ib\rho}(\sin(\rho))^{a-1}d\rho, \quad b=-2n, \quad a = 2\alpha k+1.$$

It follows from the integral representation (2.5.3) that

$$\int_0^{2\pi} e^{-i\theta n}\sin^{2\alpha k}(\theta/2)d\theta = \frac{2\pi}{2^{2\alpha k}}\frac{(-1)^n}{(2\alpha k+1)B(\alpha k-n+1,\alpha k+n+1)}, \tag{4.2.6}$$

Now, using the relation $\Gamma(z+1) = z\Gamma(z)$ together with (4.2.5) and (4.2.6) we get that

$$G_{\alpha,\beta}(n,t) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k 4^{\alpha k}t^{\beta k}}{\Gamma(\beta k+1)}\frac{2\pi}{2^{2\alpha k}}\frac{(-1)^n}{(2\alpha k+1)B(\alpha k-n+1,\alpha k+n+1)}$$

$$=(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k t^{\beta k}}{\Gamma(\beta k+1)} \frac{\Gamma(2\alpha k+2)}{(2\alpha k+1)\Gamma(\alpha k-n+1)\Gamma(\alpha k+n+1)}.$$

We have shown that

$$G_{\alpha,\beta}(n,t) = (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t^\beta)^k}{k!}. \tag{4.2.7}$$

From the definition of the generalized Wright function (2.4.11), we can deduce

$$G_{\alpha,\beta}(n,t) = (-1)^n \ {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ & & \\ (1,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right].$$

Furthermore, notice that by (2.4.1), we have

$$tE_{\beta,2}(-(-J(z))^\alpha t^\beta) = \int_0^t E_{\beta,1}(-(-J(z))^\alpha x^\beta)dx. \tag{4.2.8}$$

We shall use this fact to derive the expression $H_{\alpha,\beta}$. In fact, by (4.2.8), we have

$$\begin{aligned} H_{\alpha,\beta}(n,t)) :=& \mathcal{F}^{-1}(tE_{\beta,2}(-(-J(z))^\alpha t^\beta)) \\ =&(-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)(\beta k+1)} \frac{(-1)^k t^{\beta k+1}}{k!} \\ =&(-1)^n t \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t^\beta)^k}{k!} \\ =&(-1)^n t \ {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ (2,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^\beta \right]. \end{aligned} \tag{4.2.9}$$

Analogously to the computations of $G_{\alpha,\beta}(n,t)$, we obtain

$$\begin{aligned} L_{\alpha,\beta}(n,t)) :=& \mathcal{F}^{-1}(t^{\beta-1}E_{\beta,\beta}(-(-J(z))^\alpha t^\beta))(n) \\ =& (-1)^n t^{\beta-1} \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t^\beta)^k}{k!} \\ =& (-1)^n t^{\beta-1} \ {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ & & \\ (\beta,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^k \right]. \end{aligned} \tag{4.2.10}$$

We have shown that the solution $u(n,t)$ of (4.0.1) has the representation (4.2.1). In addition since $\kappa = \beta - 1 > -1$ in each of the series (4.2.7), (4.2.9) and (4.2.10), it follows from Theorem 2.4.3(i) that the series converge uniformly and that $G_{\alpha,\beta}(n,z), H_{\alpha,\beta}(n,z)$, and $L_{\alpha,\beta}(n,z)$ are entire functions of z.

Step 2. We show uniqueness. Assume that the system (4.0.1) has two solutions u_1 and u_2 with the same initial data φ, ϕ, and set $v := u_1 - u_2$. Then v is a solution of the following

$$\mathbb{D}_t^\beta \hat{v}(z,t) = -(-J(z))^\alpha \hat{v}(z,t), \;\; \hat{v}(z,0) = 0, \;\; \hat{v}_t(z,0) = 0.$$

Since the above ordinary differential equation has zero has its unique solution, we have that $\hat{v}(z,t) = 0$. The uniqueness of the inverse discrete Fourier transform implies that $v(n,t) = 0$ for every $n \in \mathbb{Z}$ and $t \geq 0$. Hence, $u_1 = u_2$.

Step 3. Next, we show that $G_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$, for every $t > 0$. Indeed, it follows from the representation (4.2.7) that

$$G_{\alpha,\beta}(n,t) = G_{\alpha,\beta}(-n,t) \;\text{ for every }\; n \in \mathbb{Z} \;\text{ and }\; t \geq 0.$$

Therefore, using the Fubini-Tonelli Theorem we get that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |G_{\alpha,\beta}(n,t)| &\leq \sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!} \\ &\leq 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!} \\ &\leq 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}.\end{aligned}$$

Let $c_k = \dfrac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)|\Gamma(\alpha k-n+1)|}\dfrac{1}{k!}$. Using (2.4.12), we obtain that, for k sufficiently large,

$$|c_k| \sim \frac{(\alpha k)^n}{(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^\beta}{\alpha^\alpha}\right)^{-k}. \tag{4.2.11}$$

Therefore, for k_0 sufficiently large, it follows from (4.2.11) that

$$2\sum_{k=k_0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^\beta)^k}{k!}$$

$$
\begin{aligned}
&\sim 2\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}\sum_{n=0}^{\infty}\frac{(\alpha k)^n}{n!}\\
&=2\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}e^{\alpha k}\\
&=\frac{2}{(\pi\beta)^{\frac{1}{2}}}\sum_{k=k_0}^{\infty}\frac{k^{-(\beta-\alpha)k}}{k^{\frac{1}{2}}}(et)^{\beta k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}.
\end{aligned}
$$

Since $\beta > \alpha$ then, applying the root test, we can deduce that $G_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$, for every $t > 0$.

Step 4. Next we show $H_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$ for every $t > 0$. For $t > 0$, we have that

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}}|H(n,t)| &\le t\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^{\beta})^k}{k!}\\
&\le 2t\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^{\beta})^k}{k!}\\
&\le 2t\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^{\beta})^k}{k!}.
\end{aligned}
$$

Letting $d_k = \dfrac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)|\Gamma(\alpha k-n+1)|}\dfrac{1}{k!}$ and using (2.4.12), we obtain that, for k sufficiently large,

$$
|d_k| \sim \frac{(\alpha k)^n}{k(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}. \tag{4.2.12}
$$

Therefore, for k_0 sufficiently large, it follows from (4.2.12) that

$$
\begin{aligned}
&2t\sum_{k=k_0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(2+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^{\beta})^k}{k!}\\
&\sim 2t\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{k(\pi\beta k)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}\sum_{n\in\mathbb{N}_0}\frac{(\alpha k)^n}{n!}\\
&=\frac{2t}{(\pi\beta)^{\frac{1}{2}}}\sum_{k=k_0}^{\infty}\frac{k^{-(\beta-\alpha)k}}{k^{\frac{3}{2}}}(et)^{\beta k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}.
\end{aligned}
$$

Since $\beta > \alpha$, then using the root test again, we get that $H_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$, for every $t > 0$.

Step 5. Finally we show that $L_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$ for every $t > 0$. Proceeding as above, for each $t > 0$, we have that

$$
\sum_{n\in\mathbb{Z}}|L(n,t)| \le t^{\beta-1}\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|}\frac{(t^{\beta})^k}{k!}
$$

$$\leq 2t^{\beta-1}\sum_{k=0}^{\infty}\sum_{n\in\mathbb{N}_0}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^{\beta})^k}{k!}.$$

Letting $r_k = \dfrac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)|\Gamma(\alpha k-n+1)|}\dfrac{1}{k!}$ and using (2.4.12) again, we obtain that, for k sufficiently large,

$$|r_k| \sim \frac{(\alpha k)^n}{k^{\beta-\frac{1}{2}}(\pi\beta)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}. \tag{4.2.13}$$

Therefore, for k_0 sufficiently large (by using (4.2.13)) we have that

$$\begin{aligned}
&2t^{\beta-1}\sum_{k=k_0}^{\infty}\sum_{n\in\mathbb{N}_0}\frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(\beta+\beta k)\Gamma(n+1)|\Gamma(\alpha k-n+1)|}\frac{(t^{\beta})^k}{k!}\\
&\sim 2t^{\beta-1}\sum_{k=k_0}^{\infty}\frac{t^{\beta k}}{k^{\beta-\frac{1}{2}}(\pi\beta)^{\frac{1}{2}}}\left(\frac{k}{e}\right)^{-(\beta-\alpha)k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}\sum_{n\in\mathbb{N}_0}\frac{(\alpha k)^n}{n!}\\
&= \frac{2t^{\beta-1}}{(\pi\beta)^{\frac{1}{2}}}\sum_{k=k_0}^{\infty}\frac{k^{-(\beta-\alpha)k}}{k^{\beta-\frac{1}{2}}}(et)^{\beta k}\left(\frac{2\beta^{\beta}}{\alpha^{\alpha}}\right)^{-k}.
\end{aligned}$$

Since $\beta > \alpha$, we can deduce that $L_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$, for every $t > 0$.

Next, we prove that the solution $u(n,t)$ belongs to $\ell^1(\mathbb{Z})$, for each $t > 0$. Indeed, from Fubini-Tonelli Theorem and the Young inequality, we have

$$\begin{aligned}
||u(\cdot,t)||_{\ell^\infty(\mathbb{Z})} \leq{}& ||G_{\alpha,\beta}(\cdot,t)||_{\ell^1(\mathbb{Z})}||\varphi(\cdot,t)||_{\ell^\infty(\mathbb{Z})} + ||H_{\alpha,\beta}(\cdot,t)||_{\ell^1(\mathbb{Z})}||\phi(\cdot,t)||_{\ell^\infty(\mathbb{Z})}\\
&+\int_0^t ||L_{\alpha,\beta}(\cdot,s)||_{\ell^1(\mathbb{Z})}||g(\cdot,t-s)||_{\ell^\infty(\mathbb{Z})}\,ds.
\end{aligned}$$

It is clear that the first and the second terms are finite. For the third term, using the hypothesis for the function $g(n,t)$, we get

$$\int_0^t ||L_{\alpha,\beta}(\cdot,t-s)||_{\ell^1(\mathbb{Z})}||g(\cdot,s)||_{\ell^\infty(\mathbb{Z})}\,ds \leq \sup_{s\in[0,t]}||g(\cdot,s)||_{\ell^\infty(\mathbb{Z})}\int_0^t ||L_{\alpha,\beta}(\cdot,t-s)||_{\ell^1(\mathbb{Z})}\,ds < \infty.$$

Therefore, the solution $u(\cdot,t) \in \ell^\infty(\mathbb{Z})$. The proof of the theorem is finished. □

Remark 4.2.2. We mention that the fundamental solution for the continuous models similar to the semi-discrete problem that is under study in this chapter was investigated in [49] by Mainardi, Luchko and Pagnini. Here we have defined the fundamental solution for the equation of order $\beta \in (1,2]$ *by requiring that the initial value* φ *be the sequence* $\varphi(n)$ *such that* $\varphi(0) = 1$ *and* $\varphi(n) = 0$ *for all* $n \neq 0$ *and the initial velocity is* $\phi \equiv 0$. *In the study of the wave equation, it is sometimes the* *choice* $\varphi \equiv 0$ *and* $\phi(n) = \delta_0^n$ *(the Kronecker symbol) that is used. It is easy to see that when* $\beta = 2$ *then* $H_{\alpha,\beta}(n,t)$ *and* $L_{\alpha,\beta}(n,t)$ *coincide. In fact, in the study of the second order non-homogeneous evolution equation*

$$\begin{cases} w''(t) = Aw(t) + g(t), \quad t > 0, \\ w(0) = x, \\ w'(0) = y, \end{cases} \tag{4.2.14}$$

in a Banach space X*, if we assume that the problem is well posed, thus* A *is the infinitesimal generator of a strongly continuous cosine function* $(C(t))_{t\in\mathbb{R}}$*, denoting the associated sine function by* $(S(t))_{t\geq 0}$*, the mild solution is given by:*

$$w(t) = C(t)x + S(t)y + \int_0^t S(t-s)g(s)ds,\ t \geq 0.$$

Therefore, two operator families suffice to describe the solutions of the problem. This subject is treated in [7, Section 3.14]. In the fractional case with $1 < \beta < 2$ *three operator families are needed in the representation of the solutions (see e.e. [34, 35]). However, we have the following relations between* $G_{\alpha,\beta}(n,t)$*,* $H_{\alpha,\beta}(n,t)$ *and* $L_{\alpha,\beta}(n,t)$*.*

Remark 4.2.3. *Let* $t \geq 0$ *and* $n \in \mathbb{Z}$*. Then*

$$H_{\alpha,\beta}(n,t) = \int_0^t G_{\alpha,\beta}(n,s)\, ds$$

and

$$L_{\alpha,\beta}(n,t) = \frac{1}{\Gamma(\beta-1)}\int_0^t (t-s)^{\beta-2}G_{\alpha,\beta}(n,s)ds = g_{\beta-1} \star G_{\alpha,\beta}(n,t),$$

where for any $\gamma > 0$ *and* $t > 0$*. We recall that* $g_\gamma(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}$*.*

Remark 4.2.4. *From the representation series* (4.2.7) *and using the definition for* g_α *we obtain the following identity*

$$G_{\alpha,\beta}(n,t) = \sum_{j=0}^{\infty} K^{\alpha j}(n)\frac{(-t^\beta)^j}{\Gamma(1+\beta j)} = \sum_{j=0}^{\infty}(-1)^j K^{\alpha j}(n) g_{\beta j+1}(t).$$

Interpreting the above series as an internal product, we observe that we can separate the variables n *and* t *in two sequences indexed by* j*. One depending on* α *and* n *and the other depending on* β *and* t*. These sequences tell us the different role that play* α *and* β *in the equation. Since* α *is linked to*

$K^{\alpha j}(n)$, it means that this sequence is the responsible to control the lattice-behavior of the equation i.e. the discrete fractional Laplacian operator and, on the other hand, β is linked to $\frac{(-t^\beta)^j}{\Gamma(1+\beta j)}$ which means that this term expresses the behavior of the time-fractional derivative. For example, $\beta = 2$ produces $\frac{(-1)^j t^{2j}}{(2j)!}$ which corresponds to the coefficient in the series of the cosine function, i.e. the wave equation, and for $\beta = 1$ we have $\frac{(-1)^j t^j}{j!}$ that corresponds to the coefficient in the series of the exponential function, i.e. the heat equation. For instance, $G_{\alpha,2}(n,t) = \sum_{j=0}^{\infty} K^{\alpha j}(n)\frac{(-1)^j t^{2j}}{(2j)!}$ represents, when reading the coefficients in such way, the wave equation combined with the discrete fractional Laplacian of order α. We also note that

$$H_{\alpha,\beta}(n,t) = \sum_{j=0}^{\infty} K^{\alpha j}(n)\frac{(-1)^j t^{\beta j+1}}{\Gamma(2+\beta j)},$$

and

$$L_{\alpha,\beta}(n,t) = \sum_{j=0}^{\infty} K^{\alpha j}(n)\frac{(-1)^j t^{\beta j+\beta-1}}{\Gamma(\beta+\beta j)},$$

thanks to the semigroup property $g_\gamma \star g_\delta = g_{\gamma+\delta}$ valid for all $\gamma, \delta > 0$.

The following subordination principle is a direct consequence of Theorem 4.2.1.

Corollary 4.2.5. *Let $0 < \alpha \leq 1$ and $1 < \beta < 2$. Then the fundamental solution $G_{\alpha,\beta}(n,t)$ of* (4.0.1) *has an integral representation given by*

$$G_{\alpha,\beta}(n,t) = \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) G_{\alpha,2}(n, \tau t^{\frac{\beta}{2}}) d\tau, \quad t > 0,\ n \in \mathbb{Z}, \tag{4.2.15}$$

where $\Phi_{\frac{\beta}{2}}$ is the Wright function defined by (2.4.4)*.*

Proof. Indeed, using (4.2.7) we obtain that for every $t > 0$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
G_{\alpha,\beta}(n,t) &= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)\Gamma(k+1)}{\Gamma(1+\beta k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t^\beta)^k}{k!} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)(-1)^k t^{\beta k}}{\Gamma(1+2k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{\Gamma(2k+1)}{\Gamma(\beta k+1)} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)(-1)^k t^{\beta k}}{\Gamma(1+2k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau)\tau^{2k}\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau)(-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)(-1)^k}{\Gamma(1+2k)\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)}(\tau^2 t^\beta)^k \, d\tau \\
&= \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) G_{\alpha,2}(n, \tau t^{\frac{\beta}{2}}) \, d\tau.
\end{aligned}
$$

We have shown (4.2.15) and the proof is finished. □

In this section we now discuss several special cases of interest which are a direct consequence of Theorem 4.2.1.

4.2.1 The semi-discrete wave equation

This corresponds to the case $(\alpha, \beta) = (1, 2)$. The corresponding equation is:

$$
\begin{cases}
u_{tt}(n,t) = \Delta_d u(n,t) + g(n,t), \quad t > 0, & n \in \mathbb{Z} \\
u(n,0) = \varphi(n) & n \in \mathbb{Z}, \\
u_t(n,0) = \phi(n) & n \in \mathbb{Z}.
\end{cases}
\tag{4.2.16}
$$

We derive the following complementary result.

Corollary 4.2.6. *Let $\varphi, \phi \in \ell^\infty(\mathbb{Z})$ and $g : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{C}$ be such that, for each $t > 0$, $g(\cdot, t) \in \ell^\infty(\mathbb{Z})$ and $\sup_{s\in[0,t]} ||g(\cdot, t)||_\infty < \infty$. Then the wave equation (4.2.16) admits a unique solution in $\ell^1(\mathbb{Z})$ given by*

$$
\begin{aligned}
u(n,t) = &\sum_{m\in\mathbb{Z}} J_{2(n-m)}(2t)\varphi(m) + \sum_{m\in\mathbb{Z}} \int_0^t J_{2(n-m)}(2x)\phi(m) \, dx \\
&+ \sum_{m\in\mathbb{Z}} \int_0^t \left(\int_0^{t-s} J_{2(n-m)}(2x) \, dx \right) g(m,s) \, ds.
\end{aligned}
\tag{4.2.17}
$$

Proof. Taking $\alpha = 1$ and $\beta = 2$ in (4.2.7), we get that

$$\begin{aligned}
G_{1,2}(n,t) &= (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2k+1)(-1)^k}{\Gamma(2k+1)\Gamma(k+n+1)\Gamma(k-n+1)} t^{2k} \\
&= (-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+n+1)\Gamma(k-n+1)} t^{2k} \\
&= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{\Gamma(k+2n+1)k!} t^{2(k+n)} \\
&= J_{2n}(2t).
\end{aligned} \tag{4.2.18}$$

Since $H_{1,2}(n,t) = L_{1,2}(n,t)$, it follows from (4.2.8) that $H_{1,2}(n,t) = \int_0^t G_{1,2}(n,x)dx$. The proof is finished. $\square$

We define the following operator family, called semidiscrete wave cosine function [46, Formula (11)]

$$C_t f(n) := \sum_{m\in\mathbb{Z}} G_{1,2}(n-m,t)f(m) = \sum_{m\in\mathbb{Z}} J_{2(n-m)}(2t)f(m). \tag{4.2.19}$$

As observed in [46, Theorem 1.2 (i)], we note that $C_t \in \mathcal{B}(\ell^2(\mathbb{Z}))$. In fact

$$\sum_{n\in Z} |G_{1,2}(n,t)| = \sum_{n\in Z} |J_{2n}(2t)| = \sum_{n\in\mathbb{Z}} I_{2n}(2t) \leq \sum_{n\in\mathbb{Z}} I_n(2t) = e^t,$$

and the claim follows from Young's inequality.

The corresponding representation of the solution can be found in [10, Formula (4.2)] in the case where $g \equiv 0$. The author in [10] provides a wealth of concrete situations, e.g. weather prediction by numerical processes, the theory of elasticity among others where the equations under consideration here (heat and wave type equations) play an important role.

From the relation $J_k(-t) = (-1)^k J_k(t)$, we see that we can extend $(C_t)_{t\geq 0}$ to $(C_t)_{t\in\mathbb{R}}$ by setting $C_t = C_{-t}$. We shall prove the following result.

Theorem 4.2.7. *Let $f \in \ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$. Then the family $\{C_t\}_{t\geq 0}$ satisfies the following properties.*

(i) $C_0 f = f$.

(ii) $C_{t+s}f + C_{t-s}f = 2C_tC_sf.$

(iii) $\lim_{t\to 0} C_tf = f$ *in* $\ell^1(\mathbb{Z})$.

Proof. (i) Using (2.4.16) and the definition of C_t we get that

$$\begin{aligned} C_0f(n) &= \sum_{m\in\mathbb{Z}} J_{2m}(0)f(n-m) \\ &= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+2m+1)}(0)^{2(k+m)}f(n-m) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-k+1)}f(n+k). \end{aligned}$$

Since if $m+k \neq 0$, all those terms in the sum are 0, whence only appear the term $k=-m$, obtaining the last expression. Now we note that for $k \geq 1$, the terms in the last sum are 0. Therefore, only appears the term when $k=0$, obtaining $C_tf(n) = f(n)$.

(ii) We prove the cosine property of C_t. Using (2.4.17), (2.4.19) and the Fubini-Tonelli Theorem, we obtain

$$\begin{aligned} C_{t+s}f(n) + C_{t-s}f(n) &= \sum_{m\in\mathbb{Z}} J_{2(n-m)}(t+s)f(m) + \sum_{m\in\mathbb{Z}} J_{2(n-m)}(t-s)f(m) \\ &= \sum_{m\in\mathbb{Z}} \big(J_{2(n-m)}(t+s) + J_{2(n-m)}(t-s)\big)f(m) \\ &= \sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} J_{2(n-m)-l}(2t)J_k(2s) + \sum_{l\in\mathbb{Z}}(-1)^l J_{2(n-m)-l}(2t)J_l(2s)\right) f(m) \\ &= 2\sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} J_{2(n-m-l)}(2t)J_{2l}(2s)\right) f(m) \\ &= 2\sum_{m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} J_{2(n-l)}(2t)J_{2(l-m)}(2s)\right) f(m) \\ &= 2\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} J_{2(n-l)}(2t)J_{2(l-m)}(2s)f(m) \\ &= 2\sum_{l\in\mathbb{Z}} J_{2(n-l)}(2t)\sum_{m\in\mathbb{Z}} J_{2(l-m)}(2s)f(m) \\ &= 2\sum_{l\in\mathbb{Z}} J_{2(n-l)}(2t)C_sf(l) = 2C_t(C_sf(n)) \end{aligned}$$

(iii) Let $\mathcal{M}_t(n) = J_{2n}(2t)$ and $\delta_0(n) = 1$ if $n = 0$ and 0 in other case. We notice that

$$(C_t f - f)(n) = \sum_{k\in\mathbb{Z}} (\mathcal{M}_t(n-k) - \delta_0(n-k)) f(k) = ((\mathcal{M}_t - \delta_0) * f)(n). \tag{4.2.20}$$

From the Young inequality, it is enough to estimate $||\mathcal{M}_t - \delta_0||_{\ell^1(\mathbb{Z})}$. It follows from (2.4.20) that

$$|J_{2n}(2t)| = \frac{2t^{2n}}{\sqrt{\pi}\Gamma(2n+\frac{1}{2})} \leq \frac{2t^{2n}}{\sqrt{\pi}\Gamma(2n+1)} \frac{\Gamma(2n+1)}{\Gamma(2n+\frac{1}{2})}.$$

Using Stirling's formula (2.5.2) for $|z|$ large enough, we have that

$$\frac{\Gamma(2n+1)}{\Gamma(2n+\frac{1}{2})} = \frac{\Gamma(\frac{4n+1}{2})}{\Gamma(2n+1)} \sim \frac{\left(\frac{4n+1}{2}\right)^{\frac{4n+1}{2e}}}{\left(\frac{2n+1}{e}\right)^{2n+1}} \frac{\sqrt{(4n+1)\pi}}{\sqrt{(2n+1)2\pi}} \leq \left(\frac{4n+1}{2n+1}\right)^{\frac{4n+1}{2}} \frac{\sqrt{e}}{(2n+1)^{\frac{1}{2}}} 2^{-\frac{4n+1}{2}}.$$

Since $\frac{4n+1}{2n+1} < 2$, it follows that

$$\left(\frac{4n+1}{2n+1}\right)^{\frac{4n+1}{2}} \frac{\sqrt{e}}{(2n+1)^{\frac{1}{2}}} 2^{-\frac{4n+1}{2}} < 2^{\frac{4n+1}{2}} \sqrt{e} 2^{-\frac{4n+1}{2}} = \sqrt{e}.$$

Therefore,

$$|J_{2n}(2t)| \leq M \frac{t^{2n}}{\Gamma(2n+1)}, \;\; M > 0. \tag{4.2.21}$$

Using (2.4.17) and (4.2.21), we get that

$$\begin{aligned} ||\mathcal{M}_t - \delta_0||_{\ell^1(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} |(\mathcal{M}_t - \delta_0)(n)| = \sum_{n\neq 0} |J_{2n}(2t)| + |J_0(2t) - 1| \\ &= 2\sum_{n=1}^{\infty} |J_{2n}(2t)| + |J_0(2t) - 1| \\ &\leq 2\sum_{n=1}^{\infty} M \frac{t^{2n}}{\Gamma(2n+1)} + |J_0(2t) - 1| \\ &= 2M(\cosh(t) - 1) + |J_0(2t) - 1|. \end{aligned}$$

Thus $||\mathcal{M}_t - \delta_0||_{\ell^1(\mathbb{Z})} \to 0$ when $t \to 0$ and we have shown that $\lim_{t\to 0} C_t f = f$, in $\ell^1(\mathbb{Z})$. The proof is finished. □

Remark 4.2.8. *We mention the following facts.*

(a) From Theorem 4.2.7 (iii), we note that $||C_t|| \leq Me^t, \; t > 0,$ *since* $|J_0(2t)| \leq 1, t \in \mathbb{R}$.

(b) Theorem 4.2.7 follows from [46, Theorem 1.2 (i)] and is the counterpart of [14, Proposition 1] where the case of the heat semigroup W_t *is proved. We stress that the point is to have a*

concrete verification using the properties of the special functions of mathematical physics, for the corresponding properties follow from the general theory of differential equations. The case of the heat equation is covered by semigroup theory whereas the case of the wave equation is governed by cosine function theory (see e.g. [7, Section 3.14] and [19]). The latter reference also covered some semigroup theory and the relation to Hadamard's principle is discussed.

4.2.2 The super diffusive case

In this subsection, we analyze the case $\alpha = 1$ and $1 < \beta \leq 2$. This is given by the system

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = \Delta_d u(n,t) + g(n,t), & t > 0, \quad n \in \mathbb{Z}, \\ u(n,0) = \varphi(n), & \qquad\quad\; n \in \mathbb{Z}, \\ u_t(n,0) = \phi(n), & \qquad\quad\; n \in \mathbb{Z}. \end{cases} \tag{4.2.22}$$

The following result shows that the fundamental solution can be represented in terms of the Wright function.

Corollary 4.2.9. *Let $1 < \beta \leq 2$. Then the fundamental solution $G_{1,\beta}(n,t)$ has an integral representation given by*

$$G_{1,\beta}(n,t) = \int_0^{\infty} \Phi_{\frac{\beta}{2}}(\rho) J_{2n}(2\rho t^{\beta}) d\rho, \quad t > 0, \; n \in \mathbb{Z}, \tag{4.2.23}$$

where Φ_{ν} is the Wright function defined by (2.4.4).

Proof. The result follows by applying Corollaries 4.2.5 and 4.2.6. □

Example 4.2.10. (Transverse vibrations of an infinite light string) The partial differential/difference equation

$$u_{tt}(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t) + g(n,t), \quad n \in \mathbb{Z}, \quad t > 0, \tag{4.2.24}$$

was studied by Bateman [11, Section 4.9] with forcing term $g(n,t) = -\delta_0(n)f'(t)$, where $f \in W^{1,q}(\mathbb{R}^+),\ 1 < q < \infty$, and

$$\delta_0(n) = \begin{cases} 1, & n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.25}$$

Physically, this means that the motion of the particle labeled 0 is forced while the motion of the other particles is free. We consider their super diffusive version, i.e. equation (4.2.22) with g as just described. Assuming $\sup_{s\in[0,t]} |f'(s)| < \infty$ for all $t > 0$ and zero initial conditions, we obtain the following explicit representation of the solution

$$u(n,t) = -\int_0^t (g_{\beta-1} \star G_{1,\beta})(n, t-s) f'(s) ds,$$

where $1 < \beta \leq 2$. In the border case $\beta = 2$ we get after integration by parts

$$u(n,t) = \int_0^t J_{2n}(2(t-s)) f(s) ds,$$

so that we recover the explicit solution of (4.2.24) proposed by Bateman in [11, p.644] and give new insight on the representation in the super diffusive case.

4.3 Explicit solution in the subdiffusive case

In this section, we give an explicit representation of solution for the following time/space fractional diffusion equation

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0 \\ u(n,0) = \varphi(n) & n \in \mathbb{Z}, \end{cases} \tag{4.3.1}$$

where $0 < \alpha, \beta \leq 1$. Note that this problem was studied in [34].

Theorem 4.3.1. *Let $\varphi \in \ell^\infty(\mathbb{Z})$ and $g : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{C}$ be such that for each $t > 0$, $g(\cdot,t) \in \ell^\infty(\mathbb{Z})$ and $\sup_{s\in[0,t]} ||g(\cdot,t)||_\infty < \infty$. Then the function*

$$u(n,t) = \sum_{m\in\mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m) + \sum_{m\in\mathbb{Z}} \int_0^t L_{\alpha,\beta}(n-m,t-s) g(m,s) ds. \tag{4.3.2}$$

is the unique solution of (4.3.1), *where here*

$$G_{\alpha,\beta}(n,t) := (-1)^n \; {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ & & \\ (1,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^{\beta} \right] \tag{4.3.3}$$

and

$$L_{\alpha,\beta}(n,t)) := (-1)^n t^{\beta-1} \; {}_2\Psi_3 \left[\begin{matrix} (1,2\alpha) & (1,1) & \\ & & \\ (\beta,\beta) & (n+1,\alpha) & (-n+1,\alpha) \end{matrix} \middle| -t^{\beta} \right]. \tag{4.3.4}$$

Furthermore, for each $t \in \mathbb{R}_+$, $G_{\alpha,\beta}(\cdot,t)$ *and* $L_{\alpha,\beta}(\cdot,t)$ *belong to* $\ell^1(\mathbb{Z})$

Proof. As in the proof of Theorem 4.2.1, taking the discrete Fourier transform of (4.3.1) we get that

$$\begin{cases} \mathbb{D}_t^{\beta}\hat{u}(z,t) = -(-J(z))^{\alpha}\hat{u}(z,t) + \hat{g}(z,t), \quad z \in \mathbb{T},\ t > 0, \\ \hat{u}(z,0) = \hat{\varphi}(z). \end{cases}$$

In the same way that we obtained (4.2.3), we get

$$\hat{u}(z,t) = E_{\beta,1}\left(-(-J(z))^{\alpha}t^{\beta}\right)\hat{u}(z,0) + t^{\beta-1}E_{\beta,\beta}\left(-(-J(z))^{\alpha}t^{\beta}\right) \star \hat{g}(z,t). \tag{4.3.5}$$

Now taking the inverse discrete Fourier transform we get that for $t > 0$ and $n \in \mathbb{Z}$,

$$u(n,t) = \sum_{m\in\mathbb{Z}} G_{\alpha,\beta}(n-m,t)\varphi(m) + \sum_{m\in\mathbb{Z}} \int_0^t L_{\alpha,\beta}(n-m,t-s)g(m,s)ds.$$

Next, we show that $G_{\alpha,1}(\cdot,t) \in \ell^1(\mathbb{Z})$ for every $t > 0$. Since

$$G_{\alpha,1}(n,t) = (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)}{\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)} \frac{(-t)^k}{k!},$$

then using that $G_{\alpha,1}(n,t) = G_{\alpha,1}(-n,t)$ and Fubini-Tonelli Theorem, we can deduce that

$$\begin{aligned} \sum_{n\in\mathbb{Z}} |G_{\alpha,1}(n,t)| &= \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k+n+1)||\Gamma(\alpha k-n+1)|} \frac{t^k}{k!} \\ &\leq 2 \sum_{n\in\mathbb{N}_0} \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)|n!} \frac{t^k}{k!} = 2 \sum_{k=0}^{\infty} \sum_{n\in\mathbb{N}_0} \frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)|n!} \frac{t^k}{k!}. \end{aligned}$$

Let $c_k := \dfrac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)|}\dfrac{1}{k!}$. Then, using (2.4.12) with $\kappa = -\alpha$, $\delta = 2^{-2\alpha}\alpha^{\alpha}$, $\mu = -n$, $A = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{2\alpha}}{\alpha^{-n+\frac{1}{2}}}$), we get that for k sufficiently large

$$|c_k| \sim \frac{(\alpha k)^n}{\sqrt{\pi}}\left(\frac{k}{e}\right)^{-(1-\alpha)k}\frac{(-2)^{2\alpha k}\alpha^{\alpha k}}{k^{\frac{1}{2}}}.$$

Thus for k_0 large enough, we have that

$$\begin{aligned} 2\sum_{k=k_0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(2\alpha k+1)}{|\Gamma(\alpha k-n+1)|n!}\frac{t^k}{k!} &\sim \frac{2}{\sqrt{\pi}}\sum_{k=k_0}^{\infty}\left(\frac{k}{e}\right)^{-(1-\alpha)k}\frac{(4\alpha)^{\alpha k}}{k^{\frac{1}{2}}}t^k\sum_{n\in\mathbb{N}_0}\frac{(\alpha k)^n}{n!} \\ &= \frac{2}{\sqrt{\pi}}\sum_{k=k_0}^{\infty}\left(\frac{k}{e}\right)^{-(1-\alpha)k}\frac{(4\alpha)^{\alpha k}}{k^{\frac{1}{2}}}t^k e^{\alpha k} \\ &= \frac{2}{\sqrt{\pi}}\sum_{k=k_0}^{\infty}k^{-(1-\alpha)k-\frac{1}{2}}e^k(4\alpha)^{\alpha k}t^k. \end{aligned} \tag{4.3.6}$$

Hence $G_{\alpha,1}(.,t) \in \ell^1(\mathbb{Z})$ for each $t > 0$.

Now, using the subordination principle on Wright function we get that

$$G_{\alpha,\beta}(n,t) = \int_0^{\infty}\Phi_\beta(s)G_{\alpha,1}(n,st^\beta)ds. \tag{4.3.7}$$

This implies

$$\sum_{n\in\mathbb{Z}}|G_{\alpha,\beta}(n,t)| = \sum_{n\in\mathbb{Z}}\left|\int_0^{\infty}\Phi_\beta(s)G_{\alpha,1}(n,st^\beta)\,ds\right| \le \int_0^{\infty}\Phi_\beta(s)\sum_{n\in\mathbb{Z}}\left|G_{\alpha,1}(n,st^\beta)\right|ds.$$

Let k_0 be sufficiently large. Let $K_t = \sum_{k=0}^{k_0-1}c_k t^k < \infty$ and $b_k = \dfrac{2}{\sqrt{\pi}}k^{-(1-\alpha)k-\frac{1}{2}}e^k(4\alpha)^{\alpha k}t^k$ in the above estimate for $G_{\alpha,1}(n,t)$. Then

$$\begin{aligned} \int_0^{\infty}\Phi_\beta(s)\sum_{n\in\mathbb{Z}}|G_{\alpha,1}(n,st^\beta)|ds &\le \int_0^{\infty}\Phi_\beta(s)K_t\,ds + \int_0^{\infty}\sum_{k=k_0}^{\infty}b_k(st^\beta)^k\Phi_\beta(s)\,ds \\ &= K_t + \sum_{k=k_0}^{\infty}b_k t^{\beta k}\int_0^{\infty}s^k\Phi_\beta(s)\,ds. \end{aligned}$$

Using (2.4.5) we obtain

$$G_{\alpha,\beta}(n,t) \le C_t + \sum_{k=k_0}^{\infty}b_k t^{\beta k}\frac{\Gamma(k+1)}{\Gamma(\beta k+1)}.$$

Let $d_k = b_k \frac{\Gamma(k+1)}{\Gamma(\beta k+1)}$. Using Stirling's formula (2.5.2), we get

$$d_k = b_k \frac{\Gamma(k+1)}{\Gamma(\beta k+1)} \sim \frac{2}{\sqrt{\pi}} k^{-(1-\alpha)k-\frac{1}{2}} e^k (4\alpha)^{\alpha k} \frac{\left(\frac{k}{e}\right)^k \sqrt{2\pi k}}{\left(\frac{\beta k}{e}\right)^k \sqrt{2\pi\beta k}} = \frac{k^{-(1-\alpha)k-\frac{1}{2}}}{\beta^{k+\frac{1}{2}}} (4\alpha)^{\alpha k} e^k.$$

Therefore, applying the root test, we can conclude that $G_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$, for each $t > 0$. The proof is finished. $\square$

4.3.1 The semi-discrete heat equation

In the case $\alpha = \beta = 1$, the corresponding equation is the semi-discrete heat equation

$$\begin{cases} u_t(n,t) = \Delta_d u(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{4.3.8}$$

From the general results on ordinary differential equations in Banach spaces, the unique solution of (4.3.8) is given by $u(n,t) = e^{t\Delta_d}\varphi(n)$, where $e^{\Delta_d t} =: W_t$ is the semigroup generated by Δ_d. This semigroup, which is called the discrete heat semigroup in analogy with the classical continuous case, has the following representation:

$$W_t f(n) = \sum_{m\in\mathbb{Z}} e^{-2t} I_{n-m}(2t) f(m), \quad f \in \ell^p(\mathbb{Z}). \tag{4.3.9}$$

We refer to [14] for more information, and the derivation of the semigroup property, as well as other standard properties. We note that the representation of the discrete heat semigroup was already given by Bateman in [10, p. 496].

In the case $\alpha \in (0,1), \beta = 1$, the corresponding equation is the semi-discrete heat equation

$$\begin{cases} u_t(n,t) = -(-\Delta_d)^\alpha u(n,t) + g(n,t) & n \in \mathbb{Z},\ t > 0 \\ u(n,0) = \varphi(n) & n \in \mathbb{Z}. \end{cases} \tag{4.3.10}$$

Is clear that the fundamental solution is given by the semigroup generated by the discrete fractional Laplacian $(-\Delta_d)^\alpha$. The following proposition given us the representation in power series of this semigroup.

Proposition 4.3.2. *Let $\alpha \in (0,1]$. Then, for all $n \in \mathbb{Z}$, the following identity*

$$e^{(-\Delta_d)^\alpha t} = \int_0^\infty I_n(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda = \sum_{j=0}^{\infty} K^{\alpha j}(n)\frac{(-t)^j}{j!}$$

holds, where $f_{t,\alpha}(\lambda)$ is the stable Lévy distribution defined in (2.4.13).

Proof. First, we note that the first equality is in virtue of the subordination principle on Wright function. Now, using the discrete Fourier transform, Remark 2.4.5(i), (2.4.26) and Fubini-Tonelli Theorem, we get

$$\begin{aligned}
\mathcal{F}\left(\int_0^\infty I_n(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda\right)(z) &= \sum_{n\in\mathbb{Z}}\left[\int_0^\infty I_n(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda\right] z^n \\
&= \int_0^\infty \sum_{n\in\mathbb{Z}} I_n(2\lambda)z^n e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= \int_0^\infty e^{\left(z+\frac{1}{z}\right)\lambda} e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= \int_0^\infty e^{\left(z+\frac{1}{z}-2\right)\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= \int_0^\infty e^{J(z)\lambda} f_{t,\alpha}(\lambda)d\lambda \\
&= e^{-(-J(z))^\alpha t} \\
&= E_{1,1}(-(-J(z))^\alpha t) \\
&= \mathcal{F}(G_{\alpha,1}(n,t))(z) \\
&= \mathcal{F}\left((-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)}{\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)}\frac{(-t)^k}{k!}\right)(z).
\end{aligned}$$

By the uniqueness of the discrete Fourier transform we can deduce that

$$\int_0^\infty I_n(2\lambda)e^{-2\lambda} f_{t,\alpha}(\lambda)d\lambda = (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(2\alpha k+1)}{\Gamma(\alpha k+n+1)\Gamma(\alpha k-n+1)}\frac{(-t)^k}{k!},$$

and the claim follows from Remark 4.2.4. □

It is clear from (4.3.8) that the case $\alpha = \beta = 1$ is the discretized heat equation. Therefore, the expression in (4.3.9) represents the heat semigroup or Gauss-Weierstrauss semigroup $\{W(t)\}_{t\geq 0}$ in the present context. The semigroup property is studied in [14, Proposition 1]. Similarly to the continuous case, when $\alpha = \frac{1}{2}$ and $\beta = 1$, the corresponding semigroup is the Poisson semigroup $\{P_t\}_{t\geq 0}$. The Poisson semigroup was considerated by Ciaurri-Gillespie in [14, Remark 2] through the subordination principle with the Lévy distribution. That is,

$$(P_t f)(n) = \sum_{m\in\mathbb{Z}} p_t(n-m)f(m), \;\; f \in \ell^p(\mathbb{Z}).$$

Corollary 4.3.3. *We have the following representations for the heat and Poisson semigroups.*

(i) For the discrete heat kernel, we have that for all $n \in \mathbb{Z}$,

$$e^{-2t}I_n(2t) = \sum_{j=0}^{\infty} K^{2j}(n)\frac{(-t)^j}{j!}.$$

(ii) For the Poisson kernel, we have, for all $n \in \mathbb{Z}$,

$$p_t(n) = \frac{1}{2\sqrt{\pi}}\int_0^{\infty} I_n(2\lambda)e^{-2\lambda}\lambda^{-\frac{3}{2}}te^{-\frac{t^2}{4\lambda}}\,d\lambda = \sum_{j=0}^{\infty} K^{j/2}(n)\frac{(-t)^j}{j!}.$$

Proof. (i) From the above proof, taking $\alpha = 1$, we obtain that

$$\mathcal{F}(G_{1,1}(n,t))(z) = E_{1,1}(J(z)t) = e^{J(z)t} = \mathcal{F}(e^{-2t}I_n(2t))(z),$$

and we have shown that

$$e^{-2t}I_n(2t) = (-1)^n\sum_{k=0}^{\infty}\frac{\Gamma(2k+1)}{\Gamma(k+n+1)\Gamma(k-n+1)}\frac{(-t)^k}{k!}.$$

(ii) For $\alpha = \frac{1}{2}$, we have that $f_{t,\frac{1}{2}}(\lambda) = \frac{1}{2\sqrt{\pi}}\lambda^{-\frac{3}{2}}te^{-\frac{t^2}{4\lambda}}$ (see e.g. [2]). The result is direct consequence of the subordination principle. □

4.3.2 The subdiffusive case

In this subsection, we consider the solution of the fractional differential equation

$$\begin{cases} \mathbb{D}_t^\beta(n,t) = \Delta_d u(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{4.3.11}$$

Now, we obtain the explicit fundamental solution for the equation in power series representation. In fact, we have the following result.

Proposition 4.3.4. *Let $\beta \in (0,1]$. Then, for all $n \in \mathbb{Z}$, the identity*

$$\int_0^\infty I_n(2\lambda t^\beta) e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda = \sum_{j=0}^\infty K^{2j}(n) \frac{(-t)^j}{\Gamma(\beta j + 1)}, \tag{4.3.12}$$

holds, where $\Phi_\beta(\lambda)$ is the Wright function given in (2.4.4).

Proof. Indeed, using the discrete Fourier transform, 2.4.2(i), (2.4.26) and Fubini-Tonelli Theorem, we obtain that

$$\begin{aligned}
&\mathcal{F}\left(\int_0^\infty I_n(2\lambda t^\beta) e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda\right)(z) \\
=& \sum_{n\in\mathbb{Z}} \left[\int_0^\infty I_n(2\lambda t^\beta) e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda\right] z^n = \int_0^\infty \sum_{n\in\mathbb{Z}} I_n(2\lambda t^\beta) z^n e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda \\
=& \int_0^\infty e^{\left(z+\frac{1}{z}\right)\lambda t^\beta} e^{-2\lambda t \beta} \Phi_\beta(\lambda) d\lambda \\
=& \int_0^\infty e^{\left(z+\frac{1}{z}-2\right)\lambda t^\beta} \Phi_\beta(\lambda) d\lambda = \int_0^\infty e^{J(z)\lambda t^\beta} \Phi_\beta(\lambda) d\lambda \\
=& E_{\beta,1}(J(z)t^\beta) = \mathcal{F}(G_{1,\beta}(n,t))(z) \\
=& \mathcal{F}\left((-1)^n \sum_{k=0}^\infty \frac{\Gamma(2k+1)}{\Gamma(\beta k+1)\Gamma(k+n+1)\Gamma(k-n+1)} (-t)^k\right)(z).
\end{aligned}$$

Thus

$$\int_0^\infty I_n(2\lambda t^\beta) e^{-2\lambda t^\beta} \Phi_\beta(\lambda) d\lambda = (-1)^n \sum_{k=0}^\infty \frac{\Gamma(2k+1)}{\Gamma(\beta k+1)\Gamma(k+n+1)\Gamma(k-n+1)} (-t)^k,$$

by the uniqueness of the discrete Fourier transform. The result follows from Remark 4.2.4. □

An interesting case is when $\beta = \frac{1}{3}$, where the explicit representation of the Wright function is known. Namely,

$$\Phi_{\frac{1}{3}}(z) = 3^{\frac{2}{3}} Ai\left(-\frac{z}{3^{\frac{1}{3}}}\right),$$

where $Ai(z)$ is the *Airy function* (see e.g. [25]), whose integral representation is given by

$$Ai(x) = \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt, \;\; x \in \mathbb{R}. \tag{4.3.13}$$

This function appears in several applied problems, specially in optics (study of caustics) and the Schrödinger's equation of quantum physics. The reference [63] contains wide information on this topic.

The following is a direct consequence of Proposition 4.3.4.

Corollary 4.3.5. *The following identity*

$$\int_0^\infty I_n(2\lambda t^{\frac{1}{3}}) e^{-2\lambda t^{\frac{1}{3}}} 3^{\frac{2}{3}} Ai\left(-\frac{\lambda}{3^{\frac{1}{3}}}\right) d\lambda = \sum_{j=0}^\infty K^{2j}(n) \frac{(-t)^j}{\Gamma(\frac{j}{3}+1)},$$

holds.

4.4 Perturbation operator

In this section, we study the fundamental solution associated to semidiscrete discrete dynamical system involving the fractional perturbed discrete Laplacian. H. Bateman considered in [10, Formula 5.1] a problem related to the operator Δ_d, which describes the dynamics of surges in springs of spirals or helical type. He singles out the case of masses concentrated on a light string and obtains the equation

$$f_{tt}(n,t) = f(n+1,t) - (2+h) f(n,t) + f(n-1,t), \quad h \geq 0, \; t \geq 0. \tag{4.4.1}$$

It appears that the operator Δ_d is now replaced by $\Delta_{d,h} := \Delta_d - 2hI$ (I being the identity operator). More recently, such additive perturbations of the discrete Laplace operator have appeared in connection with well posedness of the heat equation in Hölder spaces [46, Theorem 1.7].

In the general theory of cosine families, if A is an operator that generates a cosine family $C(t)$, and $a \in \mathbb{C}$, then $A + aI$ generates a cosine family $C_a(t)$ (see [19, Chapter VI, p.167, formula 2.8]) represented by

$$C_a(t) = C(t) + \sqrt{a}t \int_0^t \frac{I_1(\sqrt{a}(t^2 - s^2)^{\frac{1}{2}})}{(t^2 - s^2)^{\frac{1}{2}}} C(s) ds. \tag{4.4.2}$$

In the case of equation (4.4.1) we will obtain a representation of the solution through the methods of the previous sections.

We start by giving the representation of the fractional powers of $\Delta_{d,h}$, similar to the one given for Δ_d in the introduction. The main tool we shall use is the *Balakrishnan formula*

$$(-A)^\alpha x = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (T(t)x - x) \frac{dt}{t^{1+\alpha}}, \ x \in D(A), \tag{4.4.3}$$

representing the fractional powers of $(-A)$, where A is the generator of a bounded semigroup $T(t)$ (see e.g. [70, p. 260, Formula 5 and Theorem 2, p. 264]).

4.4.1 Fractional powers of the perturbed discrete fractional Laplacian

In this subsection, we study the operator

$$\Delta_{d,h} f(n) := f(n+1) - (2+h)f(n) + f(n-1), \quad h \geq 0, \quad f \in \ell^p(\mathbb{Z}), \ 1 \leq p \leq \infty. \tag{4.4.4}$$

Obviously, when $h = 0$, $\Delta_{d,0} f(n) = \Delta_d f(n)$. A simple computation yields

$$\mathcal{F}(\Delta_{d,h} f(n))(z) = \left[z + \frac{1}{z} - (2+h) \right] \mathcal{F}(f)(z).$$

Therefore, we have:

$$\mathcal{F}(\Delta_{d,h})(z) = z + \frac{1}{z} - (2+h) = J(z) - h. \tag{4.4.5}$$

Note that $z + \frac{1}{z} - (2+h) = J(z) - h \leq 0$ for all $z \in \mathbb{C}, |z| = 1, h \geq 0$.

Proposition 4.4.1. *Let $\Delta_{d,h}$ be defined in (4.4.4). Then, for $0 < \alpha < 1$, the fractional power $(-\Delta_{d,h})^\alpha$ is given by*

$$(-\Delta_{d,h})^\alpha f(n) = \sum_{l\in\mathbb{Z}} K_h^\alpha(n-l)f(l), \quad n\in\mathbb{Z}, \tag{4.4.6}$$

where

$$K_h^\alpha(n) = \frac{1}{2\pi}\int_0^{2\pi} \left(4\sin^2(\theta/2)+h\right)^\alpha e^{-in\theta}d\theta, \quad n\in\mathbb{Z}. \tag{4.4.7}$$

Proof. First, note that the semigroup generated by $\Delta_{d,h}$ is: $e^{t\Delta_{d,h}} = e^{-ht}e^{t\Delta_d}$. Then, taking the discrete Fourier transform in (4.4.3), with $A = \Delta_{d,h}$, and noting that

$$\mathcal{F}(e^{t\Delta_{d,h}})(z) = e^{\left[z+\frac{1}{z}-(2+h)\right]t}\mathcal{F}(f)(z) = e^{-(4\sin^2\frac{\theta}{2}+h)t}\mathcal{F}(f)(z), \tag{4.4.8}$$

we obtain that

$$\mathcal{F}((-\Delta_{d,h})^\alpha f)(\theta) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \left(e^{-(4\sin^2\frac{\theta}{2}+h)t}-1\right)\frac{dt}{t^{1+\alpha}}\mathcal{F}(f)(\theta) = \left(4\sin^2\theta/2+h\right)^\alpha\mathcal{F}(f)(\theta).$$

The second equality follows from an integration by parts and the fact that the Laplace transform of the function $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ is given by $\hat{g}_\beta(\lambda) = \lambda^{-\beta}$. Therefore, taking the inverse discrete Fourier transform, we obtain (4.4.6). $\square$

Remark 4.4.2. *We notice that the kernel $K_h^\alpha \in \ell^1(\mathbb{Z})$. In fact, let $f(\theta) = (4\sin^2\theta/2+h)^\alpha$ and $h>0$. It is easily to check that f, f' and f'' belong to $L^1([0,2\pi])$ and are periodic with period 2π. This, together with an integration by parts implies that*

$$||K_h^\alpha||_{\ell^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |K_h^\alpha(n)| \le ||f||_{L^1([0,2\pi])} + 2\sum_{n=1}^\infty \frac{1}{n^2}||f''||_{L^1([0,2\pi])} < \infty. \tag{4.4.9}$$

From the Young's inequality, we can deduce that $(-\Delta_{d,h})^\alpha$ is a bounded operator on $\ell^p(\mathbb{Z},X)$, for any Banach space X and $1\le p\le\infty$, since

$$||(-\Delta_{d,h})^\alpha f||_{\ell^1(\mathbb{Z})} = ||K_h^\alpha * f||_{\ell^p(\mathbb{Z})} \le ||K_h^\alpha||_{\ell^1(\mathbb{Z})}||f||_{\ell^p(\mathbb{Z})}.$$

4.4.2 Explicit representation of the solution

The following result gives an explicit representation of the fundamental solution.

Theorem 4.4.3. *Let $\varphi, \phi \in \ell^\infty(\mathbb{Z})$ and $g : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{C}$ be such that for each $t > 0$, $g(\cdot, t) \in \ell^\infty(\mathbb{Z})$ and $\sup_{s\in[0,t]} ||g(\cdot, s)||_\infty < \infty$. Then the unique solution of*

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = -(-\Delta_{d,h})^\alpha u(n,t) + g(n,t), \quad n \in \mathbb{Z},\ t \geq 0, \\ u(n,0) = \varphi(n), \\ u_t(n,0) = \phi(n), \end{cases} \tag{4.4.10}$$

is given by

$$\begin{aligned} u(n,t) = &\sum_{m\in\mathbb{Z}} G^h_{\alpha,\beta}(n-m,t)\varphi(m) + \sum_{m\in\mathbb{Z}} H^h_{\alpha,\beta}(n-m,t)\phi(m) \\ &+ \sum_{m\in\mathbb{Z}} \int_0^t L^h_{\alpha,\beta}(n-m,t-s)g(m,s)ds, \end{aligned} \tag{4.4.11}$$

where

$$G^h_{\alpha,\beta}(n,t) = (-1)^n \sum_{k=0}^\infty \frac{(-t^\beta)^k}{\Gamma(\beta k+1)} \sum_{r=0}^\infty \binom{\alpha k}{r} h^r \frac{\Gamma(2\alpha k - 2r + 1)}{\Gamma(\alpha k - r - n + 1)\Gamma(\alpha k - r + n + 1)}, \tag{4.4.12}$$

and

$$H^h_{\alpha,\beta}(n,t) = (-1)^n t \sum_{k=0}^\infty \frac{(-t^\beta)^k}{\Gamma(\beta k+2)} \sum_{r=0}^\infty \binom{\alpha k}{r} h^r \frac{\Gamma(2\alpha k - 2r + 1)}{\Gamma(\alpha k - r - n + 1)\Gamma(\alpha k - r + n + 1)}, \tag{4.4.13}$$

and

$$L^h_{\alpha,\beta}(n,t) = (-1)^n t^{\beta-1} \sum_{k=0}^\infty \frac{(-t^\beta)^k}{\Gamma(\beta k+\beta)} \sum_{r=0}^\infty \binom{\alpha k}{r} h^r \frac{\Gamma(2\alpha k - 2r + 1)}{\Gamma(\alpha k - r - n + 1)\Gamma(\alpha k - r + n + 1)}. \tag{4.4.14}$$

Furthermore, $G^h_{\alpha,\beta}(\cdot,t)$, $H^h_{\alpha,\beta}(\cdot,t)$, $L^h_{\alpha,\beta}(\cdot,t) \in \ell^1(\mathbb{Z})$ for each $t > 0$,.

Proof. Proceeding as in the previous sections (by taking the discrete Fourier transform of (4.4.10)) we obtain that

$$G^h_{\alpha,\beta}(n,t) = \mathcal{F}^{-1}\left(E_{\beta,1}(-(-J(z)-h)^\alpha t^\beta)\right)(n) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^\infty \frac{(-t^\beta)^k \left(4\sin^2\frac{\theta}{2} + h\right)^{\alpha k}}{\Gamma(\beta k+1)} d\theta.$$

Proceeding in the same way as we have obtained the expression (4.2.7) in Theorem 4.2.1 and using the generalized Binomial Theorem, we obtain (4.4.12). Moreover,

$$||G^h_{\alpha,\beta}(\cdot,t)||_{\ell^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |G^h_{\alpha,\beta}(n,t)| = \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\left|\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{(-t^\beta)^k\left(4\sin^2\frac{\theta}{2}+h\right)^{\alpha k}}{\Gamma(\beta k+1)}e^{-in\theta}d\theta\right|.$$

Using Fubini-Tonelli Theorem, we get that

$$\int_0^{2\pi}\sum_{k=0}^{\infty}\frac{(-t^\beta)^k\left(4\sin^2\frac{\theta}{2}+h\right)^{\alpha k}}{\Gamma(\beta k+1)}e^{-in\theta}d\theta = \sum_{k=0}^{\infty}\frac{(-t^\beta)^k}{\Gamma(\beta k+1)}\int_0^{2\pi}\left(4\sin^2\frac{\theta}{2}+h\right)^{\alpha k}e^{-in\theta}d\theta.$$

Let $f(\theta) = (4\sin^2\frac{\theta}{2}+h)^{\alpha k}$ and $h > 0$. Notice that $f,\ f',\ f'' \in L^1([0,2\pi])$ and are periodic with period 2π. Furthermore $\mathcal{O}(f'') \leq Mk^2(4+k)^{\alpha k}$ for some $M > 0$ sufficiently large. Then, from integration by parts we can deduce that

$$||G^h_{\alpha,\beta}(\cdot,t)||_{\ell^1(\mathbb{Z})} \leq ||f||_{L^1([0,2\pi])} + 2\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{\infty}\frac{t^{\beta k}}{\Gamma(\beta k+1)}Mk^2(4+h)^{\alpha k}.$$

Using the root test and the Stirling's formula (2.5.2) we get that

$$\sum_{k=0}^{\infty}\frac{t^{\beta k}}{\Gamma(\beta k+1)}Mk^2(4+h)^{\alpha k} = C < \infty.$$

Thus

$$||G^h_{\alpha,\beta}(\cdot,t)||_{\ell^1(\mathbb{Z})} \leq ||f||_{L^1([0,2\pi])} + 2\sum_{n=1}^{\infty}\frac{1}{n^2}C = ||f||_{L^1([0,2\pi])} + \frac{C\pi^2}{3}.$$

The proof for the explicit formula and the convergence in $\ell^1(\mathbb{Z})$ of $H^h_{\alpha,\beta}(\cdot,t)$ and $L^h_{\alpha,\beta}(\cdot,t)$ are similar to the proof for $G^h_{\alpha,\beta}$. $\square$

When $h = 0$, we know that C_t given in (4.2.19) defines a cosine operator function. Then, using (4.4.2) we obtain the following new result.

Corollary 4.4.4. *The cosine operator generated by $\Delta_{d,h}$ for $(\alpha,\beta)=(1,2)$ is given by*

$$C_{-h}(t)f(n) = C_t f(n) + \sqrt{-h}t\int_0^t \frac{I_1(\sqrt{-h}(t^2-s^2)^{\frac{1}{2}})}{(t^2-s^2)^{\frac{1}{2}}} C_s f(n)\,ds,$$

where

$$C_t f(n) = \sum_{m\in\mathbb{Z}} J_{2(n-m)}(2t)f(m).$$

Chapter 5

The fractional Cauchy problem and non-local bounded generators.

In this chapter we present the results in the joint paper [23]. We saw in the chapter 4 that the discrete fractional Laplacian can be defined through the discrete convolution with a special kernel K^α defined in (4.1.2), which appears explicitly in the representation in series of the fundamental solution $G_{\alpha,\beta}$ given by (4.2.7). A natural question arises: can we obtain the fundamental solution considering a more general bounded operator?. More especifically, we consider

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases} \tag{5.0.1}$$

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n),\ \ u_t(n,0) = \phi(n) & n \in \mathbb{Z}, \end{cases} \tag{5.0.2}$$

where $Bf(n) = (b * f)(n)$, with b belonging to Banach Algebra $\ell^1(\mathbb{Z})$, $f \in \ell^p(\mathbb{Z})$, $p \in [1,\infty]$ and $\beta \in (0,2]$ is real number. We also ask: to what space the solution belongs?. In order to answer this questions, we used tools of Banach Algebras, specifically $\ell^1(\mathbb{Z})$, functional calculus and subordination principles. We recall that $\mathbb{D}_t^\beta$ denotes the Caputo fractional derivative defined in (2.5.1).

5.1 The Banach algebra framework

In this section, we recall fundamental properties of the Banach Algebra $\ell^1(\mathbb{Z})$, the generalized Mittag-Leffler function and the discrete Fourier Transform. The notation used in this section is: the Dirac measures δ_0 and δ_n are defined as $\delta_n(j) = 0$ if $n \neq j$ and $\delta_n(n) = 1$ for $n, j \in \mathbb{Z}$, and $\mathbb{T} = \{e^{i\theta} \ : \theta \in [-\pi, \pi)\}$ is the one-dimensional torus,

Given $1 \le p \le \infty$, we recall that the Banach spaces $(\ell^p(\mathbb{Z}), \| \quad \|_p)$ are formed by bi-infinite sequences $f = (f(n))_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$\begin{aligned} \|f\|_p : &= \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{\frac{1}{p}} < \infty, \qquad 1 \le p < \infty; \\ \|f\|_\infty : &= \sup_{n\in\mathbb{Z}} |f(n)| < \infty. \end{aligned}$$

We remind the natural embeddings $\ell^1(\mathbb{Z}) \hookrightarrow \ell^p(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$, for $1 \le p \le \infty$ and that the dual of $\ell^p(\mathbb{Z})$ is identified with $\ell^{p'}(\mathbb{Z})$ where $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$ and $p = 1$ if $p' = \infty$.

In the case that $f \in \ell^1(\mathbb{Z})$ and $g \in \ell^p(\mathbb{Z})$, we recall the product

$$(f * g)(n) := \sum_{j=-\infty}^{\infty} f(n-j)g(j), \qquad n \in \mathbb{Z}.$$

From Young's Inequality, it follows that $f * g \in \ell^p(\mathbb{Z})$. Note that $(\ell^1(\mathbb{Z}), *)$ is a commutative Banach algebra with identity, given by δ_0. We observe that $\delta_1 * \delta_1 = \delta_2$ and, in general, $\delta_n * \delta_m = \delta_{n+m}$ for $n, m \in \mathbb{Z}$.

The Gelfand transform associated to $(\ell^1(\mathbb{Z}), *)$, is the discrete Fourier transform $\mathcal{F} : \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ (or Fourier series) defined in (4.1.3). The spectrum of f in $\ell^1(\mathbb{Z})$, denoted as $\sigma_{\ell^1(\mathbb{Z})}(f)$, is defined by

$$\sigma_{\ell^1(\mathbb{Z})}(f) := \{\lambda \in \mathbb{C} \ : \ (\lambda\delta_0 - f)^{-1} \in \ell^1(\mathbb{Z})\}.$$

In what follows, we consider the general theory of commutative Banach algebra as framework. We collect the results that will be of our interest in the following theorem.

Theorem 5.1.1. *The following properties hold:*

(i) The spectrum Spec($\ell^1(\mathbb{Z})$) is compact and, consequently, homeomorphic to the unit complex circle $\mathbb{T}$.

(ii) $\sigma_{\ell^1(\mathbb{Z})}(f) \subset \{z \in \mathbb{C} \ ; \ |z| < \|f\|_1\}$ and

$$(\lambda\delta_0 - f)^{-1} = \sum_{n\geq 0} \lambda^{-n-1} f^n, \qquad \|f\|_1 < |\lambda|. \tag{5.1.1}$$

(iii) The algebra $\ell^1(\mathbb{Z})$ is a semi simple regular Banach algebra and the discrete Fourier transform $\mathcal{F}$ is injective.

(iv) The spectrum of $f \in \ell^1(\mathbb{Z})$ is given by

$$\sigma_{\ell^1(\mathbb{Z})}(f) = \mathcal{F}(f)(\mathbb{T}). \tag{5.1.2}$$

Proof. The first claim follows from the fact that the algebra $\ell^1(\mathbb{Z})$ has identity, see for example [41], and the second one can be found in [41, p. 116]. The proof of (ii) is straightforward. That $\ell^1(\mathbb{Z})$ is semi simple and the injectivity of $\mathcal{F}$ follows from [41, Theorem 4.7.4]. [41, Corolary 7.2.3] shows that $\ell^1(\mathbb{Z})$ is a regular Banach algebra. The assertions in (iv) are taken from [41, Theorem 3.4.1.]. $\square$

We observe that the range of the Gelfand transform is the Wiener algebra $\mathcal{A}(\mathbb{T})$, the pointwise algebra of absolutely convergent Fourier series, i.e., $F(e^{i\theta}) = \sum_{n\in\mathbb{Z}} f(n)e^{i\theta n}$, $(\theta \in \mathbb{T})$ with $f \in \ell^1(\mathbb{Z})$. For $F \in \mathcal{A}(\mathbb{T})$, we also write $F(z) = \sum_{n\in\mathbb{Z}} f(n)z^n$, for $|z| \leq 1$.

We recall that the inverse discrete Fourier transform is given by the following expressions

$$\mathcal{F}^{-1}(F)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{i\theta})e^{-in\theta}\, d\theta = \frac{1}{2\pi i}\int_{|z|=1} F(z)\frac{dz}{z^{n+1}}, \quad n \in \mathbb{Z},$$

for $F \in \mathcal{A}(\mathbb{T})$ (and for other functions in larger sets).

The classical formulation of Wiener's Lemma characterizes functions $F \in \mathcal{A}(\mathbb{T})$ which are invertible in $\mathcal{A}(\mathbb{T})$ as follows:

Given $F \in \mathcal{A}(\mathbb{T})$ where $F(e^{i\theta}) = \sum_{n\in\mathbb{Z}} f(n)e^{i\theta n}$ for $\theta \in \mathbb{T}$. Then $F(e^{i\theta}) \neq 0$ for all $\theta \in \mathbb{T}$ if and only if $1/F \in \mathcal{A}(\mathbb{T})$, i.e., $(1/F)(e^{i\theta}) = \sum_{n\in\mathbb{Z}} g(n)e^{i\theta n}$ with $(g(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$; in this case $f * g = \delta_0$. ([20, Theorem 5.5]).

We now introduce the following definition.

Definition 5.1.2. Given $\alpha, \beta > 0$, we define the vector-valued Mittag-Leffler function, $E_{\alpha,\beta} : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$, by

$$E_{\alpha,\beta}(a) := \sum_{j=0}^{\infty} \frac{a^j}{\Gamma(\alpha j + \beta)}, \qquad a \in \ell^1(\mathbb{Z}).$$

Note that

$$E_{1,1}(a) = \sum_{j=0}^{\infty} \frac{a^j}{j!} = e^a; \qquad E_{2,1}(a) = \sum_{j=0}^{\infty} \frac{a^j}{(2j)!}.$$

The set $\exp(\ell^1(\mathbb{Z})) := \{e^a \,; a \in \ell^1(\mathbb{Z})\}$ is the connected component of δ_0 in the set of regular elements in $\ell^1(\mathbb{Z})$ ([41, Theorem 6.4.1]).

We follow the usual terminology in semigroup theory: the element a is called the generator of the entire group $(e^{za})_{z\in\mathbb{C}}$; a cosine function, $\mathrm{Cos}(z,a) := E_{2,1}(z^2 a)$, and a sine function, $\mathrm{Sin}(z,a) := zE_{2,2}(z^2 a)$. We have

$$\mathrm{Sin}(z,a) = \int_{[0,z]} \mathrm{Cos}(s,a)ds, \qquad z \in \mathbb{C},$$

for $a \in \ell^1(\mathbb{Z})$, see [7, Sections 3.1 and 3.14]. Moreover, the Laplace transform of a entire group or a cosine function is connected with the resolvent of its generator as follows:

$$(\lambda - a)^{-1} \quad = \quad \int_0^{\infty} e^{-\lambda s} e^{as} ds, \qquad \lambda > \|a\|_1,$$

$$\lambda(\lambda^2 - a)^{-1} \quad = \quad \int_0^\infty e^{-\lambda s} \mathrm{Cos}(s,a)ds, \qquad \lambda > \sqrt{\|a\|_1}, \tag{5.1.3}$$

see, for example, [7, p. 213].

Example 5.1.3. For $\alpha, \beta > 0$, we have that

$$E_{\alpha,\beta}(z\delta_0) = E_{\alpha,\beta}(z)\delta_0; \qquad E_{\alpha,\beta}(z\delta_1) = \sum_{j=0}^\infty \frac{z^j \delta_j}{\Gamma(\alpha j + \beta)}.$$

In particular, $e^{z\delta_1} = \displaystyle\sum_{j=0}^\infty \frac{z^j \delta_j}{j!}$ and $\mathrm{Cos}(z, \delta_1) = \displaystyle\sum_{j=0}^\infty \frac{z^{2j}\delta_j}{(2j)!}$ are generated by δ_1.

In the next proposition, we collect some basic properties of these vector-valued Mittag-Leffler functions. As usual, we consider Bochner vector-valued integration in the Banach space $\ell^1(\mathbb{Z})$, see for example [60, Section 1.2].

Proposition 5.1.4. *For $\alpha, \beta > 0$ and $a \in \ell^1(\mathbb{Z})$, we have that*

(i) $\|E_{\alpha,\beta}(a)\|_1 \leq E_{\alpha,\beta}(\|a\|_1)$.

(ii) $\mathcal{F}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\mathcal{F}(a))$; *in particular* $\mathcal{F}(e^{az}) = e^{z\mathcal{F}(a)}$ *and* $\mathcal{F}(Cos(z,a)) = Cos(\mathcal{F}(z), a)$ *for* $z \in \mathbb{C}$.

(iii) $\sigma_{\ell^1(\mathbb{Z})}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\sigma_{\ell^1(\mathbb{Z})}(a))$.

(iv) The following Laplace transform formula holds

$$\int_0^\infty e^{-\lambda t} t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(t^\alpha a) dt = k! \lambda^{\alpha - \beta} \left((\lambda^\alpha - a)^{-1} \right)^{(k+1)}, \quad \mathrm{Re}(\lambda) > \|a\|_1^{1/\alpha}, \tag{5.1.4}$$

for $k \in \mathbb{N} \cup \{0\}$.

(v) For $0 < \gamma < 1$, $E_{\gamma,1}(a) = \displaystyle\int_0^\infty \Phi_\gamma(t) e^{ta} dt$.

Proof. Proofs of parts (i) and (ii) are straightforward. The part (iii) is the spectral mapping theorem shown in [41, Theorem 6.2.1]. Due to the algebra $\ell^1(\mathbb{Z})$ is semisimple (see Theorem 5.1.1), formulae in (iv) and (v) are direct consequences of the scalar identities, [55, Formula (180), page 21]. □

Given $a \in \ell^1(\mathbb{Z})$, the modified Mittag-Leffler function $S_{\alpha,\beta} : (0,\infty) \to \ell^1(\mathbb{Z})$, which we define by

$$S_{\alpha,\beta}(t,a) := t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}a), \qquad t > 0 \tag{5.1.5}$$

is a (g_α, g_β)-regularized resolvent family generated by a in the algebra $\ell^1(\mathbb{Z})$. For the definition of (g_α, g_β)-regularized resolvent families and more details in the general case of linear and bounded operators in a Banach space we refer the reader to [2, Section 4] and the survey [44].

We recall the scaled Wright function $\psi_{\alpha,\beta}$ defined in (2.4.8), given by

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1}\sum_{n=0}^{\infty}\frac{(-st^{-\alpha})^n}{n!\Gamma(-\alpha n+\beta)}, \qquad s,t > 0,$$

for $0 < \alpha < 1$ and $\beta > 0$.

A direct consequence of [2, Theorem 12] is the following subordination theorem.

Theorem 5.1.5. *Let $0 < \eta_1$, $0 < \eta_2$, $a \in \ell^1(\mathbb{Z})$ and S_{η_1,η_2} defined in (5.1.5). Then*

$$S_{\alpha\eta_1,\alpha\eta_2+\beta}(t,a) = \int_0^\infty \psi_{\alpha,\beta}(t,s)S_{\eta_1,\eta_2}(s,a)\,ds, \qquad t > 0,$$

for $0 < \alpha < 1$ and $\beta \geq 0$.

Note that in the case of $\eta_1 = 2$, $\eta_2 = 1$ and $\alpha = \beta = \frac{1}{2}$ in Theorem 5.1.5, we obtain the known relation between cosine and semigroups operators generated by a, known as the Weierstrass formula

$$e^{at} = \frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\frac{s^2}{4t}}\mathrm{Cos}(s,a)ds, \qquad t > 0. \tag{5.1.6}$$

for $a \in \ell^1(\mathbb{Z})$, see for example [7, Theorem 3.14.17].

A application of the classical Wiener's lemma is the invariance of spectrum for convolution operators defined on $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$. This issue is contained in the following theorem that is the key abstract result in this chapter.

Theorem 5.1.6. *Given $a \in \ell^1(\mathbb{Z})$, we define*

$$A(b)(n) := (a * b)(n), \quad n \in \mathbb{Z}, \quad b \in \ell^p(\mathbb{Z}), \tag{5.1.7}$$

then $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$ for all $1 \leq p \leq \infty$. Moreover, $\|A\| = \|a\|_1$ and, for all $1 \leq p \leq \infty$, the following identities hold:

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(A) = \sigma_{\ell^1(\mathbb{Z})}(a) = \mathcal{F}(a)(\mathbb{T}). \tag{5.1.8}$$

For all $a \in \ell^1(\mathbb{Z})$, we have that e^{za} is an entire group in $\ell^p(\mathbb{Z})$ with generator a and for all $1 \leq p \leq \infty$, the following identities hold:

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(e^{za}) = \sigma_{\ell^1(\mathbb{Z})}(e^{za}) = e^{z\mathcal{F}(a)(\mathbb{T})}, \qquad z \in \mathbb{C}. \tag{5.1.9}$$

Proof. From Young's Inequality, it follows that $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$. Since the algebra $\ell^p(\mathbb{Z})$ has the identity δ_0, the property of the norm follows. For the identities (5.1.8), we refer to [20, Corollary 5.20]. Finally, for the spectral mapping theorem (5.1.9) we use (5.1.8) and [41, Theorem 6.2.1]. $\square$

The element a in the above theorem is also called the *symbol* of the operator A.

Remark 5.1.7. It is also straightforward to check that the adjoint operator of A is again a convolution operator given by $A'(g)(n) := (\tilde{a} * g)(n)$ where

$$\tilde{a}(n) = a(-n), \qquad n \in \mathbb{Z}.$$

5.2 Some finite difference operators in $\ell^1(\mathbb{Z})$

In this section we present important cases of finite difference operators, which appear in the seminal paper of Bateman [10]). Furthermore, we show properties of these operators, such as Fourier transform, semigroup and cosine families operators, norm and spectrum, among others.

Definition 5.2.1. For $f \in \ell^p(\mathbb{Z})$, with $1 \leq p \leq \infty$, and for $n \in \mathbb{Z}$, we define the following operators:

1. $-\Delta f(n) := f(n) - f(n+1) = ((\delta_0 - \delta_{-1}) * f)(n);$
2. $\nabla f(n) := f(n) - f(n-1) = ((\delta_0 - \delta_1) * f)(n);$
3. $\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1) = ((\delta_{-1} - 2\delta_0 + \delta_1) * f)(n);$
4. $\Delta_{dd} f(n) := f(n+2) - 2f(n) + f(n-2) = ((\delta_{-2} - 2\delta_0 + \delta_2) * f)(n);$

We note that important cases of finite difference operators are given by sequences in the set

$$c_c(\mathbb{Z}) := \{a \in \ell^1(\mathbb{Z}) : \exists\, m \in \mathbb{Z}_+ : a(n) = 0, \forall\, |n| > m)\}.$$

In such case, the discrete Fourier Transform of $a \in c_c(\mathbb{Z})$ is a trigonometric polynomial

$$\mathcal{F}(a)(e^{i\theta}) = \sum_{j=-m}^{m} a(j)e^{ij\theta}. \tag{5.2.1}$$

Notice that if $\sum_{j=-m}^{m} a(j) = 0$ then $0 \in \sigma_{\ell^1(\mathbb{Z})}(a)$. This follows immediately from (5.1.8).

We remark that when considering the above defined operators in the context of numerical analysis, the operators $-\Delta$ and ∇ are related to Euler scheme of approximation, and the operator Δ_d corresponds to the second-order central difference approximation for the second order derivative. The operator Δ_{dd} appears in Bateman's paper [10, Page 506] in connection with the equations of Born and Karman on crystal lattices in vibration.

5.2.1 The operator $-\Delta$

In this subsection, we recall the *forward difference operator*, defined by

$$\Delta f(n) := f(n+1) - f(n), \quad f \in \ell^p(\mathbb{Z}).$$

In the next Theorem, we give useful properties of this operator. This is a classical operator used in approximation theory and in the theory of difference equations. Considered as an operator from $\ell^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$, our main result read as follows.

Theorem 5.2.2. *The operator $-\Delta f = a * f$ where $a := \delta_0 - \delta_{-1}$ enjoys the following properties*

1. *The norm is given by $\|\Delta\| = 2$;*
2. *The Fourier transform is $\mathcal{F}(a)(z) = 1 - z, \ |z| = 1$;*
3. *For all $1 \leq p \leq \infty$ the spectrum is given by $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(-\Delta) = \{z \in \mathbb{T} \ : |z-1| = 1\}$;*
4. *For $|\lambda+1| > 1$,*
$$(\lambda\delta_0 + a)^{-1} = \sum_{j \geq 0} \frac{\delta_{-j}}{(1+\lambda)^{j+1}}.$$
5. *The associated group is $e^{-za}(n) = e^{-z}\dfrac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0}(n)$, $z \in \mathbb{C}$, $n \in \mathbb{Z}$ and its generator is $-a$.*
6. *The norm of the group is given by $\|e^{-ta}\|_1 = 1, \quad t > 0$;*
7. *The associated cosine function is $Cos(z, -a)(n) = \frac{\sqrt{\pi}}{(-n)!}\left(\frac{z}{2}\right)^{-n+\frac{1}{2}} J_{-n-\frac{1}{2}}(z)\chi_{-\mathbb{N}_0}(n)$ where $z \in \mathbb{C}, \ n \in \mathbb{Z}$.*

Proof. (1): The Minkowski inequality shows that $\|\Delta\| \leq 2$. Then observe that $\delta_0 \in \ell^p(\mathbb{Z})$ with $\|\delta_0\|_p = 1$ satisfies $\|\Delta\delta_0\|_p = 2$, proving the claim. (2): Follows immediately from the definition of discrete Fourier transform. (3): Follows from formula (5.1.8) in Theorem 5.1.6 and (2).

To prove (4), we apply (5.1.1) to get
$$(\lambda\delta_0 + a)^{-1} = ((\lambda+1)\delta_0 - \delta_{-1})^{-1} = \sum_{j \geq 0} \frac{\delta_{-j}}{(1+\lambda)^{j+1}},$$
for $|\lambda+1| > 1$. We show (5) directly
$$e^{-za}(n) = \left(e^{z\delta_{-1}} * e^{-z\delta_0}\right)(n) = \left(e^{z\delta_{-1}} * e^{-z}\delta_0\right)(n) = e^{-z}\frac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0}(n),$$
for $z \in \mathbb{C}$ and $n \in \mathbb{Z}$. The norm $\|e^{-ta}\|_1 = 1$, for $t > 0$ is straightforward from (5). Finally to show (7), we apply Laplace transform and formula (2.4.30) to get
$$\frac{\sqrt{\pi}}{(-n)!}\int_0^\infty e^{-\lambda t}\left(\frac{t}{2}\right)^{-n+\frac{1}{2}} J_{-n-\frac{1}{2}}(t)dt = \frac{\lambda}{(\lambda^2+1)^{-n+1}}, \qquad \lambda > 1,$$

for $n \leq 0$. By (4), we have that

$$\frac{\lambda}{(\lambda^2+1)^{-n+1}} = \lambda(\lambda^2+a)^{-1}(n), \qquad n \leq 0,$$

and we apply (5.1.3) to conclude the claimed equality and identify the generator of the cosine function with $-a$. □

We remark that groups generated by Δ are treated in [1, Section 2] and cosine functions in [10, Introduction].

5.2.2 The operator ∇

In this subsection, we recall the *backward difference operator*, defined by

$$\nabla f(n) := f(n) + f(n-1), \quad f \in \ell^p(\mathbb{Z}).$$

In the next Theorem, we give useful properties of this operator.

Theorem 5.2.3. *The operator $\nabla f = a * f$ where $a := \delta_0 - \delta_1$ enjoys the following properties*

1. $\|\nabla\| = 2$;
2. $\mathcal{F}(a)(z) = 1 - \dfrac{1}{z}$;
3. *For all* $1 \leq p \leq \infty$ *we have* $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\nabla) = \{z \in \mathbb{T} \ : |z-1| = 1\}$;
4. *For* $|\lambda + 1| > 1$,
$$(\lambda\delta_0 + a)^{-1} = \sum_{j\geq 0} \frac{\delta_j}{(1+\lambda)^{j+1}}.$$
5. $e^{-za}(n) = e^{-z}\frac{z^n}{n!}\chi_{\mathbb{N}_0}(n), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}$;
6. $\|e^{-ta}\|_1 = 1, \quad t > 0$;
7. *Cos*$(z, -a) = \frac{\sqrt{\pi}}{n!}\left(\frac{z}{2}\right)^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(z)\chi_{\mathbb{N}_0}(n), \ z \in \mathbb{C}, \ n \in \mathbb{Z}$.

Proof. The assertions (1), (2), (3) and (4) follow the same lines that Theorem 5.2.2. For the assertion (5), we have

$$e^{-za}(n) = \left(e^{z\delta_1} * e^{-z}\delta_0\right)(n) = e^{-z}\frac{z^n}{n!}\chi_{\mathbb{N}_0}(n),$$

for $z \in \mathbb{C}$ and $n \in \mathbb{Z}$. The claim (6) follows from (5). Finally, we check (7) as follows: We apply Laplace transform and formula (2.4.30) to get

$$\frac{\sqrt{\pi}}{n!}\int_0^\infty e^{-\lambda t}\left(\frac{t}{2}\right)^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(t)dt = \frac{\lambda}{(\lambda^2+1)^{n+1}}, \qquad \lambda > 1.$$

for $n \geq 0$. Then we can apply (4), (5.1.3) and the uniqueness of the Laplace transform to conclude (7). □

We remark that groups generated by ∇ are treated in [1, Section 2] and cosine functions in [10, Introduction].

We observe that when considering the Fourier transform in the context of signal processing, the conversion from continuous-times systems to discrete-times systems is done through the Euler transformation $1 - \frac{1}{z}$. In such context it is important to remark that z^{-1} represents a delay in time.

5.2.3 The operator Δ_d

We recall the discrete Laplacian operator, defined by

$$\Delta_d f(n) = f(n+1) - 2f(n) + f(n-1), \quad f \in \ell^p(\mathbb{Z}).$$

Theorem 5.2.4. *The operator $\Delta_d f = a * f$ where $a := \delta_{-1} - 2\delta_0 + \delta_1$ enjoys the following properties*

1. $\|\Delta_d\| = 4$;
2. $\mathcal{F}(a)(z) = z + \dfrac{1}{z} - 2$;
3. *For all $1 \leq p \leq \infty$ we have $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\Delta_d) = [-4, 0]$;*

4. *The group* $e^{za}(n) = e^{-2z} I_n(2z), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}$ *and its generator is* a.

5. $\|e^{ta}\|_1 = 1, \quad t > 0;$

6. *For* $\lambda \in \mathbb{C} \setminus [-4, 0]$,

$$(\lambda - a)^{-1}(n) = 2^{-n} \frac{((\lambda+2) - \sqrt{\lambda^2 + 4\lambda})^n}{\sqrt{\lambda^2 + 4\lambda}}, \qquad n \in \mathbb{Z};$$

7. *Cos*$(z, a) = J_{2n}(2z), \; z \in \mathbb{C}, \; n \in \mathbb{Z}.$

Proof. The assertions (1) and (2) follows as in the previous theorems. To prove (3) observe that

$$\begin{aligned}\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\Delta_d) &= \{z \in \mathbb{C} \,:\, z = w + \frac{1}{w} - 2, \; |w| = 1\} = \{z \in \mathbb{C} \,:\, z = 2(\mathrm{Re}(w) - 1), \; |w| = 1\} \\ &= \{z \in \mathbb{C} \,:\, z = 2(\cos(\theta) - 1), \; \theta \in [0, 2\pi)\,\} = [-4, 0].\end{aligned}$$

To show (4): we proceed as in the previous theorems, obtaining

$$\begin{aligned}e^{za}(n) &= \left((e^{z\delta_1} * e^{z\delta_{-1}}) * e^{-2z}\delta_0\right)(n) = e^{-2z}(e^{z\delta_1} * e^{z\delta_{-1}})(n) \\ &= e^{-2z} \sum_{j=-\infty}^{\infty} \frac{z^{n-j}}{(n-j)!} \chi_{\mathbb{N}_0}(n-j) \frac{z^{-j}}{(-j)!} \chi_{-\mathbb{N}_0}(j) \\ &= e^{-2z} \sum_{j=0}^{\infty} \frac{z^{n+j}}{(n+j)!} \frac{z^j}{j!} = e^{-2z} I_n(2z)\end{aligned}$$

where we have used (2.4.22) in the last identity. To prove (5) we use (4) and (2.4.26). To prove (6) we apply Laplace transform and formula (2.4.28) to get

$$(\lambda - a)^{-1}(n) \;=\; \int_0^\infty e^{-\lambda t} e^{ta}(n) dt = \int_0^\infty e^{-(\lambda+2)t} I_n(2t) dt = 2^{-n} \frac{((\lambda+2) - \sqrt{\lambda^2 + 4\lambda})^n}{\sqrt{\lambda^2 + 4\lambda}},$$

for $\mathrm{Re}\,\lambda > 0$ and $n \in \mathbb{Z}$. By the principle of analytic continuation, we can extend the equality to the set $\lambda \in \mathbb{C} \setminus [-4, 0]$. Finally to show (7), we apply again Laplace transform and formula (2.4.29) to get

$$\int_0^\infty e^{-\lambda t} J_{2n}(2t) dt = 2^{-2n} \frac{\left(\sqrt{\lambda^2+4} - \lambda\right)^{2n}}{\sqrt{\lambda^2+4}} = 2^{-n} \frac{\left(\lambda^2 + 2 - \lambda\sqrt{\lambda^2+4}\right)^n}{\sqrt{\lambda^2+4}} = \lambda(\lambda^2 - a)^{-1}(n)$$

where we have applied the assertion (6) for $\mathrm{Re}\,\lambda > 0$ and $n \in \mathbb{Z}$. $\square$

We observe that groups generated by the discrete Laplacian Δ_d are treated in [14, Section 2] and cosine functions in [46, Theorem 1.2]. Here, we have presented a complete and alternative approach using the framework of Banach algebras combined with the Laplace transform method.

5.2.4 The operator Δ_{dd}

In this subsection, we present the operator Δ_{dd}, defined by

$$\Delta_{dd} f(n) := f(n+1) - 2f(n) + f(n-1), \quad f \in \ell^p(\mathbb{Z}).$$

Theorem 5.2.5. *The operator $\Delta_{dd} f = a * f$ where $a := \delta_{-2} - 2\delta_0 + \delta_2$ enjoys the following properties*

1. $\|\Delta_{dd}\| = 4$;
2. $\mathcal{F}(a)(z) = (z - \frac{1}{z})^2$;
3. *For all* $1 \leq p \leq \infty$ *we have* $\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\Delta_{dd}) = [-4, 0]$;
4. $e^{za}(n) = e^{-2z} I_{\frac{n}{2}}(2z) \chi_{2\mathbb{Z}}(n), \quad z \in \mathbb{C}, \quad n \in \mathbb{Z}$;
5. $\|e^{-ta}\|_1 = 1, \quad t > 0$;
6. *For* $\lambda \in \mathbb{C} \setminus [-4, 0]$,
$$(\lambda - a)^{-1}(n) = 2^{-\frac{n}{2}} \frac{((\lambda+2) - \sqrt{\lambda^2 + 4\lambda})^{\frac{n}{2}}}{\sqrt{\lambda^2 + 4\lambda}} \chi_{2\mathbb{Z}}(n), \qquad n \in \mathbb{Z};$$
7. *Cos*$(z, -a)(n) = J_n(2z)\chi_{2\mathbb{Z}}(n), \ z \in \mathbb{C}, \ n \in \mathbb{Z}$.

Proof. En view of the previous theorems, (1) and (2) are straightforward. To prove (3) observe that

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{Z}))}(\Delta_{dd}) = \{z \in \mathbb{C} \,:\, z = -4\sin^2\theta, \ \theta \in [0, 2\pi)\,\} = [-4, 0].$$

We show (4): we apply the discrete Fourier transform and Theorem 5.2.4 ((2) and (4)) to get

$$\mathcal{F}(e^{-2t}I_{\frac{n}{2}}(2t)\chi_{2\mathbb{Z}}(n))(z) = \sum_{j\in\mathbb{Z}} e^{-2t}I_j(2t)(z^2)^j = \mathcal{F}(e^{t\Delta_d})(z^2) = e^{t(z-\frac{1}{z})^2} = \mathcal{F}(e^{ta})(z),$$

for $z \in \mathbb{T}$ and we conclude the equality (4) by uniqueness of the discrete Fourier transform. The equality in (5) is a consequence of (4) and (2.4.26). The proof of (6) and (7) are similar to the proof of Theorem 5.2.4 (6) and (7). □

Some simple computations show linear, algebraic and dual relations between the operators defined previously, which are presented in the following result.

Proposition 5.2.6. *The discrete operators $-\Delta, \nabla, \Delta_d$ and Δ_{dd} satisfies the following properties*

(i) The following equalities hold:

$$-\Delta_d = (\nabla - \Delta) = -\Delta\nabla;$$

(ii) For $1 \le p < \infty$, the following identities hold on $\ell^p(\mathbb{Z})$:

$$(-\Delta)' = \nabla; \qquad (\nabla)' = -\Delta;$$

$$(\Delta_d)' = \Delta_d; \qquad (\Delta_{dd})' = \Delta_{dd},$$

where A' denotes the adjoint operator of A.

In the next Theorem, we present a decomposition result for the Bessel function which seems to be new. For simplicity, for $n \in \mathbb{Z}$ and $z \in \mathbb{C}$, we define

$$g_{z,-}(n) := \frac{z^n}{n!}\chi_{\mathbb{N}_0}(n), \qquad g_{z,+}(n) := \frac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0}(n).$$

Theorem 5.2.7. *The Bessel function I_n admits a factorization via convolution given by*

$$I_n(2z) = (g_{z,+} * g_{z,-})(n), \qquad n \in \mathbb{Z}, \ z \in \mathbb{C}.$$

Proof. We apply (4) in Theorem 5.2.4, Proposition 5.2.6(i) and Theorems 5.2.2 and 5.2.3 part (5), to get

$$e^{-2z}I_n(2z) = e^{\Delta_d z}(n) = e^{-z(-\Delta)}e^{-t\nabla}(n) = e^{-2z}(g_{z,+} * g_{z,-})(n),$$

for $n \in \mathbb{Z}$, $z \in \mathbb{C}$. □

5.3 Fractional powers of generators

In this section, we are going to define the fractional powers of the operators defined in 5.2.1.

5.3.1 Integral representation for fractional powers

To define fractional powers in a Banach algebra (and in operator theory) is, in general, a difficult task. Not every element in $\ell^1(\mathbb{Z})$ has fractional powers. For example δ_1 does not have square root in $\ell^1(\mathbb{Z})$. In contrast, there might be a continuous function $f \in C(\mathbb{T})$ such that $(f(z))^2 = z$ for $z \in \mathbb{T}$.

When $\sigma_{\ell^1(\mathbb{Z})}(a) \subset \mathbb{C}^+$ and $\alpha \in \mathbb{R}$, we may consider the function $F_\alpha(z) = z^\alpha$ which is holomorphic in a neighbour of $\sigma_{\ell^1(\mathbb{Z})}(a)$. By the analytic functional calculus, the element

$$F_\alpha(a) = \frac{1}{2\pi i}\int_\gamma \frac{F_\alpha(z)}{z-a}dz,$$

(where γ is a spectral contour lying in an open set $\mathcal{O}$ containing the spectrum of a) exists in the Banach algebra $\ell^1(\mathbb{Z})$ and $\mathcal{F}(F_\alpha(a)) = (\mathcal{F}(a))^\alpha$ ([41, Lemma 6.1.2]). Then $F_\alpha(a)$ is a fractional power of a of order α, and we write $F_\alpha(a) = a^\alpha$. We note that there exists a classical way to define fractional powers of generators of uniformly bounded semigroups in Banach spaces, see for example [70, p. 260–264] and [30, Example 3.4.6-7].

As the next definition shows, we may follow a general methodology to treat fractional powers of elements in $\ell^1(\mathbb{Z})$, which is analogue to the case of operators in Banach spaces see [70, p. 265].

Definition 5.3.1. Let $0 < \alpha < 1$, and $a \in \ell^1(\mathbb{Z})$, such that $(e^{ta})_{t\geq 0}$ is a uniformly bounded semigroup, i.e., $\sup_{s>0} \|e^{as}\|_1 < \infty$. Then we write $(-a)^\alpha$ by the fractional power of a given by the following integral representation,

$$(-a)^\alpha := \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{e^{sa} - \delta_0}{s^{1+\alpha}} ds.$$

Remark 5.3.2. In fact, Definition 5.3.1 is an analogous formula in $\ell^1(\mathbb{Z})$ of the well-known equality

$$z^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{e^{-zt} - 1}{t^{1+\alpha}} dt, \qquad \operatorname{Re} z > 0.$$

As an immediate consequence of this definition, we have that, for $0 < \alpha < 1$,

$$\mathcal{F}((-a)^\alpha) = (-\mathcal{F}(a))^\alpha, \qquad \sigma((-a)^\alpha) = (\sigma(-a))^\alpha, \tag{5.3.1}$$

where we have applied (5.1.8).

It is well known that the uniformly bounded semigroup $(e^{-t(-a)^\alpha})_{t\geq 0}$ is subordinated to $(e^{ta})_{t\geq 0}$ (principle of Lévy subordination) by the formula

$$e^{-t(-a)^\alpha} = \int_0^\infty f_{t,\alpha}(s) e^{as} ds = \sum_{j=0}^\infty \frac{(-t)^j}{j!} (-a)^{j\alpha}, \qquad t \geq 0, \tag{5.3.2}$$

see, for example [70, Theorem 1, p. 263]. Note that

$$\mathcal{F}(e^{-t(-a)^\alpha}) = e^{-t(-\mathcal{F}(a))^\alpha}.$$

Now we present the fractional powers of the four elements in $\ell^1(\mathbb{Z})$ given in Definition 5.2.1. For $a \in \ell^1(\mathbb{Z})$, note that $A \in \mathcal{B}(\ell^p(\mathbb{Z}))$ where $A(f) := a * f$ for $f \in \ell^p(\mathbb{Z})$ and $1 \leq p \leq \infty$. In the case that the fractional power $a^\alpha \in \ell^1(\mathbb{Z})$ for $\operatorname{Re} \alpha > 0$ and then $A^\alpha \in \mathcal{B}(\ell^p(\mathbb{Z}))$ where $A^\alpha(f) := a^\alpha * f$ for $f \in \ell^p(\mathbb{Z})$ and $1 \leq p \leq \infty$.

5.3.2 The operators $(-\Delta)^\alpha$ and ∇^α

We define the sequence

$$k^\alpha(j) := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha) n!} = (-1)^n \binom{-\alpha}{n}, \quad n \in \mathbb{N}_0,$$

for $\alpha \in \mathbb{C} \setminus \{0, -1, -2, ...\}$ see [24, Section 2] and references therein. The sequence k^α verifies the following identity [24, Proposition 3.1]

$$\sum_{j=0}^{\infty} k^\alpha(j) z^j = \frac{1}{(1-z)^\alpha}, \quad z \in \mathbb{C}, \quad |z| \le 1, z \neq 1. \tag{5.3.3}$$

The sequence k^α has been deeply investigated in several references, see for example [76, Vol I, p.77] where $A_n^{\alpha-1} = k^\alpha(n)$. For $0 < \alpha < 1$, note that $k^{-\alpha}(0) = 1$ and

$$\sum_{j=0}^{\infty} k^{-\alpha}(j) = 0, \qquad \sum_{j=1}^{\infty} k^{-\alpha}(j) = -1,$$

see also [1, Section 2], where the notation $\Lambda^\alpha \equiv k^{-\alpha}$ is employed. For j large enough, $k^{-\alpha}(j)$ is of constant sign ([76, Theorem 1.17]) for $\alpha > 0$; in particular for $0 < \alpha < 1$, $k^{-\alpha}(j) < 0$ for $j \in \mathbb{N}$.

Since $-\Delta(f) := (\delta_0 - \delta_{-1}) * f$ and $\nabla(f) := (\delta_0 - \delta_1) * f$ for $f \in \ell^p(\mathbb{Z})$, we will denote

$$K_+^\alpha := (\delta_0 - \delta_{-1})^\alpha, \qquad K_-^\alpha := (\delta_0 - \delta_1)^\alpha. \tag{5.3.4}$$

We have the following result concerning properties of the kernels K_+^α and K_-^α.

Theorem 5.3.3. *For all* $0 < \alpha < 1$ *we have*

$$\mathcal{F}(K_+^\alpha)(z) = (1 - \frac{1}{z})^\alpha, \qquad \mathcal{F}(K_-^\alpha)(z) = (1 - z)^\alpha, \tag{5.3.5}$$

for $z \in \mathbb{T}$, *and*

$$K_+^\alpha = \sum_{j=0}^{\infty} k^{-\alpha}(j) \delta_{-j}, \qquad K_-^\alpha = \sum_{j=0}^{\infty} k^{-\alpha}(j) \delta_j. \tag{5.3.6}$$

In particular $\sigma(K_+^\alpha) = \sigma(K_-^\alpha) = \{(1 - e^{i\theta})^\alpha \mid \theta \in [-\pi, \pi)\}$. *Moreover,*

$$\|K_+^\alpha\|_1 = \|K_-^\alpha\|_1 = 2,$$

for $0 < \alpha < 1$.

Proof. Using the first identity in (5.3.1) we obtain

$$\mathcal{F}(K_+^\alpha)(z) = [\mathcal{F}(K_+)]^\alpha(z) = \Big[\sum_{j=0}^{\infty} (\delta_0 - \delta_{-1})(j) z^j\Big]^\alpha = (1 - \frac{1}{z})^\alpha,$$

proving the first identity in (5.3.5). On the other hand, note that $\Big[\sum_{j=0}^{\infty} k^{-\alpha}(j)\delta_{-j}\Big](n) = k^{-\alpha}(-n)$. Therefore, taking into account (5.3.3) we obtain

$$\mathcal{F}(\Big[\sum_{j=0}^{\infty} k^{-\alpha}(j)\delta_{-j}\Big])(z) = \mathcal{F}(k^{-\alpha}(-\cdot))(z) = (1-\frac{1}{z})^{\alpha}.$$

Then, by uniqueness of the Fourier transform, we conclude the first identity in (5.3.6). The proof of the second identities in (5.3.5) and (5.3.6) is analogous. The property of the spectrum follows from the second identity in (5.3.1). Finally, we have that

$$\|K_+^{\alpha}\|_1 = \sum_{j=0}^{\infty} |k^{-\alpha}(j)| = 1 - \sum_{j=1}^{\infty} k^{-\alpha}(j) = 2,$$

and we conclude the proof. □

Now we consider the groups generated by the fractional powers K_+^{α} and K_-^{α} as elements of the Banach algebra $\ell^1(\mathbb{Z})$.

Theorem 5.3.4. *For $0<\alpha<1$, and $z\in\mathbb{C}$, we have*

$$\begin{aligned} e^{zK_+^{\alpha}} &= e^z\delta_0 + \sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{z^n}{n!}k^{-\alpha n}(j)\right)\delta_{-j}, \\ e^{zK_-^{\alpha}} &= e^z\delta_0 + \sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{z^n}{n!}k^{-\alpha n}(j)\right)\delta_{j}. \end{aligned}$$

In particular for $z\in\mathbb{C}$ and $w\in\mathbb{T}$, we have that

$$\mathcal{F}(e^{zK_+^{\alpha}})(w) = e^{z(1-\frac{1}{w})^{\alpha}}, \qquad \mathcal{F}(e^{zK_-^{\alpha}})(w) = e^{z(1-w)^{\alpha}},$$

and $\sigma(e^{zK_+^{\alpha}}) = \sigma(e^{zK_-^{\alpha}}) = \{e^{z(1-e^{i\theta})^{\alpha}} \mid \theta\in[-\pi,\pi)\}$.

Moreover the semigroups $(e^{-tK_+^{\alpha}})_{t\geq 0}$ and $(e^{-tK_-^{\alpha}})_{t\geq 0}$ are uniformly bounded and

$$e^{-t} + \sum_{j=1}^{\infty}\left|\sum_{n=1}^{\infty}\frac{(-t)^n}{n!}k^{-\alpha n}(j)\right| \leq 1, \qquad t>0.$$

Proof. Note that

$$e^{zK_+^{\alpha}} = \delta_0 + \sum_{n=1}^{\infty}\frac{z^n}{n!}(\delta_0-\delta_{-1})^{n\alpha} = \delta_0 + \sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{j=0}^{\infty}k^{-\alpha n}(j)\delta_{-j}$$

$$= \quad e^z\delta_0 + \sum_{j=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{z^n}{n!}k^{-\alpha n}(j)\right)\delta_{-j},$$

for $z \in \mathbb{C}$.

In the case that $0 < \alpha < 1$, since $-(\delta_0 - \delta_{-1})$ and $-(\delta_0 - \delta_1)$ generate uniformly bounded semigroups, then the fractional power $-K_+^\alpha$ and $-K_-^\alpha$ also generate uniformly bounded semigroups. Then

$$\begin{aligned} e^{-t} + \sum_{j=1}^{\infty}\left|\sum_{n=1}^{\infty}\frac{(-t)^n}{n!}k^{-\alpha n}(j)\right| &= \|e^{-tK_+^\alpha}\|_1 \\ \leq \int_0^\infty f_{t,\alpha}(s)\|e^{-(\delta_0-\delta_{-1})s}\|_1 ds &\leq \int_0^\infty f_{t,\alpha}(s)ds = 1, \end{aligned}$$

for $t \geq 0$, where we have applied [70, Proposition 3, p.262]. $\square$

Now we apply the Lévy subordination principle (5.3.2) to $a = \delta_{-1} - \delta_0$ or $a = \delta_1 - \delta_0$ and from Theorem 5.3.4 to obtain the following result. Note that this formula is also obtained from (5.3.2).

Corollary 5.3.5. *Let $0 < \alpha < 1$ and $f_{s,\alpha}$ is the Lévy stable distribution defined by* (2.4.13)*.*

$$\sum_{j=1}^{\infty} k^{-\alpha j}(n)\frac{(-t)^j}{j!} = \int_0^\infty f_{t,\alpha}(s)\frac{e^{-s}s^n}{n!}ds, \quad t > 0,\, n \geq 1.$$

In particular, when $\alpha = \frac{1}{2}$, we obtain

$$\sum_{j=1}^{\infty} k^{\frac{-j}{2}}(n)\frac{(-t)^j}{j!} = \int_0^\infty \frac{t}{\sqrt{4\pi s^3}}e^{\frac{-t^2}{4s}}e^{-s}s^n ds, \qquad n \geq 1.$$

5.3.3 The operator $(-\Delta_d)^\alpha$

In this subsection, we study important properties of the discrete fractional Laplacian $(-\Delta_d)^\alpha$ for $0 < \alpha \leq 1$, defined in (4.1.1). In [46, Section 3], the sequence $(-\delta_{-1} + 2\delta_0 - \delta_1)^\alpha$ is denoted by K_d^α. We recall that

$$K_d^\alpha(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi}(4\sin^2(\theta/2))^\alpha e^{-in\theta}d\theta = \frac{(-1)^n\Gamma(2\alpha+1)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha-n)},$$

for $n \in \mathbb{Z}$ and $\alpha > 0$, [46, Formula (22)]. In the case that $1+\alpha+n \in -\mathbb{N}_0$, $K_d^\alpha(n) = 0$. Then $|K_d^\alpha(n)| \sim \frac{\Gamma(2\alpha+1)}{\pi}|n|^{-2\alpha-1}$ for $n \to \pm\infty$.

We summarize the main properties of the kernel K_d^α in the following result.

Theorem 5.3.6. *For $0 < \alpha < 1$ we have*

$$\mathcal{F}(K_d^\alpha)(z) = (2-(z+\frac{1}{z}))^\alpha = (4\sin^2(\frac{\theta}{2}))^\alpha, \qquad z = e^{i\theta} \in \mathbb{T}, \tag{5.3.7}$$

and

$$K_d^\alpha = K_+^\alpha * K_-^\alpha. \tag{5.3.8}$$

In particular:

$$K_d^\alpha = \sum_{j=0}^{\infty}(k_-^{-\alpha} * k^{-\alpha})(j)\delta_j, \tag{5.3.9}$$

where $k_-^{-\alpha}(n) := k^{-\alpha}(-n)$. Moreover, $\sigma(K_d^\alpha) = [0, 4^\alpha]$ and

$$\|K_d^\alpha\|_1 = 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}.$$

Proof. The identity (5.3.7) follows from (5.3.1). To show (5.3.8), we apply the discrete Fourier transform to obtain that

$$\mathcal{F}(K_+^\alpha * K_-^\alpha)(z) = (1-\frac{1}{z})^\alpha(1-z)^\alpha = (2-(z+\frac{1}{z}))^\alpha = \mathcal{F}(K_d^\alpha)(z),$$

for $z \in \mathbb{T}$. From the fact that the discrete Fourier transform is one to one, we obtain the equality. To prove (5.3.9) we note that the right hand side evaluated at $n \in \mathbb{Z}$ is equal to $(k_-^{-\alpha} * k^\alpha)(n)$ and we have

$$(k_-^{-\alpha} * k^\alpha)(n) = \sum_{j=0}^{n} k^{-\alpha}(-(n-j))k^{-\alpha}(j) = \sum_{j=0}^{n} K_+^\alpha(n-j)K_-^\alpha(j) = (K_+^\alpha * K_-^\alpha)(n),$$

and the result follows from (5.3.8). The spectrum is given in ([46, Theorem 1.3 (iii)] and the norm of K^α is calculated in [46, Lemma 3.2]. □

An interesting consequence is the following corollary, that seems to be a new formula for binomials of non-integer entries.

Corollary 5.3.7. *Let $\alpha \in (0,1)$ and $n \in \mathbb{N} \cup \{0\}$. The following equality holds,*

$$\binom{2\alpha}{\alpha+n} = \sum_{j=0}^{\infty} \binom{\alpha}{j+n}\binom{\alpha}{j}$$

Proof. The combinatorial equality is a straightforward consequence of the explicit expression of the kernels convolutions K_+^α, K_-^α and K_d^α. □

The following result collect the main results on the fractional discrete semigroup. For other results see also [46].

Theorem 5.3.8. *For any $0 < \alpha < 1$ we have that the fractional discrete semigroup, generated by $-K_d^\alpha$, is given*

$$e^{-zK_d^\alpha}(n) = (-1)^n \sum_{k=1}^{\infty} (-1)^k \frac{z^k}{k!} \frac{\Gamma(2k\alpha+1)}{\Gamma(1+k\alpha+n)\Gamma(1+k\alpha-n)} + \delta_0(n),$$

for $n \in \mathbb{Z}$ and $z \in \mathbb{C}$. Moreover,

(i) The discrete Fourier transform of $e^{-zK_d^\alpha}$ is given by

$$\mathcal{F}(e^{-zK_d^\alpha})(e^{i\theta}) = e^{-z(4\sin^2(\frac{\theta}{2}))^\alpha}, \qquad \theta \in [-\pi,\pi), \quad z \in \mathbb{C}.$$

(ii) $e^{-tK_d^\alpha}(n) \geq 0$, and $\|e^{-tK_d^\alpha}\|_1 = 1$ for $n \in \mathbb{Z}$ and $t \geq 0$, i.e., is a Markovian semigroup.

(iii) $\sigma(e^{-zK_d^\alpha}) = \{e^{-z(4\sin^2(\frac{\theta}{2}))^\alpha} : \theta \in [-\pi,\pi)\}$.

Proof. The fractional discrete semigroup, generated by $-K_d^\alpha$ is given in [46, Theorem 1.3]. There, the entire group $(e^{-zK_d^\alpha})_{z\in\mathbb{C}}$ is written by L_z^α. The assertion (i) follows from Proposition 5.1.4(ii) combined with (5.3.7) in Theorem 5.3.6. The proof of (ii) is contained in [46, Theorem 1.3 (v)]. Finally, to prove (iii) we use (5.1.9) in Theorem 5.1.6 and (5.3.7) in Theorem 5.3.6. □

We recall that we applied the Lévy subordination principle (5.3.2) to the semigroup generated to $a = \delta_{-1} - 2\delta_0 + \delta_1$ in Proposition 4.3.2 and Corollary 4.3.3.

5.3.4 The operator $(-\Delta_{dd})^\alpha$

Since, the element $(\delta_2 - 2\delta_0 + \delta_{-2})$ generates a uniformly bounded C_0-semigroup, then we consider the fractional power $(\delta_2 - 2\delta_0 + \delta_{-2})^\alpha$ for $0 < \alpha < 1$. For simplicity, we write K^α_{dd} instead of $(\delta_2 - 2\delta_0 + \delta_{-2})^\alpha$.

Theorem 5.3.9. *For $0 < \alpha < 1$.*

(i) We have

$$K^\alpha_{dd}(n) = \frac{\Gamma(2\alpha+1)}{\Gamma(1+\alpha+\frac{n}{2})\Gamma(1+\alpha-\frac{n}{2})}\cos(\frac{n}{2}\pi), \qquad n \in \mathbb{Z}.$$

(ii) $K^\alpha_{dd}(2n) = K^\alpha_d(n)$ *and* $K^\alpha_{dd}(2n-1) = 0$ *for* $n \in \mathbb{Z}$.

(iii) $\|K^\alpha_{dd}\|_1 = 2\dfrac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}$.

(iv) $\mathcal{F}(K^\alpha_{dd})(e^{i\theta}) = (4\sin^2(\theta))^\alpha$ *for* $\theta \in [-\pi, \pi)$ *and* $\sigma(K^\alpha_{dd}) = [0, 4^\alpha]$.

Proof. (i) For $n \in \mathbb{Z}$, we have that

$$\begin{aligned} K^\alpha_{dd}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(4\sin^2(\theta))^\alpha e^{-in\theta}d\theta = \frac{4^\alpha}{\pi}\int_0^\pi \sin^{2\alpha}(\theta)\cos(n\theta)d\theta \\ &= \frac{\Gamma(2\alpha+1)}{\Gamma(1+\alpha+\frac{n}{2})\Gamma(1+\alpha-\frac{n}{2})}\cos(\frac{n}{2}\pi) \end{aligned}$$

where we have applied [29, Formula 3.631 (8)]. Parts (ii), (iii) and (iv) are straigthforward from part (i). □

Now we consider the entire group $(e^{-zK^\alpha_{dd}})_{z\in\mathbb{C}}$ generated by $-K^\alpha_{dd}$.

Theorem 5.3.10. *For $0 < \alpha < 1$ we have*

(i) We have

$$e^{-zK^\alpha_{dd}}(n) = \cos(\frac{n}{2}\pi)\sum_{k=1}^{\infty}(-1)^k\frac{z^k}{k!}\frac{\Gamma(2k\alpha+1)}{\Gamma(1+k\alpha+\frac{n}{2})\Gamma(1+k\alpha-\frac{n}{2})} + \delta_0(n), \quad n \in \mathbb{Z}.$$

(ii) $e^{-zK^{\alpha}_{dd}}(2n) = e^{-zK^{\alpha}_{d}}(n)$ *and* $e^{-zK^{\alpha}_{dd}}(2n-1) = 0$ *for* $n \in \mathbb{Z}$.

(iii) $e^{-tK^{\alpha}_{dd}}(n) \geq 0$, *and* $\|e^{-tK^{\alpha}_{dd}}\|_1 = 1$ *for* $n \in \mathbb{Z}$ *and* $t \geq 0$, *i.e., is a Markovian semigroup.*

(iv) $\mathcal{F}(e^{-zK^{\alpha}_{dd}})(e^{i\theta}) = e^{-z(4\sin^2(\theta))^{\alpha}}$, *for* $\theta \in [-\pi, \pi)$ *and*

$$\sigma(e^{-zK^{\alpha}_{dd}}) = \sigma(e^{-zK^{\alpha}_{dd}}) = \{e^{-zu^{\alpha}} \mid u \in [0,4]\}.$$

Proof. By [70, Theorem 1, p. 263], we have that

$$e^{-tK^{\alpha}_{dd}}(n) = \int_0^{\infty} f_{t,\alpha}(s)e^{-2s}I_{\frac{n}{2}}(2s)ds\chi_{2\mathbb{Z}}(n) = e^{-tK^{\alpha}_{d}}(\frac{n}{2})\chi_{2\mathbb{Z}}(n),$$

and we conclude the equalities (i) and (ii). Parts (iii) and (iv) are proved from similar properties of $e^{-zK^{\alpha}_{d}}$. □

Remark 5.3.11. The spectrum of the discrete fractional Laplacian $-(-\Delta_d)^{\alpha}$ is determined by $\sigma(-(-\Delta_d)^{\alpha}) = [-4^{\alpha}, 0]$, recovering the result of Lizama and Roncal in [46], Theorem 1.3, iii). A similar result is obtained for the operator $-(-\Delta_{dd})^{\alpha}$ given by $\sigma(-(-\Delta_{dd})^{\alpha}) = [-4^{\alpha}, 0]$. For the discrete fractional difference operators $-\Delta$ and ∇ we have $\sigma((-\Delta)^{\alpha}) = \sigma(\nabla^{\alpha}) = [-(1+e^{i\mathbb{T}})^{\alpha}]$.

5.4 Fundamental solutions

In this section, we consider the convolution operator $Bf(n) := (b * f)(n)$, with $b \in \ell^1(\mathbb{Z})$, $f \in \ell^p(\mathbb{Z})$, $p \in [1, \infty]$ and $n \in \mathbb{Z}$. Our objective is to obtain an explicit representation of the solution for the following semi-discrete fractional evolution equation:

$$\begin{cases} \mathbb{D}^{\beta}_t u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0; \\ u(n,0) = \varphi(n), \ \ u_t(n,0) = \phi(n), & n \in \mathbb{Z}. \end{cases}$$

Here, $\beta \in (0, 2]$ is real number. For a sufficiently regular function v, we denote by $\mathbb{D}^{\beta}_t$ the Caputo derivative of order β given by

$$\mathbb{D}^{\beta}_t v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v'(s)ds = (g_{1-\beta} * v')(t), \qquad t > 0,$$

for $0 < \beta < 1$ and

$$\mathbb{D}_t^\beta v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s) ds = (g_{2-\beta} * v'')(t), \qquad t > 0,$$

for $1 < \beta < 2$. For $\beta = 1$ and $\beta = 2$, we consider the usual first and second order derivative. Note that

$$\lim_{\beta \to 1^-} \mathbb{D}_t^\beta v(t) = v'(t), \qquad \lim_{\beta \to 2^-} \mathbb{D}_t^\beta v(t) = v''(t), \qquad t > 0,$$

however,

$$\lim_{\beta \to 0^+} \mathbb{D}_t^\beta v(t) = v(t) - v(0), \qquad \lim_{\beta \to 1^+} \mathbb{D}_t^\beta v(t) = v'(t) - v'(0), \qquad t > 0, \tag{5.4.1}$$

see, for example [12, 28].

To begin with, we consider the semi discrete Cauchy problem

$$\begin{cases} \partial_t u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z}, \ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases} \tag{5.4.2}$$

and their fundamental solution that is obviously given by Duhamel's formula

$$u(n,t) = e^{Bt}\varphi(n) + \int_0^t e^{B(t-s)} g(n,s) ds \quad n \in \mathbb{Z}, \quad t \geq 0.$$

Analogously, in the case of the second order semi discrete Cauchy problem:

$$\begin{cases} \partial_{tt} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z}, \ \ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \\ u(n,0) = \psi(n), & n \in \mathbb{Z}, \end{cases} \tag{5.4.3}$$

we have that the fundamental solution is given by D'Alembert formula

$$u(n,t) = \mathrm{Cos}(t,B)\varphi(n) + \mathrm{Sin}(t,B)\psi(n) + \int_0^t \mathrm{Sin}(t-s,B) f(s) ds,$$

where $\mathrm{Cos}(t,B)$ and $\mathrm{Sin}(t,B)$ are generated by B. We first study some concrete examples.

5.4.1 The discrete Nagumo equation

Let us consider the linear part of the discrete Nagumo equation, which can be written as follows:

$$\begin{cases} \partial_t u(n,t) = \Delta_d u(n,t) - k u(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases} \tag{5.4.4}$$

where $0 < k < 1/2$. The discrete Nagumo equation is used as a model for the spread of genetic traits and for the propagation of nerve pulses in a nerve axon, neglecting recovery, see [75] and references therein. Using Theorem 5.2.4(3) we obtain

$$\sigma(e^{t(\Delta_d - kI)}) = e^{t\sigma(\Delta_d - kI)} = \{e^{ts} \,:\, t \geq 0,\, -4-k \leq s \leq -k\}.$$

It implies that the unique solution of equation (5.4.4) is uniformly asymptotically stable, i.e.

$$u(n,t) = e^{t(\Delta_d - kI)}\varphi(n) \to 0 \text{ as } t \to \infty.$$

Moreover, using Theorem 5.2.4(4) and the semigroup property, we can obtain a representation of the fundamental solution as follows:

$$\begin{aligned} u(n,t) = e^{-tkI}e^{t\Delta_d}\varphi(n) &:= (e^{-tkI} * e^{t\Delta_d} * \varphi)(n) = \sum_{j=0}^{n}(e^{-tkI} * e^{t\Delta_d})(n-j)\varphi(j) \\ &= e^{-2t}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\frac{(-kt)^l}{l!}I_{n-j-l}(2t)\varphi(j). \end{aligned}$$

Since $\sigma(-(-\Delta_d)^\alpha) = [-4^\alpha, 0]$ (see Remark 5.3.11) we have that the same asymptotic behavior also holds for the fundamental solution of the fractional Laplacian version for the discrete Nagumo equation [46, Section 7]:

$$\begin{cases} \partial_t u(n,t) = -(-\Delta_d)^\alpha u(n,t) - k u(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases}$$

5.4.2 The semi-discrete transport equation.

Let us consider the semi discrete transport equation

$$\begin{cases} \partial_t u(n,t) = {}_r\Delta u(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{5.4.5}$$

where $r > 0$ and ${}_r\Delta$ denote the r-forward difference operator defined by ${}_r\Delta f(n) := f(n+1) - rf(n)$, see [3, Section 5.5]. Observe that ${}_r\Delta = \Delta + (1-r)I$ where I denote the identity operator. Then, by perturbation semigroup theory, the unique solution of (5.4.5) has the form $u(n,t) = e^{t(\Delta+(1-r)I)}\varphi(n), n \in \mathbb{Z}$. By the spectral mapping (5.1.9) in Theorem 5.1.6, we obtain that

$$\sigma(e^{t(\Delta+(1-r)I)}) = e^{t\sigma(\Delta+(1-r)I))}.$$

Hence, by Theorem 5.2.2 part (3) we deduce that $\sigma(\Delta+(1-r)I) = \{z \in \mathbb{T} : |z+r| = 1\}$. Therefore, for any $r > 1$ we have that the upper bound of the spectrum of $B = \Delta + (1-r)I$ is negative i.e.

$$\omega_\sigma(\Delta + (1-r)I) := \sup\{\operatorname{Re} z \,:\, z \in \sigma(\Delta + (1-r)I)\} < 0,$$

and consequently we obtain that for any $r > 1$ the unique solution of equation (5.4.5) is uniformly asymptotically stable, i.e.

$$\|e^{{}_r\Delta t}\varphi\| \to 0 \quad \text{as} \quad t \to \infty,$$

uniformly with respect to $\|\varphi\| \leq 1$. Of course, this result can be also directly deduced from Theorem 5.2.2 part (5). Analogously, using the fact that

$$\operatorname{Re}(r - e^{it}) > 0 \text{ implies } \operatorname{Re}((r - e^{it})^\alpha) > 0,$$

for any $0 < \alpha < 1$ and $r > 1$, we can deduce from Theorem 5.3.4 that the same property of asymptotic stability remains true for the unique solution of the fractional semi-discrete transport equation:

$$\begin{cases} \partial_t u(n,t) = -(-{}_r\Delta)^\alpha u(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{5.4.6}$$

5.4.3 The De Juhasz equation

We consider the following semi discrete equation:

$$\begin{cases} \partial_{tt}u(n,t) = \Delta_d u(n,t) - 2ku(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}, \\ u_t(n,0) = \psi(n), & n \in \mathbb{Z}, \end{cases} \tag{5.4.7}$$

where $k > 0$. This equation can be found in the seminal paper of Bateman [10] in connection with surges in springs and connected systems of springs. We have named here as De Juhasz equation because, according Bateman's paper, De Juhasz deduced by the first time the modeling of such equation in mechanical theory. Following Bateman's paper, this semi discrete equation is obtained when the concentrated masses on a light string are mounted on springs arranged either along a straight line or on the circumference of a circle or helix [10, Section 5, formula (5.1)]. Applying Theorem 5.2.4 and considering the operator $B = \Delta_d - 2kI$ we obtain

$$\sigma(\Delta_d - 2kI) = [-4 - 2k, -2k],$$

and, therefore

$$\sigma(C(t,B)) = \{\cos(t\sqrt{s})\ :\ s \in [2k, 4+2k]\}.$$

In particular, it implies that on the Hilbert space $\ell^2(\mathbb{Z})$ we have $\|C(t)\| \le 1$ and consequently the unique solution of (5.4.7) when $\psi \equiv 0$ must be bounded. This extends a previous result of Bateman [10, Section 5] that studied (5.4.7) with the initial conditions $\psi \equiv 0$ and $\varphi(n) = \delta_0(n)$.

After this concrete examples, we can easily generalize the above argument to more complexes dynamics. This is the content of the next subsection.

5.4.4 Fractional in time generalizations

Given $0 < \beta \le 1$, we first consider the equation

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{Z},\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{Z}. \end{cases} \tag{5.4.8}$$

We recall that function $E_{\alpha,\beta}(b)$ (with $b \in \ell^1(\mathbb{Z})$) is the vector-valued Mittag-Leffler function given in Definition 5.1.2. The main result is the following Theorem.

Theorem 5.4.1. *Let $\varphi, \phi \in \ell^p(\mathbb{Z})$, and $g : \mathbb{Z}\times\mathbb{R}^+ \to \mathbb{C}$ be such that, for each $t \in \mathbb{R}^+$, $g(\cdot, t) \in \ell^p(\mathbb{Z})$ and $\sup_{s\in[0,t]} ||g(\cdot, s)||_p < \infty$ with $1 \le p \le \infty$.*

(i) For $0 < \beta < 1$, the function

$$\begin{aligned} u(n,t) =& (E_{\beta,1}(t^\beta b) * \varphi)(n) \\ &+ \int_0^t (t-s)^{\beta-1} \left(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot, s)\right)(n)ds,\ n \in \mathbb{Z}, \end{aligned}$$

is the unique solution of the initial value problem (5.4.8). Moreover, $u(\cdot, t)$ belong to $\ell^p(\mathbb{Z})$ for $t > 0$.

(ii) For $1 < \beta < 2$, the function

$$\begin{aligned} u(n,t) =& (E_{\beta,1}(t^\beta b) * \varphi)(n) + t(E_{\beta,2}(t^\beta b) * \phi)(n) \\ &+ \int_0^t (t-s)^{\beta-1} \left(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot, s)\right)(n)ds,\ n \in \mathbb{Z}, \end{aligned}$$

is the unique solution of the initial value problem (5.0.2). Moreover, $u(\cdot, t)$ belong to $\ell^p(\mathbb{Z})$ for $t > 0$.

Proof. Due to the algebra $\ell^1(\mathbb{Z})$ is semisimple (see Theorem 5.1.1), the formulae in (i) and (ii) are direct consequences of the scalar identities, which in case $0 < \alpha < 1$ can be found in [44, Section 3.3, formula (8)] combined with [44, Section 1.2]. The case $1 < \alpha < 2$ follows from [44, Section 3.3, formula (11)]. See also the references therein. □

Remark 5.4.2. Now we consider the behavior of the solution when β tends to the integer parameter, i.e, $\beta = 1, 2$. For simplicity, we consider the homogeneous case, $g = 0$. When $\beta \to 1^-$, the solution of equation (5.0.1) converges to semigroup family operators $E_{1,1}(tb)$, and for the case $\beta \to 2^-$, the solution of equation (5.0.2),

$$u(\cdot, t) = E_{\beta,1}(t^\beta b) * \varphi + tE_{\beta,2}(t^\beta b) * \phi, \qquad t > 0,$$

converges to unique mild solution of second order Cauchy problem, i.e. the sum of a cosine function and a sine function generated by b, see [7, Corollary 3.14.8].

However, as in the scalar case, when $\beta \to 1^+$ the solution of the equation (5.0.2) converges to

$$u(\cdot, t) = E_{1,1}(bt) + tE_{1,2}(tb), \qquad t > 0.$$

Note that this function is the solution of the following first order modified Cauchy problem

$$\begin{cases} v'(n,t) = Bv(n,t) + \phi(n), & n \in \mathbb{Z},\ t > 0, \\ v(n,0) = \varphi(n), & n \in \mathbb{Z}, \end{cases}$$

for $\phi, \varphi \in \ell^p(\mathbb{Z})$. This fact is in accordance with the interpolation property of the Caputo fractional derivative, see (5.4.1).

The fundamental solution $u_{\beta,1}$ for systems (5.0.1) and (5.0.2) are obtained by requiring that the initial value ψ and the initial velocity ϕ be the sequences $\psi = \delta_0$ and $\phi = 0$. In the case $1 < \beta \leq 2$ (included the wave equation), a second fundamental solution $u_{\beta,2}$ is given by $\psi = 0$ and $\phi = \delta_0$, see [21, Remark 3.2]. A consequence of Theorems 5.4.1 and 5.1.5 is the following subordination theorem for fundamental solutions which extends [21, Corollary 3.5].

Corollary 5.4.3. *Let $u_{\beta,1}$ and $u_{\beta,2}$ be the fundamental solutions of problems* (5.0.1) *and* (5.0.2) *and Ψ_α the Wright function defined by* (2.4.4).

(i) Let $0 < \beta < 1$. Then,

$$u_{\beta,1}(n,t) = \int_0^\infty \Phi_\beta(\tau) u_{1,1}(n, \tau t^\beta) d\tau, \qquad n \in \mathbb{Z},\ t > 0.$$

(ii) Let $1 < \beta < 2$. Then

$$\begin{aligned} u_{\beta,1}(n,t) &= \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) u_{2,1}(n, \tau t^{\frac{\beta}{2}}) d\tau, \\ u_{\beta,2}(n,t) &= \int_0^t \frac{(t-u)^{\frac{-\beta}{2}}}{\Gamma(1-\frac{\beta}{2})} \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) u_{2,2}(n, \tau u^{\frac{\beta}{2}}) d\tau\, du, \end{aligned}$$

for $n \in \mathbb{Z}$ and $t > 0$.

Remark 5.4.4. We recall that the Wright function $\Phi_{\frac{1}{3}}$ can be expressed in terms of the Airy function, $Ai(z)$, i.e.,

$$\Phi_{\frac{1}{3}}(z) = 3^{\frac{2}{3}} Ai\left(\frac{z}{3^{\frac{1}{3}}}\right), \qquad z \in \mathbb{C},$$

By Corollary 5.4.3(i), we conclude that

$$E_{\frac{1}{3},1}(t^{\frac{1}{3}}b)(n) = 3^{\frac{2}{3}} \int_0^\infty Ai(\frac{\tau}{3^{\frac{1}{3}}}) e^{\tau t^{\frac{1}{3}} b}(n) d\tau, \qquad n \in \mathbb{Z},\ t > 0,$$

for $b \in \ell^1(\mathbb{Z})$.

The particular case of Theorem 5.4.1 with $B = -(-A)^\alpha$, where A is the infinitesimal generator of an uniformly bounded C_0-semigroup in $\mathcal{B}(\ell^p(\mathbb{Z}))$ has received a special attention. In [36, Theorem 3.3] and [21, Theorem 3.1] the time/space fractional evolution equations (5.0.1) and (5.0.2) of order $0 < \beta \leq 1$ and $1 < \beta \leq 2$ respectively are solved where $B = -(-\Delta_d)^\alpha$ and Δ_d is the discrete Laplacian operator. Both proofs rest about the explicit expressions of vector-valued Mittag-Leffler functions $E_{\beta,1}(-t^\beta K_d^\alpha)$, $E_{\beta,2}(-t^\beta K_d^\alpha)$ and $E_{\beta,\beta}(-t^\beta K_d^\alpha)$. As a consequence of Section (5.3) we can easily give a general version which extends both results.

Corollary 5.4.5. *Let $\varphi, \phi \in \ell^p(\mathbb{Z})$, and $g : \mathbb{Z}\times\mathbb{R}^+ \to \mathbb{C}$ be such that, for each $t \in \mathbb{R}^+$, $g(\cdot, t) \in \ell^p(\mathbb{Z})$ and $\sup_{s\in[0,t]} ||g(\cdot, s)||_p < \infty$ with $1 \leq p \leq \infty$. Take $a \in \ell^1(\mathbb{Z})$ such that generates a uniformly continuous semigroup in $\ell^1(\mathbb{Z})$, we write $(-a)^\alpha$ the fractional powers given in Definition 5.3.1 and $B(f) := -(-a)^\alpha * f$ for $f \in \ell^p(\mathbb{Z})$ and $0 < \alpha < 1$. Then the same representation of the fundamental solutions given in Theorem 5.4.1 with $b = -(-a)^\alpha$ holds.*

5.5 Applications to special functions

In this section, we present some new formulae obtained as applications of the results proved in the last sections. We give the expressions of generalized Mittag-Leffler functions for concrete fractional powers. We also interpret some known formulas in terms of subordination Weierstrass formula. Finally the application of subordination principle on Wright function of some concrete difference operators allow to obtain some new integral formulae for Bessel and Wright functions.

5.5.1 Generalized Mittag-Leffler functions for fractional powers

As we have seen in Theorem 5.4.1, combinations of vector-valued Mittag-Leffler functions in $\ell^1(\mathbb{Z})$ give the solutions of fractional evolution equations (5.0.1) and (5.0.2) with $b \in \ell^1(\mathbb{Z})$. In the case of fractional powers of elements in $\ell^1(\mathbb{Z})$, i.e., $b = -(-a)^\alpha$, we have proved Corollary 5.4.5. Taking into account subsections 5.3.2, 5.3.3 and 5.3.4, we can give some particular representations of the associated Mittag-Leffler functions.

Theorem 5.5.1. *The generalized Mittag-Leffler functions $E_{\gamma,\beta}$ for $K_+^\alpha, K_-^\alpha, K_d^\alpha$ and K_{dd}^α with $\gamma, \beta > 0$ and $0 < \alpha < 1$ can be represented as follows*

(i) $E_{\gamma,\beta}(-t^\gamma K_+^\alpha)(n) = \displaystyle\sum_{j=0}^{\infty}(-1)^j \frac{t^{\gamma j}}{\Gamma(\gamma j+\beta)} k^{-\alpha j}(-n)\chi_{-\mathbb{N}_0}(n);$

(ii) $E_{\gamma,\beta}(-t^\gamma K_-^\alpha)(n) = \displaystyle\sum_{j=0}^{\infty}(-1)^j \frac{t^{\gamma j}}{\Gamma(\gamma j+\beta)} k^{-\alpha j}(n)\chi_{\mathbb{N}_0}(n);$

(iii) $E_{\gamma,\beta}(-t^\gamma K_d^\alpha)(n) = (-1)^n \displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j t^{\gamma j}}{\Gamma(\gamma j+\beta)} \frac{\Gamma(2\alpha j+1)}{\Gamma(\alpha j+n+1)\Gamma(\alpha j-n+1)};$

(iv) $E_{\gamma,\beta}(-t^\gamma K_{dd}^\alpha)(n) = \cos(\dfrac{n}{2}\pi) \displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j t^{\gamma j}}{\Gamma(\gamma j+\beta)} \frac{\Gamma(2\alpha j+1)}{\Gamma(\alpha j+\frac{n}{2}+1)\Gamma(\alpha j-\frac{n}{2}+1)}.$

5.5.2 Weierstrass formula

The relation between cosine functions and semigroups generated by an element $a \in \ell^1(\mathbb{Z})$ is established by the Weierstrass formula in (5.1.6). Now we apply this formula to concrete finite difference operators.

1. For $a = \delta_{-1} - \delta_0$ or $a = \delta_1 - \delta_0$, we obtain that
$$\frac{1}{\sqrt{t}} \int_0^\infty e^{-\frac{s^2}{4t}} \left(\frac{s}{2}\right)^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(s) ds = t^n e^{-t}, \qquad n \in \mathbb{N}_0,\ t > 0.$$
This formula is a special case of the general formula
$$\int_0^\infty e^{-p^2t^2} t^{\nu+1} J_\nu(at) dt = \frac{a^\nu}{(2p^2)^{\nu+1}} e^{-\frac{a^2}{4p^2}}, \qquad p \in \mathbb{R}, a > 0,$$
for $\mathrm{Re}(\nu) > -1$, [66, Section 13.3, Formula (4)].

2. For $a = \delta_{-1} - 2\delta_0 + \delta_1$ or $a = \delta_{-2} - 2\delta_0 + \delta_2$, we have that
$$\frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} J_{2n}(2s) ds = e^{-2t} I_n(2t), \qquad n \in \mathbb{Z},\ t \in \mathbb{R}.$$
As it is commented in [46, Remark 3], this formula is a particular case of the general equality
$$\int_0^\infty e^{-pt^2} J_\nu(at) dt = \frac{1}{2}\sqrt{\frac{\pi}{p}} e^{-\frac{a^2}{8p}} I_{\frac{\nu}{2}}(\frac{a^2}{8p}), \qquad p > 0, a > 0,$$
for $\mathrm{Re}(\nu) > -1$, [66, Section 13.3, Formula (5)].

5.5.3 Subordination principle on Wright function

In this subsection, we apply Corollary 5.4.3 to finite difference operators. We obtain some known formulae but others seem to be new, see (5.5.1), (5.5.2) and (5.5.3) below. They give some interesting new connections between the Wright and Bessel functions.

Take $a = \delta_{-1} - \delta_0$ or $a = \delta_1 - \delta_0$ in Corollary 5.4.3.

(i) For $0 < \beta < 1$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have
$$E^{(n)}_{\beta,1}(t) = \sum_{j=0}^\infty \frac{(j+n)!}{j!} \frac{t^j}{\Gamma(\beta(j+n)+1)} = \int_0^\infty \Phi_\beta(\tau) e^{\tau t} \tau^n d\tau.$$
For $n = 0$, we obtain the formula i), remark 2.4.2; and $t = 0$, the formula (2.4.5). For $\beta = \frac{1}{3}$, we obtain the following integral formula for the Airy function
$$E^{(n)}_{\frac{1}{3},1}(t) = \sum_{j=0}^\infty \frac{(j+n)!}{j!} \frac{t^j}{\Gamma(\frac{j+n}{3}+1)} = \int_0^\infty 3^{\frac{2}{3}} Ai\left(\frac{\tau}{3^{\frac{1}{3}}}\right) e^{\tau t} \tau^n d\tau,$$
for $n \in \mathbb{N}_0$, and $t \in \mathbb{C}$. For $t = 0$, we get [63, Formula (3.83)].

(ii) For $1 < \beta < 2$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$(2t)^{n-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j (j+n)!}{j!} \frac{t^{2j}}{\Gamma(\beta(j+n)+1)} = \frac{\sqrt{\pi}}{2} \int_0^{\infty} \Phi_{\frac{\beta}{2}}(\tau) \tau^{n+\frac{1}{2}} J_{n-\frac{1}{2}}(\tau t)\, d\tau. \tag{5.5.1}$$

In the case $n = 0$ and $n = 1$, we obtain formulae (2.4.3).

Now take $a = \delta_{-1} - 2\delta_0 + \delta_1$ or $a = \delta_{-2} - 2\delta_0 + \delta_2$.

(i) For $0 < \beta < 1$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{j+n}}{\Gamma(\beta(j+n)+1)} = \int_0^{\infty} \Phi_{\beta}(\tau) e^{-2\tau t} I_n(2\tau t) d\tau. \tag{5.5.2}$$

In particular, when $\beta = \frac{1}{3}$, we get the integral formula for Airy function,

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{j+n}}{\Gamma(\frac{j+n}{3}+1)} = \int_0^{\infty} 3^{\frac{2}{3}} Ai\left(\frac{\tau}{3^{\frac{1}{3}}}\right) e^{-2\tau t} I_n(2\tau t) d\tau,$$

for $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

(ii) For $1 < \beta < 2$, $t \in \mathbb{C}$ and $n \in \mathbb{N}_0$, we have

$$\sum_{j=0}^{\infty} (-1)^j \binom{2(j+n)}{j} \frac{t^{2(j+n)}}{\Gamma(\beta(j+n)+1)} = \int_0^{\infty} \Phi_{\frac{\beta}{2}}(\tau) J_{2n}(2\tau t) d\tau. \tag{5.5.3}$$

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