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INVARIANT MEASURES IN SYMBOLIC DYNAMICS: A TOPOLOGICAL, COMBINATORIAL AND GEOMETRICAL APPROACH.

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Invariant measures in symbolic dynamics: a topological, combinatorial and geometrical approach.

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Abstract

In this work we study some dynamical properties of symbolic dynamical systems, with particular emphasis on the rol played by the invariant probability measures of such systems. We approach the study of the set of invariant measures from a topological, combinatorial and geometrical point of view.

From a topological point of view, we focus on the problem of *orbit equivalence* and *strong orbit equivalence* between dynamical systems given by minimal actions of $\mathbb{Z}$, through the study of an algebraic invariant, namely the *dynamical dimension group*. Our work presents a description of the dynamical dimension group for two particular classes of subshifts: *S-adic* subshifts and *dendric* subshifts.

From a combinatorial point of view, we are interested in the problem of *balance* in minimal uniquely ergodic systems given by actions of $\mathbb{Z}$. We investigate the behavior regarding balance for substitutive, S-adic and dendric subshifts. We give necessary conditions for a minimal substitutive system with rational frequencies to be balanced on its factors, obtaining as a corollary the unbalance in the factors of length at least 2 in the subshift generated by the Thue–Morse sequence.

Finally, from the geometrical point of view, we investigate the problem of realization of Choquet simplices as sets of invariant probability measures associated to systems given by minimal actions of amenable groups on the Cantor set. We introduce the notion of *congruent monotileable amenable* group, we prove that every virtually nilpotent amenable group is congruent monotileable, and we show that for a discrete infinite group G with this property, every Choquet simplex can be obtained as the set of invariant measures of a minimal G-subshift.

Key words: subshift, invariant probability measure, orbit equivalence, dimension group, balance, Choquet simplex, amenable group.

Resumé

Dans ce travail nous étudions quelques propriétés des systèmes symboliques, avec un accent particulier mis sur le rôle joué par les mesures invariantes de tels systèmes. Nous nous attachons à l'étude des mesures invariantes d'un point de vue topologique, combinatoire et géométrique.

Du point de vue topologique, nous nous concentrons sur le problème de *l'équivalence orbitale* et *l'équivalence orbitale forte* entre des système dynamiques donnés par des actions minimales de $\mathbb{Z}$, par l'étude d'un invariant algébrique, à savoir, le *groupe de dimension dynamique*. Notre travail donne une description du groupe de dimension dynamique pour deux classes particulières de sous-shifts : les sous-shifts *S-adiques* et les sous-shifts *dendriques*.

Du point de vue combinatoire, nous nous intéressons au problème de *l'équilibre* des sous-shifts minimaux et uniquement ergodiques donnés par des actions de $\mathbb{Z}$. Nous étudions le comportement concernant l'équilibre pour des sous-shifts substitutifs, S-adiques et dendriques. Nous établissons des conditions nécessaires pour qu'un sous-shift substitutif minimal avec des fréquences rationnelles soit équilibré par rapport à ses facteurs, en obtenant comme corollaire le déséquilibre des facteurs de longueur supérieure à 2 dans le sous-shift engendré par la substitution de Thue–Morse.

Enfin, du point de vue géométrique, nous étudions le problème de réalisation des simplexes de Choquet comme des ensembles de mesures de probabilité invariantes associés à des systèmes donnés par des actions minimales des groupes moyennables sur l'ensemble de Cantor. Nous introduisons la notion de groupe *moyennable congruent-monopavable*, nous montrons que tout groupe moyennable virtuellement nilpotent est congruent-monopavable, et que pour un group discret infini G avec cette propriété, tout simplexe de Choquet peut s'obtenir comme l'ensemble des mesures invariantes d'un G-sous-shift minimal.

Mots clés : sous-shift, mesure de probabilité invariante, équivalence orbitale, groupe de dimension, équilibre, simplexe de Choquet, groupe moyennable.

Resumen

En este trabajo estudiamos algunas propiedades dinámicas de sistemas simbólicos, con especial énfasis en el rol que juegan las medidas de probabilidad invariantes de tales sistemas. Nuestra aproximación al estudio de las medidas invariantes se realiza desde tres ángulos: topológico, combinatorio y geométrico.
Desde el punto de vista topológico, nos enfocamos en el problema de la *equivalencia orbital* y *equivalencia orbital fuerte* entre sistemas dinámicos dados por acciones minimales de $\mathbb{Z}$, a través del estudio de un invariante algebraico, a saber, el *grupo de dimensión dinámico.* Nuestro trabajo presenta una descripción del grupo de dimensión dinámico para dos clases particulares de *subshifts* minimales: los *subshifts S-ádicos* y los *subshifts déndricos.*
Desde el punto de vista combinatorio, nos interesamos en el problema del *equilibrio* en *subshifts* minimales y únicamente ergódicos dados por acciones de $\mathbb{Z}$. Investigamos el comportamiento en relación al equilibrio para *subshifts* substitutivos, S-ádicos y déndricos. Establecemos condiciones necesarias para que un *subshift* substitutivo minimal con frecuencias racionales sea equilibrado en sus factores, obteniendo como corolario el desequilibrio en los factores de largo mayor o igual a 2 en el *subshift* generado por la substitución de Thue–Morse.
Finalmente, desde el punto de vista geométrico, investigamos la posibilidad de realizar símplices de Choquet como conjuntos de medidas de probabilidad invariantes asociados a sistemas dados por acciones minimales de grupos promediables sobre el Cantor. Introducimos la noción de grupo *promediable congruente-monoembaldosable*, probamos que todo grupo promediable virtualmente nilpotente es congruente-monoembaldosable, y mostramos que para un grupo discreto e infinito G con esta propiedad, todo símplice de Choquet puede obtenerse como el conjunto de medidas invariantes de un G-*subshift* minimal.

Palabras clave: subshift, medida de probabilidad invariante, equivalencia orbital, grupo de dimensión, equilibrio, símplex de Choquet, grupo promediable.

Résumé étendu

Le but de cette thèse de doctorat est d'étudier plusieurs propriétés dynamiques des systèmes symboliques, avec un accent particulier mis sur le rôle joué par les mesures invariantes de ces systèmes. Nous abordons l'étude de l'ensemble des mesures invariantes d'un point de vue topologique, combinatoire et géométrique, en fonction de la nature du système symbolique sous-jacent.

Dans cette introduction, nous rappelons brièvement les principales notions liées à la dynamique symbolique, aux mesures invariantes et aux propriétés dans lesquelles elles entrent en jeu. Nous résumons également les résultats obtenus dans la thèse et présentons l'organisation du texte. Les systèmes dynamiques symboliques sont à l'origine un outil pour étudier les systèmes dynamiques généraux par la discrétisation de l'espace et du temps. En gros, l'idée est de discrétiser le temps et de *coder* les trajectoires continues d'un système continu donné, en utilisant différents symboles, pour obtenir un nouveau système discret fait de trajectoires symboliques. Ensuite, on récupère des informations pertinentes sur le système original en regardant le second, qui est la plupart du temps plus simple. Le début de cette approche remonte aux premiers travaux de Hadamard ([Ha98]), Thue ([Th12]) et Morse ([Mor21]), entre autres, bien que l'étude de la dynamique symbolique de manière systématique n'ait pas été initiée avant les travaux fondateurs de Morse et Hedlund [HM38], [HM40] dans les années 1940. En termes précis, étant donné un ensemble fini de symboles $\mathcal{A}$, appelé un *alphabet*, un *système dynamique symbolique* ou simplement un *système symbolique* avec des symboles dans $\mathcal{A}$ est un système dynamique topologique (X, S, G) où G est un groupe localement compact infini, X est un subspace de $\mathcal{A}^G$ et S est une action continue gauche de G sur X. Nous utilisons la notation S^g pour faire référence à l'action de l'élément $g \in G$ sur X. Ici, l'espace $\mathcal{A}^G$ est muni de la topologie produit de la topologie discrète sur $\mathcal{A}$. C'est un espace de Cantor. La dynamique symbolique ne concernait à l'origine que les $\mathbb{Z}$-actions sur l'espace de Cantor, les éléments des systèmes symboliques classiques étant des séquences de symboles ou des *mots infinis* ; c'est pourquoi la dynamique symbolique est étroitement liée à l'étude de la combinatoire des mots, des langages formels et du codage. Voir par exemple [LM95] pour une

exposition détaillée sur le sujet. Plus récemment, le champ s'est étendu à des actions de groupes plus générales sur les espaces topologiques, en particulier sur l'espace de Cantor.
Les *sous-shifts* forment une classe importante de systèmes symboliques. Étant donné l'alphabet $\mathcal{A}$ et le groupe G, considérons $X = \mathcal{A}^G$ et l'action de G sur X donnée par $S^g((x_h)_{h\in G}) = ((g.x)_h)_{h\in G} = (x_{g^{-1}h})_{h\in G}$. Le triplet (X, S, G) est appelé dans ce cas le G-*shift* sur $\mathcal{A}$. Si $Y \subseteq X$ est un sous-espace fermé S-invariant de X, $(Y, S|_Y, G)$ s'appelle un *sous-shift* sur $\mathcal{A}$. Quand $G = \mathbb{Z}$, on appelle le $\mathbb{Z}$-*shift* ou simplement le *shift* sur $\mathcal{A}$ le triple $(\mathcal{A}^{\mathbb{Z}}, T, \mathbb{Z})$, où $T = S^{-1}$, soit l'application du shift classique. Un $\mathbb{Z}$-*sous-shift* est un sous-espace fermé T invariant de $\mathcal{A}^{\mathbb{Z}}$.

Les mesures invariantes jouent un rôle important dans l'étude de plusieurs propriétés des systèmes topologiques dynamiques en général et des systèmes symboliques en particulier. Nous les utilisons dans ce travail comme fil conducteur pour l'étude des propriétés liées à l'*équilibre*, à la réalisation de *simplexes de Choquet* et aux *groupes de dimension*, ces deux derniers en rapport avec la notion d'*équivalence orbitale*. Nous décrivons chacun de ces sujets plus loin dans cette introduction.
Rappelons que, étant donné un système dynamique topologique (X, S, G), une *mesure invariante* de (X, S, G) est une mesure borélienne de probabilité μ sur X telle que pour tout $g \in G$, $\mu(S^g(A)) = \mu(A)$, pour chaque sous-ensemble borélien $A \subseteq X$. L'ensemble de toutes les mesures invariantes de (X, S, G) est désigné par $\mathcal{M}(X, S, G)$. Un élément $\mu \in \mathcal{M}(X, S, G)$ est dit *ergodique* si chaque fois que $S^g(A) = A$ pour tout $g \in G$ pour un ensemble de Borel $A \subseteq X$, soit $\mu(A) = 0$ ou $\mu(A) = 1$. Le système (X, S, G) est dit *uniquement ergodique* si $\mathcal{M}(X, S, G)$ est un singleton. Ce sont des notions classiques qui appartiennent au domaine de la Théorie Ergodique (voir par exemple [W82] pour plus de détails sur ce sujet). Selon le théorème de Bogolyubov [Bog39], un groupe est moyennable (voir Chapitre 5 pour une définition) si et seulement si pour toute action continue de G sur un espace métrique compact X, il existe une mesure de probabilité sur X qui est invariante sous l'action de G. Cela a été prouvé à l'origine pour $G = \mathbb{R}$ dans [BogK37] et ensuite pour les groupes moyennables en général dans [Bog39]. Plus tard, Giordano et de la Harpe ont montré dans [GdH97] qu'un groupe G est moyennable si et seulement si une action continue de G sur l'ensemble Cantor a au moins une mesure de probabilité invariante. Ainsi, en particulier, lorsque G est un groupe moyennable, $\mathcal{M}(X, S, G)$ n'est pas vide. Pour plus de détails historiques sur le sujet de la moyenabilité, voir par exemple [CSS17, Chapitre 9] ou [Ju15, Chapitre 1].

Mesures invariantes et fréquences. Dans les systèmes symboliques donnés par des actions de

$\mathbb{Z}$, les mesures invariantes sont liées à la notion de *fréquence* d'une lettre ou d'un mot fini avec des lettres dans $\mathcal{A}$. Étant donné un système symbolique $(X, S, \mathbb{Z})$, notons $\mathcal{L}_X$ le *langage* de X, c'est-à-dire l'ensemble de tous les mots finis ou *facteurs* dans le monoide libre $\mathcal{A}^*$ apparaissant dans des éléments de X. La *fréquence* $f_w(x)$ d'un facteur $w \in \mathcal{L}_X$ dans un mot infini x est définie comme la limite suivante, quand elle existe,

$$f_w(x) = \lim_{n\to\infty} \frac{|x_{-n} \cdots x_0 \cdots x_n|_w}{2n+1}.$$

Un mot infini $x \in \mathcal{A}^{\mathbb{Z}}$ est dit avoir des *fréquences uniformes* si pour chaque facteur $w \in \mathcal{L}_X$, le rapport $\frac{|x_k \cdots x_{k+2n}|_w}{2n+1}$ converge vers $f_w(x)$ quand n tend vers l'infini, de façon uniforme en k. Si $\mu \in \mathcal{M}(X, S, \mathbb{Z})$ est une mesure ergodique, le quadruple $(X, S, \mathcal{B}, \mu)$ où $\mathcal{B}$ est la tribu de Borel sur X, est un système dynamique mesuré ergodique. Pour tous les facteurs $w \in \mathcal{L}_X$ on définit le *cylindre* de w par

$$[w] = \{x \in \mathcal{A}^{\mathbb{Z}} : x_0 \cdots x_{|w|-1} = w\},$$

et on peut appliquer le théorème ergodique [W82, Section 1.6] à la fonction indicatrice $\chi_{[w]}$ pour obtenir que pour μ-presque chaque point $x \in X$ et pour tout facteur w, la fréquence $f_w(x)$ existe. De même, l'unique ergodicité de $(X, S, \mathbb{Z})$ est équivalente au fait que chaque $x \in X$ a des fréquences uniformes. Dans le cas de systèmes symboliques minimaux, l'unique ergodicité est équivalent en effet à l'existence de fréquences pour tous les facteurs (voir section 1.4.2 pour plus de détails).

Étant donné un système symbolique uniquement ergodique $(X, S, \mathbb{Z})$ on peut s'interroger sur la vitesse de convergence des sommes de Birkhoff vers les fréquences des mots finis. Si la convergence est assez rapide, on dit que le mot infini $x \in X$ est *équilibré* sur un facteur donné. Nous détaillons cette notion dans le paragraphe suivant.

Équilibre. En termes combinatoires, $x \in X$ est équilibré sur le facteur $w \in \mathcal{L}_X$ s'il existe une constante C_w telle que pour chaque paire (u, v) de facteurs de x de même longueur, la différence entre le nombre d'occurrences de w dans u et v diffère d'au plus C_w, c'est-à-dire $||u|_w - |v|_w| \leq C_w$ lorsque $|u| = |v|$, où $|u|_w$ représente le nombre d'occurrences de w dans v et $|u|$ la longueur de u. Si $(X, S, \mathbb{Z})$ est un système minimal, chaque mot infini a le même langage et donc l'équilibre sur un facteur donné est une propriété du système entier.

L'étude de l'équilibre est d'abord apparue dans les travaux de Morse et Hedlund ([HM38], [HM40]) sous la forme de 1-équilibre pour les lettres des mots infinis définis sur un alphabet de deux lettres,

c'est-à-dire, quand w est une lettre et $C_w = 1$. Il a été montré que les mots infinis qui sont 1-équilibrés sur un alphabet de deux lettres sont exactement les mots sturmiens, c'est-à-dire les codages binaires des trajectoires de rotations irrationnelles sur le cercle unitaire (voir Exemple 1.5). Plus tard, le concept a été étendu aux facteurs et au C_w-équilibre dans des mots infinis avec des symboles dans des alphabets plus grands.
Les mots définis sur un alphabet plus grand qui sont 1-équilibrés ont été caractérisés dans [Hu00]. Il a été prouvé dans [FV02] que les mots sturmiens sont équilibrés sur tous leurs facteurs : les auteurs ont montré que la constante C_w ci-dessus correspond exactement à $|w|$, c'est-à-dire, chaque fois que $|u| = |v|$, $||u|_w - |v|_w| \leq |w|$, ce qui généralise leur comportement sur l'équilibre des lettres.
Comme indiqué dans la proposition 1.27, lorsqu'un système symbolique minimal uniquement ergodique avec une mesure unique μ est équilibré sur un facteur donné w, la mesure $\mu([w])$ est une valeur propre topologique additive du système. Cette connexion a été exploitée en Théorie Ergodique pour prouver le déséquilibre (voir par exemple [CFM08]).

Nous consacrons une partie de cette thèse de doctorat à l'étude du comportement de l'équilibre dans les sous-shifts dendriques et ultimement dendriques (voir Chapitre 4, Section 4.4). Ce type de sous-shifts est précisément défini dans la Section 4.1. Pour la classe des sous-shifts ultimement dendriques nous prouvons que deux mesures invariantes μ et μ' sont égales si et seulement si elles coincident sur les cylindres de facteurs de longueur n, pour tout n avec $n \leq m+1$ (Théorème 4.12), où m est le *seuil* du système. Nous prouvons également qu'un sous-shift ultimement dendrique avec seuil m est équilibré sur les facteurs de longueur $m+1$ si et seulement s'il est équilibré sur tous ses facteurs (Théorème 4.19). Ce n'est pas le cas des systèmes symboliques arbitraires. En effet, nous donnons également des conditions nécessaires pour l'équilibre dans les systèmes substitutifs à fréquences rationnelles dans le Chapitre 3, Section 3.5 (Théorème 3.50) et utilisons ceci pour prouver le déséquilibre pour tous les facteurs de longueur d'au moins 2 dans la séquence Thue–Morse (Corollaire 3.53), qui est équilibrée sur les lettres.
La plupart des résultats précédents sur l'équilibre sont publiés dans [BCB18].

Si nous sortons du cas uniquement ergodique, il est intéressant d'étudier l'ensemble de toutes les mesures invariantes d'un point de vue géométrique. Nous introduisons ce sujet dans le paragraphe suivant.

Le simplexe de Choquet des mesures invariantes. D'un point de vue géométrique, on sait que l'ensemble des mesures de probabilité invariantes d'une action continue d'un groupe moyennable sur un espace métrique compact est un simplexe de Choquet, c'est-à-dire, un sous-ensemble compact convexe métrizable K d'un espace vectoriel réel localement convexe tel que pour chaque $v \in K$ il existe une unique mesure de probabilité m supporté sur $ext(K)$ avec $\int_{ext(K)} xdm(x) = v$, dont les points extrémaux sont les mesures ergodiques (voir par exemple [Gl03], ou [BR10, Chapitre 7] pour une preuve dans le cas des $\mathbb{Z}$-actions). Ici, nous considérons $\mathcal{M}(X, T, G)$ comme un sous-espace du dual $C(X, \mathbb{R})$ muni de la topologie faible*, où $C(X, \mathbb{Z})$ désigne le groupe additif des fonctions continues à valeurs réelles définies sur X. Une question naturelle est donc de savoir si l'inverse est vrai, c'est-à-dire, que si l'on donne un simplexe de Choquet K et un groupe moyennable G, il est possible de réaliser K comme l'ensemble des mesures de probabilité invariantes d'une action continue de G sur un espace métrique compact. Dans [Do91] Downarowicz a répondu pour la première fois à cette question dans le cas $G = \mathbb{Z}$, montrant que chaque simplexe de Choquet peut être réalisé comme l'ensemble des mesures de probabilité invariantes d'un $\mathbb{Z}$-sous-shift de Toeplitz (voir Exemple 1.6). L'extension de ce résultat à tout groupe moyennable résiduellement fini (voir Exemple 1.9 pour la définition) a été montrée dans [CP14].
En général, il se trouve que les propriétés de G en tant que groupe imposent certaines restrictions à la possibilité de réalisation de tout simplexe de Choquet comme l'ensemble des mesures invariantes d'une action continue de G sur un espace métrique compact. Par exemple, on sait que le simplexe de Poulsen est le simplexe des mesures invariantes associées à l'action du G-shift pour n'importe quel groupe dénombrable moyennable G sur l'espace de Cantor $\{0,1\}^G$, et que si G a la Propriété T, alors pour chaque action continue de G sur un espace métrique compact X, $\mathcal{M}(X, S, G)$ est soit vide, soit un simplexe de Bauer, c'est-à-dire un simplexe dans lequel l'ensemble des points extrémaux est fermé (voir [GW97]).

Dans cette thèse de doctorat, nous abordons le problème de réalisation des simplexes de Choquet dans le contexte d'actions de groupes moyennables (voir Chapitre 5). Dans [W01], Weiss a introduit le concept de groupe moyennable *monopavable*, qui est une généralisation de la notion de groupe moyennable résiduellement fini, au sens que les monotuiles utilisées pour paver un groupe monopavable jouent le rôle des domaines fondamentaux des sous-groupes d'indices finis dans les groupes résiduellement finis (voir Section 5.1.2 pour plus de détails). On ne sait pas s'il existe des groupes qui ne sont pas monopavables. Nous introduisons ici le concept de groupe moyennable congruent-monopavable (voir

Section 5.2), qui comprend tous les groupes moyennables résiduellement finis. Nous montrons que la classe des groupes moyennables congruent-monopavables est plus grande que la classe des groupes moyennables résiduellement finis en prouvant que chaque groupe moyennable virtuellement nilpotent est congruent-monopavable (Théorème 5.23), et que tout simplexe de Choquet peut être réalisé comme l'ensemble des mesures invariantes d'une action minimale de n'importe quel groupe moyennable congruent-monopavable G. Cette action est libre si G est virtuellement abélien (Theorem 5.34). En conséquence directe, nous obtenons que pour tout simplexe de Choquet K, il existe un $\mathbb{Q}$-sous-shift libre et minimal $(X, T, \mathbb{Q})$ tel que $\mathcal{M}(X, T, \mathbb{Q})$ est affine homéomorphe à K (Corollaire 5.39).
Les résultats précédents sur les groupes moyennables congruent-monopavables et la réalisation de simplexes de Choquet sont inclus dans [CC18].

Le problème de réalisation de simplexes de Choquet en tant qu'ensembles de mesures invariantes décrit ci-dessus est lié à la notion d'*équivalence orbitale topologique* entre des systèmes de Cantor minimaux. Dans le paragraphe suivant, nous rappelons cette notion d'équivalence entre des systèmes dynamiques et introduisons les groupes de dimensions qui leur sont associés, qui correspondent à des invariants algébriques pour l'équivalence orbitale.

Équivalence orbitale et groupes de dimension. Deux systèmes dynamiques minimaux (X_1, S_1, G_1) et (X_2, S_2, G_2) sont dits *topologiquement orbitalement équivalents* s'il existe un homéomorphisme $\phi : X_1 \to X_2$ qui envoie les orbites de l'action de G_1 sur les orbites de l'action de G_2. Dans le cas de deux actions minimales de $\mathbb{Z}$, l'équivalence orbitale implique l'existence de deux applications $n_1 : X_1 \to \mathbb{Z}$ et $n_2 : X_2 \to \mathbb{Z}$ (uniquement définies par minimalité) de sorte que, pour tout $x \in X_1$,

$$\phi \circ S_1(x) = S_2^{n_1(x)} \circ \phi(x) \text{ et } \phi \circ S_1^{n_2(x)}(x) = S_2 \circ \phi(x).$$

Les applications n_1 et n_2 sont appelées les *cocycles* de S_1 et S_2 respectivement.
Deux systèmes dynamiques minimaux $(X_1, S_1, \mathbb{Z})$ et $(X_2, S_2, \mathbb{Z})$ sont dits (topologiquement) fortement orbitalement équivalents si n_1 et n_2 ont au plus un point de discontinuité. L'équivalence orbitale et l'équivalent orbitale forte sont des notions *a priori* plus faibles que la conjugaison qui ont été introduites dans [GPS95] afin d'obtenir des résultats similaires à ceux obtenus dans le cadre des systèmes dynamiques mesurés. Deux systèmes dynamiques mesurés $(X_1, S_1, \mathcal{B}_1, \mu_1, G_1)$ et $(X_2, \mathcal{B}_2, \mu_2, G_2)$ sont orbitalement équivalents s'il existe une bijection bimesurable $\phi : X_1 \to X_2$ qui envoie les orbites de

l'action de G_1 sur les orbites de l'action de G_2. C'est un théorème de Dye [Dye59] que deux actions ergodiques de $\mathbb{Z}$ sur des espaces de probabilité non-atomiques sont orbitalement équivalentes (en mesure). Ce résultat a par la suite été étendu aux actions ergodiques de groupes moyennables dans [OW80] : deux actions ergodiques de groupes moyennables sur des espaces de probabilité non atomiques sont orbitalement équivalentes (en mesure). (Voir [Ga10] pour une étude complète sur l'équivalence orbitale en mesure).

Dans le cadre topologique, la situation est très différente : parmi les actions minimales de $\mathbb{Z}$ sur l'ensemble Cantor, il existe d'innombrables classes d'équivalence orbitale. Ceci peut être vu comme une conséquence du fait que l'équivalence orbitale topologique et l'équivalence orbitale forte sont caractérisées ([GPS95]) dans le cas d'actions minimales de $\mathbb{Z}$ sur l'espace de Cantor, en utilisant ce qui est appelé le *groupe de dimension dynamique* associé au système : $(X_1, S_1, \mathbb{Z})$ et $(X_2, S_2, \mathbb{Z})$ sont fortement orbitalement équivalents si et seulement si leurs groupes de dimensions dynamiques sont isomorphes comme groupes ordonnés avec unité ; ils sont orbitalement équivalents si et seulement si leurs groupes de dimensions dynamiques réduits sont isomorphes comme groupes ordonnés avec unité. En gros, le groupe de dimension dynamique d'un système $(X, S, \mathbb{Z})$ est un groupe ordonné avec unité dont la partie groupe est donnée par $H(X, S, \mathbb{Z}) = C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$, où $C(X, \mathbb{Z})$ est le groupe additif de fonctions continues de X dans $\mathbb{Z}$ et $\beta C(X, \mathbb{Z})$ est l'image de $C(X, \mathbb{Z})$ sous *l'application de cobord* $\beta : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$, définie par $\beta f(x) = f \circ S(x) - f(x)$ pour tout $x \in X$. Le groupe de dimension dynamique réduit de $(X, S, \mathbb{Z})$ est un autre groupe ordonné avec une unité dont la partie groupe est le quotient $H(X, S, \mathbb{Z})/\{[f] \in H(X, S, \mathbb{Z}) : \int_X f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X, S, \mathbb{Z})\}$ (voir Section 1.5.3 pour plus d'information sur les groupes ordonnés avec unité et groupes de dimension). Un résultat similaire a été prouvé pour les actions de $\mathbb{Z}^d$ dans [GPMS10] : deux systèmes dynamiques $(X_1, S_1, \mathbb{Z}^{d_1})$ et $(X_2, S_2, \mathbb{Z}^{d_2})$ sur l'espace de Cantor sont orbitalement équivalents si et seulement si leurs groupes de dimensions dynamiques réduits sont isomorphes comme groupes ordonnés avec unité, où le groupe de dimension dynamique réduit correspond par définition au quotient $C(X, \mathbb{Z})/\{f \in C(X, \mathbb{Z}) : \int_X f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X, S, \mathbb{Z}^d)\}$.

Ainsi, le groupe de dimension dynamique réduit est un invariant total pour l'équivalence orbitale topologique parmi les actions minimales de $\mathbb{Z}^d$ sur l'espace Cantor. C'est aussi un invariant (pas nécessairement total) por l'équivalence orbitale entre les actions minimales de groupes moyennables sur l'espace de Cantor. D'autre part, l'espace de *traces* de ce groupe (voir Section 1.5.2 pour la définition) est affine homéomorphe à l'ensemble des mesures invariantes $\mathcal{M}(X, S, G)$. Ainsi, un groupe moyennable G avec la propriété que chaque simplexe de Choquet métrisable peut être réalisé comme l'ensemble des mesures de probabilité invariantes d'une action minimale libre de G sur l'ensemble de Cantor, admet au moins autant de classes d'équivalence orbitale topologique qu'il existe de simplexes métrisables de Choquet.

Il n'est pas difficile de voir que lorsque X_1 ou X_2 sont connexes, l'équivalence orbitale topologique implique l'existence d'un isomorphisme de groupe $\varphi : G_1 \to G_2$ tel que pour tout $g \in G_1$, $\phi \circ S_1^g = S_2^{\varphi(g)} \circ \phi$, soit les deux systèmes sont conjugués. C'est une des raisons pour lesquelles, dans l'étude de l'équivalence orbitale topologique, il est intéressant d'examiner des espaces de phase totalement déconnectés, comme l'espace de Cantor.

La notion d'équivalence orbitale forte est en quelque sorte naturelle puisqu'il a été montré dans [Bo83] que si n_1 (et donc n_2) ou n_2 (et donc n_1) ci-dessus sont continues, alors les deux systèmes sont *flip* conjugués, c'est-à-dire que $(X_1, S_1, \mathbb{Z})$ est conjugué soit à $(X_2, S_2, \mathbb{Z})$ ou à $(X_2, S_2^{-1}, \mathbb{Z})$.

Une partie de ce travail de doctorat est consacrée à la description du groupe de dimension dynamique, et donc à létude des classes d'équivalence orbitale, des sous-shifts dendriques et d'un type particulier de sous-shifts ultimement dendriques, appelés *sous-shifts spéculaires* (voir Section 4.1.1 pour plus de détails). Ceci est présenté au Chapitre 4, Section 4.5 et a été partiellement réalisé dans le cadre d'un travail en collaboration avec F. Dolce, F. Durand, J. Leroy, D. Perrin et S. Petite ([BCD+18]). Les principaux résultats sur ce sujet sont le Théorème 4.24 (groupe de dimension des sous-shifts dendriques) et le Théorème 4.25 (groupe de dimension des sous-shifts spéculaires). En utilisant ces deux résultats, nous obtenons le Corollaire 4.26, qui dit que deux sous-shifts dendriques (resp. spéculaires) minimaux sont fortement orbitalement équivalents si et seulement s'ils ont le même simplexe de fréquences des lettres (resp. le même simplexe engendré par les fréquences de facteurs de longueur 2), et deux sous-shifts dendriques (resp. spéculaires) minimaux et uniquement ergodiques sont fortement orbitalement équivalents si et seulement s'ils ont le même groupe additif de fréquences de lettres (resp. le même groupe additif engendré par les fréquences de facteurs de longueur 2). Nous établissons également une relation entre l'équilibre et le groupe de dimension dynamique dans le cas de systèmes uniquement ergodiques ayant des fréquences de lettres rationnellement indépendantes (Théorème 3.42), qui indique que si la partie groupe dans le groupe de dimension dynamique est abélien libre de rang d, où d est la cardinalité de l'alphabet, l'équilibre des lettres passe à l'équilibre sur les facteurs de longueur arbitraire. Enfin, nous décrivons le groupe de dimension dynamique pour certains sous-shifts S-adiques (voir Section 3.1 pour plus de détails), qui sont une généralisation des systèmes substitutifs, obtenus à partir d'une composition infinie de substitutions (Théorème 3.39).

Ce texte est organisé en cinq chapitres. Le premièr est consacré aux définitions générales, à l'historique et à fixer les notations que nous utilisons dans le texte. Dans le second, nous présentons en détail les concepts liés aux *partitions en tours* pour des actions minimales de $\mathbb{Z}$ sur l'espace de Cantor ; nous donnons quelques résultats sur le rapport entre certaines suites bien choisies de partitions en tours et le sous-groupe image, le groupe des infinitésimaux et le groupe de dimension dynamique d'un sous-shift minimal de $\mathbb{Z}$. Au Chapitre 3, nous appliquons les résultats du Chapitre 2 à l'étude de certaines propriétés des systèmes substitutifs et S-adiques, à savoir les groupes de dimensions et l'équilibre, et au Chapitre 4, nous faisons de même pour les sous-shifts dendriques et ultimement dendriques. Au Chapitre 5, nous étudions le problème de la réalisation des simplexes de Choquet en tant qu'ensembles de mesures invariantes des actions de groupes moyennables congruent-monopavables ; nous introduisons la notion de groupe moyennable congruent-monopavable, nous prouvons que cette classe de groupes est plus grande que celle des groupes moyennables résiduellement finis et que pour tout simplexe de Choquet K et tout groupe moyennable congruent-monopavable G, il existe un G-sous-shift minimal (X, S, G) tel que $\mathcal{M}(X, S, G)$ soit affine homéomorphe à K.

Contents

Introduction

The aim of this PhD thesis is to study several dynamical properties of symbolic systems, with particular emphasis on the role played by invariant measures of such systems. We approach the study of the set of invariant measures from a topological, combinatorial and geometrical point of view, depending on the nature of the underlying symbolic system.
In this introduction we briefly recall the main notions related to symbolic dynamics, invariant measures and properties in which they come into play. We also sumarize the results we have obtained and present the organization of the text.

Symbolic dynamical systems are originally a tool to study general dynamical systems by the discretization of the space and time. Roughly speaking, the idea is to discretize the time and *code* continuous trajectories of a given continuous system by using different *symbols*, to obtain a new discrete system made of *symbolic trajectories*. Then, one recovers relevant information about the original system by looking at the second one, which most of the time is simpler. The begining of this approach goes back to the early works of Hadamard ([Ha98]), Thue ([Th12]) and Morse ([Mor21]), among others, although the study of symbolic dynamics in a systematic way was not initiated until the seminal works of Morse and Hedlund [HM38], [HM40] in the 1940's. In precise words, given a finite set of symbols $\mathcal{A}$, called an *alphabet*, a *symbolic dynamical system* or simply a *symbolic system* with symbols in $\mathcal{A}$ is a topological dynamical system (X, S, G) where G is an infinite locally compact group, X is some subspace of $\mathcal{A}^G$ and S is a continuous left action of G on X. We use the notation S^g to refer the action of the element $g \in G$ on X. Here, the space $\mathcal{A}^G$ is endowed with the product topology of the discrete topology on $\mathcal{A}$. It is a Cantor space. Symbolic dynamics were originally concerned only with $\mathbb{Z}$-actions on the Cantor space, the elements in classical symbolic systems being sequences of symbols or *infinite words*; this is the reason why symbolic dynamics is closely related to the study of combinatorics on words, formal languages and coding. See for example [LM95] for a detailed exposition of the subject. More recently, the field has been extended to consider more general group actions on topological spaces, particularly in the Cantor space. An important class of symbolic systems are the *subshifts*. Given the alphabet $\mathcal{A}$ and the group G, consider $X = \mathcal{A}^G$ and the action of G on X given by $S^g((x_h)_{h\in G}) = ((g.x)_h)_{h\in G} = (x_{g^{-1}h})_{h\in G}$. The triple (X, S, G) is called in this case the *G-fullshift* on $\mathcal{A}$. If $Y \subseteq X$ is any closed S-invariant subspace of X, $(Y, S|_Y, G)$ is called a *subshift* on $\mathcal{A}$. When $G = \mathbb{Z}$, we call the *$\mathbb{Z}$-fullshift* or simply the *fullshift* on $\mathcal{A}$ the triple $(\mathcal{A}^{\mathbb{Z}}, T, \mathbb{Z})$, where $T = S^{-1}$, that is, the classical *shift map*. A *$\mathbb{Z}$-subshift* is any closed T-invariant subspace of $\mathcal{A}^{\mathbb{Z}}$.

Invariant measures play an important role in the study of several properties of topological dynamical systems in general and symbolic systems in particular. We use them in this work as a guiding thread to study properties related to *balance*, realization of *Choquet simplices* and *dimension groups*, the latter two in connection with the notion of *orbit equivalence*. We describe each of these topics later in this introduction.

Recall that, given a topological dynamical system (X, S, G), an *invariant measure* of (X, S, G) is a probability Borel measure μ on X such that for all $g \in G$, $\mu(S^g(A)) = \mu(A)$, for every Borel subset $A \subseteq X$. The set of all invariant measures of (X, S, G) is denoted $\mathcal{M}(X, S, G)$. An element $\mu \in \mathcal{M}(X, S, G)$ is said to be *ergodic* if whenever $S^g(A) = A$ for all $g \in G$ for some Borel set $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. The system (X, S, G) is said to be *uniquely ergodic* if $\mathcal{M}(X, S, G)$ is a singleton. These are classical notions which belong to the field of Ergodic Theory (see for example [W82] for details in this subject). It is a theorem by Bogolyubov [Bog39] that a group is amenable (see Chapter 5 for a definition) if and only if for all continuous action of G on a compact metric space X, there exists a probability masure on X which is invariant under the action of G. This was originally proved for $G = \mathbb{R}$ in [BogK37] and then for amenable groups in general in [Bog39]. Later, Giordano and de la Harpe showed in [GdH97] that a group G is amenable if and only if any continuous action on the Cantor set has an invariant probability measure. Thus, in particular, whenever G is an amenable group, $\mathcal{M}(X, S, G)$ is non-empty. For more historical details in the subject of amenability, see for example [CSS17, Chapter 9] or [Ju15, Chapter 1].

Invariant measures and frequencies. In symbolic systems given by $\mathbb{Z}$-actions, invariant measures are related to the notion of *frequency* of a letter or a finite word with letters in $\mathcal{A}$. Given a symbolic system $(X, S, \mathbb{Z})$, let $\mathcal{L}_X$ denote the *language* of X, that is, the set of all finite words or *factors* in the free monoid $\mathcal{A}^*$ appearing in the elements of X. The *frequency* $f_w(x)$ of a factor $w \in \mathcal{L}_X$ in an infinite word x is defined as the following limit, when it exists,

$$f_w(x) = \lim_{n\to\infty} \frac{|x_{-n}\cdots x_0 \cdots x_n|_w}{2n+1}.$$

An infinite word $x \in \mathcal{A}^{\mathbb{Z}}$ is said to have *uniform frequencies* if for every factor $w \in \mathcal{L}_X$, the ratio $\frac{|x_k\cdots x_{k+2n}|_w}{2n+1}$ converges to $f_w(x)$ when n tends to infinity, uniformly in k. If $\mu \in \mathcal{M}(X, S, \mathbb{Z})$ is an ergodic measure, the quadruple $(X, S, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra on X, is an ergodic measure-theoretic dynamical system. For all factor $w \in \mathcal{L}_X$ one defines the *cylinder* of w by

$$[w] = \{x \in \mathcal{A}^{\mathbb{Z}} : x_0 \cdots x_{|w|-1} = w\},$$

and one can apply the Ergodic Theorem [W82, Section 1.6] to the characteristic function $\chi_{[w]}$ to obtain that for μ-almost every point $x \in X$ and for any factor w, the frequency $f_w(x)$ exists. Similarly, the unique ergodicity of $(X, S, \mathbb{Z})$ is equivalent to the fact that every $x \in X$ has uniform frequencies. In the case of minimal symbolic systems, unique ergodicity is indeed equivalent to the existence of frequencies for all factors (see Section 1.4.2 for details).

Given a uniquely ergodic symbolic system $(X, S, \mathbb{Z})$ one can ask about the speed of convergence of Birkhoff sums toward frequencies of finite words. If the convergence is fast enough, one says that the infinite word $x \in X$ is *balanced* on a given factor. We detail this notion in the following paragraph.

Balance. In combinatorial terms, $x \in X$ is balanced on the factor $w \in \mathcal{L}_X$ if there exists a constant C_w such that for every pair (u, v) of factors of x with the same length, the difference between the number of occurrences of w in u and v differs by at most C_w, that is, $||u|_w - |v|_w| \leq C_w$ whenever $|u| = |v|$, where the notation $|u|_w$ stands for the number of occurrences of w in v and $|u|$, for its length. If $(X, S, \mathbb{Z})$ is a minimal system, every infinite word has the same language and thus balance on a given factor is a property of the whole system.
The study of balance first occurred in the works of Morse and Hedlund ([HM38], [HM40]) in the form of 1-balance for letters in infinite words defined over a two-letter alphabet, that is, when w is a letter and $C_w = 1$. It was shown that infinite words that are 1-balanced over a two-letter alphabet are exactly the Sturmian words, that is, the binary codings of trajectories of irrational rotations on the unit circle (see Example 1.5). Later, the concept was extended to factors and C_w-balance in infinite words with symbols in larger alphabets.

Words over a larger alphabet that are 1-balanced have been characterized in [Hu00]. Sturmian words have been proved to be balanced on all their factors in [FV02]: the authors have shown that the constant C_w above corresponds exactly to $|w|$, that is, whenever $|u| = |v|$, $||u|_w - |v|_w| \leq |w|$, which generalizes their behavior on letter balance.

The notion of C-balance for letters was extended to the multidimensional framework in [BT02]: given a configuration of $\mathbb{Z}^n$ with symbols in an alphabet $\mathcal{A}$, $x \in \mathcal{A}^{\mathbb{Z}^n}$, x is said to be balanced on the letter $i \in \mathcal{A}$ if there exists a constant C such that the difference between the number of positions $\mathbf{z} \in \mathbb{Z}^n$ verifying $x(\mathbf{z}) = i$ in two any sub-blocks of $\mathbb{Z}^n$ of the same size is bounded by C. In this article, the authors characterize 1-balanced multidimensional words in $\mathbb{Z}^n$ and study conditions of unbalance for some 2-dimensional words, like Sturmian 2-dimensional words.

This notion is also related to properties of *tilings*. We refer to [Sol97] or [Sad07] for details on this subject. Roughly speaking, a *tiling* of the space $\mathbb{R}^n$, or an *n-dimensional tiling* is a covering of $\mathbb{R}^n$ by *tiles*, such that tiles only intersect on their boundaries. A *tile* is a translation of a *prototile*, which is a subset of $\mathbb{R}^n$ homeomorphic to the closed unit ball. The tiling is thus the union of tiles obtained by translating a finite number of prototiles, and each tile is labeled according to the prototile it comes from. One then considers the space of all tilings obtained from a given set of prototiles (or a specific subspace of this), and the natural action of $\mathbb{R}^n$ by translations on it. This construction gives immediately an analogy between infinite sequences on a finite alphabet $\mathcal{A}$ and tilings of the real line constructed from a finite set of prototiles: given a sequence, one can construct a tiling by associating to each letter $a \in \mathcal{A}$ a prototile defined as an interval of some length ℓ_a and then translating the prototiles in the order given by the sequence. Conversely, given a tiling of the line, one can associate a letter to each prototile and then construct a sequence by concatenating the letters in the order given by the tiling. This extends in a natural way to multidimensional sequences and multidimensional tilings, and suggests that it could be a relation between some properties of a subshift $(X, T, \mathbb{Z}^n)$ and the associated tiling space together with the $\mathbb{R}^n$-action. Balance properties have been studied from this viewpoint in [Sad15], where C-balance on infinite sequences for letters and words have been related to what is called *plasticity* and *total plasticity* of the associated tilings. Since there is a vast theory developed for multidimensional tilings, this gives the idea of studying the relation between balance of multidimensional words and cohomological properties of the associated tilings.

As is stated in Proposition 1.27, when a minimal uniquely ergodic symbolic system with unique measure μ is balanced on a given factor w, the measure $\mu([w])$ is an additive topological eigenvalue of the system. This connection has been exploited in Ergodic Theory to prove unbalance (see for ex-

ample [CFM08]). The example of *Arnoux-Rauzy* sequences is of particular interest on this matter. Arnoux-Rauzy sequences are infinite sequences defined over a d-letter alphabet, $d \geq 2$, which generalize combinatorial properties of Sturmian sequences. In combinatorial terms, Sturmian sequences are defined as aperiodic sequences of *minimal complexity*: they are the binary sequences having $n+1$ factors of length n for all $n \in \mathbb{N}$. They were proved to be exactly the codings of irrational rotations on the circle in [CH73]. Arnoux-Rauzy sequences were thus introduced in [AR91] as sequences over a three-letter alphabet having $2n+1$ factors of length n for all $n \in \mathbb{N}$, with exactly one right special factor and one left special factor of each length (see Section 4.1 for the definitions), and then generalized to larger size alphabets as the sequences having $(d-1)n+1$ factors of length n for all $n \in \mathbb{N}$ ($d = |\mathcal{A}|$), with exactly one right special factor and one left special factor of each length. Arnoux-Rauzy sequences were conjectured to correspond exactly to the codings of rotations on the torus of higer dimensions, as well as Sturmian sequences correspond to rotations on the circle, but this conjecture was disproved in [CFZ00] by exhibiting an unbalanced Arnoux-Rauzy sequence. We refer to [CFZ00] and [CFM08] for more on this subject.
It is interesting to note that balance has been used to very concrete applications like in the field of *optimal routing* (see for example [AGH00]).

We devote a part of this PhD thesis to the study of balance behavior in dendric and eventually dendric subshifts (see Chapter 4, Section 4.4). This kind of subshifts is precisely defined in Section 4.1. We briefly recall the definition here: given a minimal subshift $(X, T, \mathbb{Z})$ on the alphabet $\mathcal{A}$ and a factor $w \in \mathcal{L}_X$, the *extensions* of w are

$$\begin{aligned} L(w) &= \{a \in \mathcal{A} \mid aw \in \mathcal{L}_X\} \\ R(w) &= \{a \in \mathcal{A} \mid wa \in \mathcal{L}_X\} \\ B(w) &= \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}\}. \end{aligned}$$

The *extension graph* $\mathcal{E}(w)$ of w is the undirected bipartite graph whose set of vertices is the disjoint union of $L(w)$ and $R(w)$ and whose edges are the pairs $(a,b) \in B(w)$. The subshift is said to be *eventually dendric* if there exists $m \in \mathbb{N}$ such that for all $w \in \mathcal{L}_X$ with $|w| \geq m$ the extension graph of w is a tree, that is, connected and without cycles. In this case, m is called the *threshold* of $(X, T, \mathbb{Z})$. If one can choose $m = 0$, $(X, T, \mathbb{Z})$ is called a *dendric subshift*.
For the class of eventually dendric subshifts we prove that two invariant measures μ and μ' are equal if and only if they coincide on the cylinders of factors of length n, for all $n \leq m+1$ (Theorem 4.12). We

also prove that an eventually dendric subshift with threshold m is balanced on the factors of length $m+1$ if and only if it is balanced on all factors (Theorem 4.19). This is not the case of arbitrary symbolic systems. Indeed, we also give necessary conditions for balance in substitutive systems with rational frequencies in Chapter 3, Section 3.5 (Theorem 3.50) and use this to prove unbalance for all factors of length at least 2 in the Thue–Morse sequence (Corollary 3.53), which is known to be balance on letters.

Most of the previous results about balance are published in [BCB18].

If we go out of the uniquely ergodic case, it is interesting to study the set of all invariant measures from a geometric point of view. We introduce this subject in the following paragraph.

The Choquet simplex of invariant measures. Geometrically speaking, it is known that the set of invariant probability measures of a continuous action of an amenable group on a compact metric space is a Choquet simplex, that is, a compact convex metrizable subset K of a locally convex real vector space such that for each $v \in K$ there is a unique probability measure m supported on $ext(K)$ with $\int_{ext(K)} xdm(x) = v$, whose extreme points are the ergodic measures (see for example [Gl03], or [BR10, Chapter 7] for a proof in the case of $\mathbb{Z}$-actions). Here, we consider $\mathcal{M}(X, T, G)$ as a subspace of the dual $C(X, \mathbb{R})$ endowed with the weak* topology, where $C(X, \mathbb{Z})$ denotes the additive group of continued real-valued functions defined on X. A natural question is thus whether the converse is true, i.e, if given a Choquet simplex K and an amenable group G, it is possible to realize K as the set of invariant probability measures of a continuous action of G on a compact metric space. In [Do91] Downarowicz answered for the first time this question in the case $G = \mathbb{Z}$, showing that every Choquet simplex can be realized as the set of invariant probability measures of a Toeplitz $\mathbb{Z}$-subshift (see Example 1.6). The extension of this result to any amenable residually finite group (see Example 1.9 for the definition) was shown in [CP14]. In the recent work [FH16], the authors show that every face in the simplex of invariant measures of a zero-dimensional free dynamical system given by an action of an amenable group, can be realized as the entire simplex of invariant measures on some other zero-dimensional dynamical system with a free action of the same group. Thus, if the Poulsen simplex (see Section 1.4.1 for the definition) could be obtained as the set of invariant measures of a free action of some prescribed amenable group G, the same would hold for any Choquet simplex.

In general, it turns out that the properties of G as a group impose some restrictions to the possibility of realization of any Choquet simplex as the set of invariant measures of a continuous action of G on a compact metric space. For example, it is known that the Poulsen simplex is the simplex of invariant measures associated to the full G-shift action of any countable amenable group G on the Cantor space $\{0,1\}^G$, and that if G has Property T, then for every continuous action of G on a compact metric space X, $\mathcal{M}(X,S,G)$ is either empty or a Bauer simplex, that is, a simplex in which the set of extreme points is closed (see [GW97]).

In this PhD thesis we tackle the problem of realization of Choquet simplices in the context of actions of amenable *monotileable* groups (see Chapter 5). In [W01] Weiss introduced the concept of *monotileable* amenable group, which is a generalization of the notion of residually finite amenable group, in the sense that the monotiles used to tile a monotileable group play the role of the fundamental domains of the finite index subgroups in residually finite groups (see Section 5.1.2 for details). It is still unknown if there are amenable groups which are not monotileable. We introduce here the concept of *congruent monotileable amenable group* (see Section 5.2), which includes all the amenable residually finite groups. We show that the class of congruent monotileable amenable groups is larger than the class of amenable residually finite groups by proving that every virtually nilpotent group is amenable congruent monotileable (Theorem 5.23), and that any Choquet simplex can be realized as the set of invariant measures of a minimal action of any congruent monotileable amenable group G. This action is free if G is virtually abelian (Theorem 5.34). As a direct consequence we obtain that for any Choquet simplex K there exists a minimal free $\mathbb{Q}$-subshift $(X,T,\mathbb{Q})$ such that $\mathcal{M}(X,T,\mathbb{Q})$ is affine homeomorphic to K (Corollary 5.39).
All previous results about congruent monotileable amenable groups and realization of Choquet simplices are included in [CC18].

The problem of realization of Choquet simplices as sets of invariant measures described above is related to the notion of *topological orbit equivalence* between minimal Cantor systems. In the next paragraph we recall this notion of equivalence between dynamical systems and introduce the *dimension groups* associated to them, which are algebraic invariants of orbit equivalence.

Orbit equivalence and dimension groups. Two minimal dynamical systems (X_1,S_1,G_1) and (X_2,S_2,G_2) are said to be *(topological) orbit equivalent* if there exists a homeomorphism $\phi: X_1 \to X_2$ sending orbits of the G_1-action onto orbits of the G_2-action. In the case of two minimal $\mathbb{Z}$-actions, orbit equivalence implies the existence of two maps $n_1: X_1 \to \mathbb{Z}$ and $n_2: X_2 \to \mathbb{Z}$ (uniquely defined by minimality) such that, for all $x \in X_1$,

$$\phi \circ S_1(x) = S_2^{n_1(x)} \circ \phi(x) \text{ and } \phi \circ S_1^{n_2(x)}(x) = S_2 \circ \phi(x).$$

The maps n_1 and n_2 are called the *cocycles* of S_1 and S_2 respectively.

The two minimal dynamical systems $(X_1, S_1, \mathbb{Z})$ and $(X_2, S_2, \mathbb{Z})$ are said to be *(topological) strong orbit equivalent* if n_1 and n_2 both have at most one point of discontinuity. Orbit equivalence and Strong orbit equivalent are notions weaker than conjugacy (*a priori*) which have been introduced in [GPS95] in an attempt to obtain similar results as those obtained in the measure-theoretic framework. Two measure-theoretic dynamical systems $(X_1, S_1, \mathcal{B}_1, \mu_1, G_1)$ and $(X_2, S_2, \mathcal{B}_2, \mu_2, G_2)$ are orbit equivalent if there exists a bimeasurable bijection $\phi : X_1 \to X_2$ sending orbits of the G_1-action onto orbits of the G_2-action. It is a theorem by Dye [Dye59] that any two ergodic measure-theoretic actions of $\mathbb{Z}$ on non-atomic probability spaces are (measure-theoretic) orbit equivalent. This result was later extended to ergodic actions of amenable groups in [OW80]: two ergodic measure-theoretic actions of amenable groups on non-atomic probability spaces are (measure-theoretic) orbit equivalent. (See [Ga10] for a complete survey on the subject of measure-theoretic orbit equivalence).

In the topological framework the situation is very different: among the minimal $\mathbb{Z}$-actions on the Cantor set there are uncountable many orbit equivalence classes. This can be easily seen as a consequence of the fact that (topological) orbit equivalence and strong orbit equivalence are characterized ([GPS95]) in the case of minimal $\mathbb{Z}$-actions on the Cantor space, by using what is called the *dynamical dimension group* associated to the system: $(X_1, S_1, \mathbb{Z})$ and $(X_2, S_2, \mathbb{Z})$ are strong orbit equivalent if and only if their *dynamical dimension groups* are isomorphic as ordered groups with unit; they are orbit equivalent if and only if their *reduced dynamical dimension groups* are isomorphic as ordered groups with unit. Roughly speaking, the dynamical dimension group of a system $(X, S, \mathbb{Z})$ is an ordered group with unit whose group part is given by $H(X, S, \mathbb{Z}) = C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$, where $C(X, \mathbb{Z})$ is the additive group of continuous functions from X to $\mathbb{Z}$ and $\beta C(X, \mathbb{Z})$ is the image of $C(X, \mathbb{Z})$ under the *coboundary map* $\beta : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$, defined by $\beta f(x) = f \circ S(x) - f(x)$ for all $x \in X$. The reduced dynamical dimension group of $(X, S, \mathbb{Z})$ is another ordered group with unit whose group part is the quotient $H(X, S, \mathbb{Z})/\{[f] \in H(X, S, \mathbb{Z}) : \int_X f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X, S, \mathbb{Z})\}$ (see Section 1.5.3 for details on ordered groups with unit and dimension groups). Consider for instance two Sturmian subshifts which code irrational rotations on the circle of angles α and β, with $\alpha \neq \beta$. The positive cone of the reduced dynamical dimension group associated to the first system is isomorphic to $P_{\frac{1-\alpha}{\alpha}}$, where for any angle γ, $P_\gamma = \{\mathbf{x} \in \mathbb{Z}^2 : x_1\gamma + x_2 \geq 0\}$, while that associated to the second one is isomorphic to P_β (see [DDM00]). This shows that there exist uncountably many minimal $\mathbb{Z}$-actions on the Cantor space which are not orbit equivalent.

A similar result was proved for $\mathbb{Z}^d$-actions in [GPMS10]: two dynamical systems $(X_1, S_1, \mathbb{Z}^{d_1})$ and $(X_2, S_2, \mathbb{Z}^{d_2})$ on the Cantor space are orbit equivalent if and only if their reduced dynamical dimension groups are isomorphic as ordered groups with unit, where the reduced dynamical dimension group corresponds by definition to the quotient $C(X,\mathbb{Z})/\{f \in C(X,\mathbb{Z}) : \int_X f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X,S,\mathbb{Z}^d)\}$. Thus, the reduced dynamical dimension group is a total invariant for topological orbit equivalence among minimal actions of $\mathbb{Z}^d$ on the Cantor space. It is also an invariant (not necessarily total) of orbit equivalence among minimal actions of amenable groups on the Cantor space. On the other hand, the space of *traces* of this group (see Section 1.5.2 for the definition) is affine homeomorphic to the set of invariant measures $\mathcal{M}(X,S,G)$. Thus, an amenable group G with the property that every metrizable Choquet simplex can be realized as the set of invariant probability measures of a minimal free G-action on the Cantor set, admits at least as many topological orbit equivalence classes as metrizable Choquet simplices there are.

It is not difficult to see that when X_1 or X_2 are connected, topological orbit equivalence implies the existence of a group isomorphism $\varphi : G_1 \to G_2$ such that for all $g \in G_1$, $\phi \circ S_1^g = S_2^{\varphi(g)} \circ \phi$, that is, both systems are conjugate. This is one reason why in the study of topological orbit equivalence it is intereseting to look at totally disconnected phase spaces, like the Cantor space.

The notion of strong orbit equivalence is somehow natural since it was shown in [Bo83] that if n_1 (and thus n_2) or n_2 (and thus n_1) above are continuous, then the two systems are *flip conjugate*, that is, $(X_1, S_1, \mathbb{Z})$ is conjugate to either $(X_2, S_2, \mathbb{Z})$ or to $(X_2, S_2^{-1}, \mathbb{Z})$.

The set of invariant measures, the dynamical dimension group and the reduced dynamical dimension group are not the only algebraic invariants used to study orbit equivalence. Let (X,S,G) be a topological dynamical system, and consider the group of homeomorphisms of X with the composition, $(Hom(X), \circ)$. The *full group* of S is defined by

$$[S] = \{\phi \in Hom(X) : \forall x \in X \exists g \in G : \phi(x) = S^g(x)\},$$

that is, the subgroup of homeomorphisms of X which locally coincide with the action of G. Note that $[S]$ trivially contains an isomorphic copy of the group G via the injection $g \mapsto S^g$. If the action of G on

X is minimal and aperiodic, each $\phi \in [S]$ defines a map $n_\phi : X \to G$ by the relation $\phi(x) = S^{n_g(x)}(x)$. The *topological full group* of S is the subgroup of $[S]$ defined by

$$[[S]] = \{\phi \in [S] : n_g \text{ is continuous}\}.$$

Note that $[[S]]$ also contains an isomorphic copy of G. Indeed, for all $g \in G$, $n_{S^g} : X \to G$ is the constant function with value g. We thus have $G \subseteq [[S]] \subseteq [S]$.
It was shown in [GPS99] that two minimal $\mathbb{Z}$-actions $(X_1, S_1, \mathbb{Z})$ and $(X_2, S_2, \mathbb{Z})$ on the Cantor space have isomorphic full groups if and only if they are orbit equivalent. Later, Medynets extended this result in 2011 ([Med11]) by showing that two aperiodic minimal systems (X_1, S_1, G_1) and (X_2, S_2, G_2) are orbit equivalent if and only if $[S_1] \cong [S_2]$. Full groups and topological full groups of Cantor minimal systems are very interesting objects from the point of view of Group Theory. They provide examples of *non-elementaary amenable groups* when the group acting is amenable. We do not work with this kind of groups in this PhD thesis. We recommend [Ju15, Chapter 4] for a survey on this subject.

It is a part of this PhD work to describe the dynamical dimension group, and thus to study orbit equivalence classes, of dendric subshifts and a special kind of eventually dendric subshifts, called *specular subshifts* (see Section 4.1.1 for details). This is presented on Chapter 4, Section 4.5 and was partially done in the context of a joint work with F. Dolce, F. Durand, J. Leroy, D. Perrin and S. Petite ([BCD+18]). The main results on this subject are Theorem 4.24 (dimension group of dendric subshifts) and Theorem 4.25 (dimension group of specular subshifts). Using these two results we obtain Corollary 4.26, which says that two minimal dendric (resp. specular) subshifts are strong orbit equivalent if and only they have the same simplex of letter frequencies (resp. the same simplex generated by frequencies of factors of length 2), and two minimal and uniquely ergodic dendric (resp. specular) subshifts are strong orbit equivalent if and only if they have the same additive group of letter frequencies (resp. the same additive group generated by the frequencies of factors of length 2). We also state a relation between balance and the dynamical dimension group in the case of uniquely ergodic systems having rationally idependent letter frequencies (Theorem 3.42), which states that if the group part in the dynamimcal dimension group is free abelian of rank d, where d is the cardinality of the alphabet, then balance on letters pass to balance on factors of arbitrary length. Finally, we describe the dynamical dimension group for some S*-adic* subshifts (see Section 3.1 for details), which are a generalization of substitutive systems, obtained from an infinte composition of substitutions (Theorem 3.39).

Given a topological dynamical system (X, S, G), there exists a way to decompose the space X into *towers*, which is extremely useful when dealing with invariant measures and continuous functions defined on X. This decomposition is originally attributed to Kakutani and Rokhlin in the case $G = \mathbb{Z}$. It is the fundamental tool we use in this work to approach both the study of dimension groups in the case of minimal $\mathbb{Z}$-actions and the problem of realization of Choquet simplex as the set of invariant measures of congruent monotileable amenable groups. We briefly describe the subject in the next paragraph.

Tower partitions. A *tower partition* of the system (X, S, G) is a partition of X of the form

$$\mathcal{P} = \{T^{u^{-1}}(B_k) : u \in F_k, 1 \leq k \leq d\},$$

where $d \in \mathbb{N}$, $F_k \subset G$ are finite subsets, and $B_k \subseteq X$ are clopen for all $1 \leq k \leq d$. The sets of the form $T^{u^{-1}}(B_k)$ are calle the *atoms* of $\mathcal{P}$. For a given $1 \leq k \leq d$, the set $\bigcup_{u \in F_k} T^{u^{-1}}(B_k)$ is called the *kth tower* of $\mathcal{P}$, B_k is its *base* and $|F_k|$ its *height*. In the case of $\mathbb{Z}$-actions, tower partitions have the form

$$\mathcal{P} = \{T^j B_i : 1 \leq i \leq d, 0 \leq j < h_i\}.$$

It is classical (see for instance [Ver81],[Pu89, Section 3],[DHP18, Section 2.9], [BR10, Section 6.4.1]) that tower partitions always exist for Cantor minimal systems given by $\mathbb{Z}$-actions. Moreover, given a Cantor minimal system $(X, S, \mathbb{Z})$ and a point $x_0 \in X$, there always exists a sequence of tower partitions

$$(\mathcal{P}_n)_{n \in \mathbb{N}} = \{T^j B_{i,n} : 1 \leq i \leq d_n, 0 \leq j < h_{i,n}\}$$

verifying the following conditions,

(KR1) $\bigcap_{1 \geq i \geq d_n} B_{i,n} = \{x_0\}$,

(KR2) $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ for all $n \in \mathbb{N}$.

(KR3) $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ generates the topology of X.

Given a sequence of tower partitions $(\mathcal{P}_n)_{n \in \mathbb{N}}$ of X, the nth-incidence matrix of $(\mathcal{P}_n)_{n \in \mathbb{N}}$ is the following

$d_{n+1} \times d_n$-integer matrix,

$$Q_n(i,j) = |\{0 \le \ell < h_{i,n+1} : T^{\ell} B_{i,n+1} \subseteq B_{j,n}\}| \qquad \forall 1 \le i \le d_{n+1}, \quad \forall 1 \le j \le d_n.$$

The existence of *appropriate* tower partition sequences (satisfying conditions similar to (KR1)-(KR3)) is also true for actions of congruent monotileable amenable groups, as it is proved in Proposition 5.27, and we use it in Chapter 5 to, given any Choquet simplex K and any congruent monotileable group G, construct a minimal G-subshift whose set of invariant measures is affine homeomorphic to K.
In the case of $\mathbb{Z}$-actions, we have carefully analyzed the relevance of conditions (KR1)-(KR3) in the computations related to the dynamical dimension group and we have replaced (KR3) for a weaker condition which we show sufficient to almost every computations (see Proposition 2.3 for details).
As we will see in Chapter 2, the dynamical dimension group of a system (X, T) corresponds to an inductive limit of a suitable sequence of tower partitions, also called Kakutani-Rokhlin partitions. We point out the article [KW04], where the authors develope an algorithm, called *TA* (tower algorithm), to compute the dynamical dimension group and the infnitesimal subgroup of Cantor minimal systems, given an appropriate sequence of tower partitions.
Tower partitions are also useful in the study of the full group and topological full group of a system. See for instance [BK00], where the authors show how to completely describe the topological full group of a Cantor minimal system by looking at some convinient sequence of tower partitions (see [BK00, Theorem 2.2]).

The notion of tower partition is closely related to that of a *Bratteli diagram*. A Bratteli diagram is a special kind of infinite graph introduced by Bratteli in [Br72]. It is a pair (V, E) where the set of vertices V and the set of edges E can be written as a countable union of non-empty finite sets,

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \quad \text{and} \quad E = E_1 \cup E_2 \cup \cdots,$$

with the property that V_0 is a single point x_0 and there exists a *range map* $r : E \to V$ and a *source map* $s : E \to V$ so that $r(E_n) \subseteq V_n$ and $s(E_n) \subseteq V_{n-1}$. Also, we assume that $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$. An *ordered Bratteli diagram* $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E such that $e, e' \in E$ are $\geq$-comparable if and only if $r(e) = r(e')$. An ordered Bratteli diagram determines a sequence of *incidence matrices*, (M_n), where each M_n has $|V_n|$ rows and $|V_{n-1}|$ columns, and $M_n(i,j)$ is the number of edges of E_n going from $v_j 1 V_{n-1}$ to $v_i \in V_n$.

The diagram is called *stationary* if all V_n's have a constant cardinality k and the incidence matrices are the same $k \times k$ matrix for all levels. A diagram which has a uniformly bounded number of vertices at each level is called a diagram of *finite rank*.

Given an ordered Bratteli diagram $B = (V, E, \geq)$, it is possible to put a dynamic on it in the following way (this is an original idea of Versik, [Ver85]): Let X_B denote the space of infinite paths on E, that is,

$$X_B = \{(e_1, e_2, \cdots) : e_i \in E_i, r(e_i) = s(e_{i+1}) \quad \forall i \in \mathbb{N}\}.$$

(We assume that X_B is infinite). We endow X_B with a topology by giving a basis of open sets, namely the family of *cylinder sets*,

$$[e_1, e_2, \cdots, e_k]_B = \{(f_1, f_2, \cdots) \in X_B : f_i = e_i \text{ for all } 1 \leq i \leq k\}.$$

The cylinder sets are also closed. The space X_B endowed with this topology is called the *Bratteli compactum* associated to B. A *minimal path* of B is an element $x = (e_1, e_2, \cdots)$ of X_B such that for all $n \in \mathbb{N}$, e_n is minimal according to $\geq$. We define analogously a *maximal path* of B. Under certain conditions (we refer [GPS95, Section 3] for details), the compactum associated to an ordered Bratteli diagram is a Cantor space and it contains exactly one minimal path and one maximal path. In this case, we denote x_{min} and x_{max} the minimal and maximal paths respectively, and we can define the *Versik map* V_B on X_B as follows: $V_B(x_{max}) = x_{min}$; if $x = (e_1, e_2, \cdots)$ is not the maximal path, let k be the smallest integer such that e_k is not a maximal edge, let f_k be the sucesor of e_k on E_k, and define $V_B(x) = (f_1, f_2, \cdots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \cdots)$, where $(f_1, \cdots, f_{k-1})$ is the minimal finite path on $E_1 \circ E_2 \circ \cdots$ with range equal to $s(f_k)$. The system $(X_B, V_B, \mathbb{Z})$ is called a *Bratteli-Vershik dynamical system.*

The key tool to prove the characterization of orbit equivalence and strong orbit equivalence on [GPS95] is what is known as the Bratteli-Vershik Model Theorem: every Cantor minimal system given by an action of $\mathbb{Z}$ is conjugate to a Bratteli-Vershik dynamical system, that is, for every Cantor minimal system $(X, S, \mathbb{Z})$ there exists a (simple) ordered Bratteli diagram B such that $(X_B, V_B, \mathbb{Z})$ is conjugate to $(X, S, \mathbb{Z})$. This was proved in [HPS92]. Given a Cantor minimal system $(X, S, \mathbb{Z})$, the idea is to obtain the Bratteli-Vershik system conjugate to it by constructing a sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ where, at each level, the number of vertices of the diagram corresponds to the number of towers of $\mathcal{P}_n$, and the edges between succesive levels are determined and ordered according to the incidence matrices of $(\mathcal{P}_n)_{n\in\mathbb{N}}$.

It is shown in [DHS99, Theorem 1] that the family of Bratteli-Vershik dynamical systems associated to stationary, prperly ordered Bratteli diagrams is the disjoint union of the family of substitution minimal systems and the family of stationary odometer systems.
Bratteli diagrams are also a useful tool to describe the invariant measures of dynamical systems. See for example [BR10, Section 6.8], and the articles [BKMS10] and [BKMS12], where the authors study the set of invarant measures on stationary and finite rank Bratteli diagrams.
When dealing with minimal $\mathbb{Z}$-actions on the Cantor space one can decide to work directly with tower partitions or to pass to Bratteli-Vershik models. We do not adopt the approach of Bratteli diagrams in this work. We refer to [HPS92],[GPS95],[BR10, Chapter 6] or [DHS99] for details on this interesting subject and on the interaction between tower partitions and Bratteli diagrams. We also recommend the survey [BKa15] for more on Bratteli diagrams and invariant measures.

This text is organized in five chapters. The first one is devoted to general definitions and background and to fix the notation we use through the text. In the second one we present in detail the concepts related to tower partitions for minimal $\mathbb{Z}$-actions on the Cantor space; we give some results about the relation between certain well-chosen sequences of tower partitions and the image subgroup, the group of infinitesimals and the dynamical dimension group of a minimal $\mathbb{Z}$-subshift. In Chapter 3 we apply results of Chapter 2 to the study of some properties of substitutive and S-adic systems, namely dimension groups and balance, and in Chapter 4 we do the same for dendric and eventually dendric subshifts. In Chapter 5 we study the problem of realization of Choquet simplices as sets of invariant measures of actions of congruent monotileable amenable groups; we introduce the notion of congruent monotileable amenable group, we prove that this class of groups is larger than that of residually finite amenable groups and that for any Choquet simplex K and any congruent monotileable amenable group G there exists a minimal G-subshift (X, S, G) such that $\mathcal{M}(X, S, G)$ is affine homeomorphic to K.

The work presented in this PhD thesis is partially included in the following three articles,

- Berthé, V., Cecchi Bernales, P., *Balancedness and coboundaries in symbolic systems*; Theoretical Computer Science, 2018, https://doi.org/10.1016/j.tcs.2018.09.012.
- Cecchi Bernales, P., Cortez, M.I.; *Invariant measures for actions of congruent monotileable amenable groups*; Groups, geometry and dynamics, 2019, DOI 10.4171/GGD/506.
- Berthé, V., Cecchi Bernales, P., Dolce, F., Durand, F. Leroy, J., Perrin, D., Petite, S., *Dimension*

group of dendric subshifts; Preprint.

Chapter 1

Definitions and background.

In this introductory chapter, we give all basic notions and background required to understand the results and examples presented in Chapters 2, 3, 4 and 5.

A *topological dynamical system* or simply a *dynamical system* is a triple (X, S, G) where X is a compact metric space, G is an infinite countable group and $S : X \times G \to X$ is a continuous left action of G on X. For all $g \in G$, S^g denotes the homeomorphism induced by the action of g on X. If we refer to a dynamical system as a pair (X, T), that means that $G = \mathbb{Z}$ and the action of an element $n \in \mathbb{Z}$ is given by $S(x, n) = T^n(x)$.
Through all the text, G will be an infinite countable group.
A dynamical system (X, S, G) is said to be *minimal* if X admits no non-trivial closed and G-invariant subset, that is, if $Y \subseteq X$ is a closed subset such that $S^g(Y) \subseteq Y$ for all $g \in G$, then either $Y = \emptyset$ or $Y = X$. Note that (X, T) is minimal if and only if the only closed T-invariant subsets of X are $\emptyset$ and X itself. Minimality is equivalent to the fact that for all $x \in X$, the orbit of x under the action of G is dense in X. We say that (X, S, G) is *free* or *aperiodic* on $Y \subseteq X$ if $S^g(x) = x$ implies $g = 1_G$, for every $x \in Y$. If $Y = X$ we just say that the subshift is free. We say that (X, S, G) is *equiconinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in X$ satisfy $d(x, y) < \delta$, $d(S^g(x), S^g(y)) < \varepsilon$ for all $g \in G$.
Let (X_1, S_1, G) and (X_2, S_2, G) be two dynamical systems given by two actions of the same group G. We say that (X_2, S_2, G) is a *factor* of (X_1, S_1, G) if there exists a continuous surjective function $\phi : X_1 \to X_2$ which commutes with the actions of G, that is, for all $x \in X$, for all $g \in G$, one has $\phi(S_1^g(x)) = S_2^g(\phi(x))$. If ϕ is also injective, the two systems are said to be *conjugate*.

1.1 Coboundaries.

Given a dynamical system (X, T), the *coboundary map* $\beta : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ is defined by

$$\beta f = f \circ T - f.$$

Elements on the image of β are called *coboundaries.* If $f, g \in C(X, \mathbb{R})$ satisfy $f - g \in \beta C(X, \mathbb{R})$, we say that f and g are *cohomologous.* Note that β maps $C(X, \mathbb{Z})$ to $C(X, \mathbb{Z})$. For any $f \in C(X, \mathbb{R})$, $x \in X$ and $n \in \mathbb{N}$, define

$$f^{(n)}(x) = f(x) + f(Tx) + \cdots + f(T^{n-1}x).$$

The family $(f^{(n)})_{n\in\mathbb{N}}$ is called the *cocycle* of f. The following classical result is a characterization of coboundaries.

Theorem 1.1 (Gotshalk-Hedlund [GH55]). *Let (X, T) be a minimal topological dynamical system. The map $f \in C(X, \mathbb{R})$ is a coboundary if and only in there exists $x_0 \in X$ such that the sequence $(f^{(n)}(x_0))_{n\in\mathbb{N}}$ is bounded.*

Corollary 1.2. *Let (X, T) be a minimal dynamical system. If $f \in C(X, \mathbb{R})$ is a non-negative coboundary, then it is identically zero.*

Proof. By Theorem 1.1, there exists $x_0 \in X$ such that $(f^{(n)}(x_0))_{n\in\mathbb{N}}$ is bounded, but by minimality, $f(T^n x_0) > \frac{1}{2}\sup f$ for infinitely many values of n, so that $\lim_{n\to\infty} f^{(n)}(x) \to \infty$ unless $\sup f$ is identically zero. □

Proposition 1.3. *([DHP18, Proposition 4.2]) Let (X, T) be a minimal dynamical system. If $f \in C(X, \mathbb{Z})$ is a coboundary, then it is the coboundary of some integer-valued function.*

Proof. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and $\pi : \mathbb{R} \to \mathbb{T}$ the canonical projection. Let $\tilde{\beta}$ denote the coboundary map defined on $C(X, \mathbb{T})$ in the same way as on $C(X, \mathbb{R})$. Note first that if $\gamma \in C(X, \mathbb{T})$ and $\tilde{\beta}\gamma = 0$, then γ is constant. Indeed, let $\tilde{c} \in \mathbb{T}$ and set $Y = \gamma^{-1}(\{\tilde{c}\})$. The subset Y is closed since γ is continuous and it is $T-$invariant since $\tilde{\beta}\gamma = 0$. The system being minimal, if Y is nonempty, it is necessarily the whole space X.

Suppose $f \in C(X, \mathbb{Z})$ is the coboundary of $g \in C(X, \mathbb{R})$. Then, $g \circ T(x) - g(x) \in \mathbb{Z}$ for all $x \in X$. This implies that $\tilde{\beta}(\pi \circ g) = 0$ and then there exists $\tilde{c} \in \mathbb{T}$ such that $\pi \circ g(x) = \tilde{c}$ for all $x \in X$. Let c be any element in $\pi^{-1}(\{\tilde{c}\})$ and define $h(x) := g(x) - c$. Since $\pi \circ h = 0$, $h \in C(X, \mathbb{Z})$, and it is clear that $\beta(h) = \beta(g) = f$. □

1.2 $\mathbb{Z}$-subshifts.

Let $\mathcal{A}$ be a finite non-empty set of cardinality d, which we call an *alphabet*. The sets $\mathcal{A}^{\mathbb{Z}}$ and $\mathcal{A}^{\mathbb{N}}$ endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$ are compact metric spaces. We refer to the elements on $\mathcal{A}^{\mathbb{Z}}$ as *infinite words* and to those on $\mathcal{A}^{\mathbb{N}}$ as *one-sided infinite words*.

The free monoid $\mathcal{A}^*$ is the set of all words with symbols in $\mathcal{A}$, including the empty word, which we denote ε. For $a \in \mathcal{A}$ and for $w \in \mathcal{A}^*$, the non-negative integer $|w|_a$ stands for the number of occurrences of the letter a in the word w, and $|w|$ stands for the length of w, that is, the total number of letters appearing on w. We use the convention $|\varepsilon| = 0$. The ith letter of w is denoted w_i by labelling indices from 0, i.e., $w = w_0 \cdots w_{|w|-1}$. A *factor* of a finite word $w \in \mathcal{A}^*$ is defined as a finite concatenation of some consecutive letters occurring in w. A factor of an infinite word or a one-sided infinite word is defined in the same way. For $v, w \in \mathcal{A}^*$ such that v is a factor of w, the non-negative integer $|w|_v$ stands for the number of occurrences of v in w. We use the notation $u \prec w$ (resp. $u \prec x \in \mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$) for u a factor of w (resp. of x). The set of factors $\mathcal{L}_x$ of $x \in \mathcal{A}^{\mathbb{Z}}$ or $\mathcal{A}^{\mathbb{N}}$ is called its *language*. More generally, any set of finite words with symbols in an alphabet $\mathcal{A}$ is called a *language* on $\mathcal{A}$. For an infinte word $x \in \mathcal{A}^{\mathbb{Z}}$ or a one-sided infinite word $x \in \mathcal{A}^{\mathbb{N}}$, for $n \geq 1$, we use the notation $x_{[0,n)}$ to refer to the word $x_0 \cdots x_{n-1}$, and $x_{(-n,n)}$ to refer to $x_{-n+1} \cdots x_{n-1}$, when $x \in \mathcal{A}^{\mathbb{Z}}$.

Recall that a non-empty compact metric space X is a *Cantor space* if it is *totally disconnected* (it has no non-trivial connected subsets) and has no isolated points. Equivalently, X is a Cantor space if it has no isolated points and a countable basis of *clopen* (closed and open) subsets. Up to homeomorphism, there exists only one Cantor space. If $\#\mathcal{A} \geq 2$, the space $\mathcal{A}^{\mathbb{Z}}$ has no isolated points and it is a Cantor space, whose countable basis of clopen subsets is given by the family of all *cylinder sets*: given any word $w \in \mathcal{A}^*$ and an integer n, the *cylinder of w of index n* is the following set,

$$[w]_n = \{x \in \mathcal{A}^{\mathbb{Z}} : x_n \cdots x_{n+|w|-1} = w\}.$$

For all $w \in \mathcal{A}^*$, we denote $[w]$ the cylinder of w of index 0 and we call it the *cylinder of w*.

Let T denote the *shift* transformation acting on $\mathcal{A}^{\mathbb{Z}}$, defined by $T((u_n)_{n\in\mathbb{Z}}) = (u_{n+1})_{n\in\mathbb{Z}}$. A *$\mathbb{Z}$-subshift* on $\mathcal{A}$ or simply a *subshift* on $\mathcal{A}$ is the dynamical system given by the pair $(X, T|_X)$, where X is a closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$, endowed with the induced topology. We usually denote (X, T) the system $(X, T|_X)$ to avoid an overcharged notation. We use the word *subshift* indistinguishly to refer to both the space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ or the dynamical system (X, T). When $X = \mathcal{A}^{\mathbb{Z}}$, we refer to (X, T) as *the*

$\mathbb{Z}$*-fullshift* on $\mathcal{A}$ or *the fullshift* on $\mathcal{A}$. From the previous discussion, every subshift on $\mathcal{A}$ is a *Cantor system*. An element $x \in X$ is said to be *eventually periodic* if there exist $N, k \in \mathbb{N}$ such that for all $n \geq N$, $x_{n+k} = x_n$.

Given any infinite word x in $\mathcal{A}^{\mathbb{Z}}$ (or infinite word in $\mathcal{A}^{\mathbb{N}}$), we define the *subshift generated by* x, (X_x, T), where

$$X_x = \{y \in \mathcal{A}^{\mathbb{Z}} : \forall w, w \prec y \Rightarrow w \prec x\}.$$

Equivalently, X_x is the closure of the orbit of x under the action of the shift T. If (X, T) is a subshift on $\mathcal{A}$, then its *language* $\mathcal{L}_X$ is defined as the set of factors of elements of X. For any $n \geq 1$, $\mathcal{L}_n(X)$ denotes the set of factors of length n of elements in X.

An infinite word $x = (x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ is said to be *uniformly recurrent* if every factor occurring in x occurs infinitely often and with bounded gaps, that is, for every $w \prec x$, there exists s such that, for every n, w is a factor of the finite word $x_n \dots x_{n+s-1}$. It is well known that the subshift (X, T) is minimal if and only if for all $x \in X$, x is uniformly recurrent (see for example [Que10, Proposition 4.7] or [BR10, Proposition 7.1.5]).

Example 1.4. Substitution subshifts.

As a first example we introduce substitution subshifts. We give here just basic definitions and properties, since they will be treated on detail in Chapter 3.

Let $\mathcal{A}$ be a finite alphabets with $|\mathcal{A}| \geq 2$. A *substitution* on $\mathcal{A}$ is a non-erasing morphism on the free monoid $\mathcal{A}^*$, that is, a map $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ satisfying $\sigma(ab) = \sigma(a)\sigma(b)$ for all letters $a, b \in \mathcal{A}$, and such that there is no letter in $\mathcal{A}$ whose image under σ is empty. A substitution σ extends to a map from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{A}^{\mathbb{Z}}$ by concatenation. The subhift generated by a substitution σ is the pair (X_σ, T), where

$$X_\sigma := \{x \in \mathcal{A}^{\mathbb{Z}} : \forall w, w \prec x \Rightarrow \exists a \in \mathcal{A}, \exists n \in \mathbb{N} : w \prec \sigma^n(a)\}.$$

Let $\mathcal{A}, \mathcal{B}$ be two finite alphabets. Given a non-erasing morphism $\sigma : \mathcal{A} \to \mathcal{B}^*$, the *incidence matrix* M_σ of σ is the $|\mathcal{B}| \times |\mathcal{A}|$-matrix whose coefficients are $M_\sigma(b, a) = |\sigma(a)|_b$, for all $a \in \mathcal{A}, b \in \mathcal{B}$. A substitution σ on $\mathcal{A}$ is said to be *primitive* if there exists a power of M_σ which is positive. Equivalently, σ is primitive if there exists a positive ineger n such that for all $a, b \in \mathcal{A}$, b occurs in $\sigma^n(a)$. The incidence matrix of a substitution does not contain all the information about it, since the order of letter occurrences is missing, but we will see thet properties of M_σ determine some of those of (X_σ, T), for example, frequencies (see Chapter 3, Section 3.3.1). It is well known that subshifts generated by primitive substitutions are minimal (see for example [Que10]).

Example 1.5. Sturmian subshifts.

Sturmian sequences were for the first time introduced with this name in [HM40]. A fundamental work on Sturmian sequences is [CH73]. There are many equivalent ways to define Sturmian subshifts (see for example [BR10], Chapter 1, for a combinatorial definition in terms of language complexity). We will explain in Chapter 3 their S-adic characterization. Here we present their characterization as codings of irrational rotations on the circle.

Let $\alpha \in (0,1)$ be an irrational value, and consider the rotation by angle α defined on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by

$$R_\alpha(x) = x + \alpha \qquad \mod 1.$$

The *itinerary* of a point $x \in \mathbb{R}/\mathbb{Z}$ under R_α is the following sequence in $\{0,1\}^{\mathbb{Z}}$

$$I_\alpha(x)_n = \begin{cases} 0 & \text{if } R_\alpha^n(x) \in [0, 1-\alpha) \\ 1 & \text{if } R_\alpha^n(x) \in [1-\alpha, 1). \end{cases}$$

Such an itinerary is called a *Sturmian sequence* or *Sturmian word*.

Given any $\alpha \in (0,1)$ irrational, the closure of the set of all itineraries of points in $\mathbb{R}/\mathbb{Z}$ under R_α, denoted X_α, is a subshift on $\mathcal{A} = \{0,1\}$ and it is called a Sturmian subshift. Indeed, since α is irrational, X_α corresponds to the orbit closure of any itinerary associated to the rotation R_α, so it is the subshift generated by any itinerary. It is known that Sturmian subshifts are minimal. As dynamical systems given by actions of $\mathbb{Z}$, $(\mathbb{T}, R_\alpha)$ is a topological factor of (X_α, T), where T is the shift. Two Sturmian subshifts (X_α, T) and (X_β, T) are conjugated if and only if $\alpha = \beta$ (see for example [BS94] or [HM40]). All previous results and their respective proofs can be found in the surveys on Sturmian words presented on [PF02, Chapter 6] or [Lo02, Chapter 2]. A typical example of a Sturmian sequence is the *Fibonacci word*: it is defined by the *Fibonacci morphism* $\sigma_F : \{0,1\} \to \{0,1\}$ given by $\sigma_F(0) = 01$ and $\sigma_F(1) = 0$. If we iteratively apply σ_F we obtain the one-sided infinite sequence

$$\sigma_F^\omega(0) = 0100101001001010010100100101001 0 \cdots,$$

which is called the *Fibonacci word*. Thus, the subshift generated by the Fibonacci word is a Sturmian subshift. This coincide with the substitutive subshift (X_{σ_F}, T).

Example 1.6. Toeplitz $\mathbb{Z}$-subshifts.

Toeplitz $\mathbb{Z}$-subshifts (or simply Toeplitz subshifts) were originally introduced in [JK69]. Since then, they have seen extensively studied and generalized to other group actions. We present the example of Toeplitz G-subshifts in Section 1.3. For more details in the subject of Toeplitz $\mathbb{Z}$-subshifts we refer to [Do95].

An infinite word $(x_n)_{n\in\mathbb{Z}}$ with symbols in the alphabet $\mathcal{A}$ is said to be a *Toeplitz word* if for all $n \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that for all $k \in \mathbb{Z}$, $x_n = x_{n+kp}$.

A $\mathbb{Z}$-subshift (X, T) is said to be a *Toeplitz subshift* if X is the orbit closure under the shift of some Toeplitz word. Toeplitz subshifts are known to be minimal, since Toeplitz sequences are uniformly recurrent (indeed, they are *regularly recurrent*, see [Do95, Section 5 and 7] for details).

1.2.1 Return words.

Given a minimal symbolic system (X, T), on the alphabet $\mathcal{A}$ and a letter $a \in \mathcal{A}$, a word w with $wa \in \mathcal{L}_X$ is said to be a *left return word* to a if a is a prefix of wa. It is said to be a *first left return word* to a if a is a prefix of wa and there are exactly two occurrences of a in wa. Similarly, a word w with $aw \in \mathcal{L}_X$ is said to be a *right return word* to a if a is a suffix of aw. It is said to be a *first right return word* to a if a is a suffix of aw and there are exactly two occurrences of a in aw. Right and left return words to a factor are defined analogously.

Since (X, T) is minimal, the language $\mathcal{L}_X$ is uniformly recurrent, which implies that for every word $w \in \mathcal{L}_X$, there exists a positive integer N_w such that every word of length grater than N_w contains at least two occurrences of w. Thus, for all $w \in \mathcal{L}_w$, the length of the first return words to w is bounded, and therefore the number of first return words to w is finite.

1.2.2 Cylinder functions

Let (X, T) be a $\mathbb{Z}$-subshift. A function $f \in C(X, \mathbb{R})$ is called a *cylinder function* if there exists $n > 0$ such that for all $x \in X$, $f(x)$ depends only on $x_{[0,n)}$.

Proposition 1.7. *([DHP18, Proposition 4.13]) Let (X, T) be a dynamical system. Every function belonging to $C(X, \mathbb{Z})$ is cohomologous to some cylinder function in $C(X, \mathbb{Z})$.*

Proof. Let $f \in C(X, \mathbb{Z})$. Since f is integer-valued, it is locally constant, and then there exists $k \in \mathbb{N}$ such that for all $x \in X$, $f(x)$ depends only on $x_{(-k,k)}$. Therefore, $g(x) := f \circ T^k(x)$ belongs to $C(X, \mathbb{Z})$ and depends only on $x_{[0,2k)}$ for all $x \in X$, i.e., it is a cylinder function in $C(X, \mathbb{Z})$. Finally, $f - g = f - f \circ T^k(x)$ is a coboundary because it is a sum of coboundaries. □

Proposition 1.8. *([DHP18]) Let (X, T) ne a dynamical system. If $f \in C(X, \mathbb{Z})$ is a cylinder function and a coboundary, then it is the coboundary of some cylinder function.*

Proof. Let $f \in C(X, \mathbb{Z})$ be a cylinder function and a coboundary, then there exists $g \in C(X, \mathbb{R})$ such that $f = \beta(g)$. Since g is locally constant and f is a cylinder function, we can choose $k \in \mathbb{N}$ large enough so that for all $x \in X$, f depends only on $x_{[0,k)}$ and g depends only on $x_{(-k,k)}$. We claim that for all $x \in X$, g depends only on $x_{[0,2k)}$. Indeed, suppose that $y, z \in X$ satisfy $y_{[0,2k)} = z_{[0,2k)}$. Since $f(x)$ depends only on $x_{[0,k)}$, $f^{(k)}(x)$ depends only on $x_{[0,2k)}$ and thus $f^{(k)}(y) = f^{(k)}(z)$. Since $(T^k y)_{(-k,k)} = (T^k z)_{(-k,k)}$, one has $g(T^k y) = g(T^k z)$. Finally, recall that since $f = \beta(g)$, then for all $s \in \mathbb{N}$ and for all $x \in X$, $f^{(s)}(x) = g(T^s x) - g(x)$, so we obtain that

$$g(y) = g(T^k y) - f^{(s)}(y) = g(T^k z) - f^{(s)}(z) = g(z).$$

Therefore, g is a cylinder function depending on the first $2k$ coordinates. □

Let (X, T) be a subshift. For $n \geq 0$, let $R_n(X)$ denote the set of continuous functions from $\mathcal{L}_n(X)$ to $\mathbb{R}$. The *symbolic coboundary map* $\beta_n : R_n(X) \to R_{n+1}(X)$ is given by

$$\phi \mapsto (\beta_n \phi)(a_0 a_1 \cdots a_n) = \phi(a_1 \cdots a_n) - \phi(a_0 \cdots a_{n-1}) \quad \forall a_0 a_1 \cdots a_n \in \mathcal{L}_{n+1}(X). \tag{1.1}$$

We say that $\psi \in R_{n+1}(X)$ is a symbolic coboundary if there exists $\phi \in R_n(X)$ such that $\psi = \beta_n \phi$. If $f \in C(X, \mathbb{R})$ is a cylinder function depending on the first n coordinates, it has an associated function $\phi_f : \mathcal{L}_n(X) \to \mathbb{R}$ which is defined by $\phi_f(a_0 a_1 \cdots a_{n-1}) = f(x)$ for any $x \in [a_0 a_1 \cdots a_{n-1}]$. It is called *the symbolic map associated to f*.

1.3 G-symbolic systems.

For any infinite countable group G and any finite alphabet $\mathcal{A}$, the set $\mathcal{A}^G$ endowed with the product topology of the discrete topology on each copy of $\mathcal{A}$ is again a compact metric space. The elements of $\mathcal{A}^G$ will be refered as *configurations*. In this context, the analogue of a word in the context of $\mathbb{Z}$-actions, is what we call a *pattern*, or a *finite configuration*: an element $P \in \mathcal{A}^F$ where F is a finite subset of G. For every pattern $P \in \mathcal{A}^F$, we define the *cylinder* of P in the same way as in the context of $\mathbb{Z}$-actions, that is,

$$[P] = \{x \in \mathcal{A}^G : x|_F = P\}.$$

The set of all cylinders associated to patterns in $\mathcal{A}^F$ forms a countable basis of clopen sets of $\mathcal{A}^G$, and $\mathcal{A}^G$ is again a Cantor space whenever $|\mathcal{A}| \geq 2$.
Let T_G denote the *G-shift action* of G on A^G, defined by $T_G^g(x)(h) = x(g^{-1}h)$, for every $g, h \in G$ and $x \in \mathcal{A}^G$. We say that a pattern P has an occurrence on the element $x \in \mathcal{A}^G$ if there exists $g \in G$ such that $T_G^g(x)|_F = P$. A *G-subshift* of $\mathcal{A}$ is the dynamical system given by the triple $(X, T_G|_X, G)$, where X is a closed subset of $\mathcal{A}^G$ which is invariant by the G-shift action. When $X = \mathcal{A}^G$, we refer to (X, T, G) as the *G-fullshift.*

To present the following examples of G-symbolic systems we introduce a notion which will be crucial in Chapter 5, namely, the notion of residually finite group. We refer to [CC10, Chapter 2] or Section 5.1.2 for details. A group G is said to be *residually finite* if for each element $g \in G$ with $g \neq 1_G$ there exists a finite group F and a homomorphism $\phi_g : G \to F$ such that $\phi_g(g) \neq 1_F$. Equivalently, there exists a sequence of finite index normal subgroups of G, $\{G_n\}_{n\in\mathbb{N}}$, such that $\bigcap_{n\in\mathbb{N}} G_n$ is trivial.

Example 1.9. Toeplitz G-subshifts.

In an analogous way as in Example 1.6, we say that a configuration $x \in \mathcal{A}^G$ is a Toeplitz configuration if for every $g \in G$ there exists a finite index subgroup Γ of G such that for all $\gamma \in \Gamma$, $x(\gamma^{-1}g) = x(g)$. A G-subshift (X, T, G) is said to be a Toeplitz subshift if X is the orbit closure under the shift of some Toeplitz configuration. Toeplitz G-subshifts are known to be minimal. We refer to [Co06], [CP08] and [CP14] for details on the dynamical properties of Toeplitz G-subshifts.
Note that if $x \in \mathcal{A}^G$ is an aperiodic Toeplitz configuration, then G is residually finite. Indeed, since for all $g \in G$ there exists a finite index subgroup Γ_g such that $x(\gamma^{-1}g) = x(g)$ for all $\gamma \in \Gamma_g$, there exists a subgroup of G defined by

$$\Gamma = \bigcap_{g\in G} \Gamma_g.$$

Note that this intersection is trivial: if there exists $\gamma \in \Gamma$ such that $\gamma \neq 1_G$, then $x(\gamma^{-1}g) = x(g)$ for all $g \in G$, since $\gamma \in \Gamma_g$ for all $g \in G$. This implies that $T^\gamma(x) = x$, that is, x is periodic, which is a contradiction. For each $g \in G$ we can assume that Γ_g is normal in G: if it is not, there exists a normal finite index subgroup H_g which is contained in Γ_g. Finally, since G is countable, $(\Gamma_g)_{g\in G}$ defines a sequence of finite index normal subgroups of G with trivial intersection, thus G is residually finite. Residually finite groups are thus the only groups that admit aperiodic Toeplitz configurations.

The next one is an example of a G-action on the Cantor space which is not a subshift.

Example 1.10. Odometers.

Let G be a residually finite group and let $(G_n)_{n\in\mathbb{N}}$ be a decreasing sequence of finite index normal subgroups of G. For all $n \in \mathbb{N}$, the map $\pi_n : G/G_{n+1} \to G/G_n$ given by $\pi_n(gG_{n+1}) = gG_n$ is a well defined epimorphism. We define the following inverse limit

$$X = \varprojlim_n (G_n, \pi_n) = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\geq 0} G/G_n : \pi_n(x_{n+1}) = x_n \text{ for all } n \geq 0\} \subseteq \prod_{n\geq 0} G/G_n.$$

The space $\prod_{n\geq 0} G/G_n$ endowed with the product topology of the discrete topology on each G/G_n is compact and X is a closed subspace of it. We define the following action of G on X,

$$S^g((x_n)_{n\in\mathbb{N}}) = (gx_n)_{n\in\mathbb{N}}.$$

The system (X, S, G) is called a G-odometer. G-odometers are free minimal eqiucontinuous systems and free Teoplitz G-subshifts are characterized as the minimal almost one-to-one extensions of G-odometers (see [Co06] and [CP08] for details).

1.4 Invariant measures.

Given a topological dynamical system (X, T, G), an *invariant measure* of (X, T, G) is a probability Borel measure μ such that for all $g \in G$, $\mu(T^g(A)) = \mu(A)$, for every Borel subset $A \subseteq X$. We denote $\mathcal{M}(X, T, G)$ the set of all invariant measures of (X, T, G). An element $\mu \in \mathcal{M}(X, T, \mu)$ is said to be an *ergodic measure* if whenever $T^g(A) = A$ for all $g \in G$ for some Borel set $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. The system (X, T, G) is said to be *uniquely ergodic* if $\mathcal{M}(X, T, G)$ is a singleton.

Recall that a group G is *amenable* if it admits a *Følner sequence* of finite subsets $(F_n)_{n\geq 0}$, that is, a sequence verifying

$$\lim_{n\to\infty} \frac{|F_n g \setminus F_n|}{|F_n|} = 0 \quad \forall g \in G.$$

(See Chapter 5 or [CC10, Chapter 4] for details). There are many characterizations of amenable groups. It is a theorem by Bogolyubov [Bog39] that a group is amenable if and only if for all continuous action of G on a compact metric space X, there exists a probability masure on X which is invariant under the action of G. Later, Giordano and de la Harpe showed in [GdH97] that for a group G to be amenable it is necessary and sufficient that any continuous action on the Cantor set has an invariant probability measure. So in particular when we deal with amenable groups we always have that $\mathcal{M}(X, T, G)$ is non-empty.

Free groups are not amenable (see Chapter 5 for a proof).

From now on we assume that every group we work with is infinite, countable and amenable.

1.4.1 The Choquet simplex of invariant measures.

A compact convex metrizable subset K of a locally convex real vector space is said to be a *Choquet simplex* or just a *simplex* if for each $v \in K$ there is a unique probability measure m supported on $ext(K)$ such that $\int_{ext(K)} xdm(x) = v$. This is a generalization of the unitary simplex in $\mathbb{R}^n$, that is, the convex hull of the canonical basis $\{e_1, \cdots, e_n\}$, whose extreme points are exactly $e_1, \cdots, e_n$. There exists a Choquet simplex whose set of extreme points is dense, this simplex is unique up to affine homeomorphism and it is called the *Poulsen Simplex*. It has the surprising property of universality: every Choquet simplex is affinely homeomorphic to a face of the Poulsen simplex. We refer to [P61], [LOS78] and [FLP] for more on the Poulsen simplex.

The set of probability measures on a compact metric space X can be identified with a convex subspace of the dual $C(X, \mathbb{R})^*$ endowed with the weak* topology (this is a consequence of the Riesz Representation Theorem). For a dynamical system (X, T, G), it is well known that the set $\mathcal{M}(X, T, G)$ is a Choquet simplex whose extreme points are the ergodic measures. This is a consequence of the Ergodic Decomposition Theorem (see for example [Gl03]). This implies in particular that if G is amenable, then every dynamical system (X, T, G) admits an ergodic invariant probabilty measure. We denote $\mathcal{E}(X, T, G)$ the set of ergodic invariant probability measures on X.

If (X, T, G) has a finite number of ergodic measures, $\mathcal{M}(X, T, G)$ is affine homeomorphic to the unitary simplex of $\mathbb{R}^n$, for some $n \in \mathbb{N}$. It is known (see [GW97]) that the Poulsen simplex is the simplex of invariant measures associated to the G-fullshift action of any countable amenable group G on the Cantor space $\{0, 1\}^G$.

1.4.2 Invariant measures and frequencies in symbolic systems.

Let (X, T) be a dynamical system given by an action of $\mathbb{Z}$. For all invariant measure $\mu \in \mathcal{M}(X, T)$, the quadruple $(X, T, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra on X, is a *measure-theoretic dynamical system* (we refer to [W82] for details): $\mathcal{B}$ is a σ-algebra on X, μ is a probability measure defined on $\mathcal{B}$, T is μ-measurable and $\mu(A) = \mu(T^{-1}A)$ for all Borel subset A. In this context we have the following classical result.

Theorem 1.11 (Birkhoff Ergodic Theorem). *Let $(X, T, \mathcal{B}, \mu)$ a measure-theoretic dynamical system. Let $f \in L^1(X, \mathbb{R})$. The sequence*

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right)_{n \geq 0}$$

converges μ-a.e. to a function $f^* \in L^1(X, \mathbb{R})$ which verifies $f^* \circ T = f^*$ μ-a.e. and $\int_X f^* d\mu = \int_X f d\mu$. If μ is ergodic, for all $f \in L^1(X, \mathbb{R})$, the above sequence converges μ-a.e. to $\int_X f d\mu$.

If the system (X, T) is uniquely ergodic, then we have the following stronger result (see for example [W82] for a proof).

Theorem 1.12. *Let (X, T) be a dynamical system. Then (X, T) is uniquely ergodic with measure μ if and only if for all continuous function $f : X \to \mathbb{R}$, the sequence*

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \right)_{n \geq 0}$$

converges uniformly to the constant value $\int_X f d\mu$.

Suppose now that (X, T) is a subshift on the alphabet $\mathcal{A}$. Let $x \in X$. The *frequency* $f_w(x)$ of a factor $w \in \mathcal{L}_X$ is defined as the following limit, when it exists,

$$f_w(x) = \lim_{n \to \infty} \frac{|x_{-n} \cdots x_0 \cdots x_n|_w}{2n + 1}.$$

Note that this quantity does not always exist. Consider for example the following one-sided infinite word on the alphabet $\mathcal{A} = \{a, b\}$,

$$x_0 = abaabbaaabbbaaaabbbb \cdots$$

and take any $x \in X_{x_0}$. It is clear that neither $f_a(x)$ nor $f_b(x)$ exists. The word x is said to have *uniform frequencies* if for every factor $w \in \mathcal{L}_X$, the ratio $\frac{|x_k \cdots x_{k+2n}|_w}{2n+1}$ converges to $f_w(x)$ when n tends to infinity, uniformly in k. This means that the frequency of w does not depend on the place of x we look at.

We know that $\mathcal{E}(X, T)$ is non-empty and for all $\mu \in \mathcal{E}(X, T)$, $(X, T, \mathcal{B}, \mu)$ is a measure-theoretic dynamical system. We thus can apply Theorem 1.11 to characteristic functions of cylinders (see

Section 1.2). Note that if w is any word in $\mathcal{L}_X$ and x any point in X, then the number of times that w occurs in $x_{[-n,n+1)}$ is

$$|x_{-n} \cdots x_{n-|w|-1}|_w = \sum_{i=-n}^{n-|w|-1} \chi_{[w]} \circ T^i(x),$$

where $\chi_{[w]}$ is the characteristic function of $[w]$. So we obtain that

$$\lim_{n\to\infty} \frac{|x_{-n} \cdots x_0 \cdots x_n|_w}{2n+1} = \lim_{n\to\infty} \frac{1}{2n+1} \left(\sum_{i=0}^{n-1} \chi_{[w]} \circ T^{-i}(Tx) + \sum_{i=0}^{n-1-|w|} \chi_{[w]} \circ T^i(x) \right).$$

By applying Theorem 1.11 we know that $\frac{1}{n}\sum_{i=0}^{n-1} \chi_{[w]} \circ T^{-i+1}$ and $\frac{1}{n}\sum_{i=0}^{n-1-|w|} \chi_{[w]} \circ T^i$ converges μ-a.e. to $\mu([w])$, so we get the following result.

Theorem 1.13. *Let (X,T) a subshift and $\mu \in \mathcal{E}(X,T)$. Then, for μ-almost every infinite word $x \in X$ and for any factor $w \in \mathcal{L}_X$, the frequency $f_w(x)$ exists and equals $\mu([w])$.*

Similarly, if we apply Theorem 1.12 to a minimal subshift, we obtain the following result.

Theorem 1.14. *Let (X,T) be a minimal subshift. Then (X,T) is uniquely ergodic if and only if for all $x \in X$, x has uniform frequencies. In this case, for all $x \in X$ and for all $w \in \mathcal{L}_X$, the frequency $f_w(x)$ is equal to $\mu([w])$, where μ is the unique invariant measure of (X,T).*

The two previous results give us the idea that there could exist minimal non-uniquely ergodic systems in which every point has frequencies in any factor but not uniform frequencies. We know that this is not the case thanks to the following result by Oxtoby [Ox52]. See [BR10, Proposition 7.2.11] for an elegant proof.

Theorem 1.15 ([Ox52])**.** *If (X,T) is a minimal non-uniquely ergodic subshift, then there exists an infinite word $x \in X$ and a factor $w \in \mathcal{L}_X$ such that $f_w(x)$ is not defined.*

So in the case of minimal subshifts, unique ergodicity is equivalent to the existence of frequencies, and if (X,T) is uniquely ergodic, then for all $x \in X$ the frequency of the factor w in x is equal to $\mu([w])$. In the latter case, we denote by μ_w the frequency of w (in any point x).

Substitution subshifts arising from primitive substitutions are known to be uniquely ergodic (see for example [Chapter 5][Que10]). In Chapter 3, Section 3.3.1, we summarize some results about how to compute frequencies in this case.

1.4.3 Balance in symbolic systems.

We now introduce the notion of *balance* for $\mathbb{Z}$-subshifts and relate it with frequencies studied in section 1.4.2.

Let (X,T) be a subshift. For an infinite word $x \in X$ and a factor $v \in \mathcal{L}_X$, we say that x is *balanced on* v if there exists a constant C_v such that for all $w, w' \in \mathcal{L}_X$ with $|w| = |w'|$, one has

$$||w|_v - |w'|_v| \leq C_v.$$

We say that x is *balanced on letters* if it is balanced on all letters in $\mathcal{A}$, and that it is *balanced on factors* (resp. *balanced on the factors of length* n) if it is balanced on all factors of X (resp. on all factors of X of length n).

If (X,T) is a minimal subshift, then since all elements have the same language, balance is a property of the language $\mathcal{L}_X$ and does not depend on the choice of a particular point. Thus, in the case of minimal systems, we will say that the system or the language is (or is not) balanced in some factor.

The following proposition (which is a rephrasing of [Adam03, Lemma 23]) states that balance is preserved when decreasing the length of factors. It is thus sufficient to prove that balance does not hold for some length to obtain that it does not hold for all larger lengths.

Proposition 1.16. *[Adam03, Lemma 23] If an infinite word x is balanced on a factor v, then it is balanced on the prefix of length $|v| - 1$ of v. If a minimal subshift (X,T) is balanced on factors of length $n+1$, then it is balanced on factors of length n.*

Proof. Let $x \in \mathcal{A}^{\mathbb{Z}}$. For every n, we consider an alphabet $\mathcal{A}_n$ and a bijection $\theta_n : \mathcal{A}_n \to \mathcal{L}_n(x)$. The word $x^{(n)} := \theta_n(x)$, defined over the alphabet $\mathcal{A}_n$, codes factors of length n according to the bijection θ_n in the same order as in x with overlaps and without repetition. The map $\theta_n \circ \pi_n \circ \theta_{n+1}^{-1}$ is a morphism from the monoid $\mathcal{A}_{n+1}^*$ to $\mathcal{A}_{n+1}^*$ that maps letters to letters: it maps the coding of a block of length $n+1$ to the coding of its prefix of length n. The word $x^{(n)}$ is thus the image by a letter-to-letter substitution of the word $x^{(n+1)}$. Indeed $x^{(n)} = \theta_n \circ \pi_n \circ \theta_{n+1}^{-1}(x^{(n+1)})$.

We conclude by noticing that the action of a letter-to-letter substitution preserves balance. □

The following proposition states a relation between balance and frequencies. Its proof can be found in [BD14, Proposition 2.4] for factors of length 1. It extends easily to factors of arbitrary length.

Proposition 1.17. *Let (X,T) be a minimal subshift. The language $\mathcal{L}_X$ is balanced in the factor v if and only if v has a frequency μ_v and there exists a constant B_v such that for any factor $w \in \mathcal{L}_X$, we* *have*

$$||w|_v - \mu_v|w|| \leq B_v$$

Equivalently, $\mathcal{L}_X$ is balanced in the factor v if and only if v has a frequency μ_v and there exists B_v such that for all $x \in X$ and for all $n \geq 1$,

$$||x_{[0,n)}|_v - \mu_v n| \leq B_v.$$

The previuos result tells us that in particular balanced minimal systems are forced to be uniquely ergodic. Balance is indeed a property which is stronger than unique ergodicity and that measures the quality of the convergence of $\frac{1}{n}|x_{[0,n)}|_v$ towards μ_v.

Example 1.18. Balance in Sturmian subshifts.

Sturmian sequences were characterized in [HM40] as binary aperiodic sequences which are 1-balanced on letters, that is, aperiodic sequences on $\mathcal{A} = \{0, 1\}$ such that given two factors of the sequence, w and w', having the same length, the difference between the number of 0's (or 1's) in w and the number of 0's (or 1's) in w' is at most 1. This behavior was observed to extend to factors in [FV02], where it was shown that every Sturmian sequence is balanced on every factor. More precisely, the authors proved that if $x \in \{0, 1\}^{\mathbb{Z}}$ is a Sturmian word, then for all factors u, v, w of x, $||u|_w - |v|_w| \leq |w|$ whenever $|u| = |v|$. However, this is not a complete characterization: not all such sequences are Sturmian.

Proposition 1.19. *Let (X, T) be a minimal and uniquely ergodic subshift and let μ denote its unique invariant measure. Given a factor $v \in \mathcal{L}_X$, define*

$$f_v = \chi_{[v]} - \mu_v \in C(X, \mathbb{R}).$$

Then, (X, T) is balanced on the factor v if and only if the map f_v is a coboundary.

Proof. It is a direct consequence of Proposition 1.17 and Theorem 1.1. □

1.5 Orbit equivalence and dimension groups.

Orbit equivalence and strong orbit equivalence are notions of equivalence between dynamical systems, which are weaker than conjugacy. Strong orbit equivalence is characterized by a total invariant called the *dynamical dimension group* of the system. We devote this section to recall some basic notions and results regarding orbit/strong orbit equivalence and dimension groups.

1.5.1 Orbit equivalence.

Two minimal dynamical systems (X_1, T_1, G_1) and (X_2, T_2, G_2) are said to be *orbit equivalent* if there exists a homeomorphism $\phi : X_1 \to X_2$ sending orbits of the G_1-action onto orbits of the G_2-action, that is, for all $x \in X_1$, one has

$$\phi(\{T_1^g(x) : g \in G_1\}) = \{T_2^h \phi(x) : h \in G_2\}.$$

When the group acting on X is $\mathbb{Z}$, orbit equivalence implies the existence of two maps $n_1 : X_1 \to \mathbb{Z}$ and $n_2 : X_2 \to \mathbb{Z}$ (uniquely defined by minimality) such that, for all $x \in X_1$,

$$\phi \circ T_1(x) = T_2^{n_1(x)} \circ \phi(x) \text{ and } \phi \circ T_1^{n_2(x)}(x) = T_2 \circ \phi(x).$$

The two minimal dynamical systems (X_1, T_1) and (X_2, T_2) are said to be *strong orbit equivalent* if n_1 and n_2 both have at most one point of discontinuity. Such notion is natural since it was shown in [Bo83] that if n_1 (or n_2) is continuous, then the two systems are *flip conjugated*, that is, (X_1, T_1) is either congujated to (X_2, T_2) or to its inverse (X_2, T_2^{-1}).

1.5.2 Dimension groups.

An *ordered group* is a pair (G, G^+) where G is a countable abelian group G and G^+ is a subset of G, called the *positive cone*, satisfying

$$G^+ + G^+ \subset G^+, \quad G^+ \cap (-G^+) = \{0\}, \quad G^+ - G^+ = G.$$

We write $a \leq b$ if $b - a \in G^+$, and $a < b$ if $b - a \in G^+$ and $b \neq a$. An *order unit* for (G, G^+) is an element u in G^+ such that, for all a in G, there exists some non-negative integer n with $a \leq nu$. We say that an ordered group is *unperforated* if $a \in G$ and $na \in G^+$ for some $a \in G$ and $n \in \mathbb{N}$ implies that $a \in G^+$.

A *dimension group* (G, G^+, u) with order unit u is an unperforated ordered group (G, G^+) satisfying the *Riesz interpolation property*: given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there exists $c \in G$ with $a_i \leq c \leq b_j$. Two dimension groups with units (G_1, G_1^+, u_1) and (G_2, G_2^+, u_2) are *isomorphic* if there exists a group isomorphism ϕ: $G_1 \to G_2$ such that $\phi(G_1^+) = G_2^+$ and $\phi(u_1) = u_2$. Given a dimension group with unit (G, G^+, u), a *trace* of (G, G^+, u) is a group homomorphism $p : G \to \mathbb{R}$ such that p is non-negative ($p(G^+) \geq 0$) and $p(u) = 1$. The collection of all traces of $(G, G^+, 1)$ is denoted by $S(G, G^+, u)$. It is known [Eff81] that $S(G, G^+, u)$ completely determines the order on G. In fact,

$$G^+ = \{a \in G : p(a) > 0, \forall p \in S(G, G^+, u)\} \cup \{0\}.$$

An *order ideal* of (G, G^+, u) is a subgroup J of G such that $J = J^+ - J^+$ (where $J^+ = G^+ \cap J$) and such that whenever $0 \leq a \leq b \in J$, $a \in J$. A dimension group (G, G^+, u) is said to be *simple* if it contains no non-trivial order ideals.

The *image subgroup* of an ordered group with unit is defined as the following subgroup of $\mathbb{R}$,

$$I(G, G^+, u) = \bigcap_{\tau \in S(G,G^+,u)} \tau(G).$$

Given a dimension group with unit (G, G^+, u), an element $a \in G$ is said to be *infinitesimal* if $p(a) = 0$ for every trace $p \in S(G, G^+, u)$. The collection of all infinitesimals of G form a subgroup, called the *infinitesimal subgroup of* G and denoted $\operatorname{Inf}(G, G^+, u)$. Note that $G/\operatorname{Inf}(G, G^+, u)$ is also a dimension group for the induced order.

1.5.3 Dimension group associated to subshifts and G-subshifts.

Let (X, T) be a dynamical system given by a $\mathbb{Z}$-action. The *dynamical dimension group* of (X, T) or simply the *dimension group* of (X, T) is the following triple,

$$K^0(X, T) = (H(X, T), H^+(X, T), [1]),$$

where $H(X, T) = C(X, \mathbb{Z})/\beta C(X, \mathbb{Z})$, $[\cdot]$ denote the class modulo $\beta C(X, \mathbb{Z})$ of an element in $H(X, T)$, $H^+(X, T)$ is the set of classes of non-negative functions and 1 is the constant function equal to 1.

Theorem 1.20. *([HPS92]) If (X, T) is a Cantor minimal system, the triple $K^0(X, T)$ is a simple dimension group. Futhermore, if (G, G^+, u) is a simple dimension group, then there exists a Cantor minimal system (X, T) such that $K^0(X, T)$ is isomorphic to (G, G^+, u) as ordered group with unit.*

The following result gives a connection between the dynamical dimension group of minimal Cantor systems given by $\mathbb{Z}$-actions and orbit equivalence.

Theorem 1.21. *([GPS95]) Let (X_1, T_1) and (X_2, T_2) two minimal Cantor systems. Then the following are equivalent.*

1. *(X_1, T_1) and (X_2, T_2) are strong orbit equivalent if and only if $K^0(X_1, T_1)$ and $K^0(X_2, T_2)$ are isomorphic as ordered groups with unit.*

2. *(X_1, T_1) and (X_2, T_2) are orbit equivalent if and only if $K^0(X_1, T_1)/\operatorname{Inf}(K^0(X_1, T_1))$ and $K^0(X_2, T_2)/\operatorname{Inf}(K^0($ are isomorphic as ordered group with unit.*

Given an invariant measure $\mu \in \mathcal{M}(X, T)$, we define the trace τ_μ on $K^0(X, T)$ by $\tau_\mu([f]) := \int f d\mu$. It is shown in [HPS92] that the correspondence $\mu \mapsto \tau_\mu$ is an affine isomorphism from $\mathcal{M}(X, T)$ to $S(K^0(X, T))$, so that traces of the dynamical dimension group $K^0(X, T)$ are the invariant measures of the system (X, T) (see also [Ho95, Section 3]). This implies that the image subgroup of $K^0(X, T)$, which is denoted $I(X, T)$, is given by

$$I(X, T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu : f \in C(X, \mathbb{Z}) \right\},$$

and the infinitesimals of $K^0(X, T)$, denoted $\operatorname{Inf}(X, T)$, are given by

$$\operatorname{Inf}(X, T) = \left\{ [f] \in H(X, T) : \int f d\mu = 0 \, \forall \mu \in \mathcal{M}(X, T) \right\}.$$

Remark 1.22. *Note that the image subgroup $I(X, T)$ coincides with the additive group generated by invariant measures of cylinders, that is*

$$I(X, T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \langle \{ \mu([w]) : w \in \mathcal{L}_X \} \rangle .$$

Indeed, the inclusion $\bigcap_{\mu \in \mathcal{M}(X,T)} \langle \{ \mu([w]) : w \in \mathcal{L}_X \} \rangle \subseteq I(X, T)$ is obvious. For the converse inclusion, let $\alpha \in I(X, T)$. By definition, for all $\mu \in \mathcal{M}(X, T)$ there exists a function $f \in C(X, \mathbb{Z})$ such that $\alpha = \int f d\mu$. Since $f \in C(X, \mathbb{Z})$, f is cohomologous to a cylinder function $g \in C(X, \mathbb{Z})$ by Proposition 1.7. One has $\alpha = \int f d\mu = \int g d\mu$. Since g is a cylinder function, there exists a positive integer n such that g can be written as the sum

$$g = \sum_{u \in \mathcal{L}_n(X)} \ell(u) \chi_{[u]},$$

where $\mathcal{L}_n(X)$ denote the set of factors of length n in $\mathcal{L}_X$, $\ell(u) \in \mathbb{Z}$ for all u, and $\chi_{[u]}$ denotes the characteristic function of the cylinder $[u]$. Thus,

$$\alpha = \int g d\mu = \sum_{u \in \mathcal{L}_n(X)} \ell(u) \mu([u]) \in \langle \{ \mu([w]) : w \in \mathcal{L}_X \} \rangle .$$

Since this is true for all $\mu \in \mathcal{M}(X,T)$, we conclude that $\alpha \in \bigcap_{\mu\in\mathcal{M}(X,T)} \langle\{\mu([w]) : w \in \mathcal{L}_X\}\rangle$.

If (X,T) is uniquely ergodic with unique T-invariant measure μ, then $H(X,T)/\operatorname{Inf}(X,T)$ is isomorphic to $(I(X,T), I(X,T)\cap\mathbb{R}^+, 1)$, via the correspondence

$$[f] + \operatorname{Inf}(X,T) \mapsto \int f d\mu.$$

Note that since the projection of a coboundary on $H(X,T)$ always belongs to $\operatorname{Inf}(X,T)$, the groups $H(X,T)/\operatorname{Inf}(X,T)$ and $C(X,\mathbb{Z})/\operatorname{Inf}(X,T)$ coincide. This motivate the definition of another dimension group associated to any dynamical system (X,T,G), called its *reduced dynamical dimension group* and denoted $\mathcal{G}(X,T,G)$, as follows,

$$\mathcal{G}(X,T,G) = C(X,\mathbb{Z})/\left\{f \in C(X,\mathbb{Z}) : \int f d\mu = 0 \quad \forall \mu \in \mathcal{M}(X,T,G)\right\}.$$

From Theorem 1.21, we know that in the case of a $\mathbb{Z}$-action, the reduced dimension group completely characterizes the orbit equivalence classes. This result was extended to $\mathbb{Z}^d$-actions in [GPMS10] with the following theorem.

Theorem 1.23. *([GPMS10]) Let $(X,T,\mathbb{Z}^d)$ and $(X',T',\mathbb{Z}^m)$ be two minimal dynamical systems on the Cantor set. Then they are orbit equivalent if and only if $\mathcal{G}(X,T,\mathbb{Z}^d)$ and $\mathcal{G}(X',T',\mathbb{Z}^m)$ are isomorphic as ordered groups with unit.*

Note that the simplex $\mathcal{M}(X,T,G)$ corresponds in the general case to the set of traces of $\mathcal{G}$. Indeed, given an invariant measure $\mu \in \mathcal{M}(X,T)$, we define the trace τ_μ on $\mathcal{G}(X,T,G)$ by $\tau_\mu([f]_{\mathrm{Inf}}) := \int f d\mu$. Conversely, given a trace τ on $\mathcal{G}(X,T,G)$ and a clopen $U \subseteq X$, define $\phi_\tau(U) = \tau(\chi_U)$. Since X is a Cantor space, there exists a unique measure μ on X such that $\mu(U) = \phi(U)$ for all U. By construction, $\tau_\mu = \tau$ and μ is T-invariant. It is not difficult to see that this defines an affine isomorphism from $\mathcal{M}(X,T)$ to $S(\mathcal{G}(X,T,G))$, so that traces of the reduced dynamical dimension group $\mathcal{G}(X,T,G)$ are the invariant measures of the system (X,T,G) and again $\mathcal{M}(X,T,G)$ is an invariant of orbit equivalence.

1.5.4 Dynamical dimension group and balance.

The following result states that when a subshift is balanced on the factors, $\operatorname{Inf}(X,T)$ is trivial.

Proposition 1.24. *Let (X,T) be a minimal and uniquely ergodic subshift. If (X,T) is balanced on its factors, then the infinitesimal subgroup $\operatorname{Inf}(X,T)$ is trivial.*

Proof. Let μ denote the unique invariant probability measure of (X,T). Thanks to Proposition 1.19, since (X,T) is balanced on the factors, for any $v \in \mathcal{L}_X$ the function $f_v := \chi_{[v]} - \mu_v$ is a coboundary. Let $f \in C(X,\mathbb{Z})$ such that $[f]$ is an infinitesimal of $K^0(X,T)$. By Proposition 1.7, f is cohomologous to a cylinder function $g \in C(X,\mathbb{Z})$. Let k be a positive integer such that g depends only on the first k coordinates. One has

$$g = \sum_{u\in\mathcal{L}_k(X)} \ell(u)\chi_{[u]}$$

for some $\ell(u) \in \mathbb{Z}$. So we obtain

$$\int f d\mu = \int g d\mu = \sum_{u\in\mathcal{L}_k(X)} \ell(u)\mu_u n = 0.$$

Therefore,

$$f(x) = \sum_{u\in\mathcal{L}_k(X)} \ell(u)(\chi_{[u]} - \mu_u) + \underbrace{\sum_{u\in\mathcal{L}_k(X)} \ell(u)\mu_u}_{=0}.$$

Since $\chi_{[u]} - \mu_u$ is a coboundary for all $u \in \mathcal{L}_k(X)$, f is an integer linear combination of coboundaries, and thus a coboundary. □

Remark 1.25. *A Cantor minimal system (X,T) is called saturated if for any two clopen sets, $A, B \subseteq X$ with $\mu(A) = \mu(B)$ for all $\mu \in \mathcal{M}(X,T)$ there exists a homeomorphism γ belonging to the topological full group of T (see Introduction for the definition) such that $\gamma(A) = B$. In [BK00] the authors show that a Cantor minimal system (X,T) is saturated if and only if every element on the inifnitesimal subgroup of the dynamical dimension group is a coboundary. Thus, we have proved in the above proposition that if (X,T) is balanced on the factors, then the system is saturated.*

1.6 Tower partitions.

Let (X,T,G) a dynamical system. A *partition in towers* of X is a clopen partition $\mathcal{P}$ of the form

$$\mathcal{P} = \{T^{u^{-1}}(B_k) : u \in F_k, 1 \leq k \leq d\},$$

where $d \in \mathbb{N}$, $F_k \subset G$ are finite subsets, and $B_k \subseteq X$ are clopen for all $1 \leq k \leq d$. The sets of the form $T^{u^{-1}}(B_k)$ are the *atoms* of $\mathcal{P}$. For a given $1 \leq k \leq d$, the set $\bigcup_{u\in F_k} T^{u^{-1}}(B_k)$ is called the *kth tower* of $\mathcal{P}$, B_k is its *base* and $|F_k|$ its *height*. The positive integer d is the *number of towers* of $\mathcal{P}$.

The terminology *tower partition* comes from the case of $\mathbb{Z}$-actions: in this case, $\mathcal{P}$ has indeed the form of a finite number of towers, each with a finite number of floors, and each floor is sended by the homeomorphism T to the next one. See Figure 1.1 for an illustration. This way of decomposing X is attributed to Kakutani and Rokhlin in the case of $\mathbb{Z}$-actions. The following result guarantees the existence of sequences of tower partitions with nice properties. It is attributed originally to Vershik ([Ver81]).

Proposition 1.26. *[BR10, Chapter 6] Let (X,T) be a minimal Cantor dynamical system, let $x \in X$. There exists a sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ verifying the following conditions,*

(KR1) $\bigcap_{1\geq i\geq d_n} B_{i,n} = \{x_0\}$,

(KR2) $\mathcal{P}_{n+1}$ *is finer than* $\mathcal{P}_n$ *for all* $n \in \mathbb{N}$.

(KR3) $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ *generates the topology of* X.

Given a sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of X of the form

$$\mathcal{P}_n = \{T^{u^{-1}}(B_{k,n}) : u \in F_{k,n}, 1 \leq k \leq d_n\},$$

let B_n denote the union of the tower basis $\{B_{k,n}\}_{k=1}^{d_n}$. We say that $\mathcal{P}_{n+1}$ *is finer than* $\mathcal{P}_n$ if each atom of $\mathcal{P}_{n+1}$ is contained in an atom of $\mathcal{P}_n$ and the sequence of bases $(B_n)_{n\in\mathbb{N}}$ is decreasing, that is, for all $n \in \mathbb{N}$, $B_{n+1} \subseteq B_n$.

1.7 Topological eigenvalues.

We consider here dynamical systems given by an action of $\mathbb{Z}$ and recall some notions and results related to their topological eigenvalues.

Let (X,T) be a topological dynamical system given by an action of $\mathbb{Z}$. A non-zero complex-value λ is said to be a *continuous eigenvalue* or a *topological eigenvalue* of (X,T) if there exists a non-zero complex-valued continuous function $f \in C(X,\mathbb{C})$ such that $\forall x \in X$, $f(Tx) = \lambda f(x)$. In this case, λ is said to be a continuous eigenvalue *associated* to the *continuous* (or *topological*) *eigenfunction* f.

If there exists a non-zero complex-valued function $f \in C(X,\mathbb{C})$ which is integrable with respect to some measure $\mu \in \mathcal{M}(X,T)$, such that $f(Tx) = \lambda f(x)$ μ-a.e., λ is called a *measurable eigenvalue.*

Given $\mu \in \mathcal{M}(X,T)$, the Koopman operator U_T defined by $U_T f = T \circ f$ is a unitary operator on

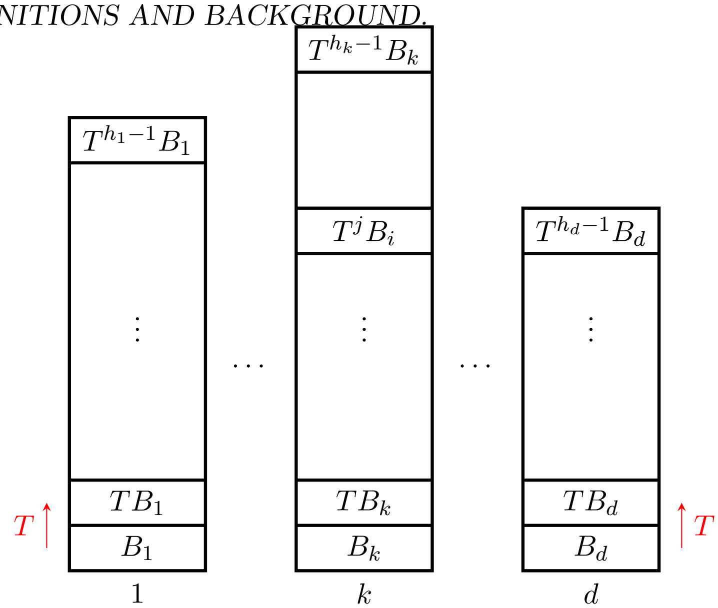

Figure 1.1: A partition in towers.

$L^2(\mu)$ (see for example [W82] for details), and since every topological eigenvalue is a measurable one for any given measure $\mu \in \mathcal{M}(X,T)$, every topological eigenvalue belongs to the unitary circle $\mathbb{S}^1 \subseteq \mathbb{C}$. If α is such that $e^{2i\pi\alpha}$ is an eigenvalue of (X,T), α is said to be an *additive topological eigenvalue*. We denote by $E(X,T)$ the set of all additive topological eigenvalues of (X,T). The set $E(X,T)$ is an additive subgroup of $\mathbb{R}$ which contains $\mathbb{Z}$ (every integer corresponds to the topological eigenvalue 1 which is associated to any constant function). We now give a relation between balance and topological eigenvalues of minimal, uniquely ergodic subshifts.

Proposition 1.27. *Let (X,T) be a minimal and uniquely ergodic subshift and let μ denote its unique invariant measure. If σ is balanced on the factor v, then μ_v is an additive topological eigenvalue of (X,T).*

Proof. Suppose that X is balanced on the factor v. By Proposition 1.19, there exists $g \in C(X,\mathbb{R})$ such that $f_v = g \circ T - g$. Note that $e^{2i\pi\chi_{[v]}(x)} = 1$ for any $x \in X$, since $\chi_{[v]}$ takes values in $\{0,1\}$. This yields

$$\exp^{2i\pi g\circ T} = \exp^{-2i\pi\mu_v} \exp^{2i\pi g}.$$

Hence, $\exp^{-2i\pi g}$ is a topological eigenfunction associated to the additive topological eigenvalue μ_v. $\square$

The previous results shows in particular that when the minimal uniquely ergodic subshift (X,T) is balanced on factors, the image subgroup $I(X,T)$ is included in $E(X,T)$ (see Section 1.5.2). It is also known that $E(X,T)$ is an additive subgroup of $I(X,T)$ ([IO07]), so we conclude that when a minimal subshift is balanced on factors, $E(X,T) = I(X,T)$.

Recall that for a given ergodic measure-theoretic dynamical system $(X,T,\mathcal{B},\mu)$, it is weakly mixing if and only if it admits no non-trivial measurable eigenvalue (see for example [W82, Section 1.7]). Thus, a minimal uniquely ergodic system (X,T) with unique measure μ which is balanced on any factor v with frequency $0 < \mu_v < 1$, defines a measure-theoretic dynamical system $(X,T,\mathcal{B},\mu)$ which cannot be weakly mixing. The absence of weak mixing is indeed a property which has been already used to prove unbalance (see for example [CFM08]).

Chapter 2

Some results on the dynamical dimension group using tower partitions.

In this chapter we describe the relation between some well-chosen sequences of tower partitions and the image subgroup, the group of infinitesimals and the dynamical dimension group of a minimal $\mathbb{Z}$-subshift. We will use the results presented here to explore some dynamical properties (image subgroup, infinitesimals, dynamical dimension group and balance) in the examples treated in Chapters 3 and 4. Our main results in this regard are Propositions 2.10, 2.11 and 2.17.

2.1 Tower partitions and inductive limits.

Let (X, T) be a minimal subshift and

$$(\mathcal{P}_n = \{T^j B_{i,n} : 1 \leq i \leq d_n, 0 \leq j < h_{i,n}\})_{n\geq 0} \tag{2.1}$$

a sequence of tower partitions of X, such that for all $n \in \mathbb{N}$, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$. Let $B_n = \bigcup_{i=1}^{d_n} B_{i,n}$. Let $C(\mathcal{P}_n)$ denote the subgroup of $C(X, \mathbb{Z})$ consisting of the set of functions which are constant on the atoms of $\mathcal{P}_n$, and $G(\mathcal{P}_n)$ the subgroup of $C(B_n, \mathbb{Z})$ consisting of the set of functions which are constant on each base $B_{i,n}$. Let $G^+(\mathcal{P}_n)$ denote the subset of $G(\mathcal{P}_n)$ consisting of non-negative functions and define $1(\mathcal{P}_n) \in G^+(\mathcal{P}_n)$ by $1(\mathcal{P}_n)(x) = h_{i,n}$ for all $x \in B_{i,n}$.

Consider the group homomorphisms $I_{\mathcal{P}_n} : C(\mathcal{P}_n) \to G(\mathcal{P}_n)$ and $R_{\mathcal{P}_n} : G(\mathcal{P}_n) \to R(\mathcal{P}_n)$ defined as follows,

$$(I_{\mathcal{P}_n} f)(x) = f^{(h_{i,n})}(x) \quad \text{for} \quad x \in B_{i,n}$$

$$(R_{\mathcal{P}_n} g)(x) = \begin{cases} g(x) & \text{if } x \in B_n \\ 0 & \text{else.} \end{cases}$$

It is not difficult to see that $I_{\mathcal{P}_n} \circ R_{\mathcal{P}_n}$ is the identity on $C(B_n, \mathbb{Z})$. By Theorem 1.1, $\ker(I_{\mathcal{P}_n})$ consists of coboundaries, and then there exists a group homomorphism $\pi_{\mathcal{P}_n} : G(\mathcal{P}_n) \to H(X,T)$ such that $\pi_{\mathcal{P}_n} \circ I_{\mathcal{P}_n} = \pi$, where π is the canonical projection from $C(X,\mathbb{Z})$ to $H(X,T)$.

Proposition 2.1 ([DHP18])**.** *The map $\pi_{\mathcal{P}_n}$ defined above is a morphism of ordered groups with unit between $(G(\mathcal{P}_n), G^+(\mathcal{P}_n), 1(\mathcal{P}_n))$ and $(H(X,T), H^+(X,T), 1_X)$.*

Since $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$,

$$I_{\mathcal{P}_{n+1},\mathcal{P}_n} := I_{\mathcal{P}_{n+1}} \circ R_{\mathcal{P}_n}$$

is a well-defined group homomorphism which maps $G(\mathcal{P}_n)$ to $G(\mathcal{P}_{n+1})$. If we identify $G(\mathcal{P}_n)$ with $\mathbb{Z}^{d_n}$ and $G(\mathcal{P}_{n+1})$ with $\mathbb{Z}^{d_{n+1}}$, the matrix associated to the morphism $I_{\mathcal{P}_{n+1},\mathcal{P}_n}$ is given by

$$Q_n(i,j) = |\{0 \le l < h_{i,n+1} : T^l B_{i,n+1} \subseteq B_{j,n}\}| \qquad \forall 1 \le i \le d_{n+1}, \quad \forall 1 \le j \le d_n \tag{2.2}$$

Let $(G(\mathfrak{S}), G^+(\mathfrak{S}), 1(\mathfrak{S}))$ be the inductive limit of the sequence $(G(\mathcal{P}_n)), G^+(\mathcal{P}_n), 1(\mathcal{P}_n))_{n\in\mathbb{N}}$ with the homomorphisms $I_{\mathcal{P}_{n+1},\mathcal{P}_n}$, where we identify $G(\mathcal{P}_n)$ with $\mathbb{Z}^{d_n}$ for all $n \ge 0$, that is, the triple $(\Delta/\Delta^0, (\Delta/\Delta^0)^+, u)$, where

$$\Delta = \{(\mathbf{x}_n)_{n\ge0} \in \prod_{n\ge0} \mathbb{Z}^{d_n} \mid \exists k \ge 0 : Q_n(\mathbf{x}_n) = \mathbf{x}_{n+1} \quad \forall n \ge k\},$$

$$\Delta^0 = \{(\mathbf{x}_n)_{n\ge0} \in \Delta \mid \exists k \ge 0 : \mathbf{x}_n = 0 \quad \forall n \ge k\},$$

$$\Delta^+ = \{(\mathbf{x}_n)_{n\ge0} \in \Delta \mid \exists k \ge 0 : \mathbf{x}_n \in \mathbb{Z}_+^{d_n} \quad \forall n \ge k\},$$

$(\Delta/\Delta^0)^+$ is the set of classes modulo Δ^0 of elements in Δ^+, and u is the class modulo Δ^0 of the sequence $(\mathbf{u}_n)_{n\ge0}$, where

$$\mathbf{u}_n = (h_{1,n}, \cdots, h_{d_n,n}).$$

Let $i_{\mathcal{P}_n} : G(\mathcal{P}_n) \to G(\mathfrak{S})$ denote the *projection on the inductive limit* defined as follows: for every

$f \in G(\mathcal{P}_n)$, $i_{\mathcal{P}_n}(f)$ is the class of the sequence

$$y_k = \begin{cases} 0 & \text{if } k < n \\ f & \text{if } k = n \\ Q_k \cdots Q_{n-1} f & \text{if } k > n \end{cases}$$

Proposition 2.2 ([DHP18])**.** *There exists a unique morphism of ordered groups with unit $\pi_{\mathfrak{S}} : G(\mathfrak{S}) \to H(X,T)$ satisfying $\pi_{\mathfrak{S}} \circ i_{\mathcal{P}_n} = \pi_{\mathcal{P}_n}$.*

(See Figure 2.1).

Proposition 2.3. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined as in (2.1), let $C(\mathfrak{S}) := \bigcup_{n\geq 0} C(\mathcal{P}_n)$. The morphism $\pi_{\mathfrak{S}}$: $G(\mathfrak{S}) \to H(X,T)$ defined in Proposition 2.2 is surjective if and only if every $f \in C(X,\mathbb{Z})$ is cohomologous to a function which belongs to $C(\mathfrak{S})$.*

Remark 2.4. *To prove this proposition we use a crucial idea presented in [Ho95, Section 4, Remark 5.b)].*

Proof. Suppose $\pi_{\mathfrak{S}} : G(\mathfrak{S}) \to H(X,T)$ is surjective. Let $f \in C(X,\mathbb{Z})$ and consider $[f] \in H(X,T)$. Since $\pi_{\mathfrak{S}}$ is surjective, there exists $\phi \in G(\mathfrak{S})$ such that $\pi_{\mathfrak{S}}(\phi) = [f]$. By definition, ϕ is the class of some sequence $(\phi_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G(\mathcal{P}_n)$ satisfying that there exists $k \in \mathbb{N}$ such that $\phi_{n+1} = Q_n(\phi_n)$ for all $n \geq k$. It is clear that $R_{\mathcal{P}_k}(\phi_k) \in C(\mathcal{P}_k)$ and $i_{\mathcal{P}_k}(\phi_k) = \phi$. Let $g = R_{\mathcal{P}_k}(\phi_k)$. By construction, $g \in C(\mathfrak{S})$. We have that

$$[f] = \pi_{\mathfrak{S}}(\phi) = \pi_{\mathfrak{S}}(i_{\mathcal{P}_k}(\phi_k)) = \pi_{\mathfrak{S}} \circ i_{\mathcal{P}_k}(I_{\mathcal{P}_k}(g)) = \pi(g) = [g].$$

We conclude that f is cohomologous to g.

Conversely, suppose that every $f \in C(X,\mathbb{Z})$ is cohomologous to some function $g \in C(\mathfrak{S})$. The morphism $\pi(\mathfrak{S})$ will be surjective if $H(X,T) = \bigcup_{n\geq 0} \pi_{\mathcal{P}_n}(G(\mathcal{P}_n))$. Since the inclusion $\bigcup_{n\geq 0} \pi_{\mathcal{P}_n}(G(\mathcal{P}_n)) \subseteq H(X,T)$ follows from the definition, it is enough to show that for any $[f] \in H(X,T)$, $[f]$ belongs to $\bigcup_{n\geq 0} \pi_{\mathcal{P}_n}(G(\mathcal{P}_n))$.

Let $[f] \in H(X,T)$ and take any representative f. By hyphotesis, f is cohomologous to a function $g \in C(\mathfrak{S})$, then, $f - g \in \beta C(X,\mathbb{Z})$ and there exists $n \in \mathbb{N}$ such that g is constant on the atoms of $\mathcal{P}_n$. Since $g \in C(\mathcal{P}_n)$, $[g] = \pi_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(g) \in \pi_{\mathcal{P}_n}(G(\mathcal{P}_n))$. But $[f] = [g]$, and then

$$[f] \in \pi_{\mathcal{P}_n}(G(\mathcal{P}_n)) \subseteq \bigcup_{n\geq 0} \pi_{\mathcal{P}_n}(G(\mathcal{P}_n))$$

□

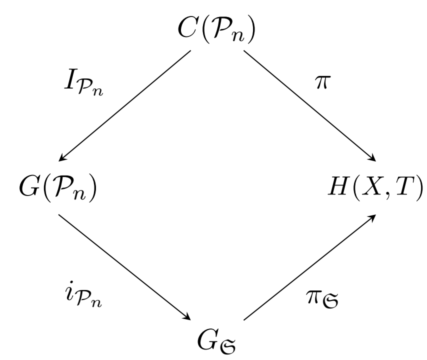

Figure 2.1: The isomorphism $\pi_{\mathfrak{S}}$, where $G_{\mathfrak{S}}$ denotes the inductive limit of the system $(G(\mathcal{P}_n), G^+(\mathcal{P}_n), \mathbf{1}_{\mathcal{P}_n})_{n\geq 0}$ with morphisms $I_{\mathcal{P}_{n+1},\mathcal{P}_n}$ and $i_{\mathcal{P}_n} : G(\mathcal{P}_n) \to G_{\mathfrak{S}}$ is the natural projection on the inductive limit.

Remark 2.5. *Note that the equivalent conditions of Proposition 2.3 are satisfied as soon as the sequence $(\mathcal{P}_n)_{n\geq 0}$ is such that for all $n \in \mathbb{N}$, $\mathcal{P}_n$ is finer than the partition in n-cylinders $\mathcal{Q}_n = \{[w] : w \in \mathcal{L}_n(X)\}$. Indeed, if every $\mathcal{P}_n$ is finer than $\mathcal{Q}_n$, every cylinder function is constant in the atoms of $\mathcal{P}_n$ for n large enough, and we know from Proposition 1.7 that every $f \in C(X,\mathbb{Z})$ is cohomologous to some cylinder function.*

Proposition 2.6. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined as in* (2.1)*, let $Y = \bigcap_{n\in\mathbb{N}} B_n$. If Y consists of only one point, the morphism $\pi_{\mathfrak{S}} : G(\mathfrak{S}) \to H(X,T)$ defined above is injective.*

Proof. Suppose $\alpha \in G(\mathfrak{S})$ belongs to $\ker(\pi_{\mathfrak{S}})$. There exists $n \in \mathbb{N}$ and $g \in C(\mathcal{P}_n)$ such that $\alpha = i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n} g$. Indeed, α is the class modulo Δ_0 of a sequence $(\mathbf{x}_i)_{i\in\mathbb{N}}$ which verifies $\mathbf{x}_i \in \mathbb{Z}^{d_i}$ and there exists $k \in \mathbb{N}$ such that for all $i \geq k$, $\mathbf{x}_{i+1} = Q_k \mathbf{x}_i$. Let $n \geq k$ and define $g \in C(\mathcal{P}_n)$ by

$$g(z) = \begin{cases} \mathbf{x}_n(j) & \text{if } \exists j : z \in \sigma_{[0,n)}([a_j]) \\ 0 & \text{otherwise} \end{cases}$$

Note that $(I_{\mathcal{P}_n} g)(y) = \mathbf{x}_n(j)$ if $y \in \sigma_{[0,n)}([a_j])$. Therefore, $i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n} g$ is the class modulo Δ_0 of the sequence

$$(0, \cdots, 0, \underbrace{\mathbf{x}_n}_{nth}, \underbrace{Q_n \mathbf{x}_n}_{(n+1)th}, \underbrace{Q_{n+1} Q_n \mathbf{x}_n}_{(n+2)th}, \cdots),$$

which is equal to α. Since $\pi_{\mathfrak{S}}(\alpha) = 0$ and $\pi(g) = \pi_{\mathfrak{S}}(\alpha)$, the function g is a coboundary. Let $h \in C(X, \mathbb{Z})$ such that $g = \beta h$. Let x_0 be the unique element belonging to Y, then for all $\ell \in \mathbb{N}$, B_ℓ is a clopen neighborhood of x_0, and since $h \in C(X, \mathbb{Z})$, it is locally constant, so for some $m \geq n$ large enough, h is constant (with value $h(x_0)$) in B_m (recall that bases B_n are nested). Since $m \geq n$, $g \in C(\mathcal{P}_m)$ and by definition $i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n} g = i_{\mathcal{P}_m} \circ I_{\mathcal{P}_m} g$, but since h is constant in B_m and $g = \beta h$, $I_{\mathcal{P}_m} g = 0$ and therefore

$$\alpha = i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n} g = i_{\mathcal{P}_m} \circ I_{\mathcal{P}_m} g = 0_{G(\mathfrak{S})}.$$

□

Let $(\mathcal{P}_n)_{n\geq 0}$ be a sequence of tower partitions of (X, T) defined as in (2.1). Let $C(\mathfrak{S})$ be as defined in Proposition 2.3 and Y as defined in Proposition 2.6. Consider the following conditions.

(C0) For all $1 \leq i \leq d_n$, the heigth $h_{i,n}$ tends to infinity when n tends to infinity.

(C1) Every $f \in C(X, \mathbb{Z})$ is cohomologous to a function which belongs to $C(\mathfrak{S})$.

(C2) The set Y consists of only one point.

(C3) There exists $m \in \mathbb{N}$ such that for all $n \geq m$, $d_n = d$ and matrices Q_n belong to $GL_d(\mathbb{Z})$.

Note that condition **(C1)** is equivalent to the surjectivity of $\pi_{\mathfrak{S}}$ thanks to Proposition 2.3, and from Remark 2.5 we know that it is satisfied if every $\mathcal{P}_n$ is finer than the partition in n-cylinders $\mathcal{Q}_n$. Note also that, thanks to Proposition 2.6, **(C2)** is a sufficient condition to have the injectivity of $\pi_{\mathfrak{S}}$.

Remark 2.7. *Conditions* **(C1)** *and* **(C2)** *have to be compared with classical conditions* **(KR1)** *and* **(KR3)** *for Kakutani-Rokhlin partitions. We follow here the notation used in [BR10, Chapter 6], where these conditions are stated as follows*

(KR1) *The set Y consists of only one point.*

(KR3) *$C(\mathfrak{S})$ generates the topology of X.*

Condition **(KR1)** *is exactly the same as our condition* **(C2)**, *but* **(KR3)** *is not the same as* **(C1)**. *It is not difficult to check that* **(KR3)** $\Rightarrow$ **(C1)**, *but there exist sequences of tower partitions which satisfy* **(C1)** *and do not satisfy* **(KR3)** *(See Example 3.13 in Section 3.2). Proposition 2.3 shows that* **(C1)** *is the optimal condition which guarantees the surjectivity of $\pi_{\mathfrak{S}}$.*

Remark 2.8. *If one thinks of the Bratteli diagram associated to a given sequence of tower partitions, condition* **(C3)** *can be interpreted as a finite rank condtion. Actually condition* **(C3)** *isa bit stronger, because we require the incidence matrices not only to be square, but also to be invertible in* $\mathbb{Z}$*.*

We now present a series of results which can be obtained if (X,T) has a sequence of tower partitions verifying some of conditions **(C0)**-**(C3)**.

2.2 Image subgroup and infinitesimals.

Lemma 2.9. *Let* (X,T) *a minimal subshift and let* $\mu \in \mathcal{M}(X,T)$*. Let* $(\mathcal{P}_n = \{T^j B_{i,n} : 1 \leq i \leq d_n, 0 \leq j < h_{i,n}\})_{n\geq 0}$ *be a sequence of tower partitions of* (X,T)*,* $(Q_n)_{n\geq 0}$ *the sequence of matrices associated to* $(\mathcal{P}_n)_{n\geq 0}$*. For all* $n \in \mathbb{N}$ *define*

$$\vec{\mu}_n = (\mu(B_{1,n}), \cdots, \mu(B_{d_n,n})).$$

If $(\mathcal{P}_n)_{n\geq 0}$ *satisfies condition* **(C3)***, then for all* $n \geq m$*,*

$$\vec{\mu_n} = ((Q_n \cdots Q_m)^t)^{-1} \vec{\mu_m}.$$

Proof. Let $\mu \in \mathcal{M}(X,T)$, $1 \leq i \leq d$ and $n \geq m$, then

$$\mu(B_{i,m}) = \mu(B_{i,m} \cap X)) = \sum_{j=1}^{d} \sum_{k=0}^{h_{j,n}-1} \mu(B_{i,m} \cap T^k B_{j,n}).$$

Since $n \geq m$, $\mathcal{P}_n$ is finer than $\mathcal{P}_m$ and then $B_{i,m} \cap T^k B_{j,n}$ is either empty or the whole atom $T^k B_{j,n}$, so we obtain

$$\mu(B_{i,m}) = \sum_{j=1}^{d} \mu(B_{i,m}) \cdot |\{0 \leq k < h_{j,n} : T^k B_{j,n+1} \subseteq B_{i,m}\}|.$$

If we rewrite this in terms of matrices,

$$\mu(B_{i,m}) = \sum_{j=1}^{d} \mu(B_{j,n})(Q_n \cdots Q_m)^t(i,j) = ((Q_n \cdots Q_m)^t \vec{\mu_n})_i.$$

Since this is true for all $1 \leq i \leq d$, we obtain that $\vec{\mu_m} = (Q_n \cdots Q_m)^t \vec{\mu_n}$ and consequently

$$\vec{\mu_n} = ((Q_n \cdots Q_m)^t)^{-1} \vec{\mu_m}.$$

□

Proposition 2.10. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of tower partitions of (X,T) satisfying conditions* **(C1)** *and* **(C3)***. Let d and m be defined as in condition* **(C3)***. Then, the image subgroup of (X,T) is the additive subgroup of $\mathbb{R}$ generated by the measures of basis $B_{i,m}$, $1 \leq i \leq d$, that is*

$$I(X,T) = \bigcap_{\mu\in\mathcal{M}(X,T)} \left\{ \sum_{i=1}^{d} \mathbb{Z}\mu(B_{i,m}) \right\}.$$

Proof. We show that the image subgroup $I(X,T)$ is included in $\bigcap_{\mu\in\mathcal{M}(X,T)} \left\{\sum_{i=1}^{d} \mathbb{Z}\mu(B_{i,m})\right\}$ (the other inclusion is obvious). Suppose $\alpha \in I(X,T)$. By definition, for all $\mu \in \mathcal{M}(X,T)$, there exists $f \in C(X,\mathbb{Z})$ such that $\alpha = \int f d\mu$. Since $(\mathcal{P}_n)_{n\geq 0}$ satisfies condition **(C1)**, f is cohomologous to a function $g \in C(X,\mathbb{Z})$ which is constant on the atoms of $\mathcal{P}_n$ for every n large enough. We have that,

$$\int g d\mu = \sum_{j=1}^{d} \sum_{i=0}^{h_{j,n}-1} \mu(T^i B_{j,n}) g \mid_{T^i B_{j,n}} .$$

Since μ is $T-$invariant,

$$\int g d\mu = \sum_{j=1}^{d} \mu(B_{j,n}) \sum_{i=0}^{h_{j,n}-1} g \mid_{T^i B_{j,n}} .$$

Define $k_j = \sum_{i=0}^{h_{j,n}-1} g \mid_{T^i B_{j,n}}$ (the sum of the map g over the $j-th$ tower of the partition $\mathcal{P}_n$). Applying Lemma 2.9, we obtain

$$\begin{aligned}
\int g d\mu &= \sum_{j=1}^{d} \sum_{i=1}^{d} \mu(B_{i,m}) (Q_n \cdots Q_m)^{-1}(i,j) k_j \\
&= \sum_{i=1}^{d} \mu(B_{i,m}) \sum_{j=1}^{d} (Q_n \cdots Q_m)^{-1}(i,j) k_j.
\end{aligned}$$

Since $\sum_{j=1}^{d} (Q_n \cdots Q_m)^{-1}(i,j) k_j$ belongs to $\mathbb{Z}$ for all $1 \leq i \leq d$, $\alpha = \int f d\mu = \int g d\mu$ belongs to

$\sum_{i=1}^{d} \mathbb{Z}\mu(B_{i,m})$. Since $\mu \in \mathcal{M}(X,T)$ was arbitrarily taken, we conclude that

$$\alpha \in \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \sum_{i=1}^{d} \mathbb{Z}\mu(B_{i,m}) \right\},$$

which ends the proof. □

In the particular case of a uniquely ergodic system with a sequence of tower partition satisfying conditions **(C1)** and **(C3)**, we know that $I(X,T) = \sum_{i=1}^{d} \mathbb{Z}\mu(B_{i,m})$. We also have the following result regarding the infinitesimal subgroup of $K^0(X,T)$.

Proposition 2.11. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of tower partitions of a minimal subshift (X,T) satisfying conditions* **(C1)** *and* **(C3)***. Let d and m be as defined in* **(C3)***. Suppose that there exists an invariant measure $\mu \in \mathcal{M}(X,T)$ such that the coordinates $\mu(B_{i,m})$ of the vector $\vec{\mu_m}$ are rationally independent. Then, the infinitesimal subgroup* $\mathrm{Inf}(X,T)$ *of $K^0(X,T)$ is trivial, that is, (X,T) is saturated.*

Proof. Let $f \in C(X,\mathbb{Z})$ and suppose that $[f]$ is an infinitesimal of $K^0(X,T)$. Take a function $g \in C(X,\mathbb{Z})$ which is cohomologous to f and constant on the atoms of $\mathcal{P}_n$ for n large enough. Such a function always exists, because the sequence of tower partitions satisfies condition **(C1)**. Let μ be an invariant measure such that the coordinates of $\vec{\mu_m}$ are rationally independent. By definition, $[f] = [g]$ and $\int f d\mu = \int g d\mu = 0$. Since g is constant on each atom of $\mathcal{P}_n$, we have

$$0 = \int g d\mu = \sum_{j=1}^{d} \sum_{i=0}^{h_{j,n}-1} \mu(T^i B_{j,n}) g \mid_{T^i B_{j,n}} .$$

Since μ is $T-$invariant,

$$0 = \int g d\mu = \sum_{j=1}^{d} \mu(B_{j,n}) k_j,$$

where $k_j = \sum_{i=0}^{h_{j,n}-1} g \mid_{T^i B_{j,n}}$. By Proposition 2.10, for all $1 \leq j \leq d$, the measure $\mu(B_{j,n})$ is an integer linear combination of the measures $\{\mu(B_{1,m}), \cdots, \mu(B_{d,m})\}$ provided $n \geq m$, and since this quantities are rationally independent by hypothesis, then we obtain that for all $n \geq m$, $\{\mu(B_{1,n}), \cdots, \mu(B_{d,n})\}$ are rationally independent as well. This means that $\sum_{j=1}^{d} \mu(B_{j,n}) k_j = 0$ implies $k_j = 0$ for all $1 \leq j \leq d$.

That is, for all $1 \leq j \leq d$, for all $n \geq m$,

$$\sum_{i=0}^{h_{j,n}-1} g \mid_{T^i B_{j,n}} = 0.$$

Fix any $n \geq m$ and take any point x belonging to the basis of $\mathcal{P}_n$. For any $N \in \mathbb{N}$, the coclycle $f^{(N)}(x)$ can be decomposed as a sum of k_j's (for $1 \leq j \leq d$) plus an error of the form

$$\epsilon = \sum_{i=0}^{l} g \mid_{T^i B_{j,n}},$$

for some $1 \leq j \leq d$, some $0 \leq l < h_{j,n}$. As we have seen before, every k_j is zero, and the error ϵ is bounded by $\sup |g| \cdot \max_{1 \leq j \leq d} h_{j,n}$. Applying Theorem 1.1, we conclude that g is a coboundary, and then so is f. Therefore, $\mathrm{Inf}(X, T)$ consists only of coboundaries. □

In [AR16], the authors prove that a system generated by a primitive aperiodic and irreducible substitution is saturated provided it satisfies an extra condition called the *common prefix* property. See [AR16, Sections 1 and 3] for details.

Remark 2.12. *There are examples of minimal systems having sequences of tower partitions satisfying* **(C1)** *and* **(C3)***, and a measure vector $\vec{\mu_m}$ with rationally dependent coordinates. See for instance [BCD+18, Remark 6.4] where we present an example of an interval exchange transformation on a three-letter alphabet (see Example 4.4) which generates a minimal uniquely ergodic dendric subshift (see Chapter 4) having rational dependence on the letter cylinder measures. In this example, the sequence of tower partitions is constructed using return words (see Section 4.2.1), $m = 0$ and the atoms of $\mathcal{P}_0$ are the cylinder of letters. This example has indeed nontrivial infinitesimals.*

2.3 The dynamical dimension group

The dynamical dimension group is related with the inductive limit of a suitable sequence of tower partitions. This relation is described in detail in the following result.

Proposition 2.13. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of tower partitions of (X, T) satisfying* **(C1)** *and*

(C2'): *For every $u \in C(Y, \mathbb{Z})$, there exists $h \in C(X, \mathbb{Z})$ such that $h|_Y = u$ and $\beta h \in C(\mathfrak{G})$.*

Then, there exists a group homomorphism $r : C(Y,\mathbb{Z}) \to G(\mathfrak{S})$ such that the sequence

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{i} C(Y,\mathbb{Z}) \xrightarrow{r} G(\mathfrak{S}) \xrightarrow{\pi_{\mathfrak{S}}} H(X,T) \xrightarrow{0} 0,$$

where $\mathbb{Z}$ is identified with the subset of constant functions on Y and i denotes the inclusion, is exact.

Remark 2.14. *Note that condition* **(C2)** *implies* **(C2')**.

Proof. Let $u \in C(Y,\mathbb{Z})$ and take $h \in C(X,\mathbb{Z})$ as in condition **(C2')**. Since $\beta h \in C(\mathfrak{S})$, there exists $n \in \mathbb{N}$ such that βh is constant in the atoms of $\mathcal{P}_n$. Define $r(u) := i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(\beta h)$. Note that $\pi_{\mathfrak{S}}(r(u)) = \pi(\beta h) = 0$, so that r is a well-defined homomorphism between $C(Y,\mathbb{Z})$ and $\ker(\pi_{\mathfrak{S}})$. On the other hand, if $\alpha \in G(\mathfrak{S})$, there exists $n \in \mathbb{N}$ and $g \in C(\mathcal{P}_n)$ such that $\alpha = i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n} g$ (see the first part of the proof of Proposition 2.6). If $\alpha \in \ker(\pi_{\mathfrak{S}})$, then $\pi(g) = 0$ and thus g is a coboundary, then there exists $\tilde{g} \in C(X,\mathbb{Z})$ such that $g = \beta\tilde{g}$. Let $u = \tilde{g}|_Y$. By definition, $r(u) = i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(\beta\tilde{g}) = i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n} g = \alpha$, and we conclude that $\alpha \in \mathrm{Im}(r)$. We have proved that $\ker(\pi_{\mathfrak{S}}) = \mathrm{Im}(r)$.
Since **(C1)** is satisfied, $\pi_{\mathfrak{S}}$ is surjective, which is equivalent to the fact that $\ker(0) = \mathrm{Im}(\pi_{\mathfrak{S}})$.
Let us prove that $r(u) = 0$ if and only if u is constant, or equivalently, that $\ker(r) = \mathrm{Im}(i)$. Suppose that $u \in C(Y,\mathbb{Z})$ verifies $r(u) = 0$. Let $h \in C(X,\mathbb{Z})$ be as defined in condition **(C2')** and $n \in \mathbb{N}$ such that $\beta h \in C(\mathcal{P}_n)$. By hypothesis, $i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(\beta h) = 0_{G(\mathfrak{S})}$, which means that there exists $m_0 \geq n$ such that for all $m \geq m_0$, $I_{\mathcal{P}_m,\mathcal{P}_n} I_{\mathcal{P}_n}(\beta h) = 0 \in \mathbb{Z}^{d_m}$, that is, $I_{\mathcal{P}_m}(\beta h) = 0$. This implies that h is constant in B_m. Indeed, is suffices to note that for any $1 \leq i \leq d_m$ and for all $x \in B_{i,m}$,

$$I_{\mathcal{P}_m}(\beta h)(x) = h \circ T^{h_{i,m}}(x) - h(x) = 0,$$

so that $h(x) = h \circ T^{h_{i,m}}(x)$. This implies that $h|_Y$ is constant, and then u is contant as well.
Conversely, suppose that $u \in C(Y,\mathbb{Z})$ is constant. Then $h|_Y$ is constant, and since h is continuous, h is constant in B_m for all $m \geq n$ large enough. This implies that $i_{\mathcal{P}_m} \circ I_{\mathcal{P}_m}(\beta h) = 0$. By definition, $i_{\mathcal{P}_m} \circ I_{\mathcal{P}_m}(\beta h) = i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(\beta h)$ and we conclude that $r(u) = 0$. This proves that $\ker(r) = \mathrm{Im}(i)$ and concludes the proof of the proposition. □

Note that when **(C1)** and **(C2)** are satisfied, we recover Propositions 2.3 and 2.6 as immediate consequences of Proposition 2.13.

Lemma 2.15. *Let $(\mathcal{P}_n)_{n\geq 0}$ be a sequence of tower partitions of a uniquely ergodic minimal subshift (X,T) satisfying conditions* **(C0)** *and* **(C3)**, *let $(Q_n)_{n\geq 0}$ be the sequence of matrices associated to*

$(\mathcal{P}_n)_{n\geq 0}$. *Let* μ *be the unique* T*-invariant measure on* X *and* d, m *as defined in* **(C3)**. *Then, for all* $1 \leq \ell \leq d$, *for all* $1 \leq j \leq d$,

$$\lim_{n\to\infty} \frac{(Q_n \cdots Q_m)(\ell, j)}{h_{\ell,n}} = \mu(B_{j,m}).$$

Proof. Note that, if $x \in B_{\ell,n}$,

$$|\{0 \leq k < h_{\ell,n} : T^k B_{\ell,n} \subseteq B_{j,m}\}| = \sum_{k=0}^{h_{\ell,n}-1} \chi_{B_{j,m}}(T^k x),$$

that is,

$$(Q_n \cdots Q_m)(\ell, j) = \sum_{k=0}^{h_{\ell,n}-1} \chi_{B_{j,m}}(T^k x).$$

Since the system is uniquely ergodic, Birkhoff's Theorem (Theorem 1.12) implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{B_{j,m}}(T^k x) = \mu(B_{j,m}).$$

Since $(\mathcal{P}_n)_{n\geq 0}$ satisfies condition **(C0)**, we obtain

$$\lim_{n\to\infty} \frac{1}{h_{\ell,n}} \sum_{k=0}^{h_{\ell,n}-1} \chi_{B_{j,m}}(T^k x) = \mu(B_{j,m}),$$

which is what we wanted to prove. □

Proposition 2.16. *Let* $(\mathcal{P}_n)_{n\in\mathbb{N}}$ *be a sequence of tower partitions of a minimal subshift* (X, T) *satisfying conditions* **(C0)**, **(C1)** *and* **(C3)**. *Then,* $G(\mathfrak{S}) \cong \mathbb{Z}^d$ *and the isomorphism* φ *can be chosen so that* $\varphi(1(\mathfrak{S})) = \mathbf{u}$, *where*

$$\mathbf{u} = (h_{1,m}, \cdots, h_{d,m}) \in \mathbb{Z}^d.$$

If moreover (X, T) *is uniquely ergodic with unique invariant measure* μ, *then* $(G(\mathfrak{S}), G^+(\mathfrak{S}), 1(\mathfrak{S}))$ *is isomorphic as ordered group with unit to* $(\mathbb{Z}^d, A, \mathbf{u})$, *where*

$$A = \{x \in \mathbb{Z}^d : \langle x, \vec{\mu_m} \rangle > 0\} \cup \{0\},$$

and $\vec{\mu_m}$ *is as defined in Lemma 2.9.*

Proof of Proposition 2.16. Define $\varphi : \Delta \to \mathbb{Z}^d$ by

$$\varphi((x_n)_{n\geq 0}) \longmapsto (Q_k \cdots Q_m)^{-1}(x_{k+1})$$

where k is any index which satisfies $k \geq m$ and $\forall n \geq k$, $Q_n(x_n) = x_{n+1}$.
It is not difficult to see that φ is a well-defined group homomorphism whose kernel is exactly Δ^0. This homomorphism is also surjective: for any $\mathbf{x} \in \mathbb{Z}^d$, consider the sequence

$$\mathbf{y}_n = \begin{cases} 0 & \text{if } n < m \\ \mathbf{x} & \text{if } n = m \\ Q_n \cdots Q_m \mathbf{x} & \text{if } n > m \end{cases}$$

then we have that $\varphi((\mathbf{y}_n)_{n\geq 0}) = \mathbf{x}$ (choose $k = m$). We call φ the induced isomorphism between Δ/Δ^0 and $\mathbb{Z}^d$ as well.
Note that $\varphi(1(\mathfrak{S})) = \mathbf{u}$. Indeed, for all $1 \leq \ell \leq d$ and for all $n \geq 0$, one has

$$h_{\ell,n+1} = \sum_{i=1}^{d_n} |\{0 \leq k < h_{\ell,n+1} : T^k B_{\ell,n+1} \subseteq B_{i,n}\}| \cdot h_{i,n},$$

which implies that for all $n \geq 0$, $Q_n(u_n) = u_{n+1}$, so we can choose $k = m$ and we obtain

$$\varphi((u_n)_{n\geq 0}) = Q_m^{-1}(u_{m+1}) = Q_m^{-1} Q_m(u_m) = u_m = \mathbf{u}.$$

Recall that the inductive limit $(G(\mathfrak{S}), G^+(\mathfrak{S}), 1(\mathfrak{S}))$ is the triple $(\Delta/\Delta^0, (\Delta/\Delta^0)^+, u)$.
We now prove that if (X, T) is uniquely ergodic, $\varphi(G^+(\mathfrak{S})) \subseteq A$. Let $x = (\mathbf{x}_n)_{n\geq 0}$ a sequence belonging to Δ^+. We want to show that either $\varphi(x) = 0$ or $\langle \varphi(x), \vec{\mu} \rangle > 0$ for all $\mu \in \mathcal{M}(X, T)$. Suppose $\varphi(x) \neq 0$, then, for all $k \geq m$ such that $Q_n(\mathbf{x}_n) = \mathbf{x}_{n+1} \forall n \geq k$,

$$(Q_k \cdots Q_m)^{-1}(\mathbf{x}_{k+1}) \neq 0.$$

This implies that $\mathbf{x}_n \neq 0$ for all $n \geq k$. Since $x \in \Delta^+$, we conclude that for all $n \geq k$ there exists

$1 \leq i \leq d$ such that $(\mathbf{x}_n)_i > 0$. Let $\mu \in \mathcal{M}(X,T)$,

$$\begin{aligned}\langle \varphi(x), \vec{\mu} \rangle &= \sum_{i=1}^{d}\sum_{i=1}^{d}(Q_k \cdots Q_m)^{-1}(\mathbf{x}_{k+1})\mu(B_{i,m}) \\ &= \sum_{i=1}^{d}\sum_{j=1}^{d}(Q_k \cdots Q_m)^{-1}(i,j)(\mathbf{x}_{k+1})_j\mu(B_{i,m}) \\ &= \sum_{j=1}^{d}(\mathbf{x}_{k+1})_j \sum_{i=1}^{d}(Q_k \cdots Q_m)^{-1}(i,j)\mu(B_{i,m}).\end{aligned}$$

From Lemma 2.9, we know that

$$\sum_{i=1}^{d}(Q_k \cdots Q_m)^{-1}(i,j)\mu(B_{i,m}) = \mu(B_{j,k+1}).$$

So we obtain that

$$\langle \varphi(x), \vec{\mu} \rangle = \sum_{j=1}^{d}(\mathbf{x}_{k+1})_j\mu(B_{j,k+1}).$$

Since the system is uniquely ergodic, the support of μ is a closed T-invariant subset of X, so by minimality every $\mu(B_{j,k+1})$ is strictly positive. Since there exists $1 \leq j \leq d$ such that $(\mathbf{x}_n)_j > 0$, we conclude that $\sum_{j=1}^{d}(\mathbf{x}_{k+1})_j\mu(B_{j,k+1}) > 0$, which implies that $\varphi(x) \in A$.

We now prove the converse inclusion: $\varphi^{-1}(A) \subseteq G^+(\mathfrak{S})$. Let $\mathbf{x} \in A$. Recall that the inverse image of $\mathbf{x}$ under φ is the class modulo Δ^0 of a sequence $(\mathbf{y}_n)_{n\geq 0}$ verifying $\mathbf{y}_n = (Q_n \cdots Q_m)(\mathbf{x})$ for all $n > m$. We want to prove that there exists $N \geq m$ such that for all $n \geq N$, $\mathbf{y}_n = (Q_n \cdots Q_m)(\mathbf{x}) \in \mathbb{Z}_+^d$. If $\mathbf{x} = 0$, then $\mathbf{y}_n = 0$ for all $n \in \mathbb{N}$ and consequently $(\mathbf{y}_n)_{n\geq 0} \in \Delta^0 \subseteq \Delta^+$. If $\mathbf{x} \neq 0$, then $\sum_{j=1}^{d}\mathbf{x}_j\mu(B_{j,m}) > 0$, since $\mathbf{x} \in A$. By Lemma 2.15, we know that for all $1 \leq l \leq d$,

$$\lim_{n\to\infty}\frac{1}{h_{\ell,n}}\sum_{j=1}^{d}(Q_n \cdots Q_m)(l,j)\mathbf{x}_j = \sum_{j=1}^{d}\mu(B_{j,m})\mathbf{x}_j > 0.$$

Since $h_{\ell,n} \geq 0$ for all $1 \leq l \leq d$, the last inequality implies that there exists $N \geq m$ such that for all $n \geq N$, and for all $1 \leq l \leq d$,

$$\sum_{j=1}^{d}(Q_n \cdots Q_m)(l,j)\mathbf{x}_j \geq 0,$$

that is, for all $n \geq N$, $(Q_n \cdots Q_m)(\mathbf{x}) \in \mathbb{Z}_+^d$. $\square$

From the previous proposition we know that if conditions **(C0)**, **(C1)** and **(C3)** are satisfied, the group $H(X,T)$ is a quotient of the additive group $\mathbb{Z}^d$. If moreover $\pi_{\mathfrak{S}}$ is injective, the inductive limit $(G(\mathfrak{S}), G^+(\mathfrak{S}), 1(\mathfrak{S}))$ is isomorphic to the dimension group $K^0(X,T)$. We deduce the following result.

Proposition 2.17. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of tower partitions of (X,T) satisfying* **(C0)**-**(C3)**. *Let $\vec{\mu}$ and $\mathbf{u}$ be as defined in Proposition 2.16. Let*

$$\tilde{A} = \{\mathbf{x} \in \mathbb{Z}^d : \langle \mathbf{x}, \vec{\mu_m} \rangle > 0 \quad \forall \mu \in \mathcal{M}(X,T)\} \cup \{0\}.$$

Then, $(H(X,T), H^+(X,T), 1_X)$ and $(\mathbb{Z}^d, \tilde{A}, \mathbf{u})$ are isomorphic as ordered groups with unit.

Proof. Let φ the group isomorphism from $G(\mathfrak{S})$ to $\mathbb{Z}^d$ defined in the proof of Proposition 2.16. We already know that $\varphi(1(\mathfrak{S})) = \mathbf{u}$. The morphism $\pi_{\mathfrak{S}} \circ \varphi^{-1}$ is then a group isomorphism satisfying $\pi_{\mathfrak{S}} \circ \varphi^{-1}(\mathbf{u}) = \pi_{\mathfrak{S}}(1(\mathfrak{S})) = [1_X]$. Note that these properties do not need the system (X,T) to be uniquely ergodic.

We now show that $\pi_{\mathfrak{S}} \circ \varphi^{-1}(\tilde{A}) \subseteq H^+(X,T)$.

Recall from section 1.5.2 that traces completely determine the positive cone of a dimension group. Since the traces of $K^0(X,T)$ corresponds to invariant measures of (X,T), the positive cone $H^+(X,T)$ is characterized as follows,

$$H^+(X,T) = \left\{ [f] \in H(X,T) : \int f d\mu > 0 \forall \mu \in \mathcal{M}(X,T) \right\} \cup \{0_{H(X,T)}\}$$

Let $\mathbf{x} \in A$, then $\varphi^{-1}(\mathbf{x})$ is the class modulo Δ^0 of the sequence

$$y_n = \begin{cases} 0 & \text{if } n < m \\ \mathbf{x} & \text{if } n = m \\ Q_n \cdots Q_m \mathbf{x} & \text{if } n > m \end{cases}$$

This corresponds to the image under $i_{\mathcal{P}_m}$ of the function $f_{\mathbf{x}} \in G(\mathcal{P}_m)$ given by $f_{\mathbf{x}}(x) = \mathbf{x}_i$ if $x \in B_{i,m}$. Therefore,

$$\pi_{\mathfrak{S}} \circ \varphi^{-1}(\mathbf{x}) = \pi_{\mathfrak{S}} \circ i_{\mathcal{P}_m}(f_{\mathbf{x}}) = \pi_{\mathfrak{S}} \circ i_{\mathcal{P}_m} \circ I_{\mathcal{P}_m}(g_{\mathbf{x}}) = \pi(g_{\mathbf{x}}),$$

where $g_{\mathbf{x}} \in C(\mathcal{P}_m)$ is given by $g_{\mathbf{x}}(x) = \mathbf{x}_i$ if $x \in B_{i,m}$ and $g_{\mathbf{x}}(x) = 0$ else. If $\mathbf{x} = 0$, then $\mathbf{x}_i = 0$ for all $1 \leq i \leq d$, so that $g_{\mathbf{x}} = 0$ and then $\pi(g_{\mathbf{x}}) = 0 \in H^+(X,T)$.

If $\mathbf{x} \neq \mathbf{0}$, let $\mu \in \mathcal{M}(X,T)$. Since $\mathbf{x} \in A$, $\sum_{i=1}^{d} \mathbf{x}_i \mu(B_{i,m}) > 0$. Note that

$$\int g_{\mathbf{x}} d\mu = \sum_{i=1}^{d} \mathbf{x}_i \mu(B_{i,m}),$$

so we conclude that $\int g_{\mathbf{x}} d\mu > 0$ for all $\mu \in \mathcal{M}(X,T)$, which implies that $[g_{\mathbf{x}}] = \pi(g_{\mathbf{x}}) \in H^+(X,T)$. This proves that $\pi_{\mathfrak{S}} \circ \varphi^{-1}(A) \subseteq H^+(X,T)$.

Finally, we show that $\varphi \circ \pi_{\mathfrak{S}}^{-1}(H^+(X,T)) \subseteq \tilde{A}$. Let $f \in C(X,\mathbb{Z})$ be a function such that $[f] \in H^+(X,T)$. Then either f is a coboundary or $\int f d\mu$ is strictly positive for every $\mu \in \mathcal{M}(X,T)$. If f is a coboundary, then $\pi_{\mathfrak{S}}^{-1}(f) = (0,0,\cdots) \mod \Delta^0$ and then $\varphi \circ \pi_{\mathfrak{S}}^{-1}(f) = 0 \in \tilde{A}$. If f is not a coboundary, let $n \geq m$ be a positive integer such that f is cohomologous to a function $g \in C(\mathcal{P}_n)$.

We know that

$$\pi_{\mathfrak{S}}(i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(g)) = \pi(h) = \pi(f) = [f],$$

that is, $\pi_{\mathfrak{S}}^{-1}(\pi(f)) = i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(g)$. On the other hand, $I_{\mathcal{P}_n}(h)$ is the function with value

$$\sum_{k=0}^{h_{i,n}-1} g \mid_{T^k B_{i,n}}$$

in the base $B_{i,n}$, for $1 \leq i \leq d_n$. Therefore, $i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(g)$ is the class modulo Δ^0 of the sequence

$$(0,\cdots,0, \underbrace{\mathbf{g}}_{\boxed{\text{n-th}}}, \underbrace{Q_n\mathbf{g}}_{\boxed{\text{n+1-th}}}, Q_{n+1}Q_n\mathbf{g}, \cdots)$$

where $\mathbf{g} = (I_{\mathcal{P}_n}(g)|_{B_{1,n}}, \cdots, I_{\mathcal{P}_n}(h)|_{B_{d_n,n}})$. Then, $\varphi(i_{\mathcal{P}_n} \circ I_{\mathcal{P}_n}(g))$ is the vector

$$(Q_n \cdots Q_m)^{-1}(\mathbf{g}).$$

Let μ be any measure in $\mathcal{M}(X,T)$

$$\begin{aligned}\left\langle \varphi\circ\pi_{\mathfrak{G}}^{-1}(\pi(f)),\vec{\mu_m}\right\rangle &= \sum_{i=1}^{d}\varphi\circ\pi_{\mathfrak{G}}^{-1}(\pi(f))(i)\mu(B_{i,m})\\ &= \sum_{i=1}^{d}\mu(B_{i,m})\sum_{j=1}^{d}(Q_n\cdots Q_m)^{-1}(i,j)\sum_{k=0}^{h_{j,n}-1} g\mid_{T^kB_{j,n}}\\ &= \sum_{j=1}^{d}\left(\sum_{i=1}^{d}\mu(B_{i,m})(Q_n\cdots Q_m)^{-1}(i,j)\right)\sum_{k=0}^{h_{j,n}-1} g\mid_{T^kB_{j,n}}.\end{aligned}$$

Finally, applying Lemma 2.9, we obtain

$$\begin{aligned}0<\int gd\mu &= \sum_{j=1}^{d}\mu(B_{j,n})\sum_{k=0}^{h_{j,n}-1} g\mid_{T^kB_{j,n}}\\ &= \sum_{j=1}^{d}\left(\sum_{i=1}^{d}\mu(B_{i,m})(Q_n\cdots Q_m)^{-1}(i,j)\right)\sum_{k=0}^{h_{j,n}-1} g\mid_{T^kB_{j,n}}\\ &= \left\langle \varphi\circ\pi_{\mathfrak{G}}^{-1}(\pi(f)),\vec{\mu_m}\right\rangle\end{aligned}$$

and we conclude that $\varphi\circ\pi_{\mathfrak{G}}^{-1}([f])\in\tilde{A}$. This proves the second inclusion and conclude the proof of the theorem. □

Remark 2.18. *All previous results about image subgroup, infinitesimals and dimension group are based on conditions* **(C0)**-**(C3)**, *which are properties of tower partitions. We construct some appropriate tower partitions in Chapters 3 and 4 to apply those results. See for example Proposition 3.31, Corollary 3.32, Theorem 3.39, Theorems 4.14, 4.15, 4.16 and 4.17, Theorems 4.24 and 4.25.*

Chapter 3

Substitutive and S-adic systems

In this chapter we apply the results of Chapter 2 to substitution and S-adic subshifts. All the definitions and many of the results related to substitution systems are well known. S-adic systems, which roughly speaking are a generalization of substitutive ones, obtained by an infinite composition of different substitutions, are a more recent subject of study and thus less understood; we present them in detail and we treat substitutive systems as a particular case of S-adic systems (see Example 3.4 below). We study the behavior of the image subgroup, infinitesimals, dynamical dimension group and balance for this kind of subshifts.

3.1 Definitions and examples.

Let $\mathcal{A}$ be a finite alphabet with $\mathrm{Card}(\mathcal{A}) \geq 2$. Recall from Example 1.4 that, given a substitution σ on $\mathcal{A}$, we can consider the substitutive subshift associated to σ, (X_σ, T), with

$$X_\sigma = \{x \in \mathcal{A}^{\mathbb{Z}} : \forall w, w \prec x \Rightarrow \exists a \in \mathcal{A}, \exists n \in \mathbb{N} : w \prec \sigma^n(a)\},$$

and that if σ is primitive, (X_σ, T) is minimal and uniquely ergodic. The *language* of σ is the language of X_σ.

There is an equivalent definition of substitutive subshifts using fixed or periodic points. Given a substitution σ on $\mathcal{A}$, a *fixed point* of σ is an element $x \in \mathcal{A}^{\mathbb{Z}}$ satisfying $\sigma(x) = x$. A *periodic point* of σ is an element $x \in \mathcal{A}^{\mathbb{Z}}$ such that there exists $k > 0$ with $\sigma^k(x) = x$.

Note that if there exist two letters $a, b \in \mathcal{A}$ with $|\sigma(a)|, |\sigma(b)| \geq 2$ and such that $\sigma(a)$ begins with a, $\sigma(b)$ ends with b, there exists a unique fixed point $x \in \mathcal{A}^{\mathbb{Z}}$ of σ satisfying $x_{-1} = b$ and $x_0 = a$.

If $ba \in \mathcal{L}_\sigma$, then x is called an *admissible fixed point* of σ and it is not difficult to check that X_σ corresponds exactly to the orbit closure of x under the shift.

If σ is primitive, then there exists a positive integer p such that σ^p has an admissible fixed point, denoted by y. In this case, X_{σ^p} corresponds to the orbit closure of y under the shift. Since σ and σ^p define the same language, $X_{\sigma^p} = X_\sigma$, and then X_σ corresponds to the orbit closure of y under the shift.

We say that a substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$ is *aperiodic* if (X_σ, T) is aperiodic for the shift map, that is,(X_σ, T) is a free subshift. In general, when we say *a periodic/aperiodic point on* X we refer to a periodic/aperiodic point for the shift map.

Let $\mathcal{A}, \mathcal{B}$ be two finite alphabets. When we refer to a morphism $\sigma : \mathcal{A} \to \mathcal{B}^*$ we always assume that σ is non-erasing.

We say that a morphism $\sigma : \mathcal{A} \to \mathcal{B}^*$ is *left proper* (resp. *right proper*) if there exists $b \in \mathcal{B}$ such that for all $a \in \mathcal{A}$, b is a prefix (resp. a sufix) of $\sigma(a)$. We say that σ is *proper* if it is both left and right proper. We will need the following lemma in Section 3.2.

Lemma 3.1. *Let $\mathcal{A}$ be a finite alphabet. If $\sigma : \mathcal{A} \to \mathcal{A}^*$ is a left proper substitution and $\theta : \mathcal{A} \to \mathcal{A}^*$ is any substitution, then the composition $\sigma\theta : \mathcal{A} \to \mathcal{A}^*$ is left proper. If σ is left proper and θ is right proper, then the composition $\sigma\theta$ is proper.*

Proof. Suppose that $\sigma : \mathcal{A} \to \mathcal{A}^*$ is a left proper substitution and $\theta : \mathcal{A} \to \mathcal{A}^*$ is any substitution. Let $\ell \in \mathcal{A}$, $u : \mathcal{A} \to \mathcal{A}^*$ such that $\sigma(a) = \ell u(a)$ for all $a \in \mathcal{A}$. Then, for all $a \in \mathcal{A}$ one has

$$\sigma\theta(a) = \sigma(\theta(a)) = \sigma(\theta(a)_0) \cdots \sigma(\theta(a)_{|\theta(a)|-1}) = \ell u(\theta(a)_0) \cdots \ell u(\theta(a)_{|\theta(a)|-1}).$$

Thus, for all $a \in \mathcal{A}$, ℓ is a prefix of $\sigma\theta(a)$ and $\sigma\theta$ is left proper.

Suppose now that σ is left proper and θ is right proper. Let $\ell, r \in \mathcal{A}$, $u, w : \mathcal{A} \to \mathcal{A}^*$ be such that for all $a \in \mathcal{A}$, $\sigma(a) = \ell u(a)$, $\theta(a) = w(a)r$. Then, for all $a \in \mathcal{A}$ one has

$$\sigma\theta(a) = \sigma(w(a))\sigma(r) = \sigma(w(a))\ell u(r) = \ell u(w(a)_0) \cdots \ell u(w(a)_{|w(a)|-1})\ell u(r)_0 \cdots u(r)_{|u(r)|-1}.$$

Thus, for all $a \in \mathcal{A}$, ℓ is a prefix of $\sigma\theta(a)$, $u(r)_{|u(r)|-1}$ is a suffix of $\sigma\theta(a)$ and $\sigma\theta$ is proper. ☐

If $\sigma : \mathcal{A} \to \mathcal{B}^*$ is a proper morphism with $\sigma(a) = \ell u(a)$ for all $a \in \mathcal{A}$, we define its *right conjugate* $\tau : \mathcal{A} \to \mathcal{B}^*$ by

$$\tau(a) = u(a)\ell.$$

Note that the right conjugate of a left proper morphism is right proper. Given a right proper morphism, we define analogously its *left conjugate*, which is left proper.

3.1.1 Recognizability

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift on $\mathcal{A}$, we say that a substitution σ is *recognizable* in X if for all $y \in \mathcal{B}^{\mathbb{Z}}$ there exists at most one pair $(k, x) \in \mathbb{N} \times X$ such that $y = T^k\sigma(x)$ and $0 \leq k < |\sigma(x_0)|$. Such a pair is called a *centered σ-representation* of y. We say that σ is *recognizable* in X for aperiodic points if for all aperiodic $y \in \mathcal{B}^{\mathbb{Z}}$ there exists at most one centered σ-representation of y. We now present the classical definition of recognizability, which is combinatorial and was first introduced in [Mo92] and [Mo96].

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subshift on $\mathcal{A}$ and $y = T^k\sigma(x)$ for some $x \in X$, $k \in \mathbb{Z}$. The set $C_\sigma(k, x)$ of *cutting points* of y is defined as follows,

$$C_\sigma(k, x) = \{|\sigma(x_{[0,\ell)})| + k : \ell > 0\} \cup \{0\} \cup \{-|\sigma(x_{[\ell,0)})| - k : \ell < 0\}.$$

Given a morphism $\sigma : \mathcal{A} \to \mathcal{B}^*$ and a point $x \in \mathcal{A}^{\mathbb{Z}}$, we say that σ is *recognizable in the sense of Mossé* for x if there exist $\ell \in \mathbb{N}$ such that for all $m \in C_\sigma(0, x)$, for all $m' \in \mathbb{Z}$,

$$\sigma(x)_{[m-\ell,m+\ell)} = \sigma(x)_{[m'-\ell,m'+\ell)} \Rightarrow m' \in C_\sigma(0, x).$$

The constant ℓ aboved is called *a constant of recognizability* for σ. The *constant of recognizability* of σ is the smallest one among all constants of recognizability for σ. This constant is computable when σ is a primitive substitution (see [DL17]). In [BSTY18, Section 2] there is a complete analysis regarding the relation between this two notions of recognizability. We present here an important one.

Theorem 3.2. *[BSTY18, Theorem 2.5] Let $\sigma : \mathcal{A} \to \mathcal{B}^+$ a morphism, $x \in \mathcal{A}^{\mathbb{Z}}$ and let (X, T) be the subshift generated by x. Then the following holds*

- *If σ is recognizable in X, then σ is recognizable in the sense of Mossé for x.*
- *If (X, T) is minimal, σ is injective on the letters and σ is recognizable in the sense of Mossé for x, then σ is recognizable in X.*

It is a theorem by B. Mossé that primitive aperiodic substitutions are recognizable for their fixed points ([Mo92], see also [Que10, Theorem 5.8]). As a consequence of this result and Theorem 3.2, if $\sigma : \mathcal{A} \to \mathcal{A}$ is primitive and aperiodic, σ is recognizable in X_σ, since for any fixed point x of σ, $X_x = X_\sigma$.

3.1.2 Directive sequences

Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of finite alphabets and $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$ morphisms. We denote $\sigma_{[n,N)}$ the composition $\sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}$. Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a sequence of morphisms. We say that $\boldsymbol{\sigma}$ is *everywhere growing* if $\min_{a\in\mathcal{A}_n}\{|\sigma_{[0,n)}(a)|\}$ tends to ∞ as $n \to \infty$, that is, if the length $|\sigma_{[0,n)}(a)|$ tends to ∞ when $n \to \infty$ for all $a \in \mathcal{A}_n$. We say that $\boldsymbol{\sigma}$ is *primitive* if for every $n \geq 0$, there exists $N \geq n$ such that $\sigma_{[n,N)}$ has a positive incidence matrix.
For $n \geq 0$, the *language of order* n $L^{(n)}_{\boldsymbol{\sigma}}$ associated with $\boldsymbol{\sigma}$ is

$$L^{(n)}_{\boldsymbol{\sigma}} = \{w \in \mathcal{A}_n^* : \exists N > n, \exists a \in \mathcal{A}_N, w \prec \sigma_{[n,N)}(a)\}.$$

For each $n \geq 0$, the set $X^{(n)}_{\boldsymbol{\sigma}}$ is the set of infinite words $x \in \mathcal{A}_n^{\mathbb{Z}}$ all whose factors belong to $L^{(n)}_{\boldsymbol{\sigma}}$. We set $X_{\boldsymbol{\sigma}} = X^{(0)}_{\boldsymbol{\sigma}}$, $L_{\boldsymbol{\sigma}} = L^{(0)}_{\boldsymbol{\sigma}}$ and call $(X_{\boldsymbol{\sigma}}, T)$ the *S-adic system generated by the directive sequence* $\boldsymbol{\sigma}$, where T is the shift transformation. For all $\ell \geq 1$, we denote by $L^{(n)}_{\boldsymbol{\sigma},\ell}$ the subset of length ℓ factors of $L^{(n)}_{\boldsymbol{\sigma}}$.

Remark 3.3. *Note that $\mathcal{L}_\ell(X^{(n)}_{\boldsymbol{\sigma}}) \subseteq L^{(n)}_{\boldsymbol{\sigma},\ell}$ and that this inclusion is strict in general: consider for instance a substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$ such that a letter $a \in \mathcal{A}$ only appears as a prefix of $\sigma(a)$. This is a non-primitive substitution where a has an occurrence in $\sigma(a)$ but it appears in no infinite word $x \in \mathcal{A}^{\mathbb{Z}}$. If there were $x \in \mathcal{A}^{\mathbb{Z}}$ such that $a \prec x$, then it would be a letter $b \in \mathcal{A}$ such that $ba \prec \sigma^n(c)$, for some $n \in \mathbb{N}$, some $c \in \mathcal{A}$, which is not possible since a only appears as a prefix of $\sigma(a)$. This is not the case in the minimal framework: if $\boldsymbol{\sigma}$ is everywhere growing and primitive, then $\mathcal{L}_\ell(X^{(n)}_{\boldsymbol{\sigma}}) = L^{(n)}_{\boldsymbol{\sigma},\ell}$ for all $\ell \geq 1$.*

Example 3.4. Substitutive subshifts

Substitutive subshifts are exactly the S-adic systems where $\mathcal{A}_n$ is equal to a constant alphabet $\mathcal{A}$ for all $n \in \mathbb{N}$ and σ_n is the same substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Therefore, properties of substitutive subshifts can be directly recovered from more general properties o S-adic subshifts. However, substitutive systems were historically studied earlier and some results in the substitutive case have been obtained using very specific properties which is not clear how to extend to the S-adic case. Note that when the S-adic system is obtain from one single substitution $\sigma : \mathcal{A} \to \mathcal{A}$, the directive sequence $\boldsymbol{\sigma}$ is primitive if and only if σ is primitive as a substitution.

Example 3.5. Uniformly recurrent subshifts

Return words provide a way to represent minimal subshifts as S-adic systems. We quote here the procedure given in [BDD+18, Section 3] to get this representation. Let (X,T) a minimal subshift defined on the alphabet $\mathcal{A}$ and let x be any element belonging to X. Recall from Section 1.2.1 that for every $a \in \mathcal{A}$, $w \in \mathcal{L}_X$ is a first left return word to a if a is a prefix of wa and there are exactly two occurrences of a in wa. Let $\mathcal{R}'(x_0)$ be the (finite) set of first left return words to the first letter of x, x_0, and consider the factorization of x in words belonging to $\mathcal{R}'(x_0)$. There exists a unique sequence $(w_k)_{k\in\mathbb{Z}} \in \mathcal{R}'(x_0)^{\mathbb{Z}}$ such that $x = \cdots w_{-2}w_{-1}w_0w_1w_2\cdots$. Now consider R the alphabet $\{1, 2, \cdots, |\mathcal{R}'(x_0)|\}$ and let $\lambda : R \to \mathcal{A}^*$ be the morphism which maps every $i \in R$ to the ith first left return word $w \in \mathcal{R}'(x_0)$ to appear in $x_{[0,\infty)}$, and which extends to R^* and $R^{\mathbb{Z}}$ by concatenation. The *derived sequence* of x is the unique sequence $\mathcal{D}(x) \in R^{\mathbb{Z}}$ such that $\lambda(\mathcal{D}(x)) = x$. The morphism λ is called the *return morphism* in [BDD+18]. Define $\mathcal{D}^0(x) = x$, $R_0 = \mathcal{A}$, $R_1 = R$, $\lambda_1 = \lambda$, and then define inductively $\mathcal{D}^n(x)$ and λ_n as follows. Given $\mathcal{D}^n(x)$, R_n, R_{n+1} and $\lambda_n : R_{n+1} \to R_n^*$, $\mathcal{D}^{n+1}(x)$ is the unique sequence in $R_{n+1}^{\mathbb{Z}}$ such that $\lambda_n(\mathcal{D}^{n+1}(x)) = \mathcal{D}^n(x)$, then $R_{n+2} = \{1, 2, \cdots, |\mathcal{R}'(\mathcal{D}^{n+1}(x)_0)|\}$ and finally $\lambda_{n+1} : R_{n+1} \to R_{n+1}^*$ the morphism which maps every $i \in R_{n+2}$ to the ith first left return word $w \in \mathcal{R}'(\mathcal{D}^{n+1}(x)_0)$ to appear in $\mathcal{D}^{n+1}(x)_{[0,\infty)}$, and which extends to R_{n+1}^* and $R_{n+1}^{\mathbb{Z}}$ by concatennation. It is not difficult to verify that the sequence of morphisms $\boldsymbol{\lambda} = (\lambda_n : R_{n+1} \to R_n^*)_{n\in\mathbb{N}}$ is a primitive directive sequence which satisfies $X_{\boldsymbol{\lambda}} = X$.

Example 3.6. Sturmian subshifts

Sturmian subshifts (see Example 1.5) can be obtained using S-adic representations. Moreover, they can be obtained by using a directive sequence where the σ_n's belong to a finite set of morphisms. Consider the morphisms $\tau_0, \tau_1 : \{0,1\} \to \{0,1\}^*$ given by

$$\tau_0 = \begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 10 \end{array} \qquad \tau_1 = \begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 1 \end{array}.$$

It is known that, given any sequence $(i_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, the limit word

$$x = \lim_{n\to\infty} \tau_0^{i_1}\tau_1^{i_2}\tau_0^{i_3}\tau_1^{i_4}\cdots\tau^{i_{n-1}}(0)$$

exists, and that if $(i_n)_{n\in\mathbb{N}}$ is not ultimately constant, then the directive sequence $(\tau_{i_n})_{n\in\mathbb{N}}$ is primitive and the subshift $X_{\boldsymbol{\tau}}$ is Sturmian. Moreover, if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, the Sturmian subshift associated to the rotation of angle α is exactly the subshift generated by the limit word

$$x = \lim_{n\to\infty} \tau_0^{j_1}\tau_1^{j_2}\tau_0^{j_3}\tau_1^{j_4}\cdots\tau_0^{j_{2n-1}}\tau_1^{j_{2n}}(0),$$

where $[0; j_1+1, j_2, j_3, \cdots]$ is the continued fraction expansion of α (see [AR91, Section 1] for details).

S-adic systems are known to be minimal provided their directive sequences of morphisms are primitive (see for example [BD14, Lemma 5.2]).
We say that the directive sequence $\boldsymbol{\sigma}$ is *recognizable* if for all $n \geq 0$, σ_n is recognizable in $X_{\boldsymbol{\sigma}}^{(n+1)}$. We say that $\boldsymbol{\sigma}$ is *eventually recognizable* if there exists $n \in \mathbb{N}$ such that for all $n \geq N$ σ_n is recognizable in $X_{\boldsymbol{\sigma}}^{(n+1)}$. There exist directive sequences which are eventually recognizable but not recognizable, and directive sequences which are not even eventually recognizable. Moreover, in both cases the directive sequences can be choosen primitive (see [BSTY18, Section 4] for examples). This shows that Mossé's Theorem ([Mo92]) cannot be extended in a natural way from the substitutive to the S-adic framework. We now list some results from [BSTY18, Section 4] which provide sufficient conditions to recognizability and eventual recognizability for sequences of morphisms.
Recall that a morphism $\sigma : \mathcal{A} \to \mathcal{B}^*$ is *left (right) permutative* if for all $a \neq b$ in $\mathcal{A}$, the first (last) letters of $\sigma(a)$ and $\sigma(b)$ are different. Two morphisms $\sigma, \tau : \mathcal{A} \to \mathcal{B}^*$ are said to be *rotationally conjugate* if there exists $w \in \mathcal{B}^*$ such that $\sigma(a)w = w\tau(a)$ for all $a \in \mathcal{A}$, or $w\sigma(a) = \tau(a)w$ for all $a \in \mathcal{A}$.

Theorem 3.7. *[BSTY18, Theorem 4.6] Let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a sequence of morphisms with $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$. For all $n \in \mathbb{N}$, let M_{σ_n} denote the incidence matrix of σ_n. If*

- $\mathrm{rk}(M_{\sigma_n}) = \#\mathcal{A}_{n+1}$ *for all $n \in \mathbb{N}$, or*
- $\#\mathcal{A}_{n+1} = 2$ *for all $n \in \mathbb{N}$, or*
- σ_n *is rotationally conjugate to a left or right permutative morphism,*

then $\boldsymbol{\sigma}$ is recognizable for aperiodic points.

Let $d \geq 2$ be an integer and $\Omega \subseteq \mathbb{R}_+^d$. Let $(\Omega_i)_{i\in I}$ be a finite or countable partition of Ω. Let $(M_i)_{i\in I}$ a family of matrices such that $M_i\Omega_i \subseteq \Omega_i$. The *$d$-dimensional continued fraction map* associated to $(M_i)_{i\in I}$ is the map $F : \Omega \to \Omega$ defined by $F(x) = M_i^{-1}(x)$ if $x \in \Omega_i$. We define $M(x) = M_i$ if $x \in \Omega_i$.

The associated *continued fraction algorithm* is the iterative application of F to a given vector $x \in \Omega$, which produces the sequence $(M(F^n(x)))_{n\in\mathbb{N}}$, called the *continued fraction expansion* of x.
Given a continued fraction expansion $(M(F^n(x)))_{n\in\mathbb{N}}$, we can associate each matrix $M(F^n(x))$ to a substitution σ_n (in a non-canonical way, since the same incidence matrix could correspond to different substitutions). Then, to a continued fraction algorithm we associate a directive sequence of substitutions and then an S-adic subshift.
Recall that a matrix M with integer coefficients is said to be *unimodular* if $\det(M) = \pm 1$.

Theorem 3.8. *[BSTY18, Proposition 4.9] Let $\boldsymbol{\sigma}$ be a directive sequence obtained from a unimodular continued fraction expansion algorithm. Then $\boldsymbol{\sigma}$ is recognizable.*

Theorem 3.9. *[BSTY18, Theorem 5.2] Let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a sequence of morphisms with $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$ such that $\liminf_{n\to\infty} \#\mathcal{A}_n < \infty$. Then, $\boldsymbol{\sigma}$ is eventually recognizable for aperiodic points.*

Through this chapter we will use the following equality

$$\sigma_{[0,n)} T^j(x) = T^{|\sigma_{[0,n)}(x_{[0,j)})|}\sigma_{[0,n)}(x) \quad \forall x \in X, \forall j \geq 1, \tag{3.1}$$

which is a consequence of the fact that

$$\sigma_{[0,n)} T^j(x) = \cdots \sigma_{[0,n)}(x_{j-1}) \cdot \sigma_{[0,n)}(x_j)\sigma_{[0,n)}(x_{j+1}) \cdots .$$

We assume henceforth that $(X_{\boldsymbol{\sigma}}, T)$ is minimal. For $w \in L_\ell(X_{\boldsymbol{\sigma}}^{(n)})$, the cylinder $[w]_n$ corresponds to the following subset of $X_{\boldsymbol{\sigma}}^{(n)}$

$$[w]_n = \{x \in X_{\boldsymbol{\sigma}}^{(n)} : x_0 \cdots x_{|w|-1} = w\}.$$

When the S-adic system is obtained from one single substitution $\sigma : \mathcal{A} \to \mathcal{A}$, we omit the subindex n in the above notation, since $X_{\boldsymbol{\sigma}}^{(n)} = X_\sigma$ for all $n \in \mathbb{N}$.

In the next section we use the condition of recognizability to construct appropriate sequences of tower partitions for S-adic systems.

3.2 Tower partitions for S-adic systems.

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a recognizable sequence of morphisms with $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$. For all $\ell \geq 1$, define the sequence of partitions $\mathcal{P}_n^{(\ell)}$ of $X_{\boldsymbol{\sigma}}$ as follows

$$\mathcal{P}_n^{(\ell)} = \{T^j \sigma_{[0,n)}([a_0 \cdots a_{\ell-1}]_n) : a_0 a_1 \cdots a_{\ell-1} \in L^{(n)}_{\boldsymbol{\sigma},\ell}, 0 \leq j < |\sigma_{[0,n)}(a_0)|\} \quad \forall n \geq 1. \tag{3.2}$$

Define also $\mathcal{P}_0^{(\ell)} = \{[a_0 \cdots a_{\ell-1}]_0 : a_0 \cdots a_{\ell-1} \in L^{(0)}_{\boldsymbol{\sigma},\ell}\}$, that is, $\mathcal{P}_0^{(\ell)}$ is the partition whose towers correspond to the cylinders $[a_0 \cdots a_{\ell-1}]_0$ and have just one floor.

Proposition 3.10. *For all $n \in \mathbb{N}$ and for all $\ell \geq 1$, $\mathcal{P}_n^{(\ell)}$ is a partition in towers of $X_{\boldsymbol{\sigma}}$ with $\mathcal{P}_{n+1}^{(\ell)}$ finer than $\mathcal{P}_n^{(\ell)}$, and $(\mathcal{P}_n^{(\ell)})_{n\in\mathbb{N}}$ satisfies condition* **(C0)** *of Section 2.1.*

Proof. The heigth of the tower with base $\sigma_{[0,n)}([a_0 \cdots a_{\ell-1}])$ is $|\sigma_{[0,n)}(a_0)|$, which tends to infinity with n, since $\boldsymbol{\sigma}$ is everywhere growing. This means that $(\mathcal{P}_n^{(\ell)})_{n\in\mathbb{N}}$ satisfies condition **(C0)**.
Let $x \in X_{\boldsymbol{\sigma}} = X^{(0)}_{\boldsymbol{\sigma}}$ and $n \in \mathbb{N}$. We know from [BSTY18, Lemma 4.2], that x admits at least one centered $\sigma_{[0,n)}$-representation in $X^{(n)}_{\boldsymbol{\sigma}}$. Since $\mathbf{s}$ is recognizable, then this is the only centered $\sigma_{[0,n)}$-representation in $X^{(n)}_{\boldsymbol{\sigma}}$ admitted by x. In other words, there exists a unique $y \in X^{(n)}_{\boldsymbol{\sigma}}$ and a unique $0 \leq k < |\sigma_{[0,n)}(y_0)|$ such that $x = T^k \sigma_{[0,n)}(y)$. Let $a_i = y_i$ for all $0 \leq i < \ell$. By definition, $y \in [a_0 a_1 \cdots a_{\ell-1}]_n$ and, since $y \in X^{(n)}_{\boldsymbol{\sigma}}$, $a_0 \cdots a_{\ell-1} \in L_\ell(X^{(n)}_{\mathbf{s}}) \subseteq L^{(n)}_{\boldsymbol{\sigma},\ell}$. We conclude that there exists $a_0 \cdots a_{\ell-1} \in L^{(n)}_{\boldsymbol{\sigma},\ell}$ and $0 \leq k < |\sigma_{[0,n)}(a_0)|$ such that $x \in T^k \sigma_{[0,n)}([a_0 \cdots a_{\ell-1}]_n)$. This shows that for all $n \geq 1$ $\mathcal{P}_n^{\ell}$ covers $X_{\boldsymbol{\sigma}}$. $\mathcal{P}_0^{(\ell)}$ trivially covers $X_{\boldsymbol{\sigma}}$.
We now prove that $\mathcal{P}_n^{(\ell)}$ is a partition. Suppose that there exist $a_0 \cdots a_{\ell-1}, b_0 \cdots b_{\ell-1} \in L^{(n)}_{\boldsymbol{\sigma},\ell}$, $0 \leq j \leq |\sigma_{[0,n)}(a_0)|$, $0 \leq k \leq |\sigma_{[0,n)}(b_0)|$ such that $x \in T^j \sigma_{[0,n)}([a_0 \cdots a_{\ell-1}]_n) \cap T^k \sigma_{[0,n)}([b_0 \cdots b_{\ell-1}]_n)$. This means that there exist $y_1 \in [a_0 \cdots a_{\ell-1}]_n$ and $y_2 \in [b_0 \cdots b_{\ell-1}]_n$ such that

$$x = T^j \sigma_{[0,n)}(y_1) = T^k \sigma_{[0,n)}(y_2).$$

Since $0 \leq j \leq |\sigma_{[0,n)}(a_0)| = |\sigma_{[0,n)}((y_1)_0)|$, (y_1, j) is a centered $\sigma_{[0,n)}$-representation of x. Since $0 \leq k \leq |\sigma_{[0,n)}(b_0)| = |\sigma_{[0,n)}((y_2)_0)|$, (y_2, k) is a centered $\sigma_{[0,n)}$-representation of x as well. By recognizability, $y_1 = y_2$ and $j = k$, so in fact $T^k \sigma_{[0,n)}([a_0 \cdots a_{\ell-1}]_n) = T^j \sigma_{[0,n)}([b_0 \cdots b_{\ell-1}]_n)$. $\mathcal{P}_0^{(\ell)}$ is trivially a partition.
Finally, let us show that for all n, $\mathcal{P}_{n+1}^{(\ell)}$ is finer that $\mathcal{P}_n^{(\ell)}$. Let $T^k \sigma_{[0,n+1)}([a_0 \cdots a_{\ell-1}]_n)$ be an atom of $\mathcal{P}_{n+1}^{(\ell)}$ and let x belong to it. This implies that $a_0 \cdots a_{\ell-1} \in \mathcal{L}_\ell(X^{(n+1)}_{\boldsymbol{\sigma}})$, $0 \leq k < |\sigma_{[0,n+1)}(a_0)|$ and there exists $y \in [a_0 \cdots a_{\ell-1}]_{n+1}$ such that $x = T^k \sigma_{[0,n+1)}(y)$. Therefore, one has $x = T^k \sigma_{[0,n)}(\sigma_n(y))$. Note

that since $y \in X_{\boldsymbol{\sigma}}^{(n+1)}$, $\sigma_n(y) \in X_{\boldsymbol{\sigma}}^{(n)}$: if $w \prec \sigma_n(y)$, then $w \prec \sigma_n(w')$ for some $w' \prec y$ and there exists $N > n+1$, $a \in \mathcal{A}_N$ such that $w' \prec \sigma_{[n+1,N)}(a)$; we have that $\sigma_n(w') \prec \sigma_n(\sigma_{[n+1,N)}(a)) = \sigma_{[n,N)}(a)$, and therefore $w \prec \sigma_n(w') \prec \sigma_{[n,N)}(a)$; since w was arbitrarily taken, we deduce that $\sigma_n(y) \in X_{\boldsymbol{\sigma}}^{(n)}$. Define $b_i = \sigma_n(y)_i$, for all $0 \leq i < \ell$. Then $b_0 \cdots b_{\ell-1} \in L_\ell(X_{\mathbf{s}}^{(n)})$ and $x \in T^k\sigma_{[0,n)}([b_0 \cdots b_{\ell-1}]_n)$.

Suppose first that $0 \leq k < |\sigma_{[0,n)}((\sigma_n(a_0)_0)|$. Note that $\sigma_n(a_0)_0 = b_0$. In this case, $0 \leq k < |\sigma_{[0,n)}(b_0)|$ and then $T^k\sigma_{[0,n)}([b_0 \cdots b_{\ell-1}]_n)$ is an atom of $\mathcal{P}_n^{(\ell)}$, so we conclude that $T^k\sigma_{[0,n+1)}([a_0 \cdots a_{\ell-1}]_{n+1})$ is included in an atom of $\mathcal{P}_n^{(\ell)}$.

Suppose now that $|\sigma_{[0,n)}((\sigma_n(a_0)_0)| \leq k < |\sigma_{[0,n+1)}(a_0)|$. Then, there exists a unique $1 \leq j < |\sigma_n(a_0)|$ such that

$$|\sigma_{[0,n)}((\sigma_n(a_0)_{[0,j)})| \leq k < |\sigma_{[0,n)}(\sigma_n(a_0)_{[0,j+1)})|.$$

Define $m = |\sigma_{[0,n)}(\sigma_n(y)_{[0,j)})|$. By (3.1), we know that

$$x = T^{k-m}\sigma_{[0,n)}T^j(\sigma_n(y)) \in T^{k-m}\sigma_{[0,n)}([\sigma_n(y)_j \cdots \sigma_n(y)_{j+\ell-1}]_n).$$

Note that $0 \leq k-m < |\sigma_{[0,n)}(\sigma_n(a_0)_j)| = |\sigma_{[0,n)}(\sigma_n(y)_j)|$, since $1 \leq j < |\sigma_n(a_0)|$. The word $\sigma_n(y)_j \cdots \sigma_n(y)_{j+\ell-1}$ belongs to $\mathcal{L}_\ell(X_{\boldsymbol{\sigma}}^{(n)})$, then $T^{k-m}\sigma_{[0,n)}([\sigma_n(y)_j \cdots \sigma_n(y)_{j+\ell-1}]_n)$ is an atom of $\mathcal{P}_\ell^{(n)}$. Thus, $T^k\sigma_{[0,n+1)}([a_0 \cdots a_{\ell-1}]_{n+1})$ is included in an atom of $\mathcal{P}_n^{(\ell)}$. We conclude by noticing that clearly $\mathcal{P}_1^{(\ell)}$ is finer than $\mathcal{P}_0^{(\ell)}$. □

Proposition 3.11. *The sequence $(\mathcal{P}_n^{(\ell)})_{n\in\mathbb{N}}$ of partitions defined in (3.2) satisfies condition* **(C1)** *for all $\ell \geq 2$.*

Proof. Let $f \in C(X_{\boldsymbol{\sigma}}, \mathbb{Z})$. From Lemma 1.7, we know that f is cohomologous to a cylinder function $g \in C(X_{\boldsymbol{\sigma}}, \mathbb{Z})$. Let k be a positive integer such that for all $x \in X_{\boldsymbol{\sigma}}$, $g(x)$ depends only on $x_{[0,k]}$. Take $n \geq 1$ large enough so that

$$|\sigma_{[0,n)}(a)| > k \quad \forall a \in \mathcal{A}_n$$

We can choose such an n because $\boldsymbol{\sigma}$ is everywhere growing. Suppose $x, y \in T^j\sigma_{[0,n)}([a_0a_1 \cdots a_{\ell-1}]_n)$, for some $a_0a_1 \cdots a_{\ell-1} \in L_{\boldsymbol{\sigma},\ell}^{(n)}$, some $0 \leq j < |\sigma_{[0,n)}(a_0)|$. Since $|\sigma_{[0,n)}(a_0)| + |\sigma_{[0,n)}(a_1)| + |\sigma_{[0,n)}(a_{\ell-1})| > \ell k$ and the heigth of the $\sigma_{[0,n)}([a_0 \cdots a_{\ell-1}]_n)$-tower is equal to $|\sigma_{[0,n)}(a_0)|$, x and y coincide on at least their $(\ell-1)k$ first coordinates, and then $g(x) = g(y)$. We conclude that g is constant on the atoms of $\mathcal{P}_n^{\ell}$, and therefore $g \in C(\mathfrak{S})$. □

Remark 3.12. *Note that the sequence (3.2) does not necessarily generate the topology of $X_{\boldsymbol{\sigma}}$. We* *illustrate this with the following example.*

Example 3.13. Thue–Morse substitution.

Consider the Thue-Morse substitution $\sigma : \{0,1\} \to \{0,1\}^+$ given by $0 \mapsto 01, 1 \mapsto 10$, and the directive sequence $\sigma_n = \sigma^n$, with $\mathcal{A}_n = \{0,1\} \quad \forall n \in \mathbb{N}$. The sequence of partitions (3.2) for $\ell = 2$ has in this case the form

$$\mathcal{P}_n^{(2)} = \{T^j \sigma^n([ab]) : ab \in \{00, 01, 10, 11\}, 0 \leq j < 2^n - 1\} \tag{3.3}$$

Consider the map $f : X_{\boldsymbol{\sigma}} \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x_{-1} = 0 \\ 1 & \text{if } x_{-1} = 1 \end{cases}$$

We claim that for any positive integer $n \geq 0$ there are two different points x and y belonging to the same atom of $\mathcal{P}_n^{(2)}$ such that $f(x) \neq f(y)$. Indeed, let $n \geq 0$ and take $x' \in \sigma^n([001])$, $y' \in \sigma^n([101])$. These two points belong to the base of the partition $\mathcal{P}_n^{(2)}$, since $[001] \subseteq [00]$ and $[101] \subseteq [10]$. Therefore, it is not difficult to check that $T^{|\sigma^n(0)|}(x')$ and $T^{|\sigma^n(1)|}(y')$ belong to $\sigma^n([01])$. Let $x = T^{|\sigma^n(0)|}(x')$, $y = T^{|\sigma^n(1)|}(y')$. Finally, note that no matter the parity of n, the last letters in the words $\sigma^n(0)$ and $\sigma^n(1)$ are always different. Therefore, x and y belong to the same atom of $\mathcal{P}_n^{(2)}$ but $f(x) \neq f(y)$.
An alterantive way to see that $\mathcal{P}_n^{(2)}$ does not generate the topology of $X_{\boldsymbol{\sigma}}$ in this case, is to note that there are two different points which belong to the same atom of $\mathcal{P}_n^{(2)}$ for all $n \geq 0$. Let $x \in \{0,1\}^{\mathbb{N}}$ the infinite word having all powers $\sigma^n(01)$ as prefixes; let y and $z \in \{0,1\}^{\mathbb{N}}$ be the infinite words having respectively all powers $\sigma^n(01)$ and $\sigma^n(10)$ as suffixes; then, the bi-infinite words $y \cdot x$ and $z \cdot x$ are different but both belong to $\sigma^n([01])$ for all $n \geq 0$.

The previous one is an example of a sequence of tower partitions satisfying **(C1)** but not **(KR3)**. Note that the sequence does not satisfy **(C2)=(KR1)** either, since the two points $y \cdot x$ and $z \cdot x$ (where the dot indicates the zero position) belong not only to the same atom of $\mathcal{P}_n^{(2)}$, but also to the *base* of $\mathcal{P}_n^{(2)}$, for all $n \geq 0$.
When $\ell = 1$, Proposition 3.11 is no longer true. We ilustrate this fact with the following example.

Example 3.14.

Consider again the Thue-Morse substitution defined in the previous example. In this case, the sequence

$(\mathcal{P}_n^{(1)})$ is given by

$$\mathcal{P}_n^{(1)} = \{T^j\sigma^n([a]) : a \in \{0,1\}, 0 \leq j < 2^n\}. \tag{3.4}$$

The groups $G(\mathfrak{S}^{(1)})$ and $H(X_{\boldsymbol{\sigma}}, T)$ are isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right]$ and $\mathbb{Z}\times\mathbb{Z}\left[\frac{1}{2}\right]$ respectively, where $\mathbb{Z}\left[\frac{1}{2}\right]$ denotes the group of dyadic rationals (see [DHP18] for a computation). Suppose we have a homomorphism $\varphi : \mathbb{Z}\left[\frac{1}{2}\right] \to \mathbb{Z}\times\mathbb{Z}\left[\frac{1}{2}\right]$, and let $\varphi(x) = (\varphi(x)_1, \varphi(x)_2)$. We claim that $\varphi(x)_1 = 0$ for all $x \in \mathbb{Z}\left[\frac{1}{2}\right]$. Let k be a positive integer,

$$2^k\varphi\left(\frac{1}{2^k}\right) = \varphi\left(\frac{1}{2^k}\right) + \cdots + \varphi\left(\frac{1}{2^k}\right) = \varphi\left(2^k\frac{1}{2^k}\right) = \varphi(1)$$

Therefore,

$$2^k\varphi\left(\frac{1}{2^k}\right)_1 = \varphi(1)_1$$

If $\varphi\left(1/2^k\right) \neq 0$, then $|\varphi(1)_1| \geq 2^k$. Since k was arbitrarily taken, we conclude that $|\varphi(1)_1| \geq 2^k$ for all $k \in \mathbb{N}$, which is not possible. So we conclude that $\varphi\left(1/2^k\right)_1 = 0$ for all $k \in \mathbb{N}$, which implies that $\varphi(1)_1 = 0$, which in turn implies that for any $x \in \mathbb{Z}\left[\frac{1}{2}\right]$, $\varphi(x)_1 = 0$.
Thus, φ cannot be surjective.

The previous example shows that the smallest ℓ for which we know $\mathcal{P}_n^{(\ell)}$ satisfies condition **(C1)** is $\ell = 2$. However, the sequence $\mathcal{P}_n^{(1)}$ is simpler to handle and will be useful if we impose some additional hypothesis to the sequence $\boldsymbol{\sigma}$. We present these hypothesis in the following. We start by making a connection between the inidence matrices of σ_n and the morphisms $I_{\mathcal{P}_{n+1},\mathcal{P}_n}$ defined in Chapter 2. Recall from Example 1.4 that the incidence matrix of a substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$ is the $|\mathcal{A}| \times |\mathcal{A}|$ integer matrix whose (i,j) coefficient is the number of occurrences of i in $\sigma(j)$.

Proposition 3.15. *Let $\mathcal{P}_n^{(1)}$ be as defined in* (3.2)*, and let $(Q_n^{(1)})_{n\in\mathbb{N}}$ be the sequence of matrices associated to the homomorphism $I_{\mathcal{P}_{n+1}^{(1)},\mathcal{P}_n^{(1)}}$. Let $(M_n)_{n\in\mathbb{N}}$ be the sequence of incidence matrices of the substitutions σ_n. Then, for all $n \in \mathbb{N}$, $Q_n^{(1)} = M_n^T$.*

Proof. Let $a \in \mathcal{A}_{n+1}$ and $b \in \mathcal{A}_n$. By definition,

$$M_n^T(a,b) = |\sigma(n)_a|_b = \#\{0 \leq j < |\sigma_n(a)| : \sigma_n(a)_j = b\}.$$

One easily checks that $\#\{0 \leq j < |\sigma_n(a)| : \sigma_n(a)_j = b\} = \#\{0 \leq j < |\sigma_n(a)| : T^j\sigma_n([a]_{n+1}) \subseteq [b]_n\}$.

We thus want to show that

$$\#\{0 \leq j < |\sigma_n(a)| : T^j\sigma_n([a]_{n+1}) \subseteq [b]_n\} = \#\{0 \leq k < |\sigma_{[0,n+1)}(a)| : T^k\sigma_{[0,n+1)}([a]_{n+1}) \subseteq \sigma_{[0,n)}([b]_n)\}.$$

Suppose there exists $0 \leq j < |\sigma_n(a)|$ such that $T^j\sigma_n([a]_{n+1}) \subseteq [b]_n$. If $j = 0$, $\sigma_n([a]_{n+1}) \subseteq [b]_n$, which implies that $\sigma_{[0,n+1)}([a]_{n+1}) \subseteq \sigma_{[0,n)}([b]_n)$, and then $T^k\sigma_{[0,n+1)}([a]_{n+1}) \subseteq \sigma_{[0,n)}([b]_n)$ for $k = 0$. If $1 \leq j < |\sigma_n(a)|$, set $k = |\sigma_{[0,n)}(\sigma_n(a)_{[0,j)})|$. One has $0 < k < |\sigma_{[0,n+1)}(a)|$. Now take $x \in [a]_{n+1}$. By (3.1), we have

$$T^k\sigma_{[0,n)}(\sigma_n(x)) = \sigma_{[0,n)}T^j(\sigma_n(x)).$$

By hypothesis, $T^j(\sigma_n(x)) \in [b]_n$, and then $T^k\sigma_{[0,n+1)}(x) \in \sigma_{[0,n)}([b]_n)$. Note also that by definition the k associated with a given j is unique, so we conclude that

$$\#\{0 \leq j < |\sigma_n(a)| : T^j\sigma([a]_{n+1}) \subseteq [b]_n\} \geq |\{0 \leq k < |\sigma_{[0,n+1)}(a)| : T^k\sigma_{[0,n+1)}([a]_{n+1}) \subseteq \sigma_{[0,n)}([b]_n)\}.$$

Conversely, suppose that there exists $0 \leq k < |\sigma_{[0,n+1)}(a)|$ such that $T^k\sigma_{[0,n+1)}([a]_{n+1})$ is included in $\sigma_{[0,n)}([b]_n)$. Let $x \in [a]_{n+1}$ and let $y = T^k\sigma_{[0,n+1)}(x)$.
We first assume $0 \leq k < |\sigma_{[0,n)}(\sigma_n(a)_0)|$. In this case, $(k, \sigma_n(x))$ is a centered $\sigma_{[0,n)}$-representation of y. By hypothesis, there exists $z \in [b]_n$ such that $y = \sigma_{[0,n)}(z)$, so that $(0, z)$ is a centered $\sigma_{[0,n)}$-representation of y as well. By recognizability, $k = 0$ and $\sigma_n(x) = z$, and thus $\sigma_n(x) \in [b]_n$. We conclude that $\sigma_n([a]_{n+1}) \subseteq [b]_n$ and then for $j = 0$ we obtain $T^j\sigma_n([a]_{n+1}) \subseteq [b]_n$.
Now we assume that $|\sigma_{[0,n)}(\sigma_n(a)_0)| \leq k < |\sigma_{[0,n+1)}(a)|$. In this case, there exists a unique $1 \leq j < |\sigma_n(a)|$ such that

$$|\sigma_{[0,n)}(\sigma_n(a)_{[0,j)})| \leq k < |\sigma_{[0,n)}(\sigma_n(a)_{[0,j+1)})|.$$

Let $m = |\sigma_{[0,n)}(\sigma_n(a)_{[0,j)})|$. Using (3.1), one has that

$$T^m\sigma_{[0,n)}(\sigma_n(x)) = \sigma_{[0,n)}T^j(\sigma_n(x)),$$

and thus $y = T^k\sigma_{[0,n)}(\sigma_n(x)) = T^{k-m}\sigma_{[0,n)}T^j(\sigma_n(x))$. On the other hand, $y \in \sigma_{[0,n)}([b]_n)$, and then, there exists $z \in [b]_n$ such that $y = \sigma_{[0,n)}(z)$. One has $0 \leq k - m < |\sigma_{[0,n)}(\sigma_n(a)_j)|$. Again, $(k - m, T^j(\sigma_n(x)))$ and $(0, z)$ are centered $\sigma_{[0,n)}$-representations of y. By recognizability, $k - m = 0$ and $T^j\sigma_n(x) = z \in [b]_n$. We conclude that $T^j\sigma_n([a]_{n+1}) \subseteq [b]_n$. Since the j associated to a given k is unique, we conclude that

$$\#\{0 \leq j < |\sigma_n(a)| : T^j \sigma_n([a]_{n+1}) \subseteq [b]_n\} \leq \#\{0 \leq k < |\sigma_{[0,n+1)}(a)| : T^k \sigma_{[0,n+1)}([a]_{n+1}) \subseteq \sigma_{[0,n)}([b]_n)\}.$$

□

Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n)_{n\in\mathbb{N}}$ be a primitive recognizable everywhere growing sequence of morphisms. For any $n \geq 0$, we consider the finite set $L^{(n)}_{\boldsymbol{\sigma},2}$ as an alphabet, and we define the substitution σ'_n on it in the following way: for every $u = u_0u_1 \in L^{(n+1)}_{\mathbf{s},2}$, $\sigma'_n(u)$ consists of the $|\sigma_n(u_0)|$ first factors of lenght 2 of $\sigma_n(u)$. For example, if $\sigma_n(u_0) = a_0a_1 \cdots a_r \in \mathcal{A}^*_n$ and $\sigma_n(u_1) = b_0b_1 \cdots a_s \in \mathcal{A}^*_n$, then

$$\sigma'_n(u) = \sigma'_n(u_0u_1) = (a_0a_1)(a_1a_2) \cdots (a_{r-1}a_r)(a_rb_0).$$

Note that $\sigma'_n : L^{(n+1)}_{\mathbf{s},2} \to L^{(n)*}_{\mathbf{s},2}$ and $\sigma'_n(L_2(X^{(n+1)}_{\mathbf{s}})) \subseteq L_2(X^{(n)}_{\mathbf{s}})^*$. For any $n < m \in \mathbb{N}$, we write $\sigma'_{[n,m)}$ to refer the composition $\sigma'_n \circ \sigma'_{n+1} \circ \cdots \circ \sigma'_{m-1}$.
For one sinlge substitution, σ' is defined on $L_2(X_\sigma)$ and it is called the *two-block extension* of σ (see [DHP18, Section 9] or [Que10, Section 5.4.1] for more on the *higer-block extensions* of a substitution). We will use it to make the computations of the dynamical dimension group of substitution systems in Section 3.4.1.
The following result is analogous to Proposition 3.15. The proof is almost identical. We include it here because we do not know alternative proofs in the literature.

Proposition 3.16. *Let $\mathcal{P}^{(2)}_n$ be as defined in* (3.2)*, and let $(Q^{(2)}_n)_{n\in\mathbb{N}}$ be the sequence of matrices associated to the homomorphism $I_{\mathcal{P}^{(2)}_{n+1},\mathcal{P}^{(2)}_n}$. Let $(M'_n)_{n\in\mathbb{N}}$ be the sequence of incidence matrices of the substitutions σ'_n. Then, for all $n \in \mathbb{N}$, $Q^{(2)}_n = M'^T_n$.*

Proof. Let $ab \in L_2(X^{(n+1)}_{\mathbf{s}})$ and $cd \in L_2(X^{(n)}_{\mathbf{s}})$. By definition,

$$M^T_n(ab, cd) = \#\{0 \leq j < |\sigma_n(a)| : \sigma'_n(ab)_j = cd\}.$$

Note that $\#\{0 \leq j < |\sigma_n(a)| : \sigma'_n(ab)_j = cd\} = \#\{0 \leq j < |\sigma_n(a)| : T^j\sigma_n([ab]) \subseteq [cd]\}$. We thus want to show that

$$\#\{0 \leq j < |\sigma_n(a)| : T^j\sigma_n([ab]) \subseteq [cd] = \#\{0 \leq k < |\sigma_{[0,n+1)}(a)| : T^k\sigma_{[0,n+1)}([ab]) \subseteq \sigma_{[0,n)}([cd])\}$$

Suppose there exists $0 \leq j < |\sigma_n'(ab)| = |\sigma_n(a)|$ such that $T^j\sigma_n([ab]) \subseteq [cd]$. If $j = 0$, $\sigma_n([ab]) \subseteq [cd]$, which implies that $\sigma_{[0,n+1)}([ab]) \subseteq \sigma_{[0,n)}([cd])$, and then $T^k\sigma_{[0,n+1)}([ab]) \subseteq \sigma_{[0,n)}([cd])$ for $k = 0$. If $1 \leq j < |\sigma_n(a)|$, set $k = |\sigma_{[0,n)}(\sigma_n(a)_{[0,j)})|$. One has $0 < k < |\sigma_{[0,n+1)}(a)|$. Now take $x \in [ab]$. By (3.1), we have $T^k\sigma_{[0,n)}(\sigma_n(x)) = \sigma_{[0,n)}T^j(\sigma_n(x))$. By hypothesis, $T^j(\sigma_n(x)) \in [cd]$, and then $T^k\sigma_{[0,n+1)}(x) \in \sigma_{[0,n)}([cd])$. Note also that by definition the k associated to a given j is unique, so we conclude that

$$\#\{0 \leq j < |\sigma_n(a)| : T^j\sigma_n([ab]) \subseteq [cd]\} \geq \#\{0 \leq k < |\sigma_{[0,n+1)}(a)| : T^k\sigma_{[0,n+1)}([ab]) \subseteq \sigma_{[0,n)}([cd])\}.$$

Conversely, suppose that there exists $0 \leq k < |\sigma_{[0,n+1)}(a)|$ such that $T^k\sigma_{[0,n+1)}([ab])$ is included in $\sigma_{[0,n)}([cd])$. Let $x \in [ab]$ and let $y = T^k\sigma_{[0,n+1)}(x)$.
We first assume $0 \leq k < |\sigma_{[0,n)}(\sigma_n(a)_0)|$. In this case, $(k, \sigma_n(x))$ is a centered $\sigma_{[0,n)}$-representation of y. By hypothesis, there exists $z \in [cd]$ such that $y = \sigma_{[0,n)}(z)$, so that $(0, z)$ is a centered $\sigma_{[0,n)}$-representation of y as well. By recognizability, $k = 0$ and $\sigma_n(x) = z$, and thus $\sigma_n(x) \in [cd]$. We conclude that $\sigma_n([ab]) \subseteq [cd]$ and then for $j = 0$ we obtain $T^j\sigma_n([ab]) \subseteq [cd]$.
Now we assume that $|\sigma_{[0,n)}(\sigma_n(a)_0)| \leq k < |\sigma_{[0,n+1)}(a)|$. In this case, there exists a unique $1 \leq j < |\sigma_n(a)|$ such that

$$|\sigma_{[0,n)}(\sigma_n(a)_{[0,j)})| \leq k < |\sigma_{[0,n)}(\sigma_n(a)_{[0,j+1)})|.$$

Let $m = |\sigma_{[0,n)}(\sigma_n(a)_{[0,j)})|$. Using (3.1), one has that $T^m\sigma_{[0,n)}(\sigma_n(x)) = \sigma_{[0,n)}T^j(\sigma_n(x))$, and thus $y = T^k\sigma_{[0,n)}(\sigma_n(x)) = T^{k-m}\sigma_{[0,n)}T^j(\sigma_n(x))$. On the other hand, $y \in \sigma_{[0,n)}([cd])$, and then, there exists $z \in [cd]$ such that $y = \sigma_{[0,n)}(z)$. One has $0 \leq k - m < |\sigma_{[0,n)}(\sigma_n(a)_j)|$. Again, $(k - m, T^j(\sigma_n(x)))$ and $(0, z)$ are centered $\sigma_{[0,n)}$-representations of y. By recognizability, $k - m = 0$ and $T^j\sigma_n(x) = z \in [cd]$. We conclude that $T^j\sigma_n([ab]) \subseteq [cd]$. Since the j associated to a given k is unique, we conclude that

$$\#\{0 \leq j < |\sigma_n(a)| : T^j\sigma_n([ab]) \subseteq [cd]\} \leq \#\{0 \leq k < |\sigma_{[0,n+1)}(a)| : T^k\sigma_{[0,n+1)}([ab]) \subseteq \sigma_{[0,n)}([cd])\}.$$

□

In the following we present a series of results which state that conditions **(C1)** and/or **(C2)** can be satisfied by $\mathcal{P}_n^{(1)}$ when regarding S-adic systems with an infinite number of left or right proper substitutions. This strategy has been inspired in the ideas presented in [DL12].

Lemma 3.17. *Let $\mathcal{A}$ be a finite alphabet, and let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a recognizable everywhere growing sequence of morphisms such that $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Suppose there exists a subsequence*

$(n_k)_{k\in\mathbb{N}}$ *such that for all* $k \in \mathbb{N}$, σ_{n_k} *is left proper with common prefix* $\ell \in \mathcal{A}$. *Then, the sequence of tower partitions* $\mathcal{P}_n^{(1)}$ *associated to* $\boldsymbol{\sigma}$ *satisfies conditions* **(C1)**.

Proof. By Proposition 1.7, it suffices to prove that for every cylinder function $g \in C(X, \mathbb{Z})$, there exists $n \in \mathbb{N}$ such that g is constant in the atoms of $\mathcal{P}_n^{(1)}$. For simplicity we denote $\mathcal{P}_n$ each partition $\mathcal{P}_n^{(1)}$. Let $g \in C(X, \mathbb{Z})$ be a cylinder function and $m \in \mathbb{N}$ such that g depends on the coordinates $[0, m)$. Let $k \geq 0$ such that $|\sigma_{[0,n_k)}(\ell)| \geq m$. We can always choose such a k since $\boldsymbol{\sigma}$ is everywhere growing. Suppose that $z \in [a]_{n_k+1}$ for some $a \in \mathcal{A}$. Since σ_{n_k} is left proper with prefix ℓ, $\sigma_{[0,n_k+1)}(z)$ begins with the prefix $\sigma_{[0,n_k+1)}(a)\sigma_{[0,n_k)}(\ell)$, and thus every element in the atom $T^j\sigma_{[0,n_k+1)}([a]_{n_k+1})$, $0 \leq j < |\sigma_{[0,n_k+1)}(a)|$, has a common prefix of length at least $|\sigma_{[0,n_k)}(\ell)|$. Since $|\sigma_{[0,n_k)}(\ell)| \geq m$, g is constant in every such an atom. □

Remark 3.18. *Note that the hypothesis of* σ_{n_k} *having a common prefix* ℓ *for all* $k \geq 0$ *can be replaced by the following: for all* $k \geq 0$, σ_{n_k} *is left proper with prefix* ℓ_k. *Indeed, since* $\mathcal{A}$ *is finite and every* ℓ_k *belongs to* $\mathcal{A}$, *this condition implies that some prefix occurs infinitely often, and thus, up to consider a subsequence of* $(n_k)_{k\in\mathbb{N}}$, *we may assume that every* σ_k *has the same prefix.*

Lemma 3.19. *Let* $\mathcal{A}$ *be a finite alphabet, and let* $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ *be a recognizable everywhere growing sequence of morphisms such that* $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ *for all* $n \in \mathbb{N}$. *Suppose there exists a subsequence* $(n_k)_{k\in\mathbb{N}}$ *such that for all* $k \in \mathbb{N}$, $\sigma_{n_{2k}}$ *is left proper and* $\sigma_{n_{2k+1}}$ *is right proper. Then, the sequence of tower partitions* $\mathcal{P}_n^{(1)}$ *associated to* $\boldsymbol{\sigma}$ *satisfies conditions* **(C1)** *and* **(C2)**.

Proof. All substitutions in the subsequence $(\sigma_{n_{2k}})_{k\in\mathbb{N}}$ are left proper, so by Lemma 3.17, the sequence $\mathcal{P}_n^{(1)}$ satisfies condition **(C1)**. Let us prove that $\mathcal{P}_n^{(1)}$ satisfies **(C2)**. For any $k \geq 0$, consider the composition $\sigma_{[n_{2k},n_{2k+1})}\sigma_{n_{2k+1}}$. By Lemma 3.1, $\sigma_{[n_{2k},n_{2k+1})}$ is left proper, and since $\sigma_{n_{2k+1}}$ is right proper, $\sigma_{[n_{2k},n_{2k+1})}\sigma_{n_{2k+1}}$ is proper. This means that for all $k \geq 0$ there exist $\ell_k, r_k \in \mathcal{A}$, $v_k : \mathcal{A} \to \mathcal{A}$ such that for all $a \in \mathcal{A}$,

$$\sigma_{[n_{2k},n_{2k+1})}\sigma_{n_{2k+1}}(a) = \ell_k v_k(a) r_k.$$

Since $\mathcal{A}$ is finite, we can reason as in Remark 3.18 to conclude that, up to take a subsequence, there exists $\ell, r \in \mathcal{A}$ such that for all $k \geq 0$ and for all $a \in \mathcal{A}$,

$$\sigma_{[n_{2k},n_{2k+1})}\sigma_{n_{2k+1}}(a) = \ell v_k(a) r.$$

Suppose $x \in Y = \bigcap_{n\in\mathbb{N}} B_n$. Then the central window of x has the form

$$\cdots \sigma_{[0,n_{2k})}(r) \cdot \sigma_{[0,n_{2k})}(\ell) \cdots$$

Since $|\sigma_{[0,n_{2k})}(\ell)|$ and $|\sigma_{[0,n_{2k})}(r)|$ are arbirtarily large, we conclude that $Y = \{x\}$. □

Lemma 3.20. *Let $\mathcal{A}$ be a finite alphabet, and let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a recognizable everywhere growing sequence of morphisms such that $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$ for all $n \in \mathbb{N}$. For all $n \geq 0$, let $\ell_n \in \mathcal{A}_n^*$ a (possibly empty) word and $u_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$ a substitution verifying $\sigma_n(a) = \ell_n u_n(a)$ for all $a \in \mathcal{A}_{n+1}$. For all $n \in \mathbb{N}$, define $\tau_n(a) = u_n(a)\ell_n$ for all $a \in \mathcal{A}_{n+1}$. Then, the following assertions hold.*

- *For all $n \geq 2$, for all $a \in \mathcal{A}_n$, $\sigma_{[0,n)}(a)$ begins with the word*

$$\sigma_{[0,n-1)}(\ell_{n-1}) \cdots \sigma_{[0,1)}(\ell_1)\ell_0,$$

  and $\tau_{[0,n)}(a)$ ends with the word

$$\ell_0 \tau_{[0,1)}(\ell_1) \cdots \tau_{[0,n-1)}(\ell_{n-1}).$$

- *For all $n \geq 1$, for all $a \in \mathcal{A}_n \cup \{\varepsilon\}$ one has*

$$\sigma_{[0,n)}(a)\sigma_{[0,n-1)}(\ell_{n-1}) \cdots \sigma_{[0,1)}(\ell_1)\ell_0 = \ell_0\tau_{[0,1)}(\ell_1) \cdots \tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(a).$$

Proof. We proceed by induction on n. For the first assertion, let $n = 2$, let $a \in \mathcal{A}_2$. One has

$$\begin{aligned}
\sigma_{[0,2)}(a) &= \sigma_{[0,1)}(\sigma_1(a)) \\
&= \sigma_{[0,1)}(\ell_1 u_1(a)) \\
&= \sigma_{[0,1)}(\ell_1)\sigma_{[0,1)}(u_1(a)) \\
&= \sigma_{[0,1)}(\ell_1)\sigma_0(u_1(a)).
\end{aligned}$$

Since ℓ_0 is a prefix of $\sigma_0(u_1(a))$, then $\sigma_{[0,1)}(\ell_1)\ell_0$ is a prerix of $\sigma_{[0,2)}(a)$.
Let $n > 2$ and suppose that for all $a \in \mathcal{A}_n$, $\sigma_{[0,n)}(a)$ begins with $\sigma_{[0,n-1)}(\ell_{n-1}) \cdots \sigma_{[0,1)}(\ell_1)\ell_0$. Let

$a \in \mathcal{A}_{n+1}$. Then one has

$$\begin{aligned}\sigma_{[0,n+1)}(a) &= \sigma_{[0,n)}(\sigma_n(a)) \\ &= \sigma_{[0,n)}(\sigma_n(a)_0)\sigma_{[0,n)}(\sigma_n(a)_1)\cdots\sigma_{[0,n)}(\sigma_n(a)_{|\sigma_n(a)|-1}) \\ &= \sigma_{[0,n)}(\ell_n)\sigma_{[0,n)}(\sigma_n(a)_1)\cdots\sigma_{[0,n)}(\sigma_n(a)_{|\sigma_n(a)|-1}).\end{aligned}$$

By inductive hypothesis, $\sigma_{[0,n)}(\sigma_n(a)_1)$ begins with $\sigma_{[0,n-1)}(\ell_{n-1})\cdots\sigma_{[0,1)}(\ell_1)\ell_0$, and therefore $\sigma_{[0,n+1)}(a)$ begins with $\sigma_{[0,n)}(\ell_n)\cdots\sigma_{[0,1)}(\ell_1)\ell_0$.

In a completely analogous way, we prove that every $\tau_{[0,n)}(a)$ ends with the word $\ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})$.

Let us prove the second assertion. Let $n = 1$. If $a = \varepsilon$, then

$$\begin{aligned}\sigma_{[0,1)}(\varepsilon)\ell_0 &= \varepsilon\ell_0 \\ &= \ell_0 \\ &= \ell_0\varepsilon \\ &= \ell_0\tau_{[0,1)}(\varepsilon).\end{aligned}$$

If $a \in \mathcal{A}_1$ is not the empty word, one has

$$\begin{aligned}\sigma_{[0,1)}(a)\ell_0 &= \sigma_0(a)\ell_0 \\ &= \ell_0 u_0(a)\ell_0 \\ &= \ell_0\tau_0(a) \\ &= \ell_0\tau_{[0,1)}(a).\end{aligned}$$

Let $n > 1$ and suppose that the assertion is true for all $a \in \mathcal{A}_n \cup \{\varepsilon\}$. If $a = \varepsilon$, then

$$\begin{aligned}\sigma_{[0,n+1)}(\varepsilon)\sigma_{[0,n)}(\ell_n)\cdots\sigma_{[0,1)}(\ell_1)\ell_0 &= \sigma_{[0,n+1)}(\varepsilon)\ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n) \\ &= \varepsilon\ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n) \\ &= \ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n) \\ &= \ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n)\varepsilon \\ &= \ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n)\tau_{[0,n+1)}(\varepsilon).\end{aligned}$$

If $a \neq \varepsilon$, for all $1 \leq k \leq n$ let $L_k = |\sigma_{[1,n+1)}(a)| - 1$. One has

$$
\begin{aligned}
\sigma_{[0,n+1)}(a)\sigma_{[0,n)}(\ell_n)\cdots\sigma_{[0,1)}(\ell_1)\ell_0 &= \sigma_{[0,n+1)}(a)\ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n)\\
&= \sigma_0(\sigma_{[1,n+1)}(a))\ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n)\\
&= \ell_0 u_0(\sigma_{[1,n+1)}(a)_0)\cdots\ell_0 u_0(\sigma_{[1,n+1)}(a)_{L_1})\ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n)}(\ell_n)\\
&= \ell_0\tau_{[0,1)}(\sigma_{[1,n+1)}(a))\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n)}(\ell_n)\\
&= \ell_0\tau_{[0,1)}(\sigma_1(\sigma_{[2,n+1)}(a)))\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n)}(\ell_n)\\
&= \ell_0\tau_{[0,1)}(\ell_1 u_1(\sigma_{[2,n+1)}(a)_0)\cdots\ell_1 u_1(\sigma_{[2,n+1)}(a)_{L_2})\tau_{[0,2)}(\ell_2)\cdots\tau_{[0,n)}(\ell_n)\\
&\vdots\\
&= \ell_0\tau_{[0,1)}(\ell_1)\tau_{[0,2)}(\ell_2)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}\sigma_{[n,n+1)}(a)\tau_{[0,n)}(\ell_n)\\
&= \ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\sigma_n(a))\tau_{[0,n)}(\ell_n)\\
&= \ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n u_n(a))\tau_{[0,n)}(\ell_n)\\
&= \ell_0\tau_{[0,1)}(\ell_1)\cdots\tau_{[0,n-1)}(\ell_{n-1})\tau_{[0,n)}(\ell_n)\tau_{[0,n+1)}(a).
\end{aligned}
$$

□

Proposition 3.21. *Let $\mathcal{A}$ be a finite alphabet, and let $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a recognizable sequence of morphisms such that $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Suppose that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, σ_{n_k} is left proper or right proper. Then, there exists a directive sequence $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_i)_{i\geq 0}$ of substitutions $\tilde{\sigma}_i : \mathcal{A} \to \mathcal{A}^*$, such that $X_{\boldsymbol{\sigma}} = X_{\tilde{\boldsymbol{\sigma}}}$ and the sequence of tower partitions $\mathcal{P}_n^{(1)}$ associated to $\tilde{\boldsymbol{\sigma}}$ satisfies conditions* **(C1)** *and* **(C2)**.

Proof. If there are infinitely many σ_{n_k}'s which are left proper and infinitely many σ_{n_k}'s which are right proper, then we can assume, modulo taking a subsequence, that for all $k \geq 0$, $\sigma_{n_{2k}}$ is left proper and $\sigma_{n_{2k+1}}$ is right proper. We thus define $\tilde{\sigma}_i = \sigma_{n_i}$ and we apply Lemma 3.19 to conclude.
Suppose that all σ_{n_k}'s are left proper. Define the new sequence $(\tilde{\sigma}_i)_{i\in\mathbb{N}}$ as follows,

$$
\tilde{\sigma}_i = \begin{cases} \sigma_{n_i} & \text{if } i \text{ is even} \\ \tau_{n_i} & \text{if } i \text{ is odd.} \end{cases}
$$

where τ_{n_i} is the right conjugate of σ_{n_i}.
The sequence $(\tilde{\sigma}_i)_{i\in\mathbb{N}}$ verifies that $\tilde{\sigma}_i$ is left proper if i is even and right proper if i is odd, so by Lemma 3.19, the sequence of tower partitions $\mathcal{P}_n^{(1)}$ associated to $\tilde{\boldsymbol{\sigma}}$ satisfies conditions **(C1)** and **(C2)**.

Let us show that $X_{\tilde{\boldsymbol{\sigma}}} = X_{\boldsymbol{\sigma}}$, for which is enough to show that $\mathcal{L}_{X_{\tilde{\boldsymbol{\sigma}}}} = \mathcal{L}_{X_{\boldsymbol{\sigma}}}$. First, note that $X_{\boldsymbol{\sigma}} = X_{\boldsymbol{\tau}}$. Indeed, if ℓ_{n_k} and u_{n_k} are as in Lemma 3.20, we know that

- For all $n_k \geq 2$, for all $a \in \mathcal{A}_{n_k}$, $\sigma_{[0,n_k)}(a)$ begins with the word

$$\sigma_{[0,n_k-1)}(\ell_{n_k-1}) \cdots \sigma_{[0,n_1)}(\ell_{n_1})\ell_{n0},$$

  and $\tau_{[0,n_k)}(a)$ ends with the word

$$\ell_{n0}\tau_{[0,n_1)}(\ell_{n_1}) \cdots \tau_{[0,n_k-1)}(\ell_{n_k-1}).$$

- For all $n_k \geq 1$, for all $a \in \mathcal{A}_n \cup \{\varepsilon\}$ one has

$$\sigma_{[0,n_k)}(a)\sigma_{[0,n_k-1)}(\ell_{n_k-1}) \cdots \sigma_{[0,n_1)}(\ell_{n_1})\ell_{n_0} = \ell_{n_0}\tau_{[0,n_1)}(\ell_{n_1}) \cdots \tau_{[0,n_k-1)}(\ell_{n_k-1})\tau_{[0,n_k)}(a).$$

Let v belong to the language of $X_{\boldsymbol{\sigma}}$. There exists some $a, b \in \mathcal{A}$ and $k \geq 0$ such that $ab \in \mathcal{L}(X_{bs}^{(n_k)})$ and $v \prec \sigma_{[0,n_k)}(ab)$, that is, $v \prec \sigma_{[0,n_k)}(a)\sigma_{[0,n_k)}(b)$. By Lemma 3.20, $\sigma_{[0,n_k)}(b)$ begins with

$$\sigma_{[0,n_k-1)}(\ell_{n_k-1}) \cdots \sigma_{[0,n_1)}(\ell_{n_1})\ell_{n_0},$$

$$= \ell_{n_0}\tau_{[0,n_1)}(\ell_{n_1}) \cdots \tau_{[0,n_k-1)}(\ell_{n_k-1}).$$

This implies that

$$v \prec \sigma_{[0,n_k)}(a)\sigma_{[0,n_k-1)}(\ell_{n_k-1}) \cdots \sigma_{[0,n_1)}(\ell_{n_1})\ell_{n_0} = \ell_{n_0}\tau_{[0,n_1)}(\ell_{n_1}) \cdots \tau_{[0,n_k-1)}(\ell_{n_k-1})\tau_{[0,n_k)}(a).$$

There exists also $c \in \mathcal{A}$ such that $\tau_{[0,n_k)}(ca) = \tau_{[0,n_k)}(c)\tau_{[0,n_k)}(a) \in \mathcal{L}_{X_{\boldsymbol{\tau}}}$. By Lemma 3.20, $\tau_{[0,n_k)}(c)$ ends with the word

$$= \ell_{n_0}\tau_{[0,n_1)}(\ell_{n_1}) \cdots \tau_{[0,n_k-1)}(\ell_{n_k-1}),$$

which implies that $v \prec \tau_{[0,n_k)}(ca)$. If there exists $d \in \mathcal{A}$ such that $\tau_{n_k}(d)$ begins with ca, then $v \prec \tau_{[0,n_k+1)}(d)$ and thus $v \in \mathcal{L}_{X_{\boldsymbol{\tau}}}$. Analogously, one proves that every factor belonging to $\mathcal{L}_{X_{\boldsymbol{\tau}}}$ belongs also to $\mathcal{L}_{X_{\boldsymbol{\sigma}}}$. This proves that $X_{\boldsymbol{\sigma}} = X_{\boldsymbol{\tau}}$.

Now, let $\tilde{\sigma}_i = \tilde{\ell}_i \tilde{u}_i$, where

$$\tilde{\ell}_i = \begin{cases} \ell_{n_i} & \text{if } i \text{ is even} \\ \varepsilon & \text{if } i \text{ is odd.} \end{cases}$$

and

$$\tilde{u}_i = \begin{cases} u_{n_i} & \text{if } i \text{ is even} \\ \tau_{n_i} & \text{if } i \text{ is odd.} \end{cases}$$

For all $i \geq 0$, let $\tilde{\tau}_i$ be defined by $\tilde{\tau}_i(a) = \tilde{u}_i \tilde{\ell}_i$. We may apply Lemma 3.20 in the same way as above to obtain that $X_{\tilde{\boldsymbol{\sigma}}} = X_{\tilde{\boldsymbol{\tau}}}$.

Finally, since for all $i \geq 0$ $\tilde{\tau}_i = \tau_{n_i}$, we have that $X_{\boldsymbol{\tau}} = X_{\tilde{\boldsymbol{\tau}}}$. We conclude that $X_{\boldsymbol{\sigma}} = X_{\boldsymbol{\tau}} = X_{\tilde{\boldsymbol{\tau}}} = X_{\tilde{\boldsymbol{\sigma}}}$.

If all σ_{n_k}'s are right proper, we define the new sequence $(\tilde{\sigma}_i)_{i \in \mathbb{N}}$ as follows,

$$\tilde{\sigma}_i = \begin{cases} \sigma_{n_i} & \text{if } i \text{ is odd} \\ \tau_{n_i} & \text{if } i \text{ is even.} \end{cases}$$

where τ_{n_i} is the left conjugate of σ_{n_i}.

The sequence $(\tilde{\sigma}_i)_{i \in \mathbb{N}}$ verifies that $\tilde{\sigma}_i$ is left proper if i is even and right proper if i is odd, so by Lemma 3.19, the sequence of tower partitions $\mathcal{P}_n^{(1)}$ associated to $\tilde{\boldsymbol{\sigma}}$ satisfies conditions **(C1)** and **(C2)**. To show that $X_{\tilde{\boldsymbol{\sigma}}} = X_{\boldsymbol{\sigma}}$, we proceed in a completely analogous way as in the case where all σ_{n_k}'s are left proper.

□

3.3 Frequencies.

Recall from Chapter 1, Section 1.4.2, that in a minimal subshift (X, T) every point has (uniform) frequencies if and only if (X, T) is uniquely ergodic, in which case the frequency μ_v of a factor $w \in \mathcal{L}_X$ is equal to $\mu([w])$, where μ is the unique invariant measure of $\mathcal{M}(X, T)$.

3.3.1 Frequencies for substitutive systems.

Substitutive systems arising from primitive substitutions are minimal and uniquely ergodic (see for example [Que10, Sections 5.2 and 5.4]). This implies that frequencies do exist and are uniform. The way to compute them is related to linear properties of the incidence matrix of the substitution. We recall here a list of results which make explicit this relation and which are taken almost literally from

[Que10, Chapter 5]. We start by recalling a classical theorem for primitive matrices, the Perron-Frobenius's Theorem.

Theorem 3.22 (Perron-Frobenius). *Let M be a primitive real matrix. Then, the following assertions hold,*

(a) M admits a unique positive eigenvalue θ verifying $\theta > |\lambda|$ for all other eigenvalue λ of M.

(b) The eigenvector associated to θ can be chosen positive.

(c) The eigenvalue θ is simple.

The eigenvalue θ above is called the *dominant eigenvalue* or the *Perron-Frobenius eigenvalue* of M. As a consequence of Theorem 3.22, there exists a unique right normalized eigenvector associated to the dominant eigenvalue, normalized meaning that the sum of its components equals 1. It is called the *right normalized dominant eigenvector* or the *right normalized Perron-Frobenius eigenvector* of M.
We define now an analogous to the 2-block extension substitution σ' in higher dimensions.
Let $\sigma : \mathcal{A} \to \mathcal{A}$ a primitive substitution and $\ell \geq 2$. Consider the finite set $\mathcal{L}_\ell(X_\sigma)$ as an alphabet, and define the substitution σ_ℓ on it in the following way: for every $u = u_0 u_1 \cdots u_{\ell-1} \in \mathcal{L}_\ell(X_\sigma)$, $\sigma_\ell(u)$ consists of the $|\sigma(u_0)|$ first factors of lenght ℓ of $\sigma(u)$. We extend σ_ℓ to $\mathcal{L}_\ell(X_\sigma)^*$ and $\mathcal{L}_\ell(X_\sigma)^\mathbb{Z}$ by concatennation.

Lemma 3.23. *[Que10, Lemma 5.3] If $\sigma : \mathcal{A} \to \mathcal{A}$ is a primitive substitution, then for all $\ell \geq 2$, $\sigma_\ell : \mathcal{L}_\ell(X_\sigma) \to \mathcal{L}_\ell(X_\sigma)^*$ is a primitive substitution as well.*

The previous lemma ensures that we can apply Theorem 3.22 to the incidence matrix M_{σ_2} of the substitution σ_2. The following result states that frequencies of letters (resp. factors of length 2) on primitive substitutive systems exist and are provided by the right normalized Perron-Frobenius eigenvector of M_σ (resp. M_{σ_2}). It is a restatement of Propositions 5.8 and 5.9 in [Que10].

Proposition 3.24. *If $\sigma : \mathcal{A} \to \mathcal{A}$ is a primitive substitution, then for all $\ell \in \{1, 2\}$ the frequencies of factors of length ℓ exists and the vector $\mathbf{f}_\ell \in \mathbb{R}^{\mathcal{L}_\ell(X_\sigma)}$ whose coordinates are given by these frequencies is equal to the normalized right Perron-Frobenius eigenvector of M_{σ_ℓ}, where $\sigma_1 = \sigma$.*

Example 3.25. Chacon substitution

The primitive Chacon substitution σ_C is defined over the alphabet $\{1, 2, 3\}$ by $\sigma_C \colon 1 \mapsto 1123, 2 \mapsto 23, 3 \mapsto 123$. The eigenvalues of M_{σ_C} are 3, 1 and 0, so the dominant eigenvalue is 3 and the letter frequency vector is $(1/3, 1/3, 1/3)$.

We refer to [Que10, Section 5.4.3] for a description of an algorithm allowing to compute the frequency of any factor $w \in \mathcal{L}(X_\sigma)$ thanks to matrices M_σ and M_{σ_2}. The next result states that eigenvalues of M_{σ_ℓ} are the same for all $\ell \geq 1$, except possibly for the aditional eigenvalue zero.

Proposition 3.26. *[Que10, Corollary 5.5] Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a primitive substitution. The eigenvalues of M_{σ_ℓ} are those of M_{σ_2}, with possibly the additional eigenvalue zero.*

3.3.2 Frequencies for S-adic systems.

We have seen that substitutive systems are uniquely ergodic provided the underlying substitution is primitive ([Que10]). The situation is different when dealing with S-adic systems: there exist uniformly recurrent (and thus primitive S-adic) subshifts which are not uniquely ergodic. This is for instance the case of the counterexample constructed in [Keane77], consisting of a regular interval exchange transformation (see Example 4.4) of 4 intervals, which has exactly two invariant measures.
The following result gives sufficient conditions for unique ergodicity of S-adic systems.

Theorem 3.27. *[BD14, Theorem 5.7] Let $X_{\boldsymbol{\sigma}}$ be an S-adic subshift with directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$, such that $\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n^*$ and $\mathcal{A}_0 = \{1, 2, \cdots, d\}$. Suppose that $\boldsymbol{\sigma}$ is everywhere growing. Let $(M_n)_{n\in\mathbb{N}}$ be the sequence of incidence matrices of σ_n. Then, the limit cone*

$$C^{(0)} = \bigcap_{n\to\infty} M_0 \cdots M_n \mathbb{R}_+^d$$

parametrizes the letter frequencies: the set of vectors $f \in C^{(0)}$ such that $f_1 + f_2 + \cdots + f_d = 1$ coincides with the image of the map which sends a shift-invariant measure μ on $X_{\boldsymbol{\sigma}}$ to the vector of letter frequencies $(\mu([1]), \mu([2]), \cdots, \mu([d]))$. In particular, $X_{\boldsymbol{\sigma}}$ has uniform letter frequencies if and only if the cone $C^{(0)}$ is one-dimensional.
If furthermore, for each k, $(\sigma_{n+k})_{n\in\mathbb{N}}$ is an everywhere growing directive sequence, and the limit cone

$$C^{(k)} = \bigcap_{n\to\infty} M_k \cdots M_n \mathbb{R}_+^d$$

is one-dimensional, then the system $(X_{\boldsymbol{\sigma}}, T)$ is uniquely ergodic.

The above condition on the convergence of the cones $C^{(k)}$ can be interpreted as a *Perron-Frobenius-*

like condition, like in the case of substitutive systems. The following theorem states then a sufficient condtion on the incidence matrices for S-adic systems to have uniform letter frequencies.

Theorem 3.28. *[Fur60] Let $d \geq 1$, let $(M_n)_{n\in\mathbb{N}}$ be a sequence of non-negative integer $d \times d$ matrices. Suppose there exists a strictly positive matrix B and indices $j_1 \leq k_1 \leq j_2 \leq k_2 \leq \cdots$ such that $B = M_{j_1} \cdots M_{k_1-1} = M_{j_2} \cdots M_{k_2-1} = \cdots$ (that is, the block B ocurrs infinitely often in the sequence of composition matrices). Then, there exists a positive vector $f \in \mathbb{R}^d_+$ such that*

$$\bigcap_{n\to\infty} M_0 \cdots M_n \mathbb{R}^d_+ = \mathbb{R}_+ f.$$

The above condition is related with the notion of *recurrence* of a matrix sequence. A sequence of square integer matrices $(M_n)_{n\in\mathbb{N}}$ is said to be *recurrent* if for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_0 \cdots M_m = M_n \cdots M_{n+m}$. This implies that every block occurs infinitely often in the sequence. A particular case of Theorem 3.28 is thus the following result, which corresponds to [Thu17, Proposition 1.5.5].

Proposition 3.29. *[Thu17, Proposition 1.5.5] Let $(M_n)_n$ be a recurrent sequence of non-negative matrices belonging to $GL_d(\mathbb{Z})$. There is a vector $\mathbf{u} \in \mathbb{R}^d_+$ satisfying*

$$\bigcap_{n\to\infty} M_0 \cdots M_n \mathbb{R}^d_+ = \mathbb{R}_+ \mathbf{u}.$$

To ensure the existence of uniform word frequencies, that is, unique ergodicity, we have the following result in the particular case we work with a constant alphabet for all σ_n's. Its proof is a direct consequence of [PF02, 5.1.21] and [Thu17, Lemma 1.5.9].

Theorem 3.30. *[Thu17, Theorem 1.5.10] Let $\mathcal{A}$ be a finite alphabet. Let $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A} \to \mathcal{A}^*)_{n\in\mathbb{N}}$ be an everywhere growing directive sequence of morphisms and $(M_n)_{n\in\mathbb{N}}$ the associated sequence of incidence matrices. If $\boldsymbol{\sigma}$ is primitive and $(M_n)_{n\in\mathbb{N}}$ is unimodular and recurrent, then $(X_{\boldsymbol{\sigma}}, T)$ is minimal and uniquely ergodic.*

Recall from Proposition 2.10 that if a subshift (X, T) is uniquely ergodic, then whenever conditions **(C1)** and **(C3)** are satisfied for some sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of (X, T), the image subgroup $I(X, T)$ is given by

$$I(X, T) = \sum_{i=1}^{d} \mathbb{Z}\mu(B_{i,m}),$$

where d and m are as defined in condition **(C3)**. Recall also from Remark 1.22 that

$$I(X,T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \langle \{\mu([w]) : w \in \mathcal{L}_X\} \rangle ,$$

so when (X,T) is uniquely ergodic, every factor frequency μ_w is an integer linear combination of $\{\mu(B_{1,m}), \cdots, \mu(B_{d,m})\}$. Suppose that an S-adic system given by a recognizable primitive directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ with $\sigma_n : \mathcal{A} \to \mathcal{A}^*$, is uniquely ergodic. Under the hypothesis of Proposition 3.21, the sequence $\mathcal{P}_n^{(1)}$ satisfies condition **(C1)**, so we obtain the following result.

Proposition 3.31. *Let $\mathcal{A}$ be a finite alphabet with $\#A = d \geq 2$, and $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a primitive recognizable directive sequence of morphisms with $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Let $(M_n)_{n\in\mathbb{N}}$ be the sequence of incidence matrices of $\boldsymbol{\sigma}$. Suppose $(X_{\boldsymbol{\sigma}}, T)$ is uniquely ergodic. If there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, σ_{n_k} is left or right proper, and there exists $m \in \mathbb{N} \setminus \{0\}$ such that for all $n \geq m$, $M_n \in GL_d(\mathbb{Z})$, then for all factor $w \in \mathcal{L}(X_{\boldsymbol{\sigma}})$, the frequency μ_w belongs to the following additive subgroup of $\mathbb{R}$,*

$$\sum_{i\in\mathcal{A}} \mathbb{Z}\mu(\sigma_{[0,m)}([a])).$$

If there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, σ_{n_k} is left or right proper, and for all $n \geq 0$, $M_n \in GL_d(\mathbb{Z})$, then for all factor $w \in \mathcal{L}(X_{\boldsymbol{\sigma}})$, the frequency μ_w belongs to

$$\sum_{i\in\mathcal{A}} \mathbb{Z}\mu([a]).$$

Corollary 3.32. *Let $\mathcal{A}$ be a finite alphabet with $\#A = d \geq 2$, and $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a primitive directive sequence of morphisms with $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Suppose that $\boldsymbol{\sigma}$ is obtained from a unimodular continuous fraction algorithm and that the sequence of incidence matrices $(M_n)_{n\in\mathbb{N}}$ is recurrent. If there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, σ_{n_k} is left or right proper, then for all factor $w \in \mathcal{L}(X_{\boldsymbol{\sigma}})$, the frequency μ_w belongs to the following additive subgroup of $\mathbb{R}$,*

$$\sum_{i\in\mathcal{A}} \mathbb{Z}\mu([a]).$$

Example 3.33. Poincaré algorithm.

Consider the classical Poincaré three-dimensional continued fractions algorithm defined on $\mathbb{R}^3_+$ (see

[No95] for details), whose associated matrices are

$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The six associated substitutions are defined by

$$\sigma_{123} = \begin{array}{rcl} 1 & \mapsto & 123 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{array} \quad \sigma_{132} = \begin{array}{rcl} 1 & \mapsto & 132 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{array} \quad \sigma_{213} = \begin{array}{rcl} 1 & \mapsto & 13 \\ 2 & \mapsto & 213 \\ 3 & \mapsto & 3 \end{array}$$

$$\sigma_{231} = \begin{array}{rcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 231 \\ 3 & \mapsto & 31 \end{array} \quad \sigma_{312} = \begin{array}{rcl} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 312 \end{array} \quad \sigma_{321} = \begin{array}{rcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 321 \end{array}$$

Note that every M_{ijk} is unimodular and all substitutions σ_{ijk} are right proper. If $\boldsymbol{\sigma}$ is a recurrent directive sequence with all σ_n's belonging to those substitutions, then by Corollary 3.32, for all factor $w \in \mathcal{L}(X_{\boldsymbol{\sigma}})$, the frequency μ_w belongs to the following additive subgroup of $\mathbb{R}$,

$$\sum_{i \in \mathcal{A}} \mathbb{Z}\mu([a]).$$

3.4 Dynamical dimension group.

3.4.1 Dynamical dimension group for substitutive systems.

In this section we present some results allowing to explicitely compute the dynamical dimension group of substitutive systems. We first introduce some definitions and notations to understand the statements and proofs.

Given a $p \times p$-real matrix M, its *eventual range* $\mathcal{R}_M$ and its *eventual kernel* $\mathcal{K}_M$ are respectively

$$\mathcal{R}_M = \bigcap_{k \geq 1} M^k \mathbb{R}^p, \qquad \mathcal{K}_M = \bigcup_{k \geq 1} \ker(M^k).$$

We have the following chain of inclusions,

$$\cdots M^k\mathbb{R}^p \subseteq M^{k-1}\mathbb{R}^p \subseteq \cdots \subseteq M^2\mathbb{R}^p \subseteq M\mathbb{R}^p \subseteq \mathbb{R}^p.$$

Since $M^{k+1}\mathbb{R}^p \subseteq M^k\mathbb{R}^p$, $\dim(M^{k+1}\mathbb{R}^p) \leq \dim(M^{k+1}\mathbb{R}^p)$, and since $\dim(\mathbb{R}^p) = p$, there are at most p strict inclusions in the chain. Note that if $M^{k+1}\mathbb{R}^p = M^k\mathbb{R}^p$ for some k, then M is invertible on $M^k\mathbb{R}^p$, and thus $M^{k+n}\mathbb{R}^p = M^k\mathbb{R}^p$ for all $n \geq 0$. This implies that $\mathcal{R}_M = M^p\mathbb{R}^p$ and M is an automorphism of $\mathcal{R}_M$.
Note also that a similar argument shows that $\mathcal{K}_M = \ker(M^p)$, and finally that $\mathbb{R}^p = \mathcal{R}_M \oplus \mathcal{K}_M$ (see for example [LM95, Section 7.4] for details).
Let

$$\Delta_M = \{v \in \mathcal{R}_M : \exists k \geq 0, M^k v \in \mathbb{Z}^p\},$$

$$\Delta_M^+ = \{v \in \mathcal{R}_M : \exists k \geq 0, M^k v \in \mathbb{Z}_+^p\},$$

and let $\mathbf{1}_M$ be the projection of the vector $(1, \cdots, 1)$ on $\mathcal{R}_M$. The triple $(\Delta_M, \Delta_M^+, \mathbf{1}_M)$ is an ordered group with unit. When we work with primitive proper substitutions, we have the following theorem, which corresponds to [DHS99, Theorem 22, (i)].

Theorem 3.34. *Let σ be a primitive aperiodic substitution defined on the alphabet $\mathcal{A}$. Let M be the incidence matrix of σ. If σ is proper, then $K^0(X_\sigma, T)$ is isomorphic to $(\Delta_M, \Delta_M^+, \mathbf{1}_M)$.*

When the substitution is not proper, the situation is more complicated. One strategy to work with non-proper substitutions is the one developed in [DHS99], where the authors associate to each primitive aperiodic substitution, a proper primitive aperiodic substitution such that the two associated subshifts are isomorphic (see [DHS99, Proposition 20]). Another strategy is the one presented in [Ho95] and recently developed in [DHP18]. We explain this second strategy here.
We know thanks to Proposition 2.3 and Proposition 3.11 that for any primitive and recognizable S-adic system, the group homomorphism $\pi_{\mathfrak{S}} : G(\mathfrak{S}) \to H(X_{\boldsymbol{\sigma}}, T)$ associated to the sequence of partitions (3.2) for $\ell = 2$ is surjective and consequently $K^0(X, T)$ is isomorphic to $(G(\mathfrak{S})/\ker(\pi_{\mathfrak{S}}), (G(\mathfrak{S})/\ker(\pi_{\mathfrak{S}}))^+, [1(\mathfrak{S})])$, where $(G(\mathfrak{S})/\ker(\pi_{\mathfrak{S}}))^+$ is the projection on $G(\mathfrak{S})/\ker(\pi_{\mathfrak{S}})$ of $G(\mathfrak{S})^+$ and $[1(\mathfrak{S})]$ is the class modulo

$\ker(\pi_{\mathfrak{S}})$ of $1(\mathfrak{S})$. This is true in particular for primitive substitutions. The following series of results, issued from [DHP18, Section 9], allows us to compute $\ker(\pi_{\mathfrak{S}})$, so that we can explicitely know the dynamical dimension group $K^0(X_\sigma, T)$.

The following proposition ([DHP18, Proposition 3.5]) gives us a convenient description of the inductive limit $\varinjlim(\mathbb{Z}^p, M)$. We include its proof because we will use the explicit form of the isomorphism in the sequel.

Proposition 3.35. *[DHP18, Proposition 3.5] Let M be a $p \times p$ real metrix. Then, the inductive limite $\varinjlim(\mathbb{Z}^p, M)$ is isomorphic to Δ_M.*

Proof. Let $\alpha \in \varinjlim(\mathbb{Z}^p, M)$. Recall that $\varinjlim(\mathbb{Z}^p, M)$ is the quotient Δ/Δ^0, where

$$\Delta = \{(\mathbf{x}_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}^p \mid \exists k \geq 0 : M\mathbf{x}_n = \mathbf{x}_{n+1} \quad \forall n \geq k\},$$

$$\Delta^0 = \{(\mathbf{x}_n)_{n\geq 0} \in \Delta \mid \exists k \geq 0 : \mathbf{x}_n = 0 \quad \forall n \geq k\}.$$

Then, α is the class modulo Δ^0 of a sequence $x = (\mathbf{x}_n)_{n\in\mathbb{N}}$ which verifies $\mathbf{x}_n \in \mathbb{Z}^p$ and there exists $k \in \mathbb{N}$ such that for all $n \geq k$, $\mathbf{x}_{n+1} = M\mathbf{x}_n$. Choosing k large enough, we may assume that $\mathbf{x}_n \in \mathcal{R}_M$. Since M, and therefore M^k, defines an automorphism of $\mathcal{R}_M$, there exists a unique $\mathbf{y} \in \mathcal{R}_M$ such that $M^k\mathbf{y} = \mathbf{x}_k$. Define $\theta : \Delta \to \Delta_M$ by $\theta(x) = \mathbf{y}$. This map is a well defined group homomorphism between Δ and Δ_M.

Suppose $\theta(x) = 0$. This implies that $M^k 0 = \mathbf{x}_k$, so that $\mathbf{x}_k = 0$ and then $\mathbf{x}_n = 0$ for all $n \geq k$, that is, $x \in \Delta^0$. Conversely, if $x \in \Delta^0$, then we can assume that $\mathbf{x}_k = 0$ and since M^k is an automorphism of $\mathcal{R}_M$, $\mathbf{y} = 0 = \theta(x)$. We conclude that $\ker(\theta) = \Delta^0$.

Finally, note that θ is surjective. Indeed, for a vector $\mathbf{y} \in \Delta_M$, consider a positive integer k such that $M^k\mathbf{y} \in \mathbb{Z}^p$. Consider the sequence

$$\mathbf{x}_n = \begin{cases} 0 & \text{if } n < k \\ M^n\mathbf{y} & \text{if } n \geq k \end{cases}$$

Then it is clear that for all $n \geq k$, $\mathbf{x}_{n+1} = M\mathbf{x}_n$, and $\mathbf{x}_k = M^k\mathbf{y}$, which implies that $\theta((\mathbf{x}_n)_{n\in\mathbb{N}}) = \mathbf{y}$. We conclude that Δ_M is isomorphic to $\Delta/\Delta^0 = \varinjlim(\mathbb{Z}^p, M)$. ☐

Remark 3.36. *Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a primitive substitution on the alphabet $\mathcal{A}$, with $|\mathcal{L}_2(X_\sigma)| = d \geq 2$. The previous proposition tells us that when we take the matrix $M_2 = M_{\sigma_2}$, which is equal* *to the transpose of* $Q_n^{(2)}$ *for all* $n \in \mathbb{N}$ *thanks to Proposition 3.16, the inductive limit* $\varinjlim_n(\mathbb{Z}^d, M_{\sigma_2})$ *corresponds to* $\Delta_{M_{\sigma_2}}$.

If $f \in C(X_\sigma, \mathbb{Z})$ *and* i *is a positive integer such that* $f \in C(\mathcal{P}_i^{(2)})$*, then the image of* f *in* $G(\mathfrak{S}) \cong \varinjlim(\mathbb{Z}^d, M_2^T)$ *is* $i_{\mathcal{P}_i^{(2)}} \circ I_{\mathcal{P}_i^{(2)}} f$*, which is the class modulo* Δ_0 *of the sequence*

$$\mathbf{x}_k = \begin{cases} 0 & \text{if } k < i \\ I_{\mathcal{P}_i^{(2)}} f & \text{if } k = i \\ (M_2^T)^{k-i} I_{\mathcal{P}_i^{(2)}} f & \text{if } k > i \end{cases}$$

If $k = i + d$*, then* $\mathbf{x}_k$ *belongs to* $\mathcal{R}_{M_2^T}$*, and thus to* $\Delta_{M_2^T}$*. Since* $(M_2^T)^k : (M_2^T)^d\mathbb{R}^d \to (M_2^T)^{k+d}\mathbb{R}^d$ *is an automorphism and* $(M_2^T)^d I_{\mathcal{P}_i^{(2)}} f$ *belongs to* $(M_2^T)^{k+d}\mathbb{R}^d$*, there exists a unique* $\mathbf{y} \in (M_2^T)^d\mathbb{R}^d$ *such that* $(M_2^T)^k \mathbf{y} = (M_2^T)^d I_{\mathcal{P}_i^{(2)}} f$*. This implies that the image* $\mathbf{y}$ *of* f *in* $\Delta_{M_2^T}$ *satisfies*

$$(M_2^T)^i \mathbf{y} = I_{\mathcal{P}_i^{(2)}} f.$$

Lemma 3.37. *[DHP18, Lemma 9.7] Let* $\sigma : \mathcal{A} \to \mathcal{A}$ *be a primitive substitution, and let* $\mathcal{P}_n^{(2)}$ *be as defined in* (3.2)*. Let* $\pi_{\mathfrak{S}} : G(\mathfrak{S}) \to H(X_\sigma, T)$ *be the unique morphism of ordered groups with unit associated to the sequence* $(\mathcal{P}_n^{(2)})$*. Then,* $\pi_{\mathfrak{S}}$ *is surjective and* $\ker(\pi_{\mathfrak{S}})$ *(seen as a subset of* $\Delta_{M_2^T}$*) is* $\Delta_{M_2^T} \cap \beta_1(R_1(X_\sigma))$*.*

As a corollary we obtain the following.

Theorem 3.38. *Let* $\sigma : \mathcal{A} \to \mathcal{A}^*$ *be a primitive substitution, and* $\sigma_2 : \mathcal{L}_2(X_\sigma) \to \mathcal{L}_2(X_\sigma)^*$ *its 2-block extension. Let* M_2 *be the incidence matrix of* σ_2*. Then, the dynamical dimension group* $K^0(X_\sigma, T)$ *is isomorphic to* $(\Delta_{M_2^T}/\Delta_{M_2^T} \cap \beta_1(R_1(X_\sigma)), (\Delta_{M_2^T}/\Delta_{M_2^T} \cap \beta_1(R_1(X_\sigma)))^+, \mathbf{1})$*, where* $(\Delta_{M_2^T}/\Delta_{M_2^T} \cap \beta_1(R_1(X_\sigma)))^+$ *is the projection of* $\Delta^+_{M_2^T}$ *on* $\Delta_{M_2^T}/\Delta_{M_2^T} \cap \beta_1(R_1(X_\sigma))$*,* $\mathbf{1}$ *denotes the class modulo* $\Delta_{M_2^T} \cap \beta_1(R_1(X_\sigma))$ *of the vector* $\theta((1(\mathcal{P}_n)_{m\in\mathbb{N}})) \in \Delta_{M_2^T}$ *and* θ *is the morphism defined on Proposition 3.35.*

3.4.2 Dynamical dimension group for S-adic systems.

From Proposition 2.17 we know that if a sequence of tower partitions satisfies conditions **(C0)-(C3)**, then the dynamical dimension group can be explicitly computed. Let $\mathcal{A}$ be a finite alphabet with $|A| = d \geq 2$, and $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a primitive recognizable directive sequence of morphisms with $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Under the hypothesis of Proposition 3.21, the sequence $(\mathcal{P}_n^{(1)})_{n\in\mathbb{N}}$ satisfies conditions **(C1)** and **(C2)**. We also know from Proposition 3.10 that this sequences satisfies always condition **(C0)**. Then, we obtain the following result.

Theorem 3.39. *Let $\mathcal{A}$ be a finite alphabet with $|A| = d \geq 2$, and $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a primitive recognizable directive sequence of morphisms with $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Let $(M_n)_{n\in\mathbb{N}}$ be the sequence of $d \times d$ integer matrices given by the incidence matrices of $\boldsymbol{\sigma}$. Suppose that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, σ_{n_k} is left or right proper, and that there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $M_n \in GL_d(\mathbb{Z})$. Let $\vec{\mu}$ and $\mathbf{u}$ be as defined in Proposition 2.16, and*

$$\tilde{A} = \{\mathbf{x} \in \mathbb{Z}^d : \langle \mathbf{x}, \vec{\mu_m} \rangle > 0 \quad \forall \mu \in \mathcal{M}(X_{\boldsymbol{\sigma}}, T)\} \cup \{0\}.$$

Then, $(H(X_{\boldsymbol{\sigma}}, T), H^+(X_{\boldsymbol{\sigma}}, T), 1_{X_{\boldsymbol{\sigma}}})$ and $(\mathbb{Z}^d, \tilde{A}, \mathbf{u})$ are isomorphic as ordered groups with unit.

Note that the sequences of matrices $(M_{\sigma_n})_{n\in\mathbb{N}}$ and $(M_{\tilde{\sigma}_n})_{n\in\mathbb{N}}$ (where $\tilde{\boldsymbol{\sigma}}$ is the directive sequence given by Proposition 3.21) are identical, since the matrix of a left proper (resp. right proper) substitution and that of its right conjugate (resp. left conjugate) are the same.

Example 3.40. Arnoux-Rauzy-Poincaré algorithm.

Consider the Arnoux-Rauzy-Poincaré three-dimensional continued fractions algorithm defined on $\mathbb{R}^3_+$ (see [BL15] for details), whose nine associated matrices are

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_{231} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The nine associated substitutions are defined by

$$\sigma_1 = \begin{array}{rcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 31 \end{array} \quad \sigma_2 = \begin{array}{rcl} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{array} \quad \sigma_3 = \begin{array}{rcl} 1 & \mapsto & 13 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{array}$$

$$\sigma_{123} = \begin{array}{rcl} 1 & \mapsto & 123 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{array} \quad \sigma_{132} = \begin{array}{rcl} 1 & \mapsto & 132 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{array} \quad \sigma_{213} = \begin{array}{rcl} 1 & \mapsto & 13 \\ 2 & \mapsto & 213 \\ 3 & \mapsto & 3 \end{array}$$

$$\sigma_{231} = \begin{array}{rcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 231 \\ 3 & \mapsto & 31 \end{array} \quad \sigma_{312} = \begin{array}{rcl} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 312 \end{array} \quad \sigma_{321} = \begin{array}{rcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 321 \end{array}$$

Note that every M_i and every M_{ijk} is unimodular and all substitutions σ_i, σ_{ijk} are right proper, so by Theorem 3.39, the dynamical dimension group of any S-adic system obtained by a directive sequence $\boldsymbol{\sigma} = (\tau_i)_{i\in\mathbb{N}}$, where $\tau_i \in \{\sigma_1, \sigma_2, \sigma_3, \sigma_{123}, \sigma_{132}, \sigma_{213}, \sigma_{231}, \sigma_{312}, \sigma_{321}\}$, is the triple

$$(\mathbb{Z}^3, \tilde{A}, \mathbf{u}),$$

where $\tilde{A}$ and $\mathbf{u}$ are defined as in Theorem 3.39.

Example 3.41. Fully subtractive algorithm.

Consider the Fully subtractive three-dimensional continued fractions algorithm defined on $\mathbb{R}^3_+$ (see [Sch00] for details), whose three associated matrices are

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The three associated substitutions are defined by

$$\sigma_1 = \begin{array}{rcl} 1 & \mapsto & 123 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \end{array} \quad \sigma_2 = \begin{array}{rcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 231 \\ 3 & \mapsto & 3 \end{array} \quad \sigma_3 = \begin{array}{rcl} 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 312 \end{array}$$

The set of dual substitutions, which correspond to the substitutions associated to the transposed matrices, is given by

$$\tilde{\sigma}_1 = \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 21 \\ 3 & \mapsto & 31 \end{array} \qquad \tilde{\sigma}_2 = \begin{array}{ccc} 1 & \mapsto & 12 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 32 \end{array} \qquad \tilde{\sigma}_3 = \begin{array}{ccc} 1 & \mapsto & 13 \\ 2 & \mapsto & 23 \\ 3 & \mapsto & 3 \end{array}$$

Note that every M_i^T is unimodular and all substitutions $\tilde{\sigma}_i$ are right proper, so by Theorem 3.39, the dynamical dimension group of any S-adic system obtained by a directive sequence $\boldsymbol{\sigma} = (\tilde{\sigma}_i)_{i\in\mathbb{N}}$, where $i \in \{1, 2, 3\}$, is the triple

$$(\mathbb{Z}^3, \tilde{A}, \mathbf{u}),$$

where $\tilde{A}$ and $\mathbf{u}$ are defined as in Theorem 3.39.

3.5 Balance in substitutive and S-adic systems.

We begin this section by stating a connection between the dynamical dimension group of a symbolic system and their balance properties. The next theorem is the main result and states that if the group part in the dynamimcal dimension group is free abelian of rank d, where d is the cardinality of the alphabet, then balance on letters pass to balance on factors of arbitrary length.

Theorem 3.42. *Let (X, T) be a minimal uniquely ergodic symbolic system with unique invariant measure μ, defined over a finite alphabet $\mathcal{A}$ of cardinality d. Suppose that letter frequencies are rationally independent and the group part of the dynamical dimension group of (X, T) is isomorphic to $\mathbb{Z}^d$. Then (X, T) is balanced on factors if and only if it is balanced on letters. In particular, if (X, T) is balanced on letters, then all the frequencies of factors are additive topological eigenvalues and all cylinders are bounded remainder sets.*

Note that we stablish an analogous result for dendric and eventually dendric subshifts in Chapter 4, Section 4.4, but without requiring the letter frequencies to be rationally independent. To prove Theorem 3.42, we use the following lemmas.

Lemma 3.43. *Let (X, T) be a minimal uniquely ergodic subshift on the alphabet $\mathcal{A}$, with unique invariant measure μ and for all $f \in C(X, \mathbb{Z})$ let $\overline{f}$ denote the class of f in the dynamical dimension group $H(X, T)$. For all $a \in \mathcal{A}$, let χ_a denote the indicator function of $[a]$. Suppose that the measures*

$\{\mu([a]) : a \in \mathcal{A}\}$ are rationally independent. Then, $\langle\{\overline{\chi_a} : a \in \mathcal{A}\}\rangle$ is a free abelian subgroup of $H(X,T)$ of rank d.

Proof. We must show that $\{\overline{\chi_a} : a \in \mathcal{A}\}$ is free and has exactly d elements. Suppose that there exist integers λ_a for all $a \in \mathcal{A}$ such that

$$\sum_{a\in\mathcal{A}} \lambda_a \overline{\chi_a} = 0_{H(X,T)}.$$

This means that there exists $g \in \beta C(X,\mathbb{Z})$ such that for all $x \in X$,

$$\sum_{a\in\mathcal{A}} \lambda_a \chi_a(x) = g(x).$$

Therefore, for all $n \in \mathbb{N}$ and for all $x \in X$,

$$\frac{1}{n}\sum_{a\in\mathcal{A}} |x_0 \cdots x_{n-1}|_a = \frac{1}{n}\sum_{i=0}^{n-1} g \circ T^i(x).$$

Thanks to Theorem 1.1, the right part of the previous equation tends to 0 as n tends to ∞, since g is a coboundary. On the other hand, $\lim_{n\to\infty} \frac{|x_0\cdots x_{n-1}|_a}{n} = \mu([a])$ for all $a \in \mathcal{A}$, since the system is uniquely ergodic. We obtain that

$$\sum_{a\in\mathcal{A}} \lambda_a \mu([a]) = 0,$$

and by rationally independence, we conclude that $\lambda_a = 0$ for all $a \in \mathcal{A}$. This proves that $\{\overline{\chi_a} : a \in \mathcal{A}\}$ is free. In particular, if $a, b \in \mathcal{A}$ are different letters such that $\overline{\chi_a} = \overline{\chi_b}$, then $\overline{\chi_a} - \overline{\chi_b} = 0_{H(X,T)}$, which contradicts the fact that $\{\overline{\chi_a} : a \in \mathcal{A}\}$ is free. □

Lemma 3.44. *Let (X,T) be a minimal subshift defined over a finite alphabet $\mathcal{A}$ of cardinality d and let μ be an invariant measure of (X,T). Suppose that letter frequencies are rationally independent and the dynamical dimension group of (X,T) is isomorphic to $\mathbb{Z}^d$. Then, for all $f \in C(X,\mathbb{Z})$ there exists $F \in \beta C(X,\mathbb{R})$ and rational numbers $\frac{p_a}{q_a}$ for all $a \in \mathcal{A}$ such that*

$$f(x) = \sum_{a\in\mathcal{A}} \frac{p_a}{q_a}\chi_a(x) + F(x) \quad \forall x \in X.$$

Proof. By Lemma 3.43, $\langle\{\overline{\chi_a} : a \in \mathcal{A}\}\rangle$ is a free abelian subgroup of $H(X,T)$ of rank d. We know that there exists a basis $\{\overline{f_a} : a \in \mathcal{A}\}$ of $H(X,T)$ and positive integers $\{\alpha_a : a \in \mathcal{A}\}$ such that $\{\alpha_a\overline{f_a} : a \in \mathcal{A}\}$ is a basis if $\langle\{\overline{\chi_a} : a \in \mathcal{A}\}\rangle$ (see [ST, Theorem 1.12]). Since $\{\alpha_a\overline{f_a} : a \in \mathcal{A}\}$ and

$\{\overline{\chi_a} : a \in \mathcal{A}\}$ are basis of $\langle\{\overline{\chi_a} : a \in \mathcal{A}\}\rangle$, there exist integer matrices A, B such that for all $a \in \mathcal{A}$,

$$\overline{\chi_a} = \sum_{b\in\mathcal{A}} A(a,b)\alpha_b\overline{f_b},$$

$$\alpha_a\overline{f_a} = \sum_{b\in\mathcal{A}} B(a,b)\overline{\chi_b}.$$

Let $f \in C(X,\mathbb{Z})$. Since $\overline{f} \in H(X,T)$ and $\{\overline{f_a} : a \in \mathcal{A}\}$ is a basis of $H(X,T)$, we know that there exist unique integer coefficients $\{k_a : a \in \mathcal{A}\}$ such that $\overline{f} = \sum_{a\in\mathcal{A}} k_a$. This implies that there exists $g \in \beta C(X,\mathbb{Z})$ such that for all $x \in X$,

$$f(x) - \sum_{a\in\mathcal{A}} k_a f_a(x) = g(x).$$

On the other hand, since $\alpha_a\overline{f_a} = \sum_{b\in\mathcal{A}} B(a,b)\overline{\chi_b}$, there exists $h \in \beta C(X,\mathbb{Z})$ such that for all $x \in X$,

$$\alpha_a f_a(x) - \sum_{b\in\mathcal{A}} B(a,b)\chi_b(x) = h(x).$$

This means that for all $x \in X$,

$$f_a(x) = \sum_{b\in\mathcal{A}}\sum_{a\in\mathcal{A}} \frac{k_a}{\alpha_a} B(a,b)\chi_b(x) + \sum_{a\in\mathcal{A}} \frac{h(x)}{\alpha_a} + g(x).$$

The sum $\sum_{a\in\mathcal{A}} \frac{k_a}{\alpha_a} B(a,b)$ is a rational number p_b/q_b which depends on b. Since h is an integer coboundary, the sum $\sum_{a\in\mathcal{A}} \frac{h(x)}{\alpha_a}$ is a real coboundary, and then defining $F = \sum_{a\in\mathcal{A}} \frac{h(x)}{\alpha_a} + g$ we obtain that $F \in \beta C(X,\mathbb{R})$ and for all $x \in X$,

$$f(x) = \sum_{a\in\mathcal{A}} \frac{p_a}{q_a}\chi_a(x) + F(x) \quad \forall x \in X.$$

□

Proof of Theorem 3.42. If (X,T) is balanced on factors, then it is balanced on letters thanks to Proposition 1.16. Suppose (X,T) is balanced on letters and let C be a constant of balancedness for all letters $a \in \mathcal{A}$. Let $v \in \mathcal{L}_X$. If $|v| = 1$, then v is a letter and (X,T) is balanced on v by hypothesis. Suppose $|v| > 1$ and let $u, w \in \mathcal{L}_X$ of length $n-1 > |v|$. Pick a bi-infinite word $x \in X$ such that $u = x_{[i,i+n)}$

and $w = x_{[j,j+n)}$ for some indices $i, j \in \mathbb{Z}$. We have

$$||u|_v - |w|_v| = \left| \sum_{\ell=i}^{i+n-1-|v|} \chi_{[v]}(T^\ell x) - \sum_{\ell=j}^{j+n-1-|v|} \chi_{[v]}(T^\ell x) \right|.$$

Now, according to Lemma 3.44, $\chi_{[v]}$ can be written as

$$\chi_{[v]} = \sum_{a \in \mathcal{A}} \frac{p_a}{q_a} \chi_{[a]} - F_v,$$

where $F_v \in \beta C(X, \mathbb{R})$. This implies that

$$\begin{aligned}
||u|_v - |w|_v| &= \left| \sum_{\ell=i}^{i+n-1-|v|} \left(\sum_{a \in \mathcal{A}} \frac{p_a}{q_a} \chi_{[a]}(T^\ell x) + F_v(T^\ell x) \right) - \sum_{\ell=j}^{j+n-1-|v|} \left(\sum_{a \in \mathcal{A}} \frac{p_a}{q_a} \chi_{[a]}(T^\ell x) + F_v(T^\ell x) \right) \right| \\
&= \left| \sum_{a \in \mathcal{A}} \frac{p_a}{q_a} \left(\sum_{\ell=i}^{i+n-1-|v|} \chi_{[a]}(T^\ell x) - \sum_{\ell=j}^{j+n-1-|v|} \chi_{[a]}(T^\ell x) \right) + \sum_{\ell=i}^{i+n-1-|v|} F_v \circ T^\ell(x) - \sum_{\ell=j}^{j+n-1-|v|} F_v \circ T^\ell(x) \right| \\
&\leq \sum_{a \in \mathcal{A}} \frac{p_a}{q_a} ||x_i \cdots x_{i+n-|v|-1}|_a - |x_j \cdots x_{j+n-|v|-1}|_a| + \left| \sum_{\ell=i}^{i+n-1-|v|} F_v \circ T^\ell(x) - \sum_{\ell=j}^{j+n-1-|v|} F_v \circ T^\ell(x) \right| \\
&\leq \sum_{a \in \mathcal{A}} \frac{p_a}{q_a} \cdot C + \left| \sum_{\ell=i}^{i+n-1-|v|} F_v \circ T^\ell(x) \right| + \left| \sum_{\ell=j}^{j+n-1-|v|} F_v \circ T^\ell(x) \right| \\
&\leq C \sum_{a \in \mathcal{A}} \frac{p_a}{q_a} + \sum_{\ell=i}^{i+n-1-|v|} |F_v \circ T^\ell(x)| + \sum_{\ell=j}^{j+n-1-|v|} |F_v \circ T^\ell(x)|.
\end{aligned}$$

Since $F_v \in \beta C(X, \mathbb{R})$, by Theorem 1.1, both $\sum_{\ell=i}^{i+n-1-|v|} |F_v \circ T^\ell(x)|$ and $\sum_{\ell=j}^{j+n-1-|v|} |F_v \circ T^\ell(x)|$ are bounded, and we obtain that $||u|_v - |w|_v| \leq K_v C + B_v$ where $K_v = \sum_{a \in \mathcal{A}} \frac{p_a}{q_a}$, and $\sum_{\ell=i}^{i+n-1-|v|} |F_v \circ T^\ell(x)|, \sum_{\ell=j}^{j+n-1-|v|} |F_v \circ T^\ell(x)| \leq B_v$. This ends the proof of the balance on v. We conclude that (X, T) is balanced in every factor $v \in \mathcal{L}_X$.

Lastly, the result on additive topological eigenvalues comes from Proposition 1.27. □

As a consequence of the previous results we can describe the balance behaviour of some substitutive and S-adic systems, as it is stated in the following two corollaries.

Corollary 3.45. *Let σ be a proper primitive aperiodic substitution on a d-letter alphabet $\mathcal{A}$. Let M be its incidence matrix. Suppose that the letter frequencies are rationally independent. If $M \in GL_d(\mathbb{Z})$, then (X_σ, T) is balanced on letters if and only if it is balanced on factors.*

Proof. When $M \in GL_d(\mathbb{Z})$, $\Delta_M \cong \mathbb{Z}^d$, and then the result follows directly from Theorem 3.34 and Theorem 3.42. It can be seen as a direct consequence of Theorem 3.39 and Theorem 3.42 as well. □

Corollary 3.46. *Let $\mathcal{A}$ be a finite alphabet with $\#\mathcal{A} = d \geq 2$, and $\boldsymbol{\sigma} = (\sigma_n)_{n\geq 0}$ be a primitive recognizable directive sequence of morphisms with $\sigma_n : \mathcal{A} \to \mathcal{A}^*$ for all $n \in \mathbb{N}$. Let $(M_n)_{n\in\mathbb{N}}$ be the sequence of $d \times d$ integer matrices given by the incidence matrices of $\boldsymbol{\sigma}$. Suppose that $\boldsymbol{\sigma}$ satisfies the hyphotesis of Proposition 3.21 and that there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $M_n \in GL_d(\mathbb{Z})$. Suppose also that $(X_{\boldsymbol{\sigma}}, T)$ is uniquely ergodic with rationally independent letter frequencies. Then, $(X_{\boldsymbol{\sigma}}, T)$ is balanced on letters if and only if it is balanced on factors.*

Proof. It follows directly from Theorem 3.39 and 3.42. □

We now work with the sequence of tower partitions given by (3.2) with $\ell = 2$ to study balance properties of substitutive systems where frequencies are known to be rational.

3.5.1 Balance in substitutions with rational frequencies.

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a primitive substitution, and let $\mathcal{P}_n^{(2)}$ be as defined in (3.2). Let $\pi_{\mathfrak{G}} : G(\mathfrak{G}) \to H(X_{\boldsymbol{\sigma}}, T)$ be the unique morphism of ordered group with unit associated to the sequence $(\mathcal{P}_n^{(2)})$. The correspondence between $G(\mathfrak{G})$ and Δ_{M_2} we explicitely presented after Proposition 3.35, together with Lemma 3.37, implies the following result.

Proposition 3.47. *Let σ be a primitive substitution. Let $f \in C(X_\sigma, \mathbb{Z})$ such that there exists $i \in \mathbb{N}$ for which f is constant in the atoms of $\mathcal{P}_i^{(2)}$. For all $\ell \geq i$, let $\phi_\ell = I_{\mathcal{P}_\ell^{(2)}} f \in \mathbb{R}^{\mathcal{L}_2(X_\sigma)}$. Let $d = |\mathcal{L}_2(X_\sigma)|$. If f is a coboundary, then $\phi_\ell \in \beta(R_1(X_\sigma))$ for all $\ell \geq i + d$.*

Proof. For simplicity we note $\mathcal{P}_n$ the partition $\mathcal{P}_n^{(2)}$ for all $n \in \mathbb{N}$. Let $k = i + d$. From Remark 3.36, we know that the image of f in $\Delta_{M_2^T}$ is the unique element $\mathbf{y} \in (M_2^T)^d\mathbb{R}^d$ which verifies $(M_2^T)^k\mathbf{y} = (M_2^T)^d\phi_i$. Since f is a coboundary, $\pi(f) = \pi_{\mathfrak{G}}(I_{Pe_k^{(2)}} f) = 0_{H(X_\sigma,T)}$, so from Lemma 3.37 we know that the image of f in $\Delta_{M_2^T}$ belongs to $\beta_1(R_1(X_\sigma))$, that is, $\mathbf{y} \in \beta(R_1(X_\sigma))$. Note that $\beta_1(R_1(X_\sigma))$ is invariant under M_2^T. Indeed, let $I : R_1(X_\sigma) \to R_1(X_\sigma)$ be given by

$$I\phi(a) = \phi(\sigma(a)_0) \quad \forall a \in \mathcal{A}.$$

Let $\phi \in R_1(X_\sigma)$, let $ab \in \mathcal{L}_2(X_\sigma)$. Then,

$$(\beta_1 \circ I)\phi(ab) = (I\phi)(b) - (I\phi)(a) = \phi(\sigma(b)_0) - \phi(\sigma(a)_0).$$

On the other hand,

$$M_2^T(\beta_1\phi)(ab) = \sum_{i=0}^{|\sigma_2(ab)|-1} \beta_1\phi(\sigma_2(ab)_i),$$

which by definition corresponds to

$$\beta_1\phi(\sigma(a)_0\sigma(a)_1) + \beta_1\phi(\sigma(a)_1\sigma(a)_2) + \cdots + \beta_1\phi(\sigma(b)_0\sigma(a)_{|\sigma(a)|-1}) = \phi(\sigma(b)_0) - \phi(\sigma(a)_0).$$

This proves that $M_2^T \circ \beta_1 = \beta_1 \circ I$, and thus $\beta_1(R_1(X_\sigma))$ is invariant under M_2^T.
This implies that $(M_2^T)^p\mathbf{y} \in \beta_1(R_1(X_\sigma))$ for all $p \geq 0$. In particular, if $\ell \geq k$,

$$\phi_\ell = (M_2^T)^{\ell-i}\phi_i = (M_2^T)^\ell\mathbf{y} \in \beta_1(R_1(X_\sigma)).$$

□

The previous proposition is a restatement of Proposition 4.6 in [BCB18]. An alternative proof can be found there. We now explore some consequences that Proposition 3.47 has for balance in substitutiv systems having rational frequencies. The next results are presented in [BCB18, Section 4], we include the proofs for completeness.

Proposition 3.48. *Let σ be a primitive substitution. Let $v \in \mathcal{L}(X_\sigma)$ having a rational frequency μ_v and $f_v = \chi_{[v]} - \mu_v \in C(X_\sigma, \mathbb{R})$. There exists $k \in \mathbb{N}$ be such that f_v is constant in the atoms of the partition $\mathcal{P}_k^{(2)}$. If (X_σ, T) is balanced on v, then $I_{\mathcal{P}_n^{(2)}} f_v \in \beta(R_1(X))$ for all $n \geq k + d$, where $d = |\mathcal{L}_2(X_\sigma)|$.*

Proof. For simplicity we note $\mathcal{P}_n$ the partition $\mathcal{P}_n^{(2)}$ for all $n \in \mathbb{N}$. We write $\mu_v = p_v/q_v$ in irreducible form. For all $n \geq 0$, the partition $\mathcal{P}_n$ verifies that all elements in any atom of $\mathcal{P}_n$ share at least their $L_n + 1$ letters, where $L_n = \min\{|\sigma^n(a)| : a \in \mathcal{A}\}$. Therefore, for all k large enough, f_v (and consequently $q_v \cdot f_v$) is constant in the atoms of $\mathcal{P}_k$. By Proposition 1.19, since (X_σ, T) is balanced in v, f_v is a coboundary, and then so is $q_v \cdot f_v$. By Proposition 3.47, $q_v \cdot I_{\mathcal{P}_n^{(2)}} f_v \in \beta_1(R_1(X_\sigma))$ for all $n \geq k + d$, and consequently $I_{\mathcal{P}_n^{(2)}} f_v \in \beta(R_1(X))$ for all $n \geq k + d$. □

Recall from Section 1.2.1 that, given a minimal symbolic system (X, T) on the alphabet $\mathcal{A}$ and a letter $a \in \mathcal{A}$, a word w with $wa \in \mathcal{L}_X$ is a *left return word* to the letter a if a is a prefix of wa. It is a first left return word if wa contains exactly two occurrences of a. Recall also that the number of first left return words to any letter is finite. In the sequel we refer to left return words as return words.

Lemma 3.49. *Let (X,T) be a minimal symbolic system defined on the alphabet $\mathcal{A}$, $a \in \mathcal{A}$ and $w = w_0 \cdots w_{|w|-1}$ be a return word to a. If $\phi \in \beta_1(R_1(X))$, then*

$$\phi(w_{|w|-1}a) + \sum_{i=1}^{|w|-1} \phi(w_{i-1}w_i) = 0.$$

Proof. One has $w_0w_1, w_1w_2, \cdots, w_{|w|-2}w_{|w|-1}, w_{|w|-1}a \in \mathcal{L}_2(X)$. The result follows directly from the definition of return words and from the fact that there exists $\varphi \in R_1(X)$ such that $\phi = \beta\varphi$. □

We now deduce from Proposition 3.48 and Lemma 3.49 necessary conditions for balance. The following theorem corresponds to Theorem 1.2 in [BCB18].

Theorem 3.50. *Let σ be a primitive substitution over the alphabet $\mathcal{A}$ and let $\mathcal{L}(X_\sigma)$ denote the language of σ. Let v be in $\mathcal{L}(X\sigma)$, and suppose that it has a rational frequency $\mu_v = p_v/q_v$ written in irreducible form. Suppose that the associated subshift (X_σ, T) is balanced on v. Then, we have the following.*

1. *For each $a \in \mathcal{A}$ and each return word w to a, q_v divides $|\sigma^n(w)|$ for all n large. In particular, if $aa \in \mathcal{L}_2(X_\sigma)$, then q_v divides $|\sigma^n(a)|$ for all n large.*
2. *Let $a \in \mathcal{A}$ and suppose that there exist $b, c \in \mathcal{A}$ such that $bac \in \mathcal{L}(X_\sigma)$ and $bc \in \mathcal{L}(X_\sigma)$. Then q_v divides $|\sigma^n(a)|$ for all n large.*

Proof. Let $\phi_{v,n} = I_{\mathcal{P}_n^{(2)}} f_v$. By Proposition 3.47, $\phi_{v,n} \in \beta_1(R_1(X_\sigma))$ for all n large. For any $ab \in \mathcal{L}_2(X_\sigma)$

$$\phi_{v,n}(ab) = \alpha_{ab}\left(1 - \frac{p_v}{q_v}\right) - (|\sigma^n(a)| - \alpha_{ab}) \cdot \frac{p_v}{q_v}, \tag{3.5}$$

where

$$\alpha_{ab} = |\{0 \leq j < |\sigma^n(a)| : T^j\sigma^n([ab]) \subseteq [v]\}|,$$

that is, α_{ab} is the number of levels in the $ab-$tower of $\mathcal{P}_n^{(2)}$ in which all elements begin with the word v. Using Lemma 3.49 and (3.5), we obtain

$$\begin{aligned} 0 \; = \; & \alpha_{w_{|w|-1}a}\,(q_v - p_v) - (|\sigma^n(w_{|w|-1})| - \alpha_{w_{|w|-1}a}) \cdot p_v + \\ & \sum_{i=1}^{|w|-1} \alpha_{w_{i-1}w_i}\,(q_v - p_v) - (|\sigma^n(w_{i-1})| - \alpha_{w_{i-1}w_i}) \cdot p_v \end{aligned}$$

which implies

$$\begin{aligned} q_v\left(\alpha_{w_{|w|-1}a}+\sum_{i=1}^{|w|-1}\alpha_{w_{i-1}w_i}\right) &= p_v\left(|\sigma^n(w_{|w|-1})|+\sum_{i=1}^{|w|-1}|\sigma^n(w_{i-1})|\right)\\ &= p_v|\sigma^n(w)|.\end{aligned}$$

The integers p_v and q_v being coprime, either

$$\left(\alpha_{w_{|w|-1}a}+\sum_{i=1}^{|w|-1}\alpha_{w_{i-1}w_i}\right)=0$$

or q_v divides $|\sigma^n(w)|$. Since $|\sigma^n(w)|\neq 0$, we conclude that q_v divides $|\sigma^n(w)|$, which ends the proof of the first assertion.

For the second assertion, let $a\in\mathcal{A}$ and assume that there exist b,c such that bac belongs to $\mathcal{L}_3(X_\sigma)$ and $bc\in\mathcal{L}_2(X_\sigma)$. Since $\phi_{v,n}\in\beta_1(R_1(X_\sigma))$ and $ba,ac,bc\in\mathcal{L}_2(X_\sigma)$, one has $\phi_n(ba)+\phi_n(ac)=\phi_n(bc)$, that is,

$$\begin{aligned} 0 &= \alpha_{ba}(q_v-p_v)-p_v(|\sigma^n(b)|-\alpha_{ba})+\alpha_{ac}(q_v-p_v)-p_v(|\sigma^n(a)|-\alpha_{ac})\\ &\quad -\alpha_{bc}(q_v-p_v)+p_v(|\sigma^n(b)|-\alpha_{bc})\\ &= (\alpha_{ba}+\alpha_{ac}-\alpha_{bc})q_v-p_v|\sigma^n(a)|.\end{aligned}$$

The integers p_v and q_v being coprime, either $\alpha_{ba}+\alpha_{ac}-\alpha_{bc}=0$ or q_v divides $|\sigma^n(a)|$. Here again $\alpha_{ba}+\alpha_{ac}-\alpha_{bc}\neq 0$, since $|\sigma^n(a)|\neq 0$, hence q_v divides $|\sigma^n(a)|$. □

Remark 3.51. *Note that Proposition 3.47 gives us the smallest n for which the conclusions of both parts of Theorem 3.50 are always true. It corresponds to $n=i+d$ and thus it can be determined in an effective way. We illustrate this through the following example.*

Example 3.52. *Consider the primitive Chacon substitution σ_C as defined in Example 3.25. We know that the letter frequency vector is $(1/3,1/3,1/3)$ and then $q_1=q_2=q_3=3$. One has $11\in\mathcal{L}_2(X_{\sigma_C})$, and then, for every $a\in\{1,2,3\}$, if the system is balanced on a, 3 divides $|\sigma_C^n(1)|$ for all $n\geq i+d$ (see Proposition 3.47 for notation). In this case, it is enough to take $i=1$; moreover one has $d=5$, so that 3 divides $|\sigma_C^6(1)|$. But $|\sigma_C^6(1)|=1093$, which is not divisible by 3. We conclude that (X_{σ_C},T) is neither balanced on letters, nor balanced on factors of any given size, by Proposition 1.16. In view of*

Proposition 1.27, this is consistent with the fact that (X_{σ_C}, T) *is weakly mixing, that is, it admits no non-trivial topological eigenvalue (see for example [PF02, Lemma 5.5.1]).*

As a consequence of the previous theorem, we have the following corollary about the Thue–Morse substitution.

Corollary 3.53 (Imbalance in Thue–Morse substitution). *Let* σ_{TM} *be the Thue–Morse substitution on* $\{0, 1\}$ *(see example 3.13). The subshift* $(X_{\sigma_{TM}}, T)$ *is unbalanced on any factor of length* $\ell \geq 2$*.*

Proof. From [Dekk77, Theorem 1], we know that the frequency μ_v of a factor v of length $\ell \geq 2$ verifies $\mu_v = \frac{1}{6}2^{-m}$ or $\mu_v = \frac{1}{3}2^{-m}$, where m is such that $2^m < \ell \leq 2^{m+1}$. Frequencies are then rational, $p_v = 1$, and $q_v \in \{3 \cdot 2^{m+1}, 3 \cdot 2^m\}$. Note that 00 belongs to $\mathcal{L}_2(X_{\sigma_{TM}})$. The result then follows from the first assertion of Theorem 3.50. □

Corollary 3.53 gives an answer to the question about balance in factors of length grater than 2 in the Thue–Morse sequence which cannot be obtained by using the criteria presented in [Adam03] and [Adam04], since the matrix M_{σ_2} admits a root of unity as eigenvalue, wchich corresponds to a critical case where linear properties of M_{σ_2} give not enough information to decide if balance does or does not hold (see for example the discussion in [Adam04, Section 5.3 and 5.4]). We also deduce from Theorem 3.50 the following.

Corollary 3.54. *Let* σ *be primitive substitution of constant length* ℓ *over the alphabet* $\mathcal{A}$ *of cardinality* d *such that its incidence matrix is symmetric and* d *is coprime with* ℓ*, or does not divide* ℓ^n*, for all* n *large. If there exists a letter* a *and a return word* w *to* a *such that* d *is coprime with* $|w|$*. Then,* (X_σ, T) *is not balanced on letters.*

Proof. The substitution matrix M_σ admits as left eigenvector (and thus as right eigenvector) associated with the eigenvalue ℓ the vector with coordinates all equal to 1. One thus has $\mu_a = 1/d$ for all a and we apply the first part of Theorem 3.50. □

3.6 Further work.

3.6.1 Dynamical dimension group in the general case.

We have constructed appropiate sequences of tower partitions (satisfying conditions **(C0)**-**(C3)**) which allow us to compute the dynamical dimension group of S-adic systems. This has been possible only by asking the directive sequences to satisfy certain conditions, namely those related to properness (see for instance Proposition 3.21), and the fact that the matrices of the partition are invertible. However, there are many examples of S-adic systems in which the directive sequence is given only by substitutions which are neither right proper nor left proper, and where the substitution matrices are not invertible.

We would like to know how to compute the dynamical dimension group of any given S-adic system only by knowing its primitive directive sequence. Since the sequence of partitions $(\mathcal{P}_n^{(\ell)})_{n\in\mathbb{N}}$ satisfies conditions **(C0)** and **(C1)** for all $\ell \geq 2$, one possibility is to replicate the strategy used in Section 3.4.1 for substitutions. In the following we describe some progress we have in this direction in the case of a recurrent directive sequence. Recall that if we take a random sequence $\boldsymbol{\sigma} \in \mathcal{S}^{\mathbb{N}}$, where $\mathcal{S}$ is a finite set of substitutions, we will see almost always (with respect to any natural measure on $\mathcal{S}^{\mathbb{N}}$) that any finite pattern of $\boldsymbol{\sigma}$ occurs infinitely often, so recurrence is not a very expensive condition.

Suppose that every alphabet $\mathcal{A}_n$ is equal to a fixed finite alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ is included in a finite set $\mathcal{S}$ of morphisms on $\mathcal{A}$. Suppose that $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ is recurrent.

Each matrix $Q_n^{(2)}$ (wich we will note Q_n for simplicity) has $|\mathcal{L}_2(X_{\mathbf{s}}^{(n+1)})|$ rows and $|\mathcal{L}_2(X_{\mathbf{s}}^{(n)})|$ columns. Suppose that every alphabet $\mathcal{L}_2(X_{\mathbf{s}}^{(n)})$ has constant cardinality ℓ. Then every Q_n is a square matrix. For any $n \geq 0$, consider the following linear transformation

$$Q_n\big|_{Q_{n-1}\cdots Q_0\mathbb{R}^\ell} : Q_{n-1}\cdots Q_0\mathbb{R}^\ell \to Q_nQ_{n-1}\cdots Q_0\mathbb{R}^\ell$$

By the rank-nullity theorem, $\dim(Q_{n-1}\cdots Q_0\mathbb{R}^\ell) \geq \dim(Q_n\cdots Q_0\mathbb{R}^\ell)$, as vector subspaces, and then we have the following chain of inequalities

$$\dim(\mathbb{R}^\ell) \geq \dim(Q_0\mathbb{R}^\ell) \geq \dim(Q_1Q_0\mathbb{R}^\ell) \geq \cdots \geq \dim(Q_nQ_{n-1}\cdots Q_0\mathbb{R}^\ell) \geq \cdots$$

Since $\dim(\mathbb{R}^\ell) = \ell$ and for all $n \geq 0$ $\dim(Q_nQ_{n-1}\cdots Q_0\mathbb{R}^\ell) \in \{0, 1, \cdots, \ell\}$, there are at most ℓ strict inclusions in the previous chain, and then there exists an $N \geq 0$ such that for all $n \geq N$, for all $j \geq 0$,

$$Q_nQ_{n-1}\cdots Q_0\mathbb{R}^\ell \cong Q_{n+j}Q_{n+j-1}\cdots Q_0\mathbb{R}^\ell \tag{3.6}$$

and the product $Q_{n+j}Q_{n+j-1}\cdots Q_{n+1}$ defines a bijection between the subspaces $Q_nQ_{n-1}\cdots Q_0\mathbb{R}^\ell$ and $Q_{n+j}Q_{n+j-1}\cdots Q_0\mathbb{R}^\ell$. Note that, in contrast to the situation we have in Section 3.4.1, the integer N is *a priori* not computable in an effective way in this case.

Let m denote the dimension of $Q_N Q_{N-1} \cdots Q_0 \mathbb{R}^\ell$, we claim that $m \geq 1$. Indeed, if $m = 0$, then for all $n \geq N$

$$Q_n Q_{n-1} \cdots Q_0 \mathbb{R}^\ell = \{0\},$$

that is, for any $v \in \mathbb{R}^\ell$, $Q_n Q_{n-1} \cdots Q_0 v = 0$. In particular, for $v = (1, \cdots, 1)$, we obtain that for all $1 \leq j \leq \ell$

$$\sum_{i=1}^{\ell} Q_n Q_{n-1} \cdots Q_0(j,i) = 0,$$

which means that for all $1 \leq j \leq \ell$, the sum of the j-th row of $Q_n Q_{n-1} \cdots Q_0$ is null, but then, since each Q_k has nonnegative entries, the whole matrix $Q_n Q_{n-1} \cdots Q_0$ is null itself, a contradiction.

Lemma 3.55. *Suppose that the sequence of matrices Q_n is recurrent, that is, for all $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that*

$$Q_{n+m} \cdots Q_n = Q_m \cdots Q_0$$

Let N be the positive integer defined in (3.6) and let $(i_k)_{k \geq 0}$ an increasing subsequence of indices verifying

$$Q_{i_k+N} Q_{i_k+N-1} \cdots Q_{i_k} = Q_N Q_{N-1} \cdots Q_0 \quad \forall k \geq 0.$$

Then, there exists a sequence of isomorphisms $\phi_k : Q_{i_k+N} Q_{i_k+N-1} \cdots Q_{i_k} \mathbb{R}^\ell \to \mathbb{R}^m$ which satisfies that for all $k \geq 0$

$$\phi_k = \phi_{k+1} \circ (Q_{i_{k+1}+N} \cdots Q_{i_k+N+1}). \tag{3.7}$$

Proof. We know that $Q_{i_k+N} Q_{i_k+N-1} \cdots Q_{i_k}$ is isomorphic to $\mathbb{R}^m$ for all $k \geq 0$. We can always assume that $(i_k)_{k \geq 0}$ is strictly increasing and $i_0 > 0$. Let f be any isomorphism between $Q_{i_0+N} \cdots Q_{i_0}$ and $\mathbb{R}^m$. We will define ϕ_k inductively. For $k = 0$, set $\phi_0 = f$. For $k > 0$, suppose we have already defined the isomorphism ϕ_k. The linear transformation

$$Q_{i_{k+1}+N} \cdots Q_{i_k+N+1}\big|_{Q_{i_k+N} \cdots Q_{i_k} \mathbb{R}^\ell} : Q_{i_k+N} \cdots Q_{i_k} \mathbb{R}^\ell \to Q_{i_{k+1}+N} \cdots Q_{i_k} \mathbb{R}^\ell$$

is surjective by definition. Its image verifies

$$Q_{i_{k+1}+N} \cdots Q_{i_k} \mathbb{R}^\ell \subseteq Q_{i_{k+1}+N} \cdots Q_{i_{k+1}} \mathbb{R}^\ell \cong \mathbb{R}^m$$

On the other hand,

$$Q_{i_{k+1}+N}\cdots Q_0\mathbb{R}^\ell \subseteq Q_{i_{k+1}+N}\cdots Q_{i_k}\mathbb{R}^\ell,$$

and then

$$m = \dim(Q_{i_{k+1}+N}\cdots Q_0\mathbb{R}^\ell) \leq \dim(Q_{i_{k+1}+N}\cdots Q_{i_k}\mathbb{R}^\ell).$$

This implies that $\dim(Q_{i_{k+1}+N}\cdots Q_{i_k}\mathbb{R}^\ell) = m$, and therefore $Q_{i_{k+1}+N}\cdots Q_{i_k}\mathbb{R}^\ell = Q_{i_{k+1}+N}\cdots Q_{i_{k+1}}\mathbb{R}^\ell$. We conclude that $Q_{i_{k+1}+N}\cdots Q_{i_k+N+1}$ is a bijection between $Q_{i_k+N}\cdots Q_{i_k}$ and $Q_{i_{k+1}+N}\cdots Q_{i_{k+1}}$. Finally, we define

$$\phi_{k+1} = \phi_k \circ (Q_{i_{k+1}+N}\cdots Q_{i_k+N+1})^{-1}$$

□

Let $(\phi_k)_{k\geq 0}$ be the sequence of isomorphisms defined in lemma 3.55, and

$$\Delta_Q := \{\mathbf{v} \in \mathbb{R}^m : \exists k \geq 0 \quad s.t \quad \phi_k^{-1}(\mathbf{v}) \in \mathbb{Z}^\ell\}.$$

This space is an analogous of Δ_M of Section 3.4.1. The next proposition shows that under appropriate hypothesis, Δ_Q and the inductive limit $\varinjlim(\mathbb{Z}^\ell, Q_n)$ coincide. This result is analogous to Proposition 3.35.

Proposition 3.56. *Let $(Q_n)_{n\geq 0}$ be the sequence of incidence matrices of the two-block extension substitutions σ'_n. Suppose that every alphabet $\mathcal{L}_2(X_{\boldsymbol{\sigma}}^{(n)})$ has constant cardinality ℓ. Suppose moreover that the sequence $(Q_n)_{n\geq 0}$ is recurrent. Then, $\varinjlim(\mathbb{Z}^\ell, Q_n) = \Delta_Q$.*

Proof. Let N be as defined in (3.6) and let $(i_k)_{k\geq 0}$ an increasing subsequence of indices such that

$$Q_{i_k+N}Q_{i_k+N-1}\cdots Q_{i_k} = Q_NQ_{N-1}\cdots Q_0 \quad \forall k \geq 0.$$

Recall that the inductive limit $\varinjlim(\mathbb{Z}^\ell, Q_n)$ is the quotient Δ/Δ^0, where we use the notation introduced in Section 2.1. Define the map $\tau : \Delta \to \Delta_Q$ as follows: for $x = (x_n)_{n\geq 0} \in \Delta$, let $k \geq 0$ such that $x_{n+1} = Q_nx_n$ for all $n \geq i_k$; $\tau(x) := \phi_k(x_{N+i_k+1})$, where $(\phi_k)_{k\geq 0}$ is the sequence of isomorphisms defined in lemma 3.55.

1. τ is well defined. In the first place, $x_{N+i_k+1} = Q_{i_k+N}Q_{i_k+N-1}\cdots Q_{i_k}(x_{i_k})$ and then $\phi_k(x_{N+i_k+1}) \in \mathbb{R}^m$. Moreover, $\phi_k^{-1}(\tau(x)) = x_{N+i_k+1} \in \mathbb{Z}^\ell$, thus $\tau(x) \in \Delta_Q$ Suppose k_1 and k_2 both verify that

$x_{n+1} = Q_n x_n$ for all $n \geq i_{k_j}$, and suppose *wlog* that $i_{k_2} > i_{k_1}$. Then we have

$$x_{N+i_{k_2}+1} = Q_{i_{k_2}+N} Q_{i_{k_2}+N-1} \cdots Q_{i_{k_1}+N+1}(x_{N+i_{k_1}+1}).$$

From (3.7), it follows easily by induction that

$$\phi_{k_1} = \phi_{k_2} \circ Q_{i_{k_2}+N} Q_{i_{k_2}+N-1} \cdots Q_{i_{k_1}+N+1}$$

and therefore

$$\phi_{k_2}(x_{N+i_{k_2}+1}) = \phi_{k_1}(x_{N+i_{k_1}+1})$$

which shows that τ is well defined.

2. τ is surjective. Let $\mathbf{v} \in \Delta_Q$. By definition this means that $\mathbf{v} \in \mathbb{R}^m$ and there exists $k \geq 0$ such that $\phi^{-1}(\mathbf{v}) \in \mathbb{Z}^\ell$. Take such a k and consider the sequence

$$x = (0, \cdots, 0, \underbrace{\phi_k^{-1}(\mathbf{v})}_{(i_k+N+1)-th}, Q_{i_k+N+1}(\phi_k^{-1}(\mathbf{v})), Q_{i_k+N+2} Q_{i_k+N+1}(\phi_k^{-1}(\mathbf{v})), \cdots)$$

Clearly $x \in \Delta$ and $\tau(x) = \phi_k(x_{i_k+N+1}) = \phi_k(\phi_k^{-1}(\mathbf{v})) = \mathbf{v}$.

3. The kernel of τ is Δ^0. Let $x \in \Delta$ and suppose $\tau(x) = 0$, then there exists $k \geq 0$ such that $\phi_k(x_{i_k+N+1}) = 0 \in \mathbb{R}^m$, which implies that $x_{i_k+N+1} = 0$, since ϕ_k is an isomorphism, and then for all $n \geq i_k + N + 1$, $x_n = 0$, thus $x \in \Delta^0$. Conversely, if $x \in \Delta^0$, there exists $n \in \mathbb{N}$ such that $x_i = 0$ for all $i \geq n$. Let $K = \min\{k \geq 0 : i_k \geq n\}$, then $x_{i_K+N+1} = 0$ and consequently $\psi(x) = \phi_K(x_{i_K+N+1}) = 0$.

The map τ is trivially a group homomorphism. We conclude that $\Delta_Q \cong \Delta/\Delta^0$. $\square$

The previous proposition shows that $H(X_{\boldsymbol{\sigma}}, T)$ is isomorphic to a quotient of Δ_Q. Indeed, since the sequence $(\mathcal{P}_n^{(2)})$ satisfies **(C1)**, $H(X_{\boldsymbol{\sigma}}, T)$ is isomorphic to a quotient of $G(\mathfrak{S})$, by Corollary 2.3. By definition, $G(\mathfrak{S})$ is equal to $\varinjlim(\mathbb{Z}^\ell, Q_n)$, which by Proposition 3.56 is isomorphic to Δ_Q. Moreover, we know that

$$H(X_{\boldsymbol{\sigma}}, T) \cong \Delta_Q / \ker(\pi_{\mathfrak{S}}),$$

so we would be able to compute the group part of the dynamical dimension group of $(X_{\boldsymbol{\sigma}}, T)$ if we could explicitly determine the space Δ_Q and the image of $\ker(\pi_{\mathfrak{S}})$ in Δ_Q.

3.6.2 Balance in the general case.

We have stated in Theorem 3.42 a relation between balance and the group part of the dynamical dimension group in uniquely ergodic minimal systems having rationally independent letter frequencies. Thus, if the previous strategy to compute $H(X_{\boldsymbol{\sigma}}, T)$ worked, we could know which is the balance behavior in some uniquely ergodic minimal S-adic systems having rationally independent letter frequencies, without requiring properness of the substitutions.

In the case of S-adic systems having rational frequencies, which can occur, for instance, when all subtstitutions in the directive sequence have the same constant length, we would like to used the same strategy of Section 3.5.1 to extend Propositions 3.47-3.48 to the S-adic setting and then to obtain an analogous to Theorem 3.50, giving necessary conditions for balance in S-adic systems having rational frequencies.

Chapter 4

Dendric and eventually dendric subshifts

In this chapter we apply the results of Chapter 2 to minimal dendric and eventually dendric subshifts. These subshifts are defined in a purely combinatorial way, by looking at what is called their *extension graph* (see the definition below). The class of dendric subshifts includes Sturmian subshifts, Arnoux-Rauzy subshifts and subshifts generated by codings of regular (also called i.d.o.c) interval exchange transformations (see Examples 4.2, 4.3 and 4.4). We study the behavior of the image subgroup, infinitesimals, dynamical dimension group and balance for this kind of systems.

4.1 Definitions and examples.

Let $\mathcal{L}$ be a language on the finite alphabet $\mathcal{A}$. We say that $\mathcal{L}$ is *factorial* if it contains the alphabet $\mathcal{A}$ and the factors of all its elements. For any factor $w \in \mathcal{L}$, the *extensions* of w are the following sets,

$$L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}\}$$
$$R(w) = \{a \in \mathcal{A} \mid wa \in \mathcal{L}\}$$
$$B(w) = \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}\}.$$

We say that $\mathcal{L}$ is *biextendable* or simply *extendable* if for all $w \in \mathcal{L}$, $|L(w)| \geq 1$ and $|R(w)| \geq 1$. It is said to be *recurrent* if for every $u, v \in \mathcal{L}$, there exists $w \in \mathcal{L}$ such that $uwv \in \mathcal{L}$, and *uniformly recurrent* if it is biextendable and for every $w \in \mathcal{L}$ there exists $n \in \mathbb{N}$ such that w is a factor of any word of $\mathcal{L}$ of length n. Given an infinite word $x \in \mathcal{A}^{\mathbb{Z}}$, it is not difficult to see that x is uniformly recurrent (in the sense of Section 1.2) if and only if its language $\mathcal{L}_x$ is uniformly recurrent.

A factor $w \in \mathcal{L}$ is said to be a *left special factor* if $|L(w)| \geq 2$, a *right special factor* if $|R(w)| \geq 2$, and a *bispecial factor* $|L(w)|, |R(w)| \geq 2$. The *extension graph* $\mathcal{E}(w)$ of w is the undirected bipartite graph whose set of vertices is the disjoint union of $L(w)$ and $R(w)$ and whose edges are the pairs $(a, b) \in B(w)$.

A language $\mathcal{L}$ is said to be *eventually dendric* if there exists $m \in \mathbb{N}$ such that for all $w \in \mathcal{L}$ with $|w| \geq m$ the extension graph of w is a tree, that is, connected and without cycles. In this case, m is called the *threshold* of $\mathcal{L}$. If one can choose $m = 0$, $\mathcal{L}$ is said to be a *dendric* set.

For an infinite word $x \in \mathcal{A}^{\mathbb{Z}}$, the language $\mathcal{L}_x$ is clearly factorial and biextendable. The word x is said to be *eventually dendric with threshold* m if there exists $m \in \mathbb{N}$ such that for all $w \in \mathcal{L}_x$ with $|w| \geq m$ the extension graph of w is a tree. If one can choose $m = 0$, x is said to be a *dendric* word. Similarly, for a minimal subshift (X, T) on the alphabet $\mathcal{A}$, if there exists $m \in \mathbb{N}$ such that for all $w \in \mathcal{L}_X$ with $|w| \geq m$ the extension graph of w is a tree, (X, T) is called an *eventually dendric* subshift with *threshold* m. If one can choose $m = 0$, (X, T) is said to be a *dendric* subshift.

Dendric sets were introduced for the first time in [BDD+15] under the name of *tree sets* and have been studied for instance in [BDD+15'], [BDD+15''],[DP17],[BDD+18]. The notion of *eventually dendric* is more recent and has been introduced in [DP18]. The class of dendric subshifts is not closed under conjugacy, while that of eventually dendric is (see)[DP18]. In [BDD+17] a very special kind of eventually dendric languages, called *specular sets*, is studied (see also Section 4.1.1).

The language of a dendric subshift (X, T) has the property that every word belonging to it is *neutral*, that is, for all $w \in \mathcal{L}_X$, $|B(w)| - |L(w)| - |R(w)| + 1 = 0$. Sets of words with this property are called *neutral sets*. The *characteristic* of a language $\mathcal{L}$ is the integer $\chi_{\mathcal{L}} = |L(\varepsilon)| + |R(\varepsilon)| - |B(\varepsilon)|$ and it corresponds to the number of connected components in the extension graph of the empty word. An eventually dendric set of characteristic c is an eventually dendric set $\mathcal{L}$ with threshold $m = 1$ and such that $\chi_{\mathcal{L}} = c$, that is, the extension graph of the empty word is a union of c trees (see also [DP17]).

The following result shows that in neutral sets (and thus in languages of dendric subshifts), the notions of recurrence and uniform recurrence coincide.

Proposition 4.1. *[DP17, Corollary 5.3] A recurrent neutral set is uniformly recurrent.*

Example 4.2. Sturmian subshifts.

Sturmian subshifts can be defined in strictly combinatorial terms as the subshifts generated by *Sturmian words*: aperiodic infinite words in which for all $n \in \mathbb{N}$, the number of factors of length n

corresponds exactly to $n+1$. It is a theorem by Morse an Hedlund [HM38] that for an infinite word x, either x is eventually periodic or $p_n(x)$ is strictly increasing, and therefore if it is the case that $p_n(x) \leq n$ for some n, then x is eventually periodic. Strumian words are then those of minimal complexity among aperiodic words. By definition, Sturmian words are always defined over a binary alphabet since $p_1(x) = 2$, and they satisfy that there exists exactly one left special factor and one right special factor of each length. Any bispecial factor $w \in \mathcal{L}_x$ satisfies in this case that $\mathcal{E}(w) = \{a \times \mathcal{A}\} \cup \{\mathcal{A} \times b\}$ some $a, b \in \mathcal{A}$, as depicted in Figure 4.1. Thus, Sturmian subshifts are dendric. See [BDD+15, Example 3.2] for more details.

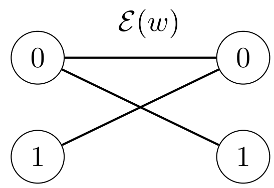

Figure 4.1: The extension graph of any bispecial factor of a Sturmian word.

Example 4.3. Arnoux-Rauzy subshifts.

This corresponds to a generalization of Sturmian subshifts for larger size alphabets: given a finite alphabet $\mathcal{A}$ with $|\mathcal{A}| = d \geq 2$, $x \in \mathcal{A}^{\mathbb{Z}}$ or $x \in \mathcal{A}^{\mathbb{N}}$ is said to be an *Arnoux-Rauzy word* if it is uniformly recurrent and for each $n \in \mathbb{N}$, it has $(d-1)n+1$ factors of length n, there exists exactly one left special factor of length n with d left extensions and exactly one right special factor of length n with d right extensions. Arnoux-Rauzy subshifts are those generated by Arnoux-Rauzy words, they were introduced in [AR91] for $d = 3$. As in Sturmian words, for any bispecial factor $w \in \mathcal{L}_x$, the extension graph verifies $\mathcal{E}(w) = \{a \times \mathcal{A}\} \cup \{\mathcal{A} \times b\}$, as depicted in Figure 4.2 (see [BDD+15, Example 3.2] for more details).

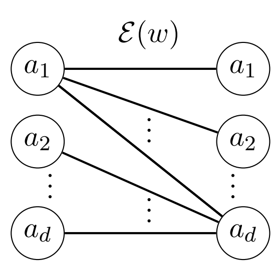

Figure 4.2: The extension graph of a factor of an Arnoux-Rauzy word.

Example 4.4. Interval exchange subshifts.

Consider a finite alphabet $\mathcal{A}$ and two orders $<_1$ and $<_2$ in the symbols of $\mathcal{A}$. Let $(I_a)_{a\in\mathcal{A}}$ a partition of the interval $[0,1)$ in semi-intervals ordered by $<_1$. See Figure 4.3 below for the example $\mathcal{A} = \{a,b,c\}$ and $a <_1 b <_1 c$.

Let λ_a be the length of I_a, let

$$\mu_a = \sum_{b\leq_1 a} \lambda_a \quad \nu_a = \sum_{b\leq_2 a} \lambda_b.$$

The *interval exchange transformation* relative to $(I_a)_{a\in\mathcal{A}}$ is the map $I : [0,1) \to [0,1)$ given by

$$I(z) = z + (\nu_a - \mu_a) \quad if \quad z \in I_a.$$

In the example of Figure 4.3, this corresponds to exchange the order of the pieces I_a, I_b and I_c on the interval, to obtain the new order J_b, J_c, J_a, as depicted in Figure 4.4.

Figure 4.3: Partition of the interval according to $<_1$.

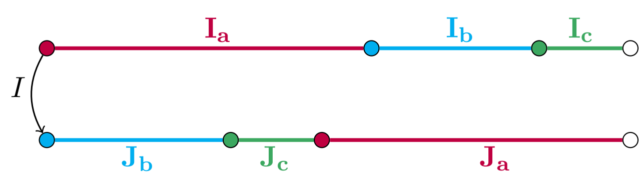

Figure 4.4: An interval exchange transformation I.

The interval exchange tranformation I is said to be *regular* (also called i.d.o.c. in the literature) if the orbits of nonzero separation points under I are infinite and disjoint, where separation points are the starting points of each I_a. Now, take a point p in the interval and consider the orbit of p under I,

$$\{I^n(p) : n \in \mathbb{Z}\}.$$

This gives an infinite sequence $(x_p(n))_{n\in\mathbb{Z}}$ on $\mathcal{A}^{\mathbb{Z}}$ satifying $x_p(n) = a$ if $I^n(p) \in I_a$. The subshift generated by x_p is called an *interval exchange subshift*. It is theorem by Keane [Keane75] that regular interval exchanges transformations produce minimal interval exchange subshifts. In [BDD+15], the authors prove that regular interval exchange subshifts are dendric (see [BDD+15, Proposition 4.2]).

4.1.1 Specular subshifts.

We introduce here a special kind of eventually dendric subshifts called *specular subshifts*. They were introduce in [BDD+17], where there is a complete description of their properties from several viewpoints. We recall here some of those related to return words, which we use later in Section 4.2.

Recall that, given a finite alphabet $\mathcal{A}$, the free group on $\mathcal{A}$, $\mathbb{F}_{\mathcal{A}}$, is the set of all *reduced* finite words in $\mathcal{A} \cup \mathcal{A}^{-1}$, that is, words on $\mathcal{A} \cup \mathcal{A}^{-1}$ which do not have factors of the form aa^{-1} or $a^{-1}a$ for $a \in \mathcal{A}$.

Let $\mathcal{A}$ be a finite alphabet and $\theta : \mathcal{A} \to \mathcal{A}$ an involution. Consider the group $G_\theta \leqslant \mathbb{F}_{\mathcal{A}}$ defined by $G_\theta = \langle a \in \mathcal{A} \mid a\theta(a) = \varepsilon \rangle$. The group G_θ is called a *specular group*. It is known that any specular group is isomorphic to $\mathbb{Z}^{*i} * (\mathbb{Z}/2\mathbb{Z})^{*j}$, where i is the number of orbits of θ with two elements and j is the number of fixed points of θ. In this case, the pair (i, j) is called the *type* of G_θ. Two specular groups are isomorphic if and only if they have the same type (see [BDD+17, Proposition 3.1]), so one refers to G_θ as *the specular group of type* (i, j). In a specular group G_θ, a *reduced word* is a word with no factors of the form $\theta(a)a$ or $a\theta(a)$ for $a \in \mathcal{A}$.

Given an alphabet $\mathcal{A}$, an involution θ and a the specular group G_θ, a *laminary set* on $\mathcal{A}$ relative to θ is a symmetric (closed under taking inverses), biextendable subset of G_θ consisting of reduced words. A *specular set* is a laminary set on $\mathcal{A}$ relative to θ which is a dendric set of characteristic 2, that is, a set of words such that the extension graph of every non-empty word is a tree, and the extension graph of the empty word is a union of two connected components. A *specular subshift* is a subshift in which the language of every element is a specular set.

Example 4.5. *[BDD+17, Example 4.2]*

Consider the substitution $\sigma : \{a, b, c, d\} \to \{a, b, c, d\}^*$ given by $a \mapsto ab$, $b \mapsto cda$, $c \mapsto cd$ and $d \mapsto abc$. The extension graph of the empty word is the depicted in Figure 4.5, so the threshold in this example is $m = 1$. It is shown in [BDD+17] that X_σ is a specular subshift.

Example 4.6. *[DP18, Example 3.6]*

Consider the Tribonacci substitution $\varphi : \{a, b, c\} \to \{a, b, c\}^*$ given by $a \mapsto ab$, $b \mapsto ac$, $c \mapsto a$. The substitutive subshift X_φ is dendric. Consider the projection morphism $\alpha : \{a, b, c\} \to \{a, c\}$ given by $a \mapsto a$, $b \mapsto a$, $c \mapsto c$. It is shown in [DP18, Example 3.6] that the image $\alpha(X_\varphi)$ is an eventually dendric subshift with threshold $m = 4$. Since there are non-empty words whose extension graph is not a tree, $\alpha(X_\varphi)$ is not a specular subshift.

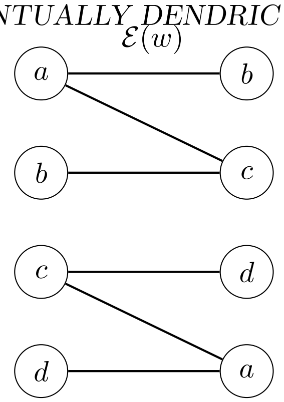

Figure 4.5: The extension graph of ε in $\mathcal{L}_{X_\sigma}$ (Example 4.5).

We now state a combinatorial lemma which will be useful in Sections 4.3 and 4.4 to study the set of invariant measures and balance properties in eventually dendric subshifts. It corresponds to Lemma 3.2 in [BCB18].

Lemma 4.7. *Let $\mathcal{T}$ be a finite tree, with a bipartition X and Y of its set of vertices, with $|X|, |Y| \geq 2$. Let E stand for its set of edges. For all $x \in X$, $y \in Y$, define*

$$Y_x := \{y \in Y : (x, y) \in E\} \qquad X_y := \{x \in X : (x, y) \in E\}.$$

Let $(G, +)$ be an abelian group and H a subgroup of G. Suppose that there exists a function $g : X \cup Y \cup E \to G$ satisfying the following conditions:

(1) $g(X \cup Y) \subseteq H$;

(2) for all $x \in X$, $g(x) = \sum_{y \in Y_x} g(x, y)$, and for all $y \in Y$, $g(y) = \sum_{x \in X_y} g(x, y)$.

Then, for all $(x, y) \in E$, $g(x, y) \in H$.

Proof. Observe first that Conditions (1) and (2) imply that the image under g of any edge connected to a leaf belongs to H. We proceed by induction on $k := \max\{|X|, |Y|\}$. First assume $k = 2$. Such as illustrated in Figure 4.6, there is only one possibility for the graph $\mathcal{T}$ (modulo a relabeling of the vertices), since $\mathcal{T}$ is connected and has no cycles, which is

$$X = \{x_1, x_2\}, Y = \{y_1, y_2\}, E = \{(x_1, y_1), (x_2, y_1), (x_2, y_2)\}.$$

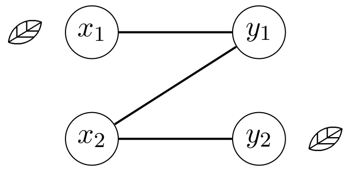

Figure 4.6: The tree $\mathcal{T}$ when $k = 2$.

Both $g(x_1, y_1)$ and $g(x_2, y_2)$ are in H because x_1 and y_2 are leaves. By Condition (2), one has $g(x_2) = g(x_2, y_1) + g(x_2, y_2)$, and then $g(x_2, y_1) = g(x_2) - g(x_2, y_2)$. Since $g(x_2) \in H$ by Condition (1) and H is a group, then $g(x_2, y_1) \in H$.

Now assume $k > 2$ and that the induction hypothesis holds for $k-1$. Suppose also *wlog* that $|X| \geq |Y|$. Note that in this case there exists a leaf in X. Indeed, if all vertices in X have degree at least 2, then

$$|E| = \sum_{x \in X} \deg(x) \geq 2|X|$$

because $\mathcal{T}$ is a bipartite graph. On the other hand, since $\mathcal{T}$ is a tree,

$$|E| = |X| + |Y| - 1 < |X| + |Y| \leq 2|X|$$

which is a contradiction. The same argument shows that if X and Y have the same cardinality, then both X and Y have at least one leaf. We distinguish two cases, namely $|X| > |Y|$ and $|X| = |Y|$.

First assume that $|X| > |Y|$. Take a leaf in X, and call it x_0. Consider the graph $\widetilde{\mathcal{T}}$ obtained from $\mathcal{T}$ by removing the vertex x_0 and the edge (x_0, y_0), where y_0 is the only vertex in Y connected with x_0. This new graph is also a tree, with bipartition of vertices $\widetilde{X} = X - \{x_0\}$, $\widetilde{Y} = Y$, and set of edges $\widetilde{E} = E - \{(x_0, y_0)\}$. Since $|\widetilde{X}| = k - 1$ and $|\widetilde{Y}| = |Y|$, then $\max\{|\widetilde{X}|, |\widetilde{Y}|\} = k - 1$.

We define $\widetilde{g}$ in $\widetilde{X} \cup \widetilde{Y} \cup \widetilde{E}$ as follows. On $(\widetilde{X} \cup \widetilde{Y} \cup \widetilde{E}) - \{y_0\}$, $\widetilde{g} = g$; on y_0, define $\widetilde{g}(y_0) = g(y_0) - g(x_0, y_0)$. Let us verify that $\widetilde{g}$ satisfies Conditions (1) and (2) with respect to $\widetilde{\mathcal{T}}$.

(1) If $x \in \widetilde{X}$, $\widetilde{g}(x) = g(x) \in H$. If $y \in \widetilde{Y}$ and $y \neq y_0$, $\widetilde{g}(y) = g(y) \in H$. If $y = y_0$, then $\widetilde{g}(y_0) = g(y_0) - g(x_0, y_0)$, but both $g(y_0)$ and $g(x_0, y_0)$ are in H, since g satisfies Conditions (1) and (2), and x_0 is a leaf. Therefore, the image under $\widetilde{g}$ of any vertex of $\widetilde{\mathcal{T}}$ is in H.

(2) We need a more precise notation here. For a vertex $x \in \widetilde{X}$, we define $Y_x^{\mathcal{T}} := \{y \in Y : (x, y) \in E\}$ and $Y_x^{\widetilde{\mathcal{T}}} := \{y \in \widetilde{Y} : (x, y) \in \widetilde{E}\}$. If $x \in \widetilde{X}$, then $Y_x^{\mathcal{T}} = Y_x^{\widetilde{\mathcal{T}}}$, and for all $y \in Y_x^{\mathcal{T}}$, $\widetilde{g}(x, y) = g(x, y)$.

Therefore,

$$\widetilde{g}(x) = g(x) = \sum_{y \in Y_x^{\mathcal{T}}} g(x, y) = \sum_{y \in Y_x^{\widetilde{\mathcal{T}}}} \widetilde{g}(x, y).$$

We use analogously the notation $X_y^{\mathcal{T}}$ and $X_y^{\widetilde{\mathcal{T}}}$ for a vertex $y \in \widetilde{Y}$. Let be $y \in \widetilde{Y}$. If $y \neq y_0$, then $X_y^{\mathcal{T}} = X_y^{\widetilde{\mathcal{T}}}$ and for all $x \in X_y^{T}$, $\widetilde{g}(x, y) = g(x, y)$. Hence,

$$\widetilde{g}(y) = g(y) = \sum_{x \in X_y^{\mathcal{T}}} g(x, y) = \sum_{x \in X_y^{\widetilde{\mathcal{T}}}} \widetilde{g}(x, y).$$

Finally, if $y \in \widetilde{Y}$ and $y = y_0$, then $X_y^{\mathcal{T}} = X_y^{\widetilde{\mathcal{T}}} \cup \{x_0\}$. We thus have

$$\begin{aligned} \widetilde{g}(y) = g(y_0) - g(x_0, y_0) \quad &= -g(x_0, y_0) + \textstyle\sum_{x \in X_y^{\mathcal{T}}} g(x, y) \\ &= -g(x_0, y_0) + g(x_0, y_0) + \textstyle\sum_{x \in X_y^{\widetilde{\mathcal{T}}}} \widetilde{g}(x, y) = \textstyle\sum_{x \in X_y^{\widetilde{\mathcal{T}}}} \widetilde{g}(x, y), \end{aligned}$$

which ends the proof of the fact that $\widetilde{g}$ satisfies Conditions (1) and (2).

By induction, for all $(x, y) \in \widetilde{E}$, $\widetilde{g}(x, y) \in H$. But in $\widetilde{E}$ one has $\widetilde{g} = g$, which implies that for all $(x, y) \in \widetilde{E}$, $g(x, y) \in H$. Since x_0 is a leaf in X, $g(x_0, y_0) \in H$, and then for all $(x, y) \in E$, $g(x, y) \in H$. This ends the case $|X| > |Y|$.

Now assume that $|X| = |Y|$. Then, both X and Y have at least one leaf; let us call them x_0 and y_0, respectively. Let x_{y_0} and y_{x_0} denote the only vertices connected with x_0 and y_0, respectively. It is not difficult to see that $y_0 \neq y_{x_0}$ and $x_0 \neq x_{y_0}$, since $\mathcal{T}$ is connected and has no cycles.

Consider the graph $\widetilde{\mathcal{T}}$ obtained from $\mathcal{T}$ by removing the vertices x_0 and y_0, and the edges (x_0, y_{x_0}) and (x_{y_0}, y_0). This new graph is again a tree, with bipartition of vertices $\widetilde{X} = X - \{x_0\}$, $\widetilde{Y} = Y - \{y_0\}$, and set of edges $\widetilde{E} = E - \{(x_0, y_{x_0}), (x_{y_0}, x_0)\}$. Since $|\widetilde{X}| = k-1$ and $|\widetilde{Y}| = k-1$, then $\max\{|\widetilde{X}|, |\widetilde{Y}|\} = k-1$.

On the new set $\widetilde{X} \cup \widetilde{Y} \cup \widetilde{E}$, define the function $\widetilde{g}$ as follows. On $(\widetilde{X} \cup \widetilde{Y} \cup \widetilde{E}) - \{x_{y_0}, y_{x_0}\}$, $\widetilde{g} = g$; on x_{y_0}, define $\widetilde{g}(x_{y_0}) = g(x_{y_0}) - g(x_{y_0}, y_0)$, and on y_{x_0}, $\widetilde{g}(y_{x_0}) = g(y_{x_0}) - g(x_0, y_{x_0})$.

Following the same strategy as in the case $|X| > |Y|$, one can see that $\widetilde{g}$ satisfies Conditions (1) and (2) in $\widetilde{\mathcal{T}}$, and since $\max\{|\widetilde{X}|, |\widetilde{Y}|\} = k - 1$, we conclude by induction that for any edge $(x, y) \in \widetilde{E}$, $\widetilde{g}(x, y)$ belongs to H, which implies that $g(x, y) \in H$. Since x_0 and y_0 are leaves in X and Y, $g(x_0, y_{x_0}), g(x_{y_0}, y_0) \in H$. We conclude that for all $(x, y) \in E$, $g(x, y) \in H$. □

4.2 Return words in dendric and specular subshifts.

Recall that, given a minimal symbolic system (X,T), on the alphabet $\mathcal{A}$ and a factor $u \in \mathcal{A}^+$, a word w with $wu \in \mathcal{L}_X$ is said to be a *left return word* to u if u is a prefix of wu. It is said to be a *first left return word* to u if u is a prefix of wu and there are exactly two occurrences of u in wu. Similarly, a word w with $uw \in \mathcal{L}_X$ is said to be a *right return word* to u if u is a sufix of au. It is said to be a *first right return word* to u if u is a sufix of uw and there are exactly two occurrences of u in aw. Dendric and specular subshifts have interesting properties regarding the set of return words of their languages. They have been mostly explored in [BDD+15] and [BDD+17]. We quote here two important results we will use in Sections 4.3 and 4.5.

Let $\mathcal{L}$ be a specular set on the alphabet $\mathcal{A}$, given by the involution θ. Since $\mathcal{L}$ is biextendable, every letter $a \in \mathcal{A}$ appers exactly twice in $\mathcal{E}(\varepsilon)$, once as a vertex in $L(\varepsilon)$ and once as a vertex in $R(\varepsilon)$. A letter is said to be *even* if these two occurrences are in the same tree of $\mathcal{E}(\varepsilon)$, it is said to be *odd* otherwise. A word $w \in \mathcal{L}$ is said to be *even* if it has an even number of odd letters, it is said to be *odd* otherwise. The *even subgroup on* $\mathbb{F}_{\mathcal{A}}$ is the subgroup of G_θ formed by the even words. It is a free subgroup of index 2 and rank $|\mathcal{A}| - 1$.

Theorem 4.8. *[BDD+15, Theorem 4.5] Let S be a (uniformly) recurrent dendric set containing the alphabet $\mathcal{A}$. Then for any non-empty $w \in S$, the set of first right return words to w is a basis of the free group $\mathbb{F}_{\mathcal{A}}$. In particular, every non-empty word has $|\mathcal{A}|$ first right return words.*

Theorem 4.9. *[BDD+17, Theorem 6.15] Let S be a (uniformly) recurrent specular set. Then for any non-empty $w \in S$, the set of first right return words to w is a basis of the even subgroup on $\mathbb{F}_{\mathcal{A}}$. In particular, every non-empty word has $|\mathcal{A}| - 1$ first rigth return words.*

The proofs of the previous results work exactly in the same way for left return words.

4.2.1 Tower partitions using return words.

Let (X,T) be a minimal subshift over the alphabet $\mathcal{A}$ with cardinality d and take any $x \in X$. For every $n \geq 1$, let $W_n(x) := \{w_{1,n}, \cdots, w_{d_n,n}\}$ be the set of first left return words to $x_{[0,n)}$, and define

$$\mathcal{P}_n = \{T^j[w_{i,n}x_{[0,n)}] : 1 \leq i \leq d_n, 0 \leq j < |w_{i,n}|\}. \tag{4.1}$$

Define also $\mathcal{P}_0 = \{[a] : a \in \mathcal{A}\}$, that is, $\mathcal{P}_0$ is the partition whose towers correspond to the cylinders $[a]$ and have just one floor (it thus has d towers). The following proposition states that $(\mathcal{P}_n)_{n\in\mathbb{N}}$ above is a nested sequence of tower partitions of (X, T). A proof can be found in [DHP18, Proposition 2.18].

Proposition 4.10. *For all $n \geq 0$, $\mathcal{P}_n$ is a tower partition of (X, T) and $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$.*

The matrices Q_n associated to partitions $\mathcal{P}_n$ correspond by definition to

$$Q_n(i,j) = |\{0 \leq j < |w_{i,n+1}| : T^j[w_{i,n+1}x_{[0,n+1)}] \subseteq [w_{j,n}x_{[0,n)}]\}|.$$

The $(i,j)th$ entry of Q_n is exactly $|w_{i,n+1}|_{w_{j,n}}$. That is, the number of occurrences of $w_{j,n}$ in $w_{i,n+1}$. This is consistent with the fact that, since $x_{[0,n)}$ is a prefix of $x_{[0,n+1)}$, any $w_{i,n+1} \in W_{n+1}(x)$ has a unique decomposition as a concatenation of elements $w_{j,n} \in W_n(x)$ (here we use the fact the $w_{j,n}$'s are *first* left return words). Note that in the case of dendric subshifts, every $W_n(x)$ is a basis of the free group $\mathbb{F}_{\mathcal{A}}$, which implies that for all $n \in \mathbb{N}$, for all $w \in W_n(x)$, w is written in a unique way as a concatenation of elements $w' \in W_{n+1}(x)$ and their inverses, which in turn implies that for all $n \in \mathbb{N}$, the matrix Q_n belongs to $GL_d(\mathbb{Z})$ with

$$Q_n^{-1}(j,i) = |w_{j,n}|_{w_{i,n+1}},$$

when now the occurrences have to be counted considering inverses. Similarly, in the case of specular sets, $W_n(x)$ is a basis of the even subgroup for all $n \geq 1$, which implies that Q_n is invertible in $\mathbb{Z}$ for all $n \geq 1$. So both dendric and specular subshifts are such that partitions $\mathcal{P}_n$ as defined in (4.1) satisfy condition **(C3)**, with $m = 0$, $d = |\mathcal{A}|$ in the case of dendric subshifts, $m = 1$, $d = |\mathcal{A}| - 1$ in the case of specular subshifts. More generally, if we have that for all $n \geq \ell$, for some positive integer ℓ, $W_n(x)$ generates the same subgroup of $\mathbb{F}_{\mathcal{A}}$, then the matrix Q_n is invertible for all $n \geq \ell$.

4.2.2 S-adic representation of dendric systems using return words.

As explained in Example 3.5, return words provide S-adic representations of every minimal subshift. Consider a minimal dendric subshift (X, T) defined on the alphabet $\mathcal{A}$ and $x \in X$. The directive sequence $(\lambda_n : R_{n+1} \to R_n^*)_{n \in \mathbb{N}}$ described in Example 3.5, obtained from the factorization of $\mathcal{D}^n(x)$ in first left return words to $\mathcal{D}^n(x)_0$, is called the Λ-adic representation of X. Note that thanks to Theorem 4.8, in the case of dendric minimal subshifts every R_n corresponds to $\mathcal{A}$. Moreover, in this particular case the Λ-adic representation has the property that every λ_n belongs to a precise set of substitutions $\mathcal{S}_e$ on $\mathcal{A}$. We introduce these substitutions in the following.

An automorphism φ of the free group $\mathbb{F}_{\mathcal{A}}$ is said to be *positive* if for all $a \in \mathcal{A}$, $\varphi(a)$ belongs to the semigroup $\mathcal{A}^+$, that is, the set of non-empty words with symbols in $\mathcal{A}$. A positive automorphism is *tame* if it belongs to the submonoid generated by the permutations of $\mathcal{A}$ and the automorphisms $\alpha_{a,b}$, $\tilde{\alpha}_{a,b}$, defined for all $a, b \in \mathcal{A}$, $a \neq b$, by

$$\alpha_{a,b}(c) = \begin{cases} ab & \text{if } c = a \\ c & \text{else} \end{cases}$$

$$\tilde{\alpha}_{a,b}(c) = \begin{cases} ba & \text{if } c = a \\ c & \text{else.} \end{cases}$$

The automorphims $\alpha_{a,b}$ and $\tilde{\alpha}_{a,b}$ together with the permutations of $\mathcal{A}$ are called the *elementary positive* automorphisms of $\mathcal{A}$ and we denote them by $\mathcal{S}_e$. A substitution on $\mathcal{A}$ which extends to a tame automorphism on $\mathbb{F}_{\mathcal{A}}$ is called a *tame substitution.*
The following result corresponds to the second part of [BDD+18, Theorem 6].

Proposition 4.11. *Let (X, T) be a minimal subshift defined on the alphabet $\mathcal{A}$, let $(\lambda_n : R_{n+1} \to R_n^*)_{n \in \mathbb{N}}$ be the Λ-adic representation of X. For all $n \in \mathbb{N}$, $\lambda_n : \mathcal{A} \to \mathcal{A}^*$ is a tame substitution. In other words, the Λ-adic representation of X provides a $\mathcal{S}_e$-adic representation of X.*

4.3 Invariant measures, image subgroup and infinitesimals.

Most part of the results of this section are presented in [BCD+18] for dendric subshifts. We present them here in its more general version, some of them for the class of all eventually dendric subshifts and some others for dendric and specular subshifts.

Theorem 4.12. *Let (X, T) be a minimal eventually dendric subshift with threshold m on a d-letter alphabet $\mathcal{A}$, and let μ and μ' be two T-invariant measures on X. If μ and μ' coincide on factors of length n for all $n \leq m + 1$, then they are equal.*

Proof. Let μ and μ' be two T-invariant measures such that $\mu[u] = \mu'[u]$ for any word $u \in \mathcal{L}_X$ with $|u| \leq m + 1$. Let us show that for any word $w \in \mathcal{L}_X$, $\mu([w]) = \mu'([w])$ by induction on the length of w.
If $|w| = m + 1$, then the result follows immediately. Let $n \geq m + 1$ and suppose that for all $v \in \mathcal{L}_X$ with $|v| \leq n$, one has $\mu([v]) = \mu'([v])$. Let w be a word with length $|w| = n + 1$. Write

$$w = w_0 \cdots w_n, \text{ and define } \quad \widetilde{w} := w_1 \cdots w_n, \quad w' := w_0 \cdots w_{n-1}, \quad \bar{w} = w_1 \cdots w_{n-1}$$

We analyze separately three cases depending on the right/left extensions of $\widetilde{w}$ and w', namely $|L(\widetilde{w})| = 1$, $|R(w')| = 1$, and lastly $|L(\widetilde{w})| \geq 2$ and $|R(w')| \geq 2$.

- We first assume $|L(\widetilde{w})| = 1$. The only left extension of $\widetilde{w}$ is w_0. By T-invariance of μ, one has $\mu([\tilde{w}]) = \sum_{a \in L(\tilde{w})} \mu([a\tilde{w}]) = \mu[w]$. Similarly, $\mu'([\tilde{w}]) = \mu'([w])$. Since $\mu([\tilde{w}])$ and $\mu'([\tilde{w}])$ coincide by induction hypothesis, we get $\mu([w]) = \mu'([w])$.

- We now assume $|R(w')| = 1$. The only right extension of w' is w_n, which yields $\mu([w']) = \sum_{b \in R(w')} \mu([w'b]) = \mu[w]$. Similarly, $\mu'([w']) = \mu'([w])$. Since $\mu(w')$ and $\mu'(w')$ coincide by induction hypothesis, we get $\mu([w]) = \mu'([w])$.

- Finally, we assume $|L(\widetilde{w})| \geq 2$ and $|R(w')| \geq 2$. Let $\mathcal{E}(\bar{w})$ be the extension graph of $\bar{w}$. It is a tree since $|\bar{w}| \geq m$, and each of the sets in its bipartition of vertices has cardinality at least two. We thus can apply Lemma 4.7 with $G = \mathbb{R}$, $H = \{0\}$ and $g : L(\bar{w}) \cup R(\bar{w}) \cup E(\bar{w}) \to \mathbb{R}$ defined as follows:
$$\begin{cases} g(a) = \mu([a\bar{w}]) - \mu'([a\bar{w}]), \text{ for } a \in L(\bar{w}), \\ g(b) = \mu([\bar{w}b]) - \mu'([\bar{w}b]), \text{ for } b \in R(\bar{w}), \\ g(a,b) = \mu([a\bar{w}b]) - \mu'([a\bar{w}b]), \text{ for } (a,b) \in E(\bar{w}). \end{cases}$$
Conditions (C1) and (C2) of Lemma 4.7 hold, and then for any biextension $a\bar{w}b$ of $\bar{w}$, $\mu([a\bar{w}b]) - \mu'([a\bar{w}b]) = 0$. In particular, since $(w_0, w_n) \in E(\bar{w})$, $\mu([w]) = \mu'([w])$.

This proves that for any word w in the language of X, $\mu([w]) = \mu'([w])$. Since th family of cylinders is a basis of the topology, we conclude that, for any clopen $U \subseteq X$, $\mu(U) = \mu'(U)$. □

We obtain the following corollary for dendric and specular subshifts.

Corollary 4.13. *Let (X,T) be a minimal dendric (resp. specular) subshift on a d-letter alphabet $\mathcal{A}$, and let μ and μ' be two T-invariant measures on X. If μ and μ' coincide on the letters (resp. on the letters and the factors of length 2), then they are equal.*

The previous result extends to the family of dendric subshifts a statement initially proved for interval exchanges in [FZ08].

Theorem 4.14. *Let (X,T) be a minimal dendric subshift and let $\mathcal{M}(X,T)$ stands for its set of*

T-invariant probability measures. Then, the image subgroup of (X,T) is

$$I(X,T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \sum_{a \in \mathcal{A}} \mathbb{Z}\mu([a]) \right\}.$$

In particular, if (X,T) is uniquely ergodic and μ is its unique T-invariant measure, then

$$I(X,T) = \sum_{a \in \mathcal{A}} \mathbb{Z}\mu([a]).$$

Proof. This is an immediate consequence of Proposition 2.10, since the sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined in (4.1) satisfies conditions **(C1)** and **(C3)** with $m = 0$ and $d = |\mathcal{A}|$. Since the bases of $\mathcal{P}_0$ are the cylinders of letters $\{[1], [2], \cdots, [d]\}$, the result follows. □

Theorem 4.15. *Let (X,T) be a minimal specular subshift and let $\mathcal{M}(X,T)$ stands for its set of T-invariant probability measures. Then, the image subgroup of (X,T) is*

$$I(X,T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \sum_{w \in W_1(x)} \mathbb{Z}\mu([wx_0]) \right\},$$

where x is any element of X. In particular, if (X,T) is uniquely ergodic and μ is its unique T-invariant measure, then

$$I(X,T) = \sum_{w \in W_1(x)} \mathbb{Z}\mu([wx_0]).$$

Proof. This is an immediate consequence of Proposition 2.10, since the sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined in (4.1) satisfies conditions **(C1)** and **(C3)** with $m = 1$ and $d = |\mathcal{A}| - 1$, and this is true for any $x \in X$. Since the bases of $\mathcal{P}_1$ are the cylinders $\{[w_{1,1}x_0], [w_{2,1}x_0], \cdots, [w_{d-1,1}x_0]\}$, the result follows. □

Theorem 4.16. *Let (X,T) be a minimal dendric subshift on a d-letter alphabet $\mathcal{A}$ and suppose that there exists a measure $\mu \in \mathcal{M}(X,T)$, such that $\{\mu([a]) : a \in \mathcal{A}\}$ are rationally independent. Then, the infinitesimal subgroup $\mathrm{Inf}(X,T)$ is trivial, that is, (X,T) is saturated.*

Proof. This is an immediate consequence of Proposition 2.11, since the sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined in (4.1) satisfies conditions **(C1)** and **(C3)** with $m = 0$ and $d = |\mathcal{A}|$. Since the bases of $\mathcal{P}_0$ are the cylinders of letters $\{[1], [2], \cdots, [d]\}$, whose measures are suppose to be rationally independent, the result follows. □

Theorem 4.17. *Let (X,T) be a minimal specular subshift on a d-letter alphabet $\mathcal{A}$, let $x \in X$. Suppose that there exists a measure $\mu \in \mathcal{M}(X,T)$ such that $\{\mu([wx_0]) : w \in W_1(x)\}$ are rationally independent. Then, the infinitesimal subgroup* $\mathrm{Inf}(X,T)$ *is trivial, that is, (X,T) is saturated.*

Proof. This is an immediate consequence of Proposition 2.11, since the sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined in (4.1) satisfies conditions **(C1)** and **(C3)** with $m = 1$ and $d = |\mathcal{A}|_1$ and this is true for any $x \in X$. Since the bases of $\mathcal{P}_0$ are the cylinders of letters $\{[w_{1,0}x_0], [w_{2,0}x_0], \cdots, [w_{d-1,0}x_0]\}$, whose measures are suppose to be rationally independent for μ, the result follows. □

4.4 Balance in eventually dendric subshifts.

Lemma 4.18. *Let (X,T) be a minimal eventually dendric subshift with threshold m on the alphabet $\mathcal{A}$. Let H be the following subset of $C(X,\mathbb{Z})$*

$$H = \left\{ \sum_{w\in\mathcal{L}_{m+1}(X)} \sum_{k\in K_w} \alpha(w,k)\chi_{T^k([w])} : K_w \subseteq \mathbb{Z}, |K_w| < \infty, \alpha(w,k) \in \mathbb{Z} \right\},$$

where χ_A denotes the characteristic function of the set A, for all $A \subseteq X$. Then, for all $v \in \mathcal{L}_X$ with $|v| \geq m+1$, the characteristic function $\chi_{[v]}$ belongs to H.

Proof. One first easily checks that H is a subgroup. We now proceed by induction on the length of v. The claim is clearly true if $|v| = m+1$, by setting $K_a = \{0\}$ and $\alpha(w,k) = 1$ if $w = v$, 0 otherwise. Now let $n \geq m+1$ and suppose that for all $u \in \mathcal{L}_X$ with $|u| \leq n$, one has $\chi_{[u]} \in H$. Let v be a word of length $n+1$. We write

$$v = v_0 \cdots v_n \text{ and define } \widetilde{v} = v_1 \cdots v_n,\ v' = v_0 \cdots v_{n-1},\ \bar{v} = v_1 \cdots v_{n-1}.$$

As in the proof of Theorem 4.12, we analyze separately three cases depending on the right/left extensions of $\widetilde{v}$ and v', namely $|L(\widetilde{v})| = 1$, $|R(v')| = 1$, and lastly $|L(\widetilde{v})| \geq 2$ and $|R(v')| \geq 2$, the latter case being handled thanks to Lemma 4.7.

- Suppose first that $|L(\widetilde{v})| = 1$. The only left extension of $\widetilde{v}$ is v_0, and thus, for all $x \in X$, $\chi_{[v]}(x) = \chi_{[\widetilde{v}]}(Tx)$. By induction hypothesis we have that $\chi_{[\widetilde{v}]}$ belongs to H, that is,

$$\chi_{[\widetilde{v}]} = \sum_{w\in\mathcal{L}_{m+1}(X)} \sum_{k\in K_w} \alpha(w,k)\chi_{T^k([w])},$$

for some $K_w \subseteq \mathbb{Z}, |K_w| < \infty, \alpha(w,k) \in \mathbb{Z}$, so we obtain that for all $x \in X$,

$$\chi_{[v]}(x) = \chi_{[\tilde{v}]}(Tx) = \sum_{w \in \mathcal{L}_{m+1}(X)} \sum_{k \in K_w} \alpha(w,k)\chi_{T^{k-1}([w])}(x).$$

Defining $K'_w := \{k-1 : k \in K_w\}$ for all $w \in \mathcal{L}_{m+1}(X)$, and $\beta(w,k) = \alpha(w,k+1)$ for all $k \in K'_w$, we conclude that, for all $x \in X$,

$$\chi_{[v]}(x) = \sum_{w \in \mathcal{L}_{m+1}} \sum_{k \in K'_w} \beta(w,k)\chi_{T^k([w])},$$

and then $\chi_{[v]}$ belongs to H.

- Now suppose that $|R(v')| = 1$. The only right extension of v' is v_n, and thus, for all $x \in X$, $\chi_{[v]}(x) = \chi_{[v']}(x)$. We conclude by applying the induction hypothesis.

- Finally, we assume $|L(\tilde{v})| \geq 2$ and $|R(v')| \geq 2$. Let $\mathcal{E}(\bar{v})$ be the extension graph of $\bar{v}$. It is a tree since $|\bar{v}| \geq m$, and each of the sets in its bipartition of vertices has cardinality at least two. Define $g : L(\bar{v}) \cup R(\bar{v}) \cup E(\bar{v}) \to G$ as follows. For $a \in L(\bar{v})$, $g(a) = \chi_{[a\bar{v}]}$, for $b \in R(\bar{v})$, $g(b) = \chi_{T^{-1}[\bar{v}b]}$, and for $(a,b) \in E(\bar{v})$, $g(a,b) = \chi_{[a\bar{v}b]}$.
Condition (1) of Lemma 4.7 holds by induction hypothesis. Let us check that (2) holds. Let $a \in L(\bar{v})$. For all $x \in X$, one has

$$\chi_{[a\bar{v}]} = \sum_{b \in R(\bar{v}),(a,b) \in E(\bar{v})} \chi_{[a\bar{v}b]}(x) \quad \text{and thus} \quad g(a) = \sum_{b \in R(\bar{v}),(a,b) \in E(\bar{v})} g(a,b).$$

Similarly, let $b \in R(\bar{v})$ and $x \in X$. One has

$$\chi_{T^{-1}[\bar{v}b]}(x) = \chi_{[\bar{v}b]}(Tx) = \sum_{a \in L(\bar{v}),(a,b) \in E(\bar{v})} \chi_{[a\bar{v}b]}(x).$$

We conclude that for all $b \in R(\bar{v})$, $g(b) = \sum_{a \in L(\bar{v}),(a,b) \in E(\bar{v})} g(a,b)$. We now can apply Lemma 4.7 which yields that $\chi_{[a\bar{v}b]} \in H$, for any biextension $a\bar{v}b$ of $\bar{v}$. In particular, since $(v_0, v_n) \in E(\bar{v})$, then $\chi_{[v]} \in H$.

□

The previous lemma allows us to prove the following theorem, which correponds to Theorem 1.1 in

[BCB18] in the case of dendric subshifts. Note that it describes a balance behavior which contrasts whit that of substitutive systems with rational frequencies (see Theorem 3.50).

Theorem 4.19. *Let (X,T) be a minimal eventually dendric subshift with threshold m. Then (X,T) is balanced on factors of length $m+1$ if and only if it is balanced on every factor. In particular, if (X,T) is balanced on factors of length $m+1$, then all the frequencies of factors are additive topological eigenvalues and all cylinders are bounded remainder sets.*

Proof. We assume that the eventually dendric subshift (X,T) is balanced on the factors of length $m+1$. Let C be a constant of balancedness for factors of length $m+1$. Let $v \in \mathcal{L}_X$. If $|v| < m+1$, then (X,T) is balanced on v thanks to Proposition 1.16. Suppose that $|v| \geq m+1$ and let n be a positive integer, u, w be two factors belonging to $\mathcal{L}_X$ of length $n-1$ with $n-1 > |v|$. Pick an infinite word $x \in X$ such that $u = x_{[i,i+n)}$ and $w = x_{[j,j+n)}$ for some indices $i, j \in \mathbb{Z}$. We have

$$||u|_v - |w|_v| = \left| \sum_{\ell=i}^{i+n-1-|v|} \chi_{[v]}(T^\ell x) - \sum_{\ell=j}^{j+n-1-|v|} \chi_{[v]}(T^\ell x) \right|.$$

Now, according to Lemma 4.18, for all $w \in \mathcal{L}_{m+1}(X)$, let K_w be a finite subset of $\mathbb{Z}$ such that, for all $k \in K_w$, there exists $\alpha(w,k) \in \mathbb{Z}$ verifying

$$\chi_{[v]} = \sum_{w \in \mathcal{L}_{m+1}(X)} \sum_{k \in K_w} \alpha(w,k) \chi_{T^k([w])}.$$

Then,

$$\begin{aligned}
||u|_v - |w|_v| &= \left| \sum_{\ell=i}^{i+n-1-|v|} \sum_{w \in \mathcal{L}_{m+1}(X)} \sum_{k \in K_w} \alpha(w,k) \chi_{T^k[w]}(T^\ell x) - \sum_{\ell=j}^{j+n-1-|v|} \sum_{w \in \mathcal{L}_{m+1}(X)} \sum_{k \in K_w} \alpha(w,k) \chi_{T^k[w]}(T^\ell x) \right| \\
&= \left| \sum_{w \in \mathcal{L}_{m+1}} \sum_{k \in K_w} \alpha(w,k) \left(\sum_{\ell=i}^{i+n-1-|v|} \chi_{T^k[w]}(T^\ell x) - \sum_{\ell=j}^{j+n-1-|v|} \chi_{T^k[w]}(T^\ell x) \right) \right| \\
&\leq \sum_{w \in \mathcal{L}_{m+1}(X)} \sum_{k \in K_w} |\alpha(w,k)| \left| \sum_{\ell=i}^{i+n-1-|v|} \chi_{T^k[w]}(T^\ell x) - \sum_{\ell=j}^{j+n-1-|v|} \chi_{T^k[w]}(T^\ell x) \right| \\
&= \sum_{w \in \mathcal{L}_{m+1}} \sum_{k \in K_w} |\alpha(w,k)| \left| \sum_{\ell=i}^{i+n-1-|v|} \chi_{[w]}(T^\ell (T^{-k} x)) - \sum_{\ell=j}^{j+n-1-|v|} \chi_{[w]}(T^\ell (T^{-k} x)) \right| \\
&= \sum_{w \in \mathcal{L}_{m+1}(X)} \sum_{k \in K_w} |\alpha(w,k)| \cdot ||(T^{-k}x)_{[i,i+n-|v|+|w|)}|_w - |(T^{-k}x)_{[j,j+n-|v|+|w|)}|_w|.
\end{aligned}$$

Note that $(T^{-k}x)_{[i,i+n-|v|+|w|)}$ and $(T^{-k}y)_{[j,j+n-|v|+|w|)}$ are two factors of length $n-1-|v|+|w|$ belonging to $\mathcal{L}_X$, and then by balance on the factors of length $m+1$, for all $w \in \mathcal{L}_{m+1}(X)$,

$$||(T^{-k}x)_{[i,i+n-|v|+|w|)}|_w - |(T^{-k}y)_{[j,j+n-|v|+|w|)}|_w| \leq C.$$

We obtain that $||u|_v - |w|_v| \leq |\mathcal{L}_{m+1}(X)|KC$, where $K = \max_{w\in\mathcal{L}_{m+1}(X)} \left\{\sum_{k\in K_w} |\alpha(w,k)|\right\}$, which ends the proof of the balance on v. We conclude that (X,T) is balanced in every factor $v \in \mathcal{L}_X$. Lastly, the result on additive topological eigenvalues comes from Proposition 1.27. □

Corollary 4.20. *Let (X,T) be a minimal dendric (resp. specular) subshift. Then (X,T) is balanced on the letters (resp. on the factors of length 2) if and only if it is balanced on every factor. In particular, if (X,T) is balanced on the letters (resp. on the factors of length 2), then all the frequencies of factors are additive topological eigenvalues and all cylinders are bounded remainder sets.*

As a consequence of Theorem 4.19 and Proposition 1.24, we otain the following corollaries.

Corollary 4.21. *Let (X,T) be a minimal eventually dendric subshift with threshold m. If (X,T) is balanced on factors os length $m+1$, then the infinitesimal subgroup* $\mathrm{Inf}(X,T)$ *is trivial, that is, $(X,T)$is saturated.*

Corollary 4.22. *Let (X,T) be a minimal dendric (resp. specular) subshift. If (X,T) is balanced on letters (resp. on factors of length 2), then the infinitesimal subgroup* $\mathrm{Inf}(X,T)$ *is trivial, that is, (X,T) is saturated.*

Example 4.23. Balance in Arnoux-Rauzy words

Arnoux-Rauzy words (see Example 4.3) can also be expressed in S-adic terms as follows. Let $\mathcal{A} = \{1,2,\ldots,d\}$. We define the set $\mathcal{S}_{AR}$ of substitutions as $\mathcal{S}_{AR} = \{\sigma_i : i \in \mathcal{A}\}$, with $\sigma_i : i \mapsto i,\ j \mapsto ji$ for $j \in \mathcal{A} \setminus \{i\}$. An infinite word $u \in \mathcal{A}^{\mathbb{Z}}$ is an Arnoux-Rauzy word if and only if its language coincides with the language of a word of the form $\lim_{n\to\infty} \sigma_{i_0}\sigma_{i_1}\cdots\sigma_{i_n}(1)$, where the sequence $\mathbf{i} = (i_n)_{n\geq 0} \in \mathcal{A}^{\mathbb{N}}$ is such that every letter in $\mathcal{A}$ occurs infinitely often in $\mathbf{i} = (i_n)_{n\geq 0}$.

In this latter case, the infinite word u is uniformly recurrent and we can associate with it the subshift $(X_{\mathbf{i}}, T)$ which contains all the bi-infinite words having the same language as u. For any given Arnoux-Rauzy word, the sequence $\mathbf{i} = (i_n)_{n\geq 0}$ is called the $\mathcal{S}_{AR}$-*directive word* of u. All the Arnoux-Rauzy words that belong to the dynamical system $(X_{\mathbf{i}}, T)$ have the same $\mathcal{S}_{AR}$-directive word. An *Arnoux-Rauzy substitution* is a finite product of substitutions in $\mathcal{S}_{AR}$. For more details on the S-adic representation of Arnou-Rauzy words, see [BCS13, Section 2].

Let σ be a primitive Arnoux-Rauzy substitution. Then, (X_σ, T) is balanced on factors. Indeed, Arnoux-Rauzy substitutions are known to be Pisot (see [AI01] or [AD15]): primitive substitutions such that the dominant eigenvalue of their substitution matrix is a Pisot number, that is, an algebraic integer whose conjugates lie strictly inside the unit disk. Thus, they generate words that are balanced on letters (see [Adam03, Section 6] for details), and consequently on factors by Theorem 4.19.

Let $(X_{\mathbf{i}}, T)$ be an Arnoux-Rauzy subshift on a three-letter alphabet with $\mathcal{S}_{AR}$-directive sequence $\mathbf{i} = (i_n)_{n\geq 0}$. If there exists some constant h such that we do not have $i_n = i_{n+1} = \cdots = i_{n+h}$ for any $n \geq 0$, then $(X_{\mathbf{i}}, T)$ is balanced on factors. Indeed, it is shown in [BCS13] that $(X_{\mathbf{i}}, T)$ is $(2h+1)$-balanced on letters. We again conclude thanks to Theorem 4.19.

4.5 Dimension group of dendric and specular subshifts.

The following result is presented in [BCD+18] for minimal dendric subshifts.

Theorem 4.24. *Let (X, T) be a minimal dendric subshift on a r-letter alphabet. Let $\mathcal{M}(X, T)$ stand for its set of invariant measures. Then, its dimension group $K^0(X, T)$ is isomorphic to*

$$(\mathbb{Z}^r, \{\mathbf{x} \in \mathbb{Z}^r \mid \langle \mathbf{x}, \mathbf{f}_\mu \rangle > 0 \textit{ for all } \mu \in \mathcal{M}(X, T)\} \cup \{0\}, \mathbf{1}),$$

where $\mathbf{f}_\mu = (\mu([\mathbf{1}]), \cdots, \mu([\mathbf{r}]))$ and $\mathbf{1} = (1, 1, \cdots, 1) \in \mathbb{Z}^d$.

Proof. This is a consequence of Proposition 2.17, since the sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined in (4.1) satisfies **(C0)-(C3)** for $m = 0$, $d = r$, and all heigths in $\mathcal{P}_0$ are equal to 1. □

Theorem 4.25. *Let (X, T) be a minimal specular subshift on a r-letter alphabet. Let $\mathcal{M}(X, T)$ stand for its set of invariant measures. Then, its dimension group $K^0(X, T)$ is isomorphic to*

$$(\mathbb{Z}^r, \{\mathbf{x} \in \mathbb{Z}^r \mid \langle \mathbf{x}, \mathbf{f}_\mu \rangle > 0 \textit{ for all } \mu \in \mathcal{M}(X, T)\} \cup \{0\}, \mathbf{u}),$$

where $\mathbf{f}_\mu = (\mu([\mathbf{w_{1,1}x_0}]), \cdots, \mu([\mathbf{w_{d-1,1}x_0}]))$ and $\mathbf{u} = (|w_{1,1}|, \cdots, |w_{d-1,1}|) \in \mathbb{Z}^r$.

Proof. This is a consequence of Proposition 2.17, since the sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined in (4.1) satisfies **(C0)-(C3)** for $m = 1$, $d = r - 1$, and this is true for all $x \in X_{\boldsymbol{\sigma}}$. The tower heigths in $\mathcal{P}_1$ corresponds to the lengths of return words to x_0. □

Corollary 4.26. *Two minimal dendric (resp. specular) subshifts are strong orbit equivalent if and only they have the same simplex of letter frequencies (resp. the same simplex generated by frequencies of factors of length* 2*). Two minimal and uniquely ergodic dendric (resp. specular) subshifts are strong orbit equivalent if and only if they have the same additive group of letter frequencies (resp. the same additive group generated by the frequencies of factors of length* 2*).*

Since on a three-letter alphabet, all dendric subshifts are uniquely ergodic, we deduce from Corollary 4.26 the following.

Corollary 4.27. *All minimal dendric subshifts on a three-letter alphabet with the same group of letter frequencies are strong orbit equivalent.*

4.6 Further work.

Some of the results we have proved for dendric and specular subshifts rely on the fact that the set of return words associated to every word (or every non-empty word) is a basis of the same subgroup of $\mathbb{F}_{\mathcal{A}}$ (for instance, Theorems 4.14 and 4.15, Theorems 4.16 and 4.17, and Theorems 4.24 and 4.25). This crucial fact is a sufficient condition for the matrices of the sequence of tower partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined in (4.1) to be invertible, and thus for $(\mathcal{P}_n)_{n\in\mathbb{N}}$ to satisfy condition **(C3)** of Chapter 2.
Note that the important thing here is not *which* subgroup of $\mathbb{F}_{\mathcal{A}}$ all these sets generate, but just the fact that it is *the same subgroup* for all (non-empty) word.
It is not the case that in an eventually dendric subshift with threshold m, the set of return words of any factor of length at least m generates the same subgroup of the free group $\mathbb{F}_{\mathcal{A}}$. This is the case of specular subshifts, where supplementary conditions have been added, apart from being eventually dendric with threshold 1.
It is thus an interesting question to know under which conditions we have that in an eventually dendric language with threshold m, the set of return words of any factor of length greater than some N (which N?) generates the same subgroup of the free group $\mathbb{F}_{\mathcal{A}}$.

Chapter 5

Subshifts of congruent monotileable amenable groups.

In this chapter we study the set of invariant measures on subshifts of a special kind of groups, namely congruent monotileable amenable groups. We recall the notions of amenability and monotileable amenable groups. We introduce the concept of *congruent* in this context, and show that any Choquet simplex can be obtained as a set of invariant measures of a minimal subshift of any congruent monotileable amenable group. We also show that this class of groups includes all virtually nilpotent groups.

5.1 Amenable groups.

We give here a brief introduction on amenable groups. We refer to [CC10, Chapter 4] for a complete survey of most important results in this topic.

5.1.1 Invariant measures on groups.

Let S be any set and denote $\mathcal{P}(S)$ the power set of S. We say that a map $\mu : \mathcal{P}(S) \to [0, 1]$ is a *finitely additive probability measure* on S if $\mu(S) = 1$ and $\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2)$ whenever B_1, B_2 are subsets of S with $B_1 \cap B_2 = \emptyset$. For any given set S, we denote by $\mathcal{PM}(S)$ the set of all finitely additive probability measures on S. Let G be a group, and consider the following left and right actions of G on $\mathcal{PM}(G)$

$$(g.\mu)(A) = \mu(g^{-1}A), \qquad (\mu.g)(A) = \mu(Ag^{-1}), \quad \forall A \subseteq G,$$

where $gA = \{ga : a \in A\}$ and $Ag = \{ag : a \in A\}$ for all $A \subseteq G$, for all $g \in G$. A measure $\mu \in \mathcal{PM}(G)$ is said to be *left-invariant* (resp. *right-invariant*) if $g.\mu = \mu$ (resp. $\mu.g = \mu$) for all $g \in G$, that is, is μ is invariant under the left (resp. right) action of G described above.

Proposition 5.1. *The group G admits a left-invariant finitely additive probability measure if and only if it admits a right-invariant probability measure.*

Proof. Suppose G admits a left-invariant finitely additive probability measure μ. Define $\bar{\mu} : \mathcal{P}(G) \to [0,1]$ by $\bar{\mu}(A) := \mu(A^{-1})$ for all $A \subseteq G$, where

$$A^{-1} = \{a^{-1} : a \in A\}.$$

Note that $\bar{\mu}$ is a right-invariant finitely additive probability measure on G. Indeed, $\bar{\mu}(G) = \mu(G^{-1}) = \mu(G) = 1$; if A, B are subsets of G with $A \cap B = \emptyset$, then

$$\bar{\mu}(A \cup B) = \mu((A \cup B)^{-1}) = \mu(A^{-1} \cup B^{-1})$$

and since $A \cap B = \emptyset$, $A^{-1} \cap B^{-1} = \emptyset$. We get $\mu(A^{-1} \cup B^{-1}) = \mu(A^{-1}) + \mu(B^{-1}) = \bar{\mu}(A) + \bar{\mu}(B)$. Finally, since μ is left-invariant, for all $g \in G$ and for all $A \subseteq G$,

$$\bar{\mu}.g(A) = \bar{\mu}(Ag^{-1}) = \mu(gA^{-1}) = \mu(A^{-1}) = \bar{\mu}(A),$$

so $\bar{\mu}$ is right-invariant and G admits a right-invariant finitely additive probability measure.
The converse implication is completely analogous. $\square$

A group G is called *amenable* if it admits a left- (or right-) invariant finitely additive probability measure. Note that finite groups are trivially amenable: let N be the cardinality of a finite group G and define

$$\mu(A) = \frac{|A|}{N} \qquad \forall A \subseteq G.$$

The map μ is a left- and right-invariant finitely additive probability measure on G.

Remark 5.2. *The equivalent conditions of Proposition 5.1 are also equivalent to the existence of a left- (or right-) invariant mean on G: a map $m : \ell^\infty(G) \to \mathbb{R}$ verifying $m(1) = 1$ and $m(x) \geq 0$ whenever $x \geq 0$, where $\ell^\infty(G)$ is the set of all bounded real sequences indexed by G endowed with the* *order $x \leq y \Leftrightarrow x(g) \leq y(g)$ for all $g \in G$, and 1 is the sequence $1(g) = 1_G$ for all $g \in G$ (see [CC10, Proposition 4.4.4]). The set $\mathcal{M}(G)$ of invariant means on G is a convex compact subset of the dual space $(\ell^\infty(G))^*$ with respect to the weak-$*$ topology (see [CC10, Theorem 4.2.1]). The group G acts continuously on $\mathcal{M}(G)$ in the following way: consider the left and right actions of G on $\mathbb{R}^G$ given by*

$$(g.x)(h) = x(g^{-1}h), \qquad (x.g)(h) = x(hg^{-1}), \quad \forall h \in G.$$

The space $\ell^\infty(G)$ is a vector subspace of $\mathbb{R}$ which is invariant under these actions. It is not difficult to show that both actions of G on $\ell^\infty(G)$ are isometric, and therefore continuous. The left (resp. right) action of G restricted to $\mathcal{M}(G)$ is affine and continuous with respect to the weak-$*$ topology (see [CC10, Proposition 4.3.1]).

Example 5.3. (The free group in two generators.)

Let us show that the free group in two generators $\mathbb{F}_2$ is not amenable. Write $\mathbb{F}_2$ in its canonical form, $\mathbb{F}_2 = \langle a, b\rangle$. Suppose there exists a left-invariant finitely additive probability measure μ on $\mathbb{F}_2$. Denote by A the subset of $\mathbb{F}_2$ consisting of all reduced words starting with a non-zero power of a. Note that $\mathbb{F}_2 = A \cup aA$ and therefore

$$\mu(\mathbb{F}_2) \leq \mu(A) + \mu(aA).$$

Since μ is left-invariant, $\mu(aA) = \mu(A)$, which implies that $2\mu(A) \geq \mu(\mathbb{F}_2) = 1$, and therefore, $\mu(A) \geq 1/2$. On the other hand, note that for all $\ell > 2$, the subsets A, bA, $b^2A, \cdots$, $b^\ell A$ are pairwise disjoints, so that

$$\mu(A) + \mu(bA) + \cdots + \mu(b^\ell A) = \mu(A \cup bA \cup b^\ell A) \leq \mu(\mathbb{F}_2) = 1.$$

Since μ is left-invariant, $\mu(A) + \mu(bA) + \cdots + \mu(b^\ell A) = (\ell + 1)\mu(A)$, which implies that $\mu(A) \leq 1/\ell$, a contradiction.

Proposition 5.4. *Every subgroup of an amenable group is amenable.*

Proof. Let G be an amenable group with a left-invariant finitely additive probability measure μ, and let $H \leqslant G$. Let R be a set of representatives of right cosets of H in G, that is, a subset $R \subseteq G$ such that G is the disjoint union of the cosets $\{Hr : r \in R\}$,

$$G = \bigcup_{r \in R} Hr.$$

Define $\widetilde{\mu} : \mathcal{P}(H) \to [0,1]$ by

$$\widetilde{\mu}(A) = \mu\left(\bigcup_{r\in R} Ar\right).$$

We claim that $\widetilde{\mu}$ belongs to $\mathcal{PM}(G)$. First, $\widetilde{\mu}(H)$ equals 1 by construction. If A, B are two disjoint subsets of H, then for all $r \in R$, $Ar \cap Br = \emptyset$, from which we deduce

$$\widetilde{\mu}(A\cup B) = mu\left(\bigcup_{r\in R}(A\cup B)r\right) = mu\left(\bigcup_{r\in R} Ar \cup \bigcup_{r\in R} Br\right) = mu\left(\bigcup_{r\in R} Ar\right) + \mu\left(\bigcup_{r\in R} Br\right) = \widetilde{\mu}(A) + \widetilde{\mu}(B).$$

Finally, since μ is left-invariant, for all $h \in H$ and for all $A \subseteq H$,

$$\widetilde{\mu}.h(A) = \widetilde{\mu}(hA) = \mu\left(\bigcup_{r\in R} hAr\right) = \mu\left(h\bigcup_{r\in R} Ar\right) = \mu\left(\bigcup_{r\in R} Ar\right) = \widetilde{\mu}(A),$$

and we conclude that $\widetilde{\mu}$ is invariant. Therefore, H admits a left-invariant finitely additive probability measure and thus it is amenable. □

Corollary 5.5. *Free groups are not amenable.*

Proof. Let $\mathbb{F}_n$ the free group on n generators,

$$\mathbb{F}_n = \langle a_0, \cdots, a_{n-1}\rangle.$$

Suppose $\mathbb{F}_n$ is amenable. Proposition 5.4 implies that every subgroup of $\mathbb{F}_n$ is amenable. Consider the subgroup H of $\mathbb{F}_n$ generated by a_0 and a_1. H is isomorphic to $\mathbb{F}_2$, but we know by 5.3 that $\mathbb{F}_2$ is not amenable, which is a contradiction. □

The proofs of the following two results are based on the properties of the set $\mathcal{M}(G)$ (see Remark 5.2). Details can be found in [CC10, Sections 4.5 and 4.6].

Proposition 5.6. *[CC10, Proposition 4.5.5] Let G be a group and $H \lhd G$. If both H and G/H are amenable, then G is amenable.*

Theorem 5.7. *[CC10, Theorem 4.6.1] Abelian groups are amenable.*

Recall that, given a group G, the *commutator* of two elements $g, h \in G$ is the element $[h,g] = hgh^{-1}g^{-1}$. If H, K are two subgroups of G, the commutator of H and K is the subgroup of G generated by all commutators $[h,k]$ where $h \in H$ and $k \in K$. Note that $[H,K]$ is a normal subgroup of G and $G/[G,G]$

is abelian. The *derived series* of G is a decreasing series of normal subgroups of G defined inductively as follows: $D^0(G) = G$, $D^1(G) = [G, G]$ and for all $n \geq 1$, $D^{n+1}(G) = D(D^n(G))$. For all $n \geq 0$, $D^{n+1}(G) \lhd D^n(G)$ and $D^{n+1}(G)/D^n(G)$ is an abelian group. The group G is said to be *solvable* if there exists a positive integer N such that $D^N(G)$ is trivial. In this case, N is called the *class* or the *degree of solvability* of G.

Theorem 5.8. *Solvable groups are amenable.*

Proof. We proceed by induction on the solvability class n. If $n = 0$ and G is a solvable group of class n, then G is trivial and thus amenable as any finite group. Suppose every solvable group of solvability class n is amenable, and let G be a solvable group with solvability class $n + 1$. The group $D(G)$ is solvable of class n, thus amenable by inductive hypothesis. The quotient group $G/D(G)$ is abelian, thus amenable by Theorem 5.7. By Proposition 5.6, G is amenable. □

Given a group G, the *lower central series* of G is a decreasing sequence of normal subgroups defined inductively as follows: $C^0(G) = G$, and for all $n \geq 0$, $C^{n+1}(G) = [C^n(G), G]$. The group is said to be *nilpotent* if there exists a positive integer N such that $C^N(G)$ is trivial. In this case, N is called the *class* or the *degree of nilpotency* of G. Every nilpotent group is easily shown to be solvable, so we obtain the following result.

Corollary 5.9. *Nilpotent groups are amenable.*

5.1.2 Residually finiteness and amenability.

Recall from Section 1.3 that a countable group G is residually finite if for every element $g \in G$ with $g \neq 1_G$ there exists a finite group F and a homomorphism $\phi_g : G \to F$ such that $\phi_g(g) \neq 1_F$.

Proposition 5.10. *Let G be a countable group. Then, G is residually finite if and only if there exists a sequence of finite index normal subgroups of G, say $\{G_n\}_{n\in\mathbb{N}}$, such that $\bigcap_{n\in\mathbb{N}} G_n$ is trivial.*

Residually finite groups are thus those groups for which we can define odometers (see Section 1.3). Finite groups are trivially residually finite: if $g \in G$ verifies $g \neq 1_G$, define $\phi_g = Id_G : G \to G$. Let $p \in \mathbb{N}$ be a prime number. The sequence $G_n = p^n\mathbb{Z}$ is a decreasing sequence of finite index normal subgroups of $\mathbb{Z}$ whose intersection is trivial, so by Proposition 5.10, $\mathbb{Z}$ is residually finite. Since every direct product of a family of residually finite groups is residually finite (see [CC10, Proposition 2.2.2]), $\mathbb{Z}^d$ is residually finite for every $d \geq 1$.

Example 5.11. (The group $GL_n(\mathbb{Z})$)

Let us prove that the linear group $GL_n(\mathbb{Z})$ is residually finite for all $n \geq 1$. Let $A \in GL_n(\mathbb{Z})$ such that $A \neq I_{n\times n}$. Choose a positive integer $\ell \in \mathbb{Z}$ such that $|A(i,j)| < \ell$ for all $1 \leq i, j \leq n$, and define $\phi_A : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/\ell\mathbb{Z})$ by

$$\phi_A(M)(i,j) = M(i,j) \mod \ell.$$

Then, ϕ_A is a group homomorphism satisfying $\phi_A(A) \neq 1_{GL_n(\mathbb{Z}/\ell\mathbb{Z})}$.
Finitely generated nilpotent groups are residually finite, so finitely generated nilpotent groups are examples of amenable residually finite groups. Classical examples of non-residually finite groups are *divisible* groups: a group is said to be *divisible* if for all $g \in G$ and all $n \geq 1$, there exists $h \in G$ such that $h^n = g$.

Example 5.12. (The additive group of rationals)

The additive group $\mathbb{Q}$ is clearly divisible: let $g \in \mathbb{Q}$ and $n \geq 1$, take $h = \frac{g}{n} \in \mathbb{Q}$. We have that

$$h^n = \underbrace{\frac{g}{n} + \cdots + \frac{g}{n}}_{n-times} = n\frac{g}{n} = g.$$

Example 5.13. (The Prüfer group)

Given a prime number p, the Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the following subgroup of the unit circle,

$$\mathbb{Z}(p^\infty) = \{\exp(i2\pi m/p^n) : 0 \leq m < p^n, n \in \mathbb{N}\},$$

that is, the set of all p^n-th roots of unity, when n runs over $\mathbb{N}$. Equivalently, $\mathbb{Z}(p^\infty)$ can be represented as the inverse limit

$$\mathbb{Z}(p^\infty) = \varinjlim(\mathbb{Z}/p^n\mathbb{Z}, i_n),$$

where $i_n : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$ is the multiplication by p. By definition, Prüfer groups are p-groups, and they are divisible. They are also infinite countable abelian groups (see [Fu15] for more details).

Theorem 5.14. *A non-trivial divisible group is not residually finite.*

Proof. Suppose G is a non-trivial divisible group and let F any finite group. Let $n = |F|$. Since G is divisible, for all $g \in G$, there exists $h \in G$ such that $h^n = g$. Let $\phi : G \to F$ a group homomorphism, then $\phi(g) = \phi(h^n) = \phi(h)^n$. Since F has n elements, $f^n = 1_F$ for all $f \in F$, and thus $\phi(g) = 1_F$. Since this is true for all $g \in G$, we conclude that ϕ is trivial, and since G is non-trivial, it cannot be residually finite. □

The previous theorem shows that $\mathbb{Q}$ and the Prüfer group are examples of countable infinite abelian non-residually finite groups. They are not finitely generated. There also exist finitely generated groups which are not residually finite (see [CC10, Section 2.6]).

5.1.3 The Følner conditions.

We now recall an equivalent definition of amenability which is the one we use in this chapter for all proofs. Given a countable group, a sequence $(F_n)_{n\geq 0}$ of finite subsets of G is called a *right Følner sequence* if for every $g \in G$,

$$\lim_{n\to\infty} \frac{|F_n g \setminus F_n|}{|F_n|} = 0.$$

A *left Følner sequence* of G is defined analogously.

Proposition 5.15. *The countable group G admits a right Følner sequence if and only if it admits a left Følner sequence.*

Proof. Suppose $(F_n)_{n\geq 0}$ is a right Følner sequence of G. For all $n \geq 0$, define $\widetilde{F}_n = F_n^{-1}$. Observe that $(\widetilde{F}_n)_{n\geq 0}$ is a left Følner sequence of G. Indeed, for any $g \in G$,

$$\lim_{n\to\infty} \frac{|g\widetilde{F}_n \setminus \widetilde{F}_n|}{|\widetilde{F}_n|} = \lim_{n\to\infty} \frac{|gF_n^{-1} \setminus F_n^{-1}|}{|F_n|} = \lim_{n\to\infty} \frac{|(F_n g^{-1} \setminus F_n^{-1})^{-1}|}{|F_n|} = \lim_{n\to\infty} \frac{|F_n g^{-1} \setminus F_n|}{|F_n|}.$$

Since $(F_n)_{n\geq 0}$ is a right Følner sequence, the previous limit equals zero and we get that $(\widetilde{F}_n)_{n\geq 0}$ is a left Følner sequence.

The converse implication is completely analogous. □

Given finite subsets K, F of G and $\varepsilon > 0$, we say that F is right (K, ε)-invariant if

$$\frac{|\{g \in F : gK \subset F\}|}{|F|} \geq (1 - \varepsilon).$$

The concept of *left* (K, ε)-invariance is defined in the same way: for finite subsets $F, K \subseteq G$ and $\varepsilon > 0$, we say that F is left (K,ε)-invariant if

$$\frac{|\{g \in F : Kg \subset F\}|}{|F|} \geq (1-\varepsilon).$$

Note that if F is right (K,ε)-invariant, then F^{-1} is left (K^{-1},ε)-invariant.
The following lemma shows that right Følner sequences are those which become more and more invariant.

Lemma 5.16. *Let G be a countable group. A sequence $(F_n)_{n\geq 0}$ is a right Følner sequence of G if and only if for every finite subset K of G and for every $\varepsilon > 0$, there exists $N \geq 0$ such that for all $n \geq N$, F_n is right (K,ε)-invariant.*

Proof. First note that

$$\{g \in F_n : gK \subset F_n\} = \bigcap_{k\in K} F_n \cap F_n k^{-1}.$$

Suppose $(F_n)_{n\geq 0}$ is a right Følner sequence, and take any finite $K \subseteq G$, $\varepsilon > 0$. Since for all $k \in K$, $\lim_{n\to\infty} \frac{|F_n \setminus F_n k^{-1}|}{|F_n|} = 0$, then for all $k \in K$, $\exists\ N_k \in \mathbb{N}$ such that for all $n \geq N_k$,

$$\frac{|F_n \setminus F_n k^{-1}|}{|F_n|} < \frac{\varepsilon}{|K|}.$$

Since K is finte, there is a $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $k \in K$, the above inequality holds. Now,

$$F_n \setminus \bigcap_{k\in K} F_n \cap F_n k^{-1} = \bigcup_{k\in K} F_n \setminus F_n k^{-1},$$

then,

$$\begin{aligned} |F_n| - |\bigcap_{k\in K} F_n \cap F_n k^{-1}| &= |F_n \setminus \bigcap_{k\in K} F_n \cap F_n k^{-1}| \\ &= |\bigcup_{k\in K} F_n \setminus F_n k^{-1}| \\ &\leq \sum_{k\in K} |F_n \setminus F_n k^{-1}| < |F_n|\varepsilon, \end{aligned}$$

and therefore,

$$|F_n| - |\{g \in F_n : gK \subset F_n\}| < \varepsilon |F_n|.$$

Conversely, suppose that for any $\varepsilon > 0$ and any finite $K \subseteq G$, there is an $N \in \mathbb{N}$ such that for all

$n \geq N$, F_n is right (K, ε)-invariant. Take $g \in G$ and consider $K = \{g\}$. Let $\varepsilon > 0$. Then, there is a $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|F_n \cap F_n g^{-1}| > (1-\varepsilon)|F_n|$$

$$\Leftrightarrow \forall n \geq N, \quad \frac{|F_n \setminus F_n \cap F_n g^{-1}|}{|F_n|} < \varepsilon$$

But $F_n \setminus F_n \cap F_n g^{-1} = F_n \setminus F_n g^{-1}$, and therefore, for all $n \geq N$,

$$\frac{|F_n \setminus F_n g^{-1}|}{|F_n|} < \varepsilon.$$

This implies that $\lim_{n\to\infty} \frac{|F_n g \setminus F_n|}{|F_n|} \leq \varepsilon$. Since ε was arbitrarily taken, we conclude that

$$\lim_{n\to\infty} \frac{|F_n g \setminus F_n|}{|F_n|} = 0.$$

□

The equivalent conditions of Proposition 5.15 are known as *the Følner conditions.* The following theorem due to Følner states the equivalence between the Følner conditions and the amenability of a group. See [CC10, Theorem 4.9.1] for a proof.

Theorem 5.17. *Let G be a countable group. Then, G is amenable if and only if G satisfies the Følner conditions.*

In the case of abelian groups, it is clear that every right Følner sequence is a left Følner sequence and *vice versa*, and the notions of right and left $(K, \varepsilon)-$invariance coincide.

5.2 Monotileable amenable groups.

Let G be a countable infinite group. A *left monotile* of G is a finite subset F of G for which there exists a subset C of G such that the collection $\mathcal{T} = \{cF : c \in C\}$ is a *partition* of G, that is, $G = \bigcup_{c\in C} cF$ and $c_1F \cap c_2F = \emptyset$ if $c_1, c_2 \in C$ are distinct elements. In this case we say that $\mathcal{T}$ is a *left monotiling.* A *right monotile* of G is a finite subset F of G for which there exists a subset C of G such that the collection $\mathcal{T} = \{Fc : c \in C\}$ is a partition of G. In this case we say that $\mathcal{T}$ is a *right monotiling.*
We say that G is *monotileable amenable* if there exists a right Følner sequence $(F_n)_{n\geq 0}$ of G such that every F_n is a left monotile of G. Note that if G is monotilable amenable, then $(F_n^{-1})_{n\geq 0}$ is a left Følner

sequence whose elements are right monotiles of G.

Given a sequence $(F_n)_{n\geq 0}$ of finite subsets of G, we say that the sequence is *congruent* if $1_G \in F_0$ and for every $n \geq 0$ the exists a set $J_n \subseteq G$ such that $1_G \in J_n$ and such that $\{cF_n : c \in J_n\}$ is a partition of F_{n+1}.

We say that G is *congruent monotileable* if it admits a congruent right Følner sequence made of left monotiles, which is also *exhaustive*, that is, $G = \bigcup_{n\geq 0} F_n$.

Lemma 5.18. *Let G be a congruent monotilable amenable group with a right Følner sequence $(F_n)_{n\geq 0}$ made of congruent left monotiles. Then, for every $m > n \geq 0$, the collection*

$$\{c_{m-1}\cdots c_n F_n : c_i \in J_i, \text{ for every } n \leq i < m\}$$

is a partition of F_m.

Proof. The proof follows directly from the definition of congruent monotileable amenable groups by using induction. □

The next lemma is the key tool to show that countable abelian and nilpotent groups are congruent monotilable. If M is any set, $\sim$ is an equivalence relation on M, $\pi : M \to M/\sim$ is the canonical projection and $H \subseteq M/\sim$, we say that $\widehat{H} \subseteq M$ is a *lifting* of H if $\pi(\widehat{H}) = H$ and π is one-to-one in $\widehat{H}$.

Lemma 5.19. *Let L, G and Q be countable discrete amenable groups such that $1 \to L \to G \to Q \to 1$ is an exact sequence. Suppose L and Q have congruent and exhaustive right Følner sequences made of left monotiles, $(U_s)_{s\geq 0}$ and $(T_s)_{s\geq 0}$ respectively. Then, G has an exhaustive right Følner sequence made of left monotiles. More precisely, there exists a sequence $(\widehat{T}_s)_{s\geq 0}$, such that each $\widehat{T}_s \subseteq G$ is a lifting of T_s, and an increasing sequence of indices, $(m_s)_{s\geq 0}$, such that $F_s = U_{m_s}\widehat{T}_s$ defines an exhaustive right Følner sequence made of letf monotiles of G. If in adittion $L \subseteq Z(G)$, then $(F_s)_{s\geq 0}$ is also congruent.*

Proof. Let $(K_s)_{s\geq 0}$ be an increasing sequence of finite subsets of G such that $G = \bigcup_{s\geq 0} K_s$. Let $(\varepsilon_s)_{s\geq 0}$ be a decreasing to zero sequence of positive reals. Let $\pi : G \to G/L \cong Q$ be the projection of G on G/L. Since $(T_s)_{s\geq 0}$ is a right Følner sequence, Lemma 5.16 tells us that, up to take an increasing subsequence, we can assume that for all $s \geq 0$, T_s is right $(\pi(K_s), \varepsilon_s/2)$-invariant.

We define inductively $(\widehat{T}_s)_{s\geq 0}$ and $(m_s)_{s\geq 0}$. For $s = 0$, define $\widehat{T}_0 = 1_G$ and $m_0 = 0$, so that $1_G \in F_0 = U_0$.

If $s > 0$, suppose we have defined $\widehat{T}_{s-1}$ and m_{s-1}. By hypothesis, there exists a finite subset $D \subseteq Q$ containing 1_Q such that

$$T_s = \bigcup_{d \in D} dT_{s-1}$$

Take any lifting $\widehat{D}$ of D, and define $\widehat{T}_s = \bigcup_{\hat{d} \in \widehat{D}} \hat{d}\widehat{T}_{s-1}$. It is straightforward to show that $\widehat{T}_s$ is a lifting of T_s and that $\widehat{T}_s = \bigcup_{\hat{d} \in \widehat{D}} \hat{d}\widehat{T}_{s-1}$.

Note that T_s^{-1} is a left $(\pi(K_s^{-1}), \varepsilon_s/2)$-invariant right monotile of Q and that $\widehat{T}_s^{-1}$ is a lifting of T_s^{-1}. Then, by [W01, Theorem 2], there exists a subset $J \subseteq L$ such that, if $U \subseteq L$ is a left $(J, \varepsilon_s/2)$-invariant right monotile of L, then $\widehat{T}_s^{-1}U$ is a left $(K_s^{-1}, \varepsilon_s)$-invariant right monotile of G.

Now, every U_s is a left monotile of L and therefore every U_s^{-1} is a right monotile of L. Since $(U_s^{-1})_{s \geq 0}$ is a letf Følner sequence, the U_s^{-1}'s are as much left invariant as we want. Pick an index $m_s > m_{s-1}$ such that $U_{m_s}^{-1}$ is left $(J, \varepsilon_s/2)$-invariant. By [W01, Theorem 2], $\widehat{T}_s^{-1}U_{m_s}^{-1}$ is a left $(K_s^{-1}, \varepsilon_s)$-invariant right monotile of G, and therefore $F_s := U_{m_s}\widehat{T}_s$ is a right (K_s, ε_s)-invariant left monotile of G. This shows that $(F_s)_{s \geq 0}$ is a right Følner sequence made of left monotiles of G.

Let us show that $(F_s)_{s \geq 0}$ is exhaustive. Let $g \in G$, then there is a s_0 such that $\pi(g) \in T_{s_0}$, since $(T_s)_{s \geq 0}$ is exhaustive. Then, there exist $l \in L$ and $\hat{t} \in \widehat{T}_{s_0}$ such that $g = l\hat{t}$. For $s \geq 0$ big enough, $l \in U_{m_s}$ and $\hat{t} \in \widehat{T}_s$, therefore $g \in F_s$.

Finally, let us see that, if $L \subseteq Z(G)$, $(F_s)_{s \geq 0}$ is congruent. Let $s \in \mathbb{N}$. Since $(m_s)_{s \geq 0}$ is increasing and $(U_s)_{s \geq 0}$ is congruent, Lemma 5.18 tells us that there exists a finite subset of L, C, such that

$$U_{m_{s+1}} = \bigcup_{c \in C} cU_{m_s}$$

On the other hand, by construction, there is a set $\widehat{E} \subseteq G$ such that

$$\widehat{T}_{s+1} = \bigcup_{\hat{e} \in \widehat{E}} \hat{e}\widehat{T}_s$$

Therefore,

$$\begin{aligned} F_{s+1} &= \bigcup_{c\in C} cU_{m_s} \bigcup_{\hat{e}\in\widehat{E}} \hat{e}\widehat{T}_s \\ &= \bigcup_{c\in C} \bigcup_{\hat{e}\in\widehat{E}} cU_{m_s}\hat{e}\widehat{T}_s \\ &= \bigcup_{c\in C} \bigcup_{\hat{e}\in\widehat{E}} c\hat{e}\underbrace{U_{m_s}\widehat{T}_s}_{=F_s} \end{aligned}$$

□

Remark 5.20. *If G is a countable amenable group having a finite index subgroup which is congruent monotilable, then so is G. Indeed, it is not difficult to see that, if L is a congruent monotilable finite index subgroup of G, R is a subset of right coset representatives, and $(U_n)_{n\geq 0}$ is a congruent right Følner sequence made of left monotiles of L, then the sequence $F_n := U_nR$ defines a congruent right Følner sequence of G. This implies that any virtually congruent monotilable group is congruent monotilable as well.*

Proposition 5.21. *Every countable abelian group is congruent monotilable.*

Note that if G is finitely generated, then it is a direct product of $\mathbb{Z}^d$ and a finite abelian group, for some positive integer d, so it is trivially residually finite. Any such a group (provided it is amenable) is congruent monotilable, as it is stated in the following result.

Proposition 5.22. *[CP14, Lemma 5] Let G be an amenable residually finite group and let $(\Gamma_n)_{n\geq 0}$ be a decreasing sequence of finite index normal subgroups of G such that $\bigcap_{n\geq 0}\Gamma_n = \{1_G\}$. There exists an increasing sequence $(n_i)_{i\geq 0}$ and a Følner sequence $(F_i)_{i\geq 0}$ of G such that*

- *$\{1_G\} \subseteq F_i \subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i} for all $i \geq 0$,*
- *$G = \bigcup_{i\geq 0} F_i$, and*
- *$F_j = \bigcup_{v\in F_j\cap\Gamma_{n_i}} vF_i$ for every $0 \leq i < j$.*

A *fundamental domain* D of a quotient group G/Γ is a subset of G containing exactly one representative element of each class in G/Γ. The sequence $(F_i)_{i\geq 0}$ is exhaustive and in this case $J_i = F_{i+1} \cap \Gamma_{n_i}$. Since each F_i is a fundamental domain of Γ_{n_i}, for all distinct elements $v_1, v_2 \in J_i$, $v_1F_i \cap v_2F_i = \emptyset$, so that $(F_i)_{i\geq 0}$ is an exhaustive congruent Følner sequence made of left monotiles.

Proof. (*proof of Proposition 5.21*) Let G be a non-finitely generated abelian group, and let us enumerate its elements as $G = \{1_G = g_0, g_1, \cdots\}$. Since G is non-finitely generated we can define an increasing sequence $(k_n)_{n\geq 0}$ as follows: let $k_0 = 0$ and for $n > 0$ let define

$$k_n = \min\{l > k_{n-1} : g_{k_n} \notin \langle\{g_0, \cdots, g_{k_{n-1}}\}\rangle\}.$$

For every $n \geq 0$ we set

$$K_n = \{g_0, g_1, \cdots, g_{k_n}\} \text{ and } G_n = \langle K_n\rangle,$$

where $\langle\cdot\rangle$ denotes the generated subgroup in G.

Since G is abelian, $G_{n-1} \lhd G_n$ and G_n/G_{n-1} is an abelian group. Moreover, this is a non trivial cyclic group. Indeed, any class gG_{n-1} has the form

$$gG_{n-1} = g_0^{l_0} \cdots g_{k_n-1}^{l_{k_n-1}} g_{k_n}^{l_{k_n}} G_{n-1} \text{ where } l_i \in \mathbb{Z}.$$

Since $g_0^{l_0} \cdots g_{k_n-1}^{l_{k_n-1}} \in G_{n-1}$ and G_n is abelian, $gG_{n-1} = g_n^{l_{k_n}} G_{n-1} \neq G_{n-1}$, so that

$$G_n/G_{n-1} = \{g_{k_n}^k G_{n-1} : k \in \mathbb{Z}\}.$$

If for all $k \neq 0$ $g_{k_n}^k \notin G_{n-1}$, then $G_n/G_{n-1} \cong \mathbb{Z}$. If there is some $k \neq 0$ such that $g_{k_n}^k \in G_{n-1}$, then $G_n/G_{n-1} \cong \mathbb{Z}/l\mathbb{Z}$, where

$$l := \min\{k \in \mathbb{Z}^+ : g_n^k \in G_{n-1}\}.$$

The rest of the proof is organized as follows:

- For every $n \geq 0$, we will inductively define $(F_s^n)_{s\geq 0}$ an exhaustive and congruent right Følner sequence made of left monotiles of G_n, a positive integer m_{n-1} and a finite subset $F_{n-1} \subseteq G$.
- We will show that $(F_{m_n}^n)_{n\geq 0}$ is an exhaustive and congruent right Følner sequence of left monotiles of G.

For $n = 0$, define $F_s^0 = \{1_G\}$, for every $s \geq 0$. This is an exhaustive and congruent right Følner sequence of left monotiles of G_0, because this group is trivial. We also put $m_{-1} = 0$ and $F_{-1} = \{1_G\}$.

Let $n > 0$. Suppose we have defined an exhaustive and congruent right Følner sequence $(F_s^{n-1})_{s\geq 0}$ of left monotiles of G_{n-1}, the positive integer m_{n-2} and the subset F_{n-2}. Because $(F_s^{n-1})_{s\geq 0}$ is exhaustive and K_{n-1} does not depend on the parameter s, we can assume that $K_{n-1} \subseteq F_s^{n-1}$ and F_s^{n-1} is right $(K_{n-1}, \varepsilon_{n-1})$-invariant, for every $s \geq 0$.

Since G_n/G_{n-1} is cyclic (and then residually finite), it admits an exhaustive and congruent right Følner sequence of left monotiles. Let us denote this sequence as $(T_s^{n-1})_{s\geq 0}$.

From Lemma 5.19, there exist a sequence of liftings, $(\widehat{T}_s^{n-1})$ and an increasing sequence of indices, $(m_{n-1,s})_{s\geq 0}$, such that $(F_{m_{n-1,s}}^{n-1}\widehat{T}_s^{n-1})_{s\geq 0}$ is an exhaustive congruent right Følner sequence made of left monotiles of G_n. We define

$$F_s^n = F_{m_{n-1,s}}^{n-1}\widehat{T}_s^{n-1} \text{ for every } s \geq 0.$$

We can assume that for every $s \geq 0$, $m_{n-1,s} > m_{n-2}$, $K_n \subseteq F_s^n$ and that F_s^n is right (K_n, ε_n)-invariant. We define $m_{n-1} = m_{n-1,0}$, and

$$F_{n-1} = F_{m_{n-1}}^{n-1}.$$

Claim: $(F_n)_{n\geq 0}$ is an exhaustive congruent right Følner sequence of G made of left monotiles.
This sequence is right Følner because for all $n \geq 0$, $F_n = F_{m_{n,0}}^n$ is right (K_n, ε_n)-invariant. Since $F_n = F_{m_{n,0}}^n$ is a left monotile of G_n, we have that F_n is a left monotile of G. Indeed, if $C_n \subseteq G_n$ is such that $\{cF_n : c \in C_n\}$ is a partition of G_n, and $\Lambda_n \subseteq G$ is a lifting of G/G_n, then $\{gcF_n : g \in \Lambda_n, c \in C_n\}$ is a partition of G. We have that $F_n = F_{m_n}^n = F_{m_{n-1,m_n}}^n$. Since $(m_{n-1,s})_{s\geq 0}$ is increasing, and $(F_s^n)_{s\geq 0}$ is congruent, every $F_{m_{n-1,s}}^{n-1}$ is a disjoint union of translated copies of $F_{m_{n-1,0}}^{n-1} = F_{n-1}$. This together with the fact that all the elements in $\widehat{T}_{m_n}^{n-1}$ are in diferent classes of G_n/G_{n-1}, imply that F_n is a disjoint union of translated copies of F_{n-1}. □

Theorem 5.23. *Every countable virtually nilpotent group is congruent monotileable.*

Proof. Suppose G is a countable nilpotent group. Again recall that if G is finitely generated, it is residually finite and then congruent monotilable by the results in [CP14]. If G is not finitely generated, we use induction on the nilpotency class of the group. If G is of class 1, then G is abelian and the result follows from Proposition 5.21. If its nilpotency class is 2, then consider the exact sequence

$$1 \to [G,G] \to G \to G/[G,G] \to 1$$

where $[G,G]$ denotes the commutator subgroup of G. Since both $[G,G]$ and $G/[G,G]$ are abelian, each has a congruent right Følner sequence made of left monotiles. Since $[G,G] \leqslant Z(G)$, we deduce from Lemma 5.19 that G is congruent monotilable.

If G has nilpotency class n grater than 2, consider the exact sequence

$$1 \to G^{n-1} \to G \to G/G^{n-1} \to 1$$

where G^i denotes the i-th subgroup in the lower central series of G. Since $G^{n-1} \leqslant Z(G)$, it is abelian, and therefore it follows from Proposition 5.21 that G^{n-1} is congruent monotilable. On the other hand, G/G^{n-1} is a group of nilpotency class $n-1$ (this follows from the fact that for all $0 \leq i \leq n-1$, $(G/G^{n-1})^i = G^i/G^{n-1}$), and then by inductive hypothesis it is congruent monotilable as well. Since $G^{n-1} \leqslant Z(G)$, we deduce from Lemma 5.19 that G is congruent monotilable. This proves that any countable nilpotent group is congruent monotilable.

From the argument above and Remark 5.20, we deduce that every countable virtually nilpotent group is congruent monotilable. □

5.3 Tower partitions using a congruent Følner sequence made of monotiles.

We assume henceforward that G is an infinite countable congruent monotileable amenable group. Given G and $(F_n)_{n\geq 0}$ a Følner sequence made of congruent monotiles of G, we construct a minimal G-subshift in several steps as follows.

Let $d_0 \geq 3$ be an integer and let $\mathcal{A} = \{0, \cdots, d_0\}$. For every $1 \leq i \leq d_0$ let define $B_{0,i} \in \mathcal{A}^{F_0}$ as

$$B_{0,i}(v) = \begin{cases} i & \text{if } v = 1_G \\ 0 & \text{if } v \in F_0 \setminus \{1_G\}. \end{cases}$$

For $n \geq 0$, let $d_{n+1} \geq 3$ be an integer and let $B_{n+1,1}, \cdots, B_{n+1,d_{n+1}}$ be different elements in $\mathcal{A}^{F_{n+1}}$ verifying the following conditions

(B1) $B_{n+1,i}(F_n) = B_{n,1}$, for every $1 \leq i \leq k_{n+1}$.

(B2) $B_{n+1,i}(cF_n) \in \{B_{n,2}, \cdots, B_{n,d_n}\}$ for every $c \in J_n \setminus \{1_G\}$.

Lemma 5.24. *Let $(B_{n,1}, \cdots, B_{n,d_n})_{n\geq 0}$ be the sequence defined above. Then for every $n \geq 0$, $B_{n,1}, \cdots, B_{n,d_n}$ satisfy the following condition*

(B3) If $g \in F_n$ and $1 \leq i, i' \leq d_n$ are such that $B_{n,i}(gv) = B_{n,i'}(v)$ for every $v \in F_n \cap g^{-1}F_n$, then $g = 1_G$ and $i = i'$.

Proof. The case $n = 0$ is clear: since $g \in F_n$, $1_g \in g^{-1}F_n$ and then in particular $B_{0,i}(g) = B_{0,i'}(1_G) = i'$, which implies that $g = 1_G$ and $i = i'$.

Suppose the assertion is true for $n \geq 0$. Let $g \in F_{n+1}$ and $1 \leq i, i' \leq k_{n+1}$ be such that

$$B_{n+1,k}(gv) = B_{n+1,k'}(v) \text{ for every } v \in F_{n+1} \cap g^{-1}F_{n+1}.$$

Let $c_n \in J_n$ and $s \in F_n$ be such that $g = c_n s$. Since $c_n F_n \subseteq F_{n+1}$, conditions (B1) and (B2) imply

$$B_{n,\ell}(su) = B_{n,1}(u) \text{ for every } u \in F_n \cap s^{-1}F_n,$$

where $1 \leq \ell \leq d_n$ is an index such that $B_{n+1,i}(c_n F_n) = B_{n,\ell}$. By inductive hypothesis we get $s = 1$ and $\ell = 1$. Conditions (B1) and (B2) imply $c_n = 1_G$, and therefore $g = 1_G$ and $i = i'$. □

Lemma 5.25. *Let $(B_{n,1}, \cdots, B_{n,d_n})_{n\geq 0}$ be a sequence satisfying conditions $(B1)$, $(B2)$ and $(B3)$ defined above. Then there exists $x_0 \in \mathcal{A}^G$ such that*

$$\bigcap_{n\geq 0} \{x \in \mathcal{A}^G : x(F_n) = B_{n,1}\} = \{x_0\}.$$

Proof. By Condition (B1) and because every set $\{x \in X : x(F_n) = B_{n,1}\}$ is compact, we have

$$\bigcap_{n\geq 0} \{x \in \mathcal{A}^G : x(F_n) = B_{n,1}\} \neq \emptyset.$$

Since the Følner sequence $(F_n)_{n\geq 0}$ is exhaustive, we deduce there exists only one element x_0 in this intersection. □

Let T denote the shift action on $\mathcal{A}^G$, and $x_0 \in \mathcal{A}^G$ be the unique element on the interesection $\bigcap_{n\geq 0}\{x \in \mathcal{A}^G : x(F_n) = B_{n,1}\}$. Consider the subspace $X = \overline{\{T^g(x_0) : g \in G\}}$. For every $n \geq 0$ and $1 \leq i \leq d_n$ we define

$$C_{n,i} = \{x \in X : x(F_n) = B_{n,i}\} \text{ and } C_n = \bigcup_{i=1}^{d_n} C_{n,i}. \tag{5.1}$$

Lemma 5.26. *Let $v \in G$. Then the following are equivalent:*

1. $T^{v^{-1}}(x_0) \in C_n$.

2. *There exist* $m > n$ *and* $c_k \in J_k$ *for every* $n \leq k < m$ *such that* $v = c_{m-1} \cdots c_n$.

Proof. Suppose that $v \in G$ is such that $T^{v^{-1}}(x_0) \in C_n$. If $v = 1_G$ then for $m = n+1$ and $c_n = 1_G$ we get the desired property. Suppose now that $v \neq 1_G$. let $m \geq 0$ be the smallest integer such that $vF_n \subseteq F_m$. Because $|vF_n| = |F_n|$, it is necessary that $m \geq n$. Suppose that $m = n$. Since $1_G \in F_n$ this implies that $v \in F_n$. By hypothesis we have

$$x_0(vs) = B_{n,l}(s) \text{ for every } s \in F_n.$$

On the other hand,

$$x_0(vs) = B_{n,1}(vs) \text{ for every } s \in v^{-1}F_n.$$

Lemma 5.24 implies that $v = 1_G$, which is a contradiction. Thus we have $m > n$.
Since $vF_n \subseteq F_m$ and $1_G \in F_m$, we have $v \in F_m$. Lemma 5.18 implies that for every $n \leq k \leq m-1$ there exist $c_k \in J_k$ such that $v \in c_{m-1} \cdots c_n F_n$. Let $s \in F_n$ such that $v = c_{m-1} \cdots c_n s$. By definition of x_0, we have

$$x_0(c_{m-1} \cdots c_n F_n) = B_{n,k}, \text{ for some } 1 \leq k \leq k_n,$$

which implies that for every $g \in s^{-1}F_n$,

$$x_0(vg) = x_0(c_{m-1} \cdots c_n sg) = B_{n,k}(sg).$$

On the other hand, for every $g \in F_n$ we have

$$x_0(vg) = B_{n,l}(g).$$

Thus

$$B_{n,l}(g) = B_{n,k}(sg) \text{ for every } g \in F_n \cap s^{-1}F_n.$$

From Lemma 5.24, $s = 1_G$ and then $v = c_{m-1} \cdots c_n$. By applying inductively Lemma 5.18, we get that $v = c_{m-1} \cdots c_n$ is a return time of x_0 to C_n. □

Let $(B_{n,1}, \cdots, B_{n,d_n})_{n\geq 0}$ be a sequence satisfying conditions $(B1)$, $(B2)$ and $(B3)$. For every $n \geq 0$, consider the collection of sets

$$\mathcal{P}_n = \{T^{v^{-1}}(C_{n,i}) : v \in F_n, 1 \leq i \leq d_n\}, \tag{5.2}$$

where $C_{n,i}$ is defined as in (5.1). For all $n \geq 0$, $\mathcal{P}_n$ defines a tower partition of $X = \overline{\{T^g(x_0) : g \in G\}}$ and $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$. We prove this in the following proposition.

Proposition 5.27. *For every $n \geq 0$, $\mathcal{P}_n$ as defined in (5.2) is a clopen partition of X and $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$.*

Proof. Suppose that $v, u \in F_n$ and $1 \leq i, j \leq d_n$ are such that

$$T^{v^{-1}}(C_{n,k}) \cap T^{u^{-1}}(C_{n,l}) \neq \emptyset,$$

then there exists $x \in X$ such that

$$x \in T^{uv^{-1}}(C_{n,i}) \cap C_{n,j}.$$

Since $x \in X$, there exists a sequence $(g_i)_{i \geq 0}$ of elements in G such that

$$\lim_{i\to\infty} T^{g_i}(x_0) = x \text{ and then } \lim_{i\to\infty} T^{vu^{-1}g_i}(x_0) = T^{vu^{-1}}(x).$$

Lemma 5.26 implies that for a large enough ℓ there exist $c_p \in J_p$ for every $n \leq p \leq m-1$, and $\tilde{c}_p \in J_p$ for every $n \leq p \leq r-1$, such that

$$g_\ell^{-1} = c_{m-1} \cdots c_n \text{ and } g_\ell^{-1} vu^{-1} = \tilde{c}_{r-1} \cdots \tilde{c}_n,$$

where $m \geq n+1$ and $r \geq n+1$ are the smallest integers such that $g_\ell^{-1} F_n \subseteq F_m$ and $g_\ell^{-1} vu^{-1} F_n \subseteq F_r$ respectively. Then,

$$c_{m-1} \cdots c_n v = \tilde{c}_{r-1} \cdots \tilde{c}_n u.$$

Suppose that $r \geq m$. From Lemma 5.18, this implies that

$$\tilde{c}_r = \cdots \tilde{c}_m = 1_G \text{ and } \tilde{c}_\ell = c_\ell \text{ for every } n \leq \ell \leq m-1.$$

We get that $u = v$ and then the sets in $\mathcal{P}_n$ are disjoint.

Let $g \in G \setminus F_n$. Let $m > n$ be such that $g \in F_m$. Then the congruency of $(F_n)_{n \geq 0}$ implies there exist

$c_i \in J_i$ for every $n \leq i < m$ such that $g = c_{m-1} \cdots c_n u$, for some $u \in F_n$. Then from Lemma 5.18 we get

$$T^{g^{-1}}(x_0) = T^{u^{-1}}(T^{(c_{m-1}\cdots c_n)^{-1}}(x_0)) \in T^{u^{-1}}(C_{n,\ell}), \text{ for some } 1 \leq \ell \leq d_n.$$

This shows that $\mathcal{P}_n$ is a covering of X.
Finally, condition (B1) and (B2) imply that $\mathcal{P}_{n+1}$ is finer that $\mathcal{P}_n$. Indeed, let $z \in T^{u^{-1}}(C_{n+1,j})$ for some $u \in F_{n+1}$, $1 \leq j \leq d_{n+1}$. There exists $y \in C_{n+1,j}$ such that $z = T^{u^{-1}}(y)$. Since $y \in C_{n+1,j}$, $y(F_{n+1}) = B_{n+1,j}(F_{n+1})$ and by condition (B1) $y(F_n) = B_{n,1}$, so that $y \in C_{n,1}$.
If $u \in F_n$, then it is clear that $z = T^{u^{-1}}(y)$ belongs to an atom of $\mathcal{P}_n$. If $u \notin F_n$, then by congruency, there exist unique $c \in J_n \setminus 1_G$ and $v \in F_n$ such that $u = cv$. By condition (B2), $y(cF_n) = B_{n,i}$, some $1 \leq i \leq d_n$. Since $y \in C_{n,1}$, $T^{c^{-1}}(y) \in C_{n,i}$, and we get

$$z = T^{u^{-1}}(y) = T^{v^{-1}}T^{c^{-1}}(y) \in T^{v^{-1}}(C_{n,i}),$$

which concludes the proof. □

Note that the sequence $(\mathcal{P}_n)_{n\geq 0}$ does not necessarily generate the topology of X. There could be points belonging to the *partition boundary* ∂X (see the definition below), that is, points which are not separated by $(\mathcal{P}_n)_{n\geq 0}$. Let ∂X be defined as

$$\partial X = \bigcup_{g\in G} \bigcap_{n\geq 0} \bigcup_{k=1}^{k_n} \bigcup_{v\in F_n\setminus F_n g} T^{v^{-1}}(C_{n,k}).$$

Proposition 5.28. *The system (X, T, G) is minimal and free on $X \setminus \partial X$. If G is virtually abelian, the system is free.*

Proof. Let $F \subseteq G$ a finite set, and let $P \in \mathcal{A}^F$ such that

$$C = \{x \in X : x(F) = P\} \neq \emptyset.$$

We will show that $R_C(x_0) := \{g \in G : T^{g^{-1}}(x_0) \in C\}$ is *syndetic*, which is enough to conclude that the subshift is minimal (see for example [Aus88, Chapter 1]). Let $g \in G$ such that $T^g(x_0) \in C$. Since the orbit of x_0 is dense in X such a g always exists. We have $x_0(g^{-1}F) = P$. Let $n > 0$ such that $g^{-1}F \subseteq F_{n-1}$. Then

$$x_0(g^{-1}F) = B_{n-1,1}(g^{-1}F) = P.$$

Condition (B1) and Lemma 5.26 imply that for every $c_{m-1} \in J_{m-1}, \cdots, c_n \in J_n$, with $m > n$, we have

$$T^{(c_{m-1}\cdots c_n)^{-1}}(x_0) \in C_n \subseteq C_{n-1,1}.$$

Thus we get

$$T^{(c_{m-1}\cdots c_n)^{-1}}(x_0)(g^{-1}F) = B_{n-1,1}(g^{-1}F) = P.$$

This shows that $c_{m-1}\cdots c_n g^{-1} \in R_C(x_0)$.

Now let $h \in G$ and $m > n$ be such that $g \in F_m$. Lemma 5.18 implies that there exist $c_{m-1} \in J_{m-1}, \cdots, c_n \in J_n$ such that $h \in c_{m-1}\cdots c_n F_n$. Then we obtain that $h \in R_C(x_0)gF_n$, which implies that $R_C(x_0)$ is syndetic.

Let $x \in X$ and $g \in G$ be such that $T^g(x) = x$. For every $n \geq 0$, let $v_n \in F_n$ be such that $x \in T^{v_n^{-1}}(C_n)$. We have $x = T^g(x) \in T^{gv_n^{-1}}(C_n)$. Since $\mathcal{P}_n$ is a partition, if there exists $n \geq 0$ such that $v_n g^{-1} \in F_n$, then $g = 1_G$. Thus if there exists $g \in G \setminus \{1_G\}$ such that $T^g(x) = x$, then $v_n \in F_n \setminus F_n g$ for every $n \geq 0$. This shows that the subshift is free on $X \setminus \partial X$.

Suppose that G is virtually abelian. Let Γ be a finite index abelian subgroup of G. Since G/Γ is finite, there exist $\ell > k \geq 1$ such that $g^\ell\Gamma = g^k\Gamma$, which implies $g^{\ell-k} \in \Gamma$. Thus we can assume that $g \in \Gamma$. On the other hand, there exist a subsequence $(v_{n_k})_{k\geq 0}$ and $v \in G$ such that $v_{n_k} \in v\Gamma$, for every $k \geq 0$. Let $\gamma_k \in \Gamma$ be such that $v_{n_k} = v\gamma_k$, for every $k \geq 0$. Since $T^{v_{n_k}}(x) \in C_{n_k}$, we have that $\lim_{k\to\infty} T^{v^{-1}v_{n_k}}(x) = \lim_{k\to\infty} T^{\gamma_k}(x) = T^{v^{-1}}(x_0)$. This implies that $\lim_{k\to\infty} T^{g\gamma_k}(x) = T^{gv^{-1}}(x_0)$ and since $T^{g\gamma_k}(x) = T^{\gamma_k g}(x) = T^{\gamma_k}(x)$, we conclude that $T^{gv^{-1}}(x_0) = T^{v^{-1}}(x_0)$. Since $x_0 \in X \setminus \partial X$, we deduce $gv^{-1} = v^{-1}$ and then the system is free. □

We now work with the incidence matrices associated to the sequence of tower partitions $\mathcal{P}_n$. For every $n \geq 0$ and $1 \leq i \leq d_n$, define

$$J_{n,k,i} = \{c \in J_n : B_{n+1,k}(cF_n) = B_{n,i}\},$$

and the matrix $Q_n \in \mathcal{M}_{d_n \times d_{n+1}}(\mathbb{Z}^+)$ as

$$Q_n(i,j) = |J_{n,j,i}|, \text{ for every } 1 \leq i \leq d_n, 1 \leq j \leq d_{n+1}.$$

Proposition 5.29. *The sequence $(Q_n)_{n\geq 0}$ corresponds to the sequence of incidence matrices associated* *to the sequence of tower partitions* $(\mathcal{P}_n)_{n\geq 0}$*, that is, for all* $1 \leq i \leq d_n$*,* $1 \leq j \leq d_{n+1}$*,*

$$Q_n(i,j) = |\{v \in F_{n+1} : T^{v^{-1}}(C_{n+1,j}) \subseteq C_{n,i}\}|.$$

Observe that

$$\sum_{i=1}^{d_n} Q_n(i,j) = |J_n| = \frac{|F_{n+1}|}{|F_n|}, \text{ for every } 1 \leq j \leq d_{n+1}.$$

This means that the sequence $(Q_n)_{n\geq 0}$ is *managed* by $(|F_n|)_{n\geq 0}$, that is:

1. Q_n has $d_n \geq 2$ rows and $d_{n+1} \geq 2$ columns;
2. $\sum_{i=1}^{d_n} Q_n(i,k) = \frac{|F_{n+1}|}{|F_n|}$, for every $1 \leq k \leq d_{n+1}$.

It is easy to check that $Q_n(\triangle(d_{n+1}, |F_{n+1}|)) \subseteq \triangle(d_n, |F_n|)$, where

$$\triangle(k,p) = \left\{ (x_1, \cdots, x_k) \in (\mathbb{R}^+)^k : \sum_{i=1}^{k} x_i = \frac{1}{p} \right\}.$$

Thus the following inverse limit is well defined.

$$\varprojlim_n (\triangle(d_n, |F_n|), Q_n) = \left\{ (z_n)_{n\geq 0} \in \prod_{n\geq 0} \triangle(d_n, |F_n|) : z_n = Q_n z_{n+1} \forall n \geq 0 \right\}.$$

Remark 5.30. *We can assume that the matrices* Q_n *are positive. Indeed, if there exists* $B_{n,i}$ *such that for every* $m > n$ *it does not appear in* $B_{m,1}$*, then the clopen set* $\{x \in X : x(F_n) = B_{n,k}\}$ *is empty. Thus we can assume that for every* $n \geq 0$ *and* $1 \leq i \leq k_n$ *there exists* $m_{n,i}$ *such that* $B_{n,i}$ *appears (as a translated copy) in* $B_{m_{n,i},1}$*. By condition (B1) we can assume that* $m_{n,i} = m_n$ *is independent on* i*. By (B1) again, the product* $Q_n \cdots Q_{m_n+1}$ *is positive.*

5.3.1 Invariant measures for (X, T, G).

We denote $\mathcal{M}(X,T,G)$ the space of all invariant probability measures of (X,T,G). Recall that, since G is amenable, this is a non-empty Choquet simplex (see Section 1.4). We say that $A \subseteq X$ is a *full measure* set of X if A^c is negligible with respect to any invariant measure of (X,T,G).
The next Lemma will allow us to show that $\mathcal{M}(X,T,G)$ is affine homeomorphic to $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$.

Lemma 5.31. *The set* $X \setminus \partial X$ *has full measure.*

Proof. Let $\mu \in \mathcal{M}(X,T,G), g \in G$ and $n \geq 0$. We have

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{d_n} \bigcup_{v \in F_n \setminus F_n g} T^{v^{-1}}(C_{n,k})\right) &= |F_n \setminus F_n g| \sum_{i=1}^{d_n} \mu(C_{n,k}) \\ &= \frac{|F_n \setminus F_n g|}{|F_n|}.\end{aligned}$$

Then

$$\mu\left(\bigcap_{n\geq 0} \bigcup_{i=1}^{d_n} \bigcup_{v \in F_n \setminus F_n g} T^{v^{-1}}(C_{n,i})\right) \leq \lim_{m\to\infty} \frac{|F_m \setminus F_m g|}{|F_m|} = 0,$$

which implies that ∂X has zero measure with respect to any invariant measure of (X,T,G). □

From Proposition 5.28 and Lemma 5.31 we obtain:

Corollary 5.32. (X,T,G) *is free on a full measure set.*

Proposition 5.33. *There is an affine homeomorphism between* $\mathcal{M}(X,T,G)$ *and the inverse limit* $\varprojlim_n(\triangle(d_n,|F_n|),M_n)$.

Proof. From Lemma 5.31, the invariant measures of (X,T,G) are supported on $X \setminus \partial X$, and every point in this set is separated by the atoms of the partitions $\mathcal{P}_n$. Thus every open set $U \subseteq X$ is a (countable) union of elements of the atoms of the partitions $\mathcal{P}_n$'s and a set in ∂X. This implies that the measure of U is completely determined by the measures of the atoms in $\mathcal{P}_n$'s. The rest of the proof follows according to [CP14, Proposition 2]. □

5.4 Invariant measures, a realization theorem.

In this section we prove that every Choquet simplex can be obtained as the set of invariant measures of a minimal G-subshift, where G is any congruent monotileable group G. More precisely, we prove the following theorem.

Theorem 5.34. *Let G be an infinite congruent monotileable amenable group. For every Choquet simplex K, there exists a minimal G-subshift, which is free on a full measure set, whose set of invariant probability measures is affine homeomorphic to K. If G is virtually abelian, the subshift is free.*

One of the key elements to prove the above theorem is the fact that, given a sequence of positive integers $(p_n)_{n\in\mathbb{N}}$ such that p_n divides p_{n+1} for all n, every Choquet simplex can be represented as an inverse limit related to that sequence. This is stated in the two following results, whose proof can be found in [CP14].

Lemma 5.35. *([CP14, Lemma 9]) Let K be a finite dimensional metrizable Choquet simplex with exactly $d \geq 1$ extreme points. Let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}, and let $k \geq \max\{2, d\}$. Then, there exist an increasing subsequence $(n_i)_{i\geq 0}$ of indices and a sequence $(M_i)_{i\geq 0}$ of square k-dimensional matrices which is managed by $(p_{n_i})_{i\geq 0}$ such that K is affine homeomorphic to $\varprojlim_i(\triangle(k, p_{n_i}), M_i)$*

Lemma 5.36. *([CP14, Lemma 12]) Let K be an ifinite dimensional metrizable Choquet simplex with exactly $d \geq 1$ extreme points. Let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}. Then, there exist an increasing subsequence $(n_i)_{i\geq 0}$ of indices and a sequence $(M_i)_{i\geq 0}$ of matrices which is managed by $(p_{n_i})_{i\geq 0}$ such that for every $i \geq 0$,*

$$k_{i+1} \leq \min\{M_i(\ell, k) : 1 \leq \ell \leq k_i, 1 \leq k \leq k_{i+1}\},$$

and such that K is affine homeomorphic to the inverse limit $\varprojlim_i(\triangle(k_{n_i}, p_{n_i}), M_i)$, where M_i has k_i rows and k_{i+1} columns, for every $i \geq 0$.

Since we work with amenable groups which are monotileable in a congruent fashion, we will use the above results setting $(p_n)_{n\in\mathbb{N}} = (|F_n|)_{n\in\mathbb{N}}$ for a given group G, where $(F_n)_{n\in\mathbb{N}}$ is a congruent right Følner sequence made of left monotiles of G. The other element to prove Theorem 5.34 is the following result, which corresponds to Proposition 22 in [CC18].

Proposition 5.37. *Let $(F_n)_{n\geq 0}$ be a congruent Følner sequence made of monotiles of a congruent monotileable amenable group G. Let $(M_n)_{n\geq 0}$ be a sequence of matrices which is managed by $(|F_n|)_{n\geq 0}$. For every $n \geq 0$, we denote by k_n the number of rows of M_n. Suppose there exists $K > 0$ such that $k_{n+1} \leq K\frac{|F_{n+1}|}{|F_n|}$, for every $n \geq 0$. Then there exists a minimal free G-subshift (X, T, G) such that $\mathcal{M}(X, T, G)$ is affine homeomorphic to the inverse limit $\varprojlim_n(\triangle(k_n, |F_n|), M_n)$.*

Proof. We will use the following lemma.

Lemma 5.38. *([CP14, Lemma 8, part (iii)]) Let $(M_n)_{n\geq 0}$ be a sequence of matrices which is managed by $(|F_n|)_{n\geq 0}$. For every $n \geq 0$, let k_n be the number of rows of M_n. Suppose there exists a constant $K > 0$ such that*

$$k_{n+1} \leq K\frac{|F_{n+1}|}{|F_n|}, \text{ for every } n \geq 0.$$

Then there exists an increasing sequence $(n_i)_{i\geq 0}$ *in* $\mathbb{Z}^+$ *such that for every* $i \geq 0$ *and for every* $1 \leq k \leq k_{n_{i+1}}$,

$$k_{n_{i+1}} < M_{n_i} \cdots M_{n_{i+1}-1}(\ell, k) \text{ for every } 1 \leq \ell \leq k_{n_i}.$$

Thanks to the above lemma, we can assume that for every $n \geq 0$,

$$k_{n+1} < \min\{M_n(i,j) : 1 \leq i \leq k_n, 1 \leq j \leq k_{n+1}\}.$$

For every $n \geq 0$, let $\tilde{M}_n$ the $(k_n+1) \times (k_{n+1}+1)$-dimensional matrix defined as follows

$$\tilde{M}_n(\cdot,1) = \tilde{M}_n(\cdot,2) = \begin{pmatrix} 1 \\ M_n(1,1)-1 \\ M_n(2,1) \\ \vdots \\ M_n(k_n,1) \end{pmatrix},$$

and

$$\tilde{M}_n(\cdot,k+1) = \begin{pmatrix} 1 \\ M_n(1,k)-1 \\ M_n(2,k) \\ \vdots \\ M_n(k_n,k) \end{pmatrix}, \text{ for every } 2 \leq k \leq k_{n+1}.$$

From [CP14, Lemma 1] and [CP14, Lemma 2] we have that the inverse limits $\varprojlim_n(\triangle(k_n,|F_n|),M_n)$ and $\varprojlim_n(\triangle(k_n+1,|F_n|),\tilde{M}_n)$ are affine homeomorphic. Observe that $(\tilde{M}_n)_{\geq 0}$ is managed by $(|F_n|)_{n\geq 0}$ and verifies that, for every $n \geq 0$,

$$3 \leq k_{n+1}+1 \leq \min\{\tilde{M}_n(i,j) : 2 \leq i \leq k_n+1, 1 \leq j \leq k_{n+1}+1\}. \tag{5.3}$$

Let ℓ_n and ℓ_{n+1} denote the number of rows and columns of $\tilde{M}_n$ respectively. Let $\mathcal{A} = \{0,\cdots,\ell_0\}$. For every $1 \leq \ell \leq \ell_0$, define $B_{0,\ell} \in \mathcal{A}^{F_0}$ by

$$B_{0,\ell}(v) = \begin{cases} \ell & \text{if } v = 1_G \\ 0 & \text{if } v \in F_0 \setminus \{1_G\}. \end{cases}$$

For $n \geq 0$, suppose that we have defined $B_{n,1}, \cdots, B_{n,\ell_n}$ different elements in $\mathcal{A}^{F_n}$. We define $B_{n+1,1}, \cdots, B_{n+1,\ell_{n+1}}$ in $\mathcal{A}^{F_{n+1}}$ as follows: for every $1 \leq \ell \leq \ell_{n+1}$

$$B_{n+1,\ell}(F_n) = B_{n,1},$$

and for every $c \in J_n \setminus \{1_G\}$,

$$B_{n+1,\ell}(cF_n) \in \{B_{n,2}, \cdots, B_{n,\ell_n}\}$$

in a way such that

$$|\{v \in J_n : B_{n+1,\ell}(cF_n) = B_{n,i}\}| = \tilde{M}_n(i, \ell),$$

for every $2 \leq i \leq \ell_n$.

Condition (5.3) ensures that it is possible to make this procedure in order that $B_{n+1,k} \neq B_{n+1,s}$ if $k \neq s$, since there are more possibilities to define each pattern $B_{n+1,\ell}$ than patterns to be defined (see [CP14, Remark 2]). By construction, $B_{n,1}, \cdots, B_{n,\ell_n}$ satisfy conditions (B1), (B2) and (B3), so that we can construct a minimal subshift (X, T, G) as we make right after Lemma 5.25, which is free in $X \setminus \partial X$ and free if G is virtually abelian, thanks to Proposition 5.28. Finally, thanks to Proposition 5.33, $\mathcal{M}(X, T, G)$ is affine homeomorphic to $\varprojlim_n(\triangle(k_n+1, |F_n|), \tilde{M}_n)$, which is in turns homeomorphic to $\varprojlim_n(\triangle(k_n, |F_n|), M_n)$. $\square$

Now he have all the elements to prove Theorem 5.34.

Proof of Theorem 5.34. Let K be any Choquet simplex, and for all $n \geq 0$ define $p_n = |F_n|$. From Lemmas 5.35 and 5.36, we know that there exists an increasing subsequence $(n_i)_{i\geq 0}$ and matrices $(M_i)_{i\geq 0}$ managed by $|F_{n_i}|$ such that K is affine homeomorphic to the inverse limit $\varprojlim_i(\triangle(k_{n_i}, |F_{n_i}|), M_i)$, where M_i has k_i rows and k_{i+1} columns, for every $i \geq 0$ (k_{n_i} is constant and equal to $\max\{2, d\}$ when K is finite dimensional and has exactly d extreme points). Then, we apply Proposition 5.37 to get that there exists a minimal free G-subshift (X, T, G) such that $\mathcal{M}(X, T, G)$ is affine homeomorphic to the inverse limit $\varprojlim_i(\triangle(k_{n_i}, |F_{n_i}|), M_i)$. We conclude that $\mathcal{M}(X, T, G)$ is affine homeomorphic to K. $\square$

Combining Theorem 5.23 and Theorem 5.34, we get that for any countable abelian group G (even those which are not residually finite), any Choquet simplex K can be seen as the simplex of invariant measures associated to a minimal free G-subshift. In particular, we obtain the following corollary.

Corollary 5.39. *For any Choquet simplex K there exists a minimal free $\mathbb{Q}$-subshift $(X, T, \mathbb{Q})$ such that $\mathcal{M}(X, T, \mathbb{Q})$ is affine homeomorphic to K.*

5.5 Further work.

We have proved that any Choquet simplex K can be seen as the set of invariant probability measures of a minimal subshift of any congruent monotileable amenable group G. In other words, if G is a congruent monotileable amenable group, then for any Choquet simplex K there exists a minimal G-subshift (X, T, G), such that $K \cong \mathcal{M}(X, T, G)$. It is not known if there are monotileable amenable groups which are not congruent monotileable. This question is interesting from the viewpoint of the problem of realization of simplices as sets of invariant measures, since there is an important class of groups which are known to be monotileable amenable, but *a priori* not necessarily in a congruent fashion, namely the class of solvable groups (see [W01, Theorem 2]). Thus, if we could show that every monotileable amenable group is congruent monotileable, we would be able to extend Theorem 5.34 to solvable groups.

Another interesting question is whether Theorem 5.34 can be extended to the larger class of (countable) amenable groups, that is, whether any Choquet simplex K can be seen as the set of invariant measures of a minimal subshift of any amenable group G on the Cantor set. While it is not known if there are amenable groups which are not monotileable or congruent monotileable, it is known that every amenable group can be tiled using a *finite* set of prototiles [DHZ]. Thus, if we could use a similar strategy to that of Section 5.3 to construct a minimal G-subshift with a given Choquet simplex, using not one but a finite number of prototiles, we could extend Theorem 5.34 to amenable groups.

It is also possible that the previous question has a negative answer, that is, that there exists amenable groups for which there is some Choquet simplex K such that for any minimal G-subshift (X, T, G), K is not homeomorphic to $\mathcal{M}(X, T, G)$ (it would be of course a group which cannot be monotiled in a congruent fashion). More generally, we are interested in the question of, given an amenable group G, to determine which are the simplices that can be realized as the set of invariant measures of a minimal G-subshift, or, given a Choquet simplex K, which are the groups that admit K as the set of invariant measures of some minimal G-subshift.

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