Universidad de Santiago de Chile
Facultad de Ciencia
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On the Geometry of Discrete sets

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Resumen

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El objetivo de la presente tesis es investigar aspectos analíticos y geométricos de conjuntos de Delone en espacios euclideanos. En una primera parte, siguiendo el trabajo de Burago y Kleiner sobre la existencia de conjuntos de Delone en el plano que no son bi-Lipschitz equivalentes al lattice estándar, demostraremos que una función acotada y positiva genérica definida sobre el cuadrado, puede ser usada para construir tales conjuntos de Delone.

En la segunda parte de esta tesis estudiaremos el problema de encontrar conjuntos de Delone repetitivos en el plano los cuales no pueden ser mapeados sobre el lattice estándar por bijecciones Lipschitz. Este estudio lo realizaremos utilizando ideas introducidas en el trabajo de Cortez y Navas sobre construcciones explícitas de conjuntos de Delone no rectificables, y mediante un análisis de las llamadas funciones Lipschitz regulares.

Keywords: Delone sets, bi-Lipschitz maps, Lipschitz regular maps.

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Introducción

Motivado por problemas provenientes de varias ramas de la matemática (por ejemplo la Teoría de Metric Embeddings [2, 20, 23], Teoría Geométrica de Grupos [14], Teoría de la información [12], Física-Matemática de cuasi-cristales [1, 18]), en los últimos años ha habido mucha actividad en el estudio de la equivalencia Lipschitz entre conjuntos discretos. En esta tesis nos enfocaremos en un aspecto particular de esta basta teoría, a saber, la equivalencia Lipschitz entre conjuntos de Delone en espacios euclideanos.

El problema de estudiar la equivalencia bi-Lipschitz entre conjuntos de Delone fue motivado de manera independiente por Furstenberg y por Gromov. Mientras que Furstenberg estaba interesado en aspectos dinámicos asociados a este problema, la motivación de Gromov viene desde la Teoría Geométrica de Grupos. En concreto, Furstenberg estaba interesado en representar el conjunto de tiempos de retorno de una $\mathbb{R}^2$-acción por una $\mathbb{Z}^2$-acción. El conjunto de tiempos de retorno de una $\mathbb{R}^2$-acción es (bajo adecuadas condiciones) un conjunto de Delone, y para obtener la deseada representación, se debe tener una equivalencia bi-Lipschitz de este conjunto con $\mathbb{Z}^2$ (se sugiere ver [4, 17] para más detalles). Por otro lado, la motivación de Gromov viene del concepto de quasi-isometría y la geometría a gran escala. Dos espacios métricos se dicen quasi-isométricos si ellos contienen conjuntos de Delone que sean bi-Lipschitz equivalentes. En este sentido, surge de manera natural la pregunta si en un espacio métrico la elección de tales conjuntos importa, es decir, si un espacio métrico puede contener conjuntos de Delone que no sean bi-Lipschitz equivalentes.

En 1998, Burago y Kleiner en [3] y, de forma independiente McMullen en [21], demostraron que existen conjuntos de Delone en el plano que no son bi-Lipschitz equivalentes a $\mathbb{Z}^2$. Tales conjuntos, de naturaleza discreta, surgen a partir de ejemplos asociados a la siguiente pregunta de naturaleza continua: *¿Toda función continua en el plano, acotada y lejos de 0, se puede escribir como el jacobiano de algún homeomorfismo bi-Lipschitz?* Varios autores han abordado esta pregunta, no solo para funciones bi-Lipchitz, sino que también en otras clases de regularidad; recomendamos leer [7, 22, 25] y las referencias que allí aparecen. Luego, en el año 2016, Cortez y Navas en [6] construyeron los primeros ejemplos concretos de conjuntos de Delone, cuyos puntos tienen coordenadas enteras, que no son bi-Lipschitz equivalentes a $\mathbb{Z}^2$. Tales ejemplos son construidos a través de una "discretización" de los argumentos y resultados de naturaleza "continua" que aparecen en [3].

Con respecto a conjuntos de Delone en el plano que sí son bi-Lipschitz equivalentes a $\mathbb{Z}^2$, Burago y Kleiner en [4] dan condiciones suficientes para que un conjunto de Delone en $\mathbb{R}^2$ sea bi-Lipschitz equivalente a $\mathbb{Z}^2$. Usando este resultado, Solomon demuestra en [26] que el conjunto de Delone que resulta al poner un punto en cada baldosa del embaldosado de Penrose, es bi-Lipschitz equivalente a $\mathbb{Z}^2$. Posteriormente, Aliste-Prieto, Coronel y Gambaudo en [5] generalizan el resultado obtenido por Burago y Kleiner en [4] a dimensiones superiores, y lo usan para demostrar que todo conjunto de Delone linealmente repetitivo en $\mathbb{R}^d$ es bi-Lipschitz equivalente a $\mathbb{Z}^d$. Con respecto al problema de la equivalencia bi-Lipschitz en espacios no-euclideanos se sugiere ver, por ejemplo, [10,19,24,28].

En esta tesis estaremos enfocados en estudiar las siguientes preguntas:

1. ¿Existen funciones positivas y acotadas definidas en $\mathbb{R}^d$ que no se pueden escribir como el Jacobiano de una aplicación bi-Lipschitz?
2. Si tales funciones existen, ¿qué tan grande es este conjunto de funciones en el espacio $(C(\mathbb{R}^d, \mathbb{R}), ||\cdot||_0)$, donde $||\cdot||_0$ representa la norma del supremo?
3. ¿Existe algún conjunto de Delone $\mathcal{D}_d \subset \mathbb{Z}^d$ repetititvo que no puede ser mapeado por biyecciones Lipschitz sobre $\mathbb{Z}^d$?

Estas tres preguntas pueden ser fácilmente respondidas negativamente en el caso unidimensional, por lo que en este trabajo las abordaremos para dimensiones superiores (cabe destacar que la pregunta 1 se aborda en los artículos [3] y [21]). Esta tesis está dividida en cinco capítulos y un apéndice, los cuales procederemos a describir.

En el **Capítulo 1** nos dedicaremos a introducir el problema de la equivalencia bi-Lipschitz entre conjuntos de Delone. Este capítulo es separado en dos partes: en la primera parte (Sección 1.1) daremos la defininción de conjunto de Delone y además introduciremos el problema de encontrar tales conjuntos en el plano que no sean bi-Lipschitz equivalentes al lattice estándar. Veremos además las primeras propiedades y observaciones asociadas a este problema. En la segunda parte (Sección 1.2) estaremos interesados en estudiar la resolución del problema propuesto por Gromov y Furstenberg. Esta segunda parte corresponde principalmente en una revisión de la construcción de Burago y Kleiner en [3].

En el **Capítulo 2** se presenta el primer resultado obtenido en esta tesis, a saber, que una función positiva genérica (respectivamente medible) $\rho : \mathbb{R}^d \to \mathbb{R}$, donde $d \geq 2$, no puede ser realizada como el Jacobiano de una aplicación bi-Lipschitz, i.e, para las cuales la ecuación

$$Jac(f) = \rho \tag{0.0.1}$$

no tiene soluciones bi-Lipschitz $f : \mathbb{R}^d \to \mathbb{R}^d$. En particular, una función continua, positiva y acotada genérica puede ser usada para construir conjuntos de Delone que no pueden ser mapeados de forma bi-Lipschitz sobre $\mathbb{Z}^2$, en concordancia con la construcción de Burago y Kleiner de tales conjuntos. Concretamente, nuestro objetivo en el Capítulo 2 será demostrar el siguiente resultado.

Una función continua, positiva genérica $\phi : \mathbb{R}^2 \to \mathbb{R}$ no puede ser escrita como el Jacobiano de un homeomorfismo bi-Lipschitz. Lo mismo ocurre para funciones genéricas positivas en L^∞.

Este capítulo está basado en el artículo *Densities non-realizable as the Jacobian of a 2-dimensional bi-Lipschitz map are generic*, el cual fue publicado en el *Journal of Topology and Analysis* (ver [27]).

En el **Capítulo 3** introducimos las llamadas aplicaciones Lipschitz regulares, como una herramienta para tratar con problemas relacionados a conjuntos discretos. Este capítulo constituye una revisón del artículo [15] de Dymond, Kaluža y Kopecká, quienes usan esta clase de funciones para responder negativamente a una pregunta propuesta por Feige (ver [20]), quien estaba interesado en saber si todo conjunto finito de n^2-elementos en el plano puede ser mapeado sobre la grilla estándar $\{1, \ldots, n\}^2$, de tal forma que las distancias entre los puntos sean estiradas lo menos posible. Esta pregunta surge como motivación de extender a dimensiones superiores el llamado *problema del ancho de banda*, el cual consiste en etiquetar los n vértices $\{v_1, \ldots, v_n\}$ de un grafo $G = (V, E)$ vía una función $f : V \to \{1, \ldots, n\}$ de tal forma que el número $\max\{|f(v_i) - f(v_j)| : (v_i v_j) \in E\}$ (el ancho de banda) sea minimizado. En [15] los autores responden negativamente a la pregunta planteada por Feige siguiendo esencialmente las mismas ideas propuestas en [3]. La diferencia fundamental con el trabajo de Burago y Kleiner es que ejemplos de conjuntos finitos que responden negativamente la pregunta de Feige, la cual es de naturaleza discreta, surgen a partir de ejemplos asociados a la siguiente pregunta de "naturaleza continua": *¿Existe una función positiva, continua $\rho : I^2 \to \mathbb{R}$ para la cual la ecuación*

$$f_{\#}(\rho\lambda) = \lambda|_{f([0,1]^2)} \tag{0.0.2}$$

no tiene solución Lipschitz regular $f : [0,1]^2 \to \mathbb{R}^2$?

La ecuación (0.0.2) corresponde a una extensión a funciones Lipschitz regulares de la ecuación (0.0.1), estudiada por Burago y Kleiner. El Capítulo 3 se divide en tres secciones: en la Sección 3.1 introduciremos el concepto de funciones Lipschitz regulares y enunciaremos algunas propiedades básicas (para más detalles asociados a esta sección se sugiere ver [9]). En la Sección 3.2 estudiamos uno de los principales resultados del artículo [15], a saber, la descomposición de aplicaciones Lipschitz regulares en "pedazos" bi-Lipschitz. Finalmente, en la Sección 3.3 estudiamos la construcción en [15] que responde negativamente a la pregunta propuesta por Feige mediante ejemplos de funciones continuas positivas $\rho : [0,1]^2 \to \mathbb{R}$ para los cuales la ecuación (0.0.2) no tiene soluciones Lipschitz regulares.

El **Capítulo 4** tiene por objetivo discutir la equivalencia Lipschitz entre conjuntos de Delone.

En concreto, estudiaremos el problema de encontrar un conjunto de Delone repetitivo en el plano que no puede ser mapeado sobre $\mathbb{Z}^2$ por alguna bijección Lipschitz. La estrategia planteada para tratar con este problema utiliza, fundamentalmente, los resultados obtenidos por Cortez y Navas en [6], como también las ideas sobre aplicaciones Lipschitz regulares, introducidas por Dymond, Kaluža y Kopecká en [15]. De manera más precisa, primero estudiaremos una extensión de los resultados en [6] para funciones Lipschitz bajo condiciones adecuadas; esto nos permitirá usar las ideas introducidas por Cortez y Navas a nuestro problema. Por otro lado, dado un conjunto de Delone $\mathcal{D} \subset \mathbb{Z}^2$ que satisface la propiedad $2\mathbb{Z}$ (esta definición será dada en el Capítulo 4) y una aplicación Lipschitz $f : \mathcal{D} \to \mathbb{Z}^2$, demostraremos que después de normalizar y pasar al límite, obtendremos una función $F : [0,1]^2 \to \mathbb{R}^2$ la cual es Lipschitz regular. Este hecho nos permitirá usar la maquinaria introducida en [15] sobre aplicaciones Lipschitz regulares para abordar nuestro problema.

En la primera parte del Capítulo 4 (Sección 4.1), revisaremos la construcción realizada por Cortez y Navas en [6] de conjuntos de Delone repetitivos no-rectificables, como también extensiones de algunos resultados obtenidos en [6] al caso de aplicaciones Lipschitz definidas sobre un conjunto de Delone $\mathcal{D} \subset \mathbb{Z}^2$ y con valores en $\mathbb{Z}^2$, las cuales son bi-Lipschitz en un subconjunto adecuado de $\mathcal{D}$. Luego, en la Sección 4.2 demostraremos algunas implicancias relacionadas a funciones Lipschitz definidas sobre conjuntos de Delone en $\mathbb{Z}^2$ los cuales satisfacen la propiedad $2\mathbb{Z}$. En la sección 4.3 discutiremos cómo abordar el problema inicial que motiva este capítulo.

En el **Capítulo 5** presentamos las conclusiones obtenidas en este trabajo de tesis y, adicionalmente, mostraremos algunos problemas abiertos surgidos en este trabajo.

Finalmente es incluído un **Apéndice**, el cual aborda de manera introductoria algunos conceptos básicos de análisis de aplicaciones Lipschitz, como también de teoría de la medida, los cuales son importantes para entender lo desarrollado en este trabajo tesis.

Chapter 1

Delone sets and Burago-Kleiner' construction

In this Chapter we will focus on bi-Lipschitz maps and Delone sets, which are uniformly discrete and coarsely dense subsets of the euclidean space. In Section 1.1 we define Delone sets and some equivalence relations on this set class. In section 1.2 we will be devoted to the Burago-Kleiner' construction of a Delone set in $\mathbb{R}^d$, $d \geq 2$, for which there is no bi-Lipschitz bijection onto $\mathbb{Z}^d$. In this Chapter we are mainly concerned with the article [3].

1.1 Delone sets and Bi-Lipschitz equivalence.

As we mentioned before, in this chapter we study discrete sets in the Euclidean space which are separated and dense in a uniform way.

Definition 1.1:
A subset $\mathcal{D}$ of a metric space (X, d) is called a Delone set (or a separated net) if there exist two positive real numbers σ, Σ such that any two points of $\mathcal{D}$ are at least at distance σ, i.e, $d(x, y) \geq \sigma$ for any $x, y \in \mathcal{D}$ with $x \neq y$ ($\mathcal{D}$ is said to be σ-separated) and so that every ball of radius Σ in X contains a point of $\mathcal{D}$, i.e, for each $z \in X$ there is $y \in \mathcal{D}$ such that $d(y, z) \leq \Sigma$ ($\mathcal{D}$ is said to be Σ-coarsely dense).

For example, $\mathbb{Z}^d \subset \mathbb{R}^d$ is a Delone set. See Figure 1.1 for a picture of a Delone set in the Euclidean space.

Remark 1.2:
Every metric space (X, d) contains a Delone set. It is sufficient to consider a subset $\mathcal{D}$ of X which is maximal with the property that, given a positive real number σ, two points $x, y \in \mathcal{D}$ are at distance at least σ. This construction produces a Delone set which is σ-separated and σ-coarsely dense.

Delone sets are important in coarse geometry (see [14]) as a *discretization* of a metric space and in Mathematical Physics, from the discovery of Quasicrystals by the Nobel prize laureate Dan

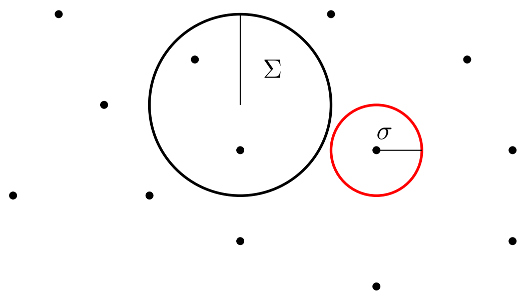

Figure 1.1: Part of a Delone set $\mathcal{D}$ (black points). Each ball of radius $\sigma > 0$ centered at a point of $\mathcal{D}$ contains at most one point of the Delone set, and each ball of radius $\Sigma > 0$ contains at least one point of $\mathcal{D}$.

Shechtman and his coworkers in the early 1980s (see [13]).

It is natural to attempt to compare two Delone sets. A Delone set $\mathcal{D}$ in $\mathbb{R}^d$ is said to be ***rectifiable*** if it is bi-Lipschitz equivalent to $\mathbb{Z}^d$ (see Section A.1 in the Appendix for a basic background on Lipschitz Analysis). The study of bi-Lipschitz equivalence between Delone sets was motivated by Gromov in [14] with a Geometric Group-theoretic motivation. We say that two metric spaces (X, d) and (Y, ρ) are quasi-isometric if and only if there exist Delone sets $\mathcal{D}_1 \subset X$ and $\mathcal{D}_2 \subset Y$ which are BL equivalent. In the analysis of the quasi-isometry equivalence relation, the following question arises naturally.

Question 1.3 (Gromov, 1993)**:**
Can a metric space contain two Delone sets that are not BL equivalent?

This question was answered in the affirmative by Burago and Kleiner in [3] and, independently by McMullen in [21]. Concretely, they prove the following theorem (see Section 1.2 below).

Theorem 1.4 (Burago & Kleiner, McMullen, 1998)**:**
For every $d \geq 2$, there exists a non-rectifiable Delone set $\mathcal{D}_d$ in $\mathbb{R}^d$.

Remark 1.5:
There is another notion of equivalence between Delone sets in a given metric space. We say that two Delone sets $\mathcal{D}_1, \mathcal{D}_2$ contained in a metric space (X, d) are bounded displacement equivalent (or that are BD-equivalent) if there exists a bijection $f : \mathcal{D}_1 \to \mathcal{D}_2$ with the property that

$$\sup_{x \in \mathcal{D}_1} d(f(x), x) < +\infty$$

Notice that BD-equivalence is a stronger condition that BL-equivalence. Indeed: let $\mathcal{D}_1, \mathcal{D}_2$ be two Delone sets in a metric space (X, d), with $\mathcal{D}_i$ being σ_i-separated and Σ_i-coarsely dense, for $i = 1, 2$. Let $f : \mathcal{D}_1 \to \mathcal{D}_2$ be a bijection such that $C := \sup_{x \in \mathcal{D}_1} d(f(x), x) < +\infty$. Observe that by the

σ_1-separability, we have that for every $x, y \in \mathcal{D}_1$

$$\begin{aligned}
d(f(x), f(y)) &\leq d(f(x), x) + d(x, y) + d(y, f(y)) \\
&\leq 2C + d(x, y) \\
&\leq \left(\frac{2C}{\sigma_1} + 1\right) d(x, y)
\end{aligned}$$

Since f^{-1} is also a bounded displacement map, the same argument applied to f^{-1} provides a lower Lipschitz constant to f. Henceforward, f is a bi-Lipschitz bijection.

Remark 1.6:
Question 1.3 can be answered negatively in the 1-dimensional case. In fact, let $\mathcal{D} \subset \mathbb{R}$ be a Delone set being σ-separated and Σ-coarsely dense. Write $\mathcal{D} = \{\ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots\}$ and suppose that $x_0 = 0$. Define $f : \mathcal{D} \to \mathbb{Z}$ by letting $f(x_n) := n$. By the Σ-density of $\mathcal{D}$, we have that every two consecutive points in $\mathcal{D}$ are at distance at most 2Σ. Then, for $n > m$

$$\begin{aligned}
|x_n - x_m| &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_{m+2} - x_{m+1}) + (x_{m+1} - x_m) \\
&\leq 2\Sigma + \ldots + 2\Sigma = 2\Sigma(n - m).
\end{aligned}$$

On the other hand, by the σ-separability, we have that

$$\begin{aligned}
|x_n - x_m| &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_{m+2} - x_{m+1}) + (x_{m+1} - x_m) \\
&\geq \sigma + \ldots + \sigma = \sigma(n - m).
\end{aligned}$$

Therefore, f is a bi-Lipschitz bijection.

Remark 1.7:
Not every Delone set in $\mathbb{R}$ is bounded displacement equivalent to $\mathbb{Z}$. For instance, $2\mathbb{Z}$ is not BD equivalent to $\mathbb{Z}$.

We point out that Burago-Kleiner and McMullen show only the existence of non-rectifiable Delone sets. The first concrete examples of non-rectifiable Delone sets contained in $\mathbb{Z}^d$, for $d \geq 2$, were constructed by Cortez & Navas in [6]; these examples are also constructed to be *repetitive* (see Section 4.1 below).

1.2 Burago-Kleiner' construction

In this section we are dedicated to the Burago-Kleiner' proof on the existence of non-rectifiable Delone sets in $\mathbb{R}^d$, for $d \geq 2$. In [3], Burago and Kleiner and (independently) McMullen in [21], answered Question 1.3 affirmatively by solving a question of analytical nature: *Is there an L^∞ positive function $\rho : \mathbb{R}^2 \to \mathbb{R}$, which can not be realized as the Jacobian of a bi-Lipschitz homeomorphism almost everywhere?* In the next paragraphs, we sketch how to produce a non-rectifiable Delone set from such a bad density map, where for simplicity, we perform all the computations in the 2-dimensional setting.

Let $\rho : [0,1]^2 \to \mathbb{R}$ be a continuous function such that $0 < \inf \rho \leq \sup \rho < +\infty$. Consider a sequence of disjoint squares $(S_k)_{k\in\mathbb{N}}$ with vertices in $\mathbb{Z}^2$ and side-lengths equal to l_k, satisfying that $l_k \to +\infty$. Moreover, consider a sequence of (unique) affine homeomorphisms $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^2$ sending the unit square onto S_k with scalar linear part. Let $(m_k)_{k\in\mathbb{N}}$ be a sequence such that m_k and l_k/m_k both tend to $+\infty$, and subdivide each square S_k into m_k^2 squares $\{T_k^i\}_{i=1}^{m_k^2}$ of side-length equal to l_k/m_k (see Figure 1.3).

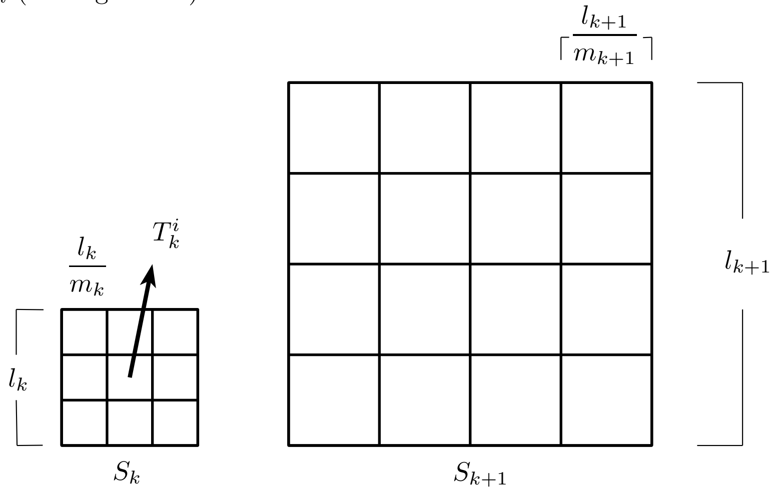

Figure 1.2: The squares S_k and S_{k+1}, and a square T_k^i.

We “implant” the density ρ into each square S_k in order to construct a set $\mathcal{D} \subset \mathbb{R}^2$ in such a way that the set $\mathcal{D}$, inside each square S_k, encodes the structure of ρ more precisely while growing $k \in \mathbb{N}$. First of all, we place one point at the center of each unit square with integer vertices which are not contained in $\bigcup_{k=1}^{+\infty} S_k$. Secondly, for every $k \in \mathbb{N}$ and $i = 1, \ldots, m_k^2$, we place $\left\lfloor \int_{T_k^i} \rho \circ \varphi_k d\lambda \right\rfloor$ points evenly inside each square T_k^i. This construction produces a discrete set $\mathcal{D} \subset \mathbb{R}^2$. Since $\inf \rho > 0$, then $\mathcal{D}$ is coarsely dense and, since $\sup \rho < +\infty$, the set $\mathcal{D}$ is separated; therefore $\mathcal{D}$ is a Delone set.

Now, suppose that $\mathcal{D}$ is a rectifiable Delone set and let $g : \mathcal{D} \to \mathbb{Z}^2$ be an L-bi-Lipschitz bijection, for some $L \geq 1$. Define $\mathcal{R}_k := \varphi_k^{-1}(\mathcal{D}) \cap [0,1]^2$ and consider the “pull-backs” $f_k : \mathcal{R}_k \to \frac{1}{l_k}\mathbb{Z}^2$ defined by:

$$f_k(x) := \frac{1}{l_k}(g \circ \varphi_k(x) - g \circ \varphi_k(x_k)),$$

where $x_k \in \mathcal{R}_k$ is some base point. It is direct that the mappings f_k are L-bi-Lipschitz and uniformly bounded. By the proof of Arzelá-Ascoli theorem, we may find a subsequence $(f_{k_i})_{i\in\mathbb{N}}$ which converges uniformly to some L-bi-Lipschitz map $f : [0,1]^2 \to \mathbb{R}^2$.

Let μ_k and ν_k be two normalized counting measures given by

$$\mu_k(A) := \frac{|A \cap \mathcal{R}_k|}{l_k^2} \qquad \text{for every } A \subset [0,1]^2,$$
$$\nu_k(B) := \frac{|B \cap \frac{1}{l_k}\mathbb{Z}^2|}{l_k^2} \qquad \text{for every } B \subset \mathbb{R}^2.$$

It is direct that $\nu_k|_{f([0,1]^2)}$ converges weakly to $\lambda|_{f([0,1]^2)}$. Besides, by Proposition $A.5$, we have that μ_k converges weakly to $\rho\lambda$ restricted to $[0,1]^2$, where:

$$\rho\lambda(A) := \int_A \rho d\lambda, \qquad \text{for every } A \subset \mathbb{R}^2.$$

Since $(f_{k_i})_{i\in\mathbb{N}}$ converges uniformly to f, by Lemma $A.7$, we have that $(f_{k_i})_\# \mu_{k_i}$ converges weakly to $f_\#(\rho\lambda)$. On the other hand, again as a consequence of Proposition $A.5$, it is shown that $(\nu_k)_{k\in\mathbb{N}}$ converges weakly to the Lebesgue measure in $f([0,1]^2)$. Hence, we have the equality $f_\#(\rho\lambda) = \lambda|_{f([0,1]^2)}$, which, by the Euclidean Area Formula (Theorem $A.3$) is equivalent to the equation

$$Jac(f) = \rho \qquad a.e.$$

This way, in [3] and [21] it is shown the following theorem.

Theorem 1.8 (Burago-Kleiner, McMullen, 1998)**:**
The following are equivalent:

i) *Every Delone set in $\mathbb{R}^d$, with $d \geq 2$, is rectifiable.*

ii) *For every continuous function $\rho : I^d \to \mathbb{R}$ with $0 < \inf \rho < \sup \rho < +\infty$ there exists a bi-Lipschitz homeomorphism $f : [0,1]^d \to \mathbb{R}^d$ so that*

$$Jac(f) = \rho \qquad a.e.$$

Remark 1.9:
The implication ii) $\Rightarrow$ i) in Theorem 1.8 was proven by McMullen in [21].

Therefore, it remains to prove that there exists a continuous function $\rho : [0,1]^2 \to \mathbb{R}$ so that $0 < \inf \rho < \sup \rho < +\infty$ which can not be written as the Jacobian of a bi-Lipschitz map. The key fact about the construction of such a bad density function, is that there are two adjacent small squares whose images by any bi-Lipschitz map defined in $[0,1]^2$, which are close in measure. In the next paragraphs we sketch how to prove the following theorem.

Theorem 1.10 (Theorem 1.2 in [3])**:**
Given $c > 0$, there exists a continuous function $\rho : [0,1]^2 \to [1, 1+c]$ for which there is no bi-Lipschitz homeomorphism $f : [0,1]^2 \to \mathbb{R}^2$ such that

$$Jac(f) = \rho \qquad a.e. \tag{1.2.1}$$

We refer to a function $\rho : [0,1]^2 \to \mathbb{R}^2$ for which the equation (1.2.1) has no bi-Lipschitz solutions as a *non-realizable density.*

Remark 1.11:
Observe that to obtain a non-realizable density it is sufficient to construct, for every $L \geq 1$ and $c > 0$, a continuous function $\rho_{L,c} : [0,1]^2 \to [1+c]$ which is not realizable as the Jacobian of an L-bi-Lipschitz map. Indeed, if we have such a family $(\rho_{n,c})_{n\in\mathbb{N}}$ of bad densities, we consider a sequence of disjoint squares $S_n \subset [0,1]^2$ converging to some point $p \in [0,1]^2$, and then define $\rho : [0,1]^2 \to [1, 1+c]$ so that $\rho|_{S_n} = \rho_{n,\min\{c,\frac{1}{n}\}} \circ \varphi_n$, where $\varphi_n : \mathbb{R}^2 \to \mathbb{R}^2$ is the unique affine homeomorphism with scalar linear part sending $[0,1]^2$ onto S_n. Outside $\bigcup_{n\in\mathbb{N}} S_n$, ρ is defined in order to obtain the continuity. This construction provides a non-realizable density ρ.

Remark 1.12:
It is sufficient to construct a non-realizable density $\rho : [0,1]^2 \to [1, 1+c]$ as in Remark 1.11 being measurable (non necessarily continuous). Indeed, if $(\rho_k)_{k\in\mathbb{N}}$ is a sequence of smoothings of the measurable function ρ converging to ρ in L^1, then by the Arzelá-Ascoli theorem, any sequence of L-bi-Lipschitz maps $\phi_k : I^2 \longrightarrow \mathbb{R}^2$ such that $Jac(\phi_k) = \rho_k$ a.e, will converge, up to a subsequence, to a bi-Lipschitz map $\phi : I^2 \longrightarrow \mathbb{R}^2$ satisfying $Jac(\phi) = \rho$ a.e. See the proof of Proposition 2.3 for a variation of this argument.

By Remarks 1.11 and 1.12, Theorem 1.10 is a consequence of the following Proposition.

Proposition 1.13 (Lemma 3.1 in [3])**:**
Given $L \geq 1$ and $c > 0$, there exists a measurable function $\rho : [0,1]^2 \to [1, 1+c]$ such that there is no L-bi-Lipschitz homeomorphism $f : [0,1]^2 \to \mathbb{R}^2$ so that

$$Jac(f) = \rho \qquad\qquad a.e.$$

Definition 1.14:
Given a positive real number A, we say that two points $x, y \in \mathbb{R}^2$ are **A-stretched** *under a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ if*

$$||f(x) - f(y)|| \geq A||x - y||.$$

For $N = N(L, c)$ sufficiently large, the specific construction of the measurable non-realizable density is based on the so-called ***checkerboard function*** $\rho_{N,c}$ defined below (see Figure 1.3).

Definition 1.15:
For $N \in \mathbb{N}$, consider the rectangle $R_N := [0,1] \times \left[0, \frac{1}{N}\right]$ and the squares $S_i := \left[\frac{i-1}{N}, \frac{i}{N}\right] \times \left[0, \frac{1}{N}\right]$, with $i = 1, \ldots, N$. Given $c > 0$, we define the function $\rho_{N,c} : R_N \to [1, 1+c]$ by letting

$$\rho_{N,c}(x) = \begin{cases} 1 & \textit{if } x \in S_i \textit{ with } i \textit{ even,} \\ \\ 1+c & \textit{if } x \in S_i \textit{ with } i \textit{ odd.} \end{cases}$$

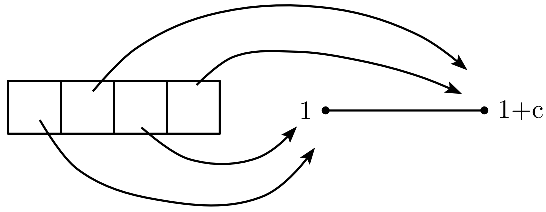

Figure 1.3: The checkerboard function $\rho_{N,c} : R_N \to [1, 1+c]$.

Using a quite long argument, Burago and Kleiner prove that for a large enough N, there exists $\epsilon = \epsilon(L, c) > 0$ such that for every L-bi-Lipschitz map $\phi : [0,1]^2 \to \mathbb{R}^2$ whose Jacobian $Jac(\phi)$ is equal to $\rho_{N,c}$ except for a set of measure less than ϵ, there must exist two points $x, y \in R_N$ that are $(1+\kappa)||\phi(0,0) - \phi(1,0)||$-stretched under ϕ for a certain $\kappa = \kappa(L, c) > 0$. Using this, as a next step they modify the function on a rectangular neighborhood U of the line segment $\overline{xy}$ by including a rescaled version of another checkerboard function. By the same argument, there are two points $x_2, y_2 \in U$ that are $(1+\kappa)^2||\phi(1,0) - \phi(0,0)||$-stretched by an homeomorphism realizing this new function as the Jacobian. Repeating this argument at smaller and smaller scales, we eventually obtain a function $\rho : R_N \to [1, 1+c]$ which cannot be the Jacobian of an L-bi-Lipschitz map. Concretely, they prove the following lemma.

Lemma 1.16 (Lemma 3.2 in [3])**:**
For every $L > 1$ and $c > 0$, there exist $\kappa > 0$, $\mu > 0$ and positive integers N_0 and M satisfying the following property: for every $N \in \mathbb{N}$ with $N \geq N_0$, there exists $\varepsilon > 0$ such that if the points $(0,0)$ and $(1,0)$ are A-stretched by an L-bi-Lipschitz homeomorphism $f : R_N \to \mathbb{R}^2$ whose Jacobian $Jac(f)$ coincides with ρ_N except in a set of measure less than ε, then there must exists $1 \leq i \leq N$ and a pair of points of the form $\left(\frac{p}{NM} + \frac{i-1}{N}, \frac{q}{NM}\right)$ and $\left(\frac{p+1}{NM} + \frac{i-1}{N}, \frac{q}{NM}\right)$ in R_N, with $p, q \in [0, M] \cap \mathbb{Z}$, which is $(1+\kappa)||f(1,0) - f(0,0)||$-stretched under f.

In order to demonstrate Lemma 1.16, Burago and Kleiner proceed by contradiction. Their proof is supported in three key claims which describe some properties of bi-Lipschitz mappings and that we proceed to explain. From now, denote by $x^i_{p,q}$ the points of the form $\left(\frac{p}{NM} + \frac{i-1}{N}, \frac{q}{NM}\right)$, where $1 \leq i \leq N$ and $p, q \in [0, M] \cap \mathbb{Z}$. We call these points *marked*. Note that the marked points are the vertices of the regular division of S_i into M^2 squares of side length $1/NM$. We write $A := ||f(1,0) - f(0,0)||$, $W := (A/N, 0)$ and $W^i_{p,q} := f(x^{i+1}_{p,q}) - f(x^i_{p,q})$. If $l \in (0,1)$ we say that $x^i_{p,q}$ is *l-regular* if the length of the projection to the x-axis of the vector $W^i_{p,q}$ is greater than $(1-l)A/N$. We say that a square S_i is *l-regular* if $x^i_{p,q}$ is regular for every $0 \leq p, q \leq M$. The three key claims in order to prove Lemma 1.16 are the following. The first claim says that there is $\kappa > 0$ such that if all pairs $(x^i_{p,q}, x^{i+1}_{p,q})$ are no more than $(1+\kappa)A$-stretched under f, then there exists regular squares.

Claim 1.17 (Claim 1 in [3])**:**
There exists $\kappa_0 = \kappa_0(l, M) > 0$ and $N_0 = N_0(l, M) \in \mathbb{N}$ such that for every $0 < \kappa \leq \kappa_0$ and for

every integer number $N \geq N_0$ there is an l-regular square S_i.

Remark 1.18:
Although in the Remark 2 of [3] *it is pointed out that Theorem 1.10 holds for Lipschitz homeomorphisms, we remark that for the proof of Claim 1.17 the lower bi-Lipschitz constant of f is necessary.*

The second claim tell us that if $l > 0$ is chosen small-enough, then every l-regular vector $W^i_{p,q}$ is extremely close to the vector W.

Claim 1.19 (Claim 2 in [3])**:**
Given $m > 0$, there exists $l_0 = l_0(m) > 0$ such that if $0 < l \leq l_0$ and $0 < \kappa < l$, then for every l-regular points $x^i_{p,q}$ we have that $||W - W^i_{p,q}|| \leq m/N$.

The third claim says that by a suitable choice of m and M, the area of $f(S_i)$ is very close to the area of $f(S_{i+1})$.

Claim 1.20 (Claim 3 in [3])**:**
There exist $m_0 > 0, M_0 \in \mathbb{N}$ such that if $0 < m \leq m_0$ and $M \leq M_0$ then the following holds: if for some $1 \leq 1 \leq N$ and for every $0 \leq p, q \leq M$ we have that $||W - W^i_{p,q}|| \leq m/N$, then

$$|\lambda(f(S_{i+1})) - \lambda(f(S_i))| < \frac{c}{2N^2}.$$

Thus, if the equation (1.2.1) holds for an L-bi-Lipschitz mapping (except for a set of measure less or equal than ε), then the Euclidean Area Formula (Theorem A.3) and Claim 1.20 force the values of $\rho : R_N \to [1 + c]$ in S_i and S_{i+1} to be very close. This yields a contradiction since ρ is chosen as a checkerboard function, whose values oscillate in two neighbouring squares S_i, S_{i+1}. Therefore, to construct a density map $\rho : [0, 1]^2 \to \mathbb{R}$ which can not be written as the Jacobian of an L-bi-Lipschitz map, it is sufficient to put a suitable checkerboard function in $[0, 1]^2$ at smaller and smaller scales, in order to stretch two marked points in a factor $(1 + \kappa)^i A$ by passing from a scale to a smaller one. The number P of rescaled checkerboard functions is chosen so that $(1 + \kappa)^P A$ is greater L, thus overcoming the bi-Lipschitz constant. See Figure 1.4 for a sketch of this construction.

Following the Burago-Kleiner' proof, Lemma 1.16 can be formulated as a dichotomy of bi-Lipschitz mappings (see [15] for more details).

Lemma 1.21 (Lemma 3.3 in [15])**:**
Let $L \geq 1$ and $\varepsilon > 0$. Then there exist parameters

$$M = M(L, \varepsilon) \in \mathbb{N}, \quad \kappa = \kappa(L, \varepsilon) \in (0, 1) \quad N_0 = N_0(L, \varepsilon) \in \mathbb{N},$$

such that for all $c > 0$, $N \in \mathbb{N}$, $N \geq N_0$ and all L-bi-Lipschitz mapping $f : [0, c] \times \left[0, \frac{c}{N}\right] \to \mathbb{R}^2$, at least one of the following statement holds:

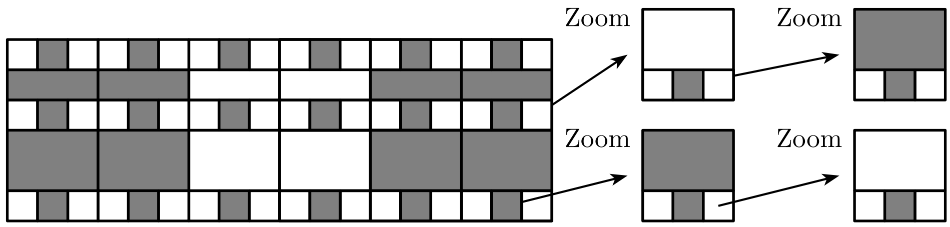

Figure 1.4: Non-realizable density. In the gray zones the density attains the value $1+c$ and in the white zones attains the value 1.

1. *There exists a set* $\Omega \subset \{1, \ldots, N-1\}$ *with* $|\Omega| \geq (1-\varepsilon)(N-1)$ *such that for all* $i \in \Omega$ *and for all* $x \in \left[\frac{(i-1)c}{N}, \frac{ic}{N}\right] \times \left[0, \frac{c}{N}\right]$,

$$\left|\left|f\left(x + \left(\frac{c}{N}, 0\right)\right) - f(x) - \frac{1}{N}(f(c,0) - f(0,0))\right|\right| \leq \frac{c\varepsilon}{N}.$$

2. *There exist* $z \in \frac{c}{NM}\mathbb{Z}^2 \cap \left(\left[0, c - \frac{c}{NM}\right] \times \left[0, \frac{c}{N} - \frac{c}{NM}\right]\right)$ *such that*

$$\frac{||f\left(z + \left(\frac{c}{NM}, 0\right)\right) - f(z)||}{\frac{c}{NM}} > (1+\kappa)\frac{||f(c,0) - f(0,0)||}{c}.$$

The first statement in Lemma 1.21 says that an L-bi-Lipschitz function maps two neighbouring cubes to $(c\varepsilon/N)$-close images (up to a translation by $(1/N)(f(c,0) - f(0,0))$). The second statement says that there exists two points which are $(1+\kappa)||f(c,0) - f(0,0)||/c$-stretched by f. As a consequence of Lemma 1.21, we have the following Proposition.

Proposition 1.22 (Lemma 3.1 in [15])**:**
Let $k \in \mathbb{N}$, $L \geq 1$ *and* $\eta, \zeta \in (0,1)$. *Then there exists* $r = r(k, L, \eta, \zeta) \in \mathbb{N}$ *such that for every non-empty open set* $U \subset \mathbb{R}^2$ *there exist finite tiled families* $\mathcal{S}_1 \ldots, \mathcal{S}_r$ *of cubes contained in* U *with the following properties:*

1. *For each* $1 \leq i < r$ *and each cube* $S \in \mathcal{S}_i$, *we have that*

$$\lambda\left(S \cap \bigcup_{j=i+1}^{r} \bigcup \mathcal{S}_j\right) \leq \eta\lambda(S).$$

2. *For any* $k-$*tuple* $(h_1, \ldots, h_k)$ *of* $L-$*bi-Lipschitz mappings* $h_j : U \to \mathbb{R}^2$ *there exist* $i \in \{1, \ldots, r\}$ *and* e_1-*adjacent cubes* $S, S' \in \mathcal{S}_i$ *such that*

$$\left|\int_S |Jac(h_j)|d\lambda - \int_{S'} |Jac(h_j)|d\lambda\right| \leq \zeta\lambda(S),$$

for all $j \in \{1, \ldots, k\}$.

Remark 1.23:

For a detailed proof of the non-realizability of the bad densities constructed by McMullen, see [16].

Chapter 2

Generic properties of bad densities.

A direct consequence of the Fundamental Theorem of Calculus is that every positive continuous function ρ defined on a compact interval $[a, b]$ can be written as the derivative of a diffeomorphism, namely

$$f(x) := \int_a^x \rho(s)ds.$$

The very same formula shows that every positive L^∞ function that is bounded away from zero can be written as the a.e. derivative of a bi-Lipschitz homeomorphism.

As was shown by Burago-Kleiner [3] and McMullen [21] in very important works (see also [16]), this is no longer true in the 2-dimensional framework: there exist positive L^∞ (even continuous) functions that cannot be written as the Jacobian of a bi-Lipschitz homeomorphism of the plane. This pure analytical result is obtained as the fundamental step of another result of a discrete nature: there exist coarsely dense, uniformly discrete sets in the plane (that can be taken as subsets of $\mathbb{Z}^2$) that are not bi-Lipschitz equivalent to the standard lattice $\mathbb{Z}^2$. Although recently a shortcut (and extension) to this last result has been produced in [6], the analytical one is interesting by itself, and deserves more attention. Based on Burago-Kleiner's version of this result (which is slightly more general than McMullen's), in this section we show that not only bad densities exist, but are also generic.

Main Theorem:
A generic positive continuous function $\phi : \mathbb{R}^2 \to \mathbb{R}$ cannot be written as the Jacobian of a bi-Lipschitz homeomorphism. The same holds for a generic positive L^∞ function.

We point out that similar results hold (with the same proof) in dimension greater than 2. The restriction to the 2-dimensional case below just allows simplifying notation and computations.

2.1 The continuous case

Recall that a subset G of a metric space M is said to be **thick** if it contains a set of the form $\bigcap_{n\in\mathbb{N}} G_n$, where each G_n is an open dense subset of M.

Definition 2.1:

If all points of a thick subset have some property, then this is said to be a **generic** *property of M.*

We now consider the space $C_+(I^2,\mathbb{R})$ of all positive continuous functions $\rho : I^2 \to \mathbb{R}$ with the norm $||f||_0 := \sup\{|f(x)| : x \in I^2\}$. In this section, we will use the Burago-Kleiner construction to prove Main Theorem in the continuous setting.

Theorem 2.2:

Let $\mathcal{C}$ be the set of all functions $\rho \in C_+(I^2,\mathbb{R})$ such that there is no bi-Lipschitz map $\phi : I^2 \to \mathbb{R}^2$ satisfying $\rho = Jac(\phi)$ a.e. Then $\mathcal{C}$ is a thick subset of $C_+(I^2,\mathbb{R})$.

This Theorem will follow from the next

Proposition 2.3:

Given $L > 1$, consider the set $\mathcal{C}_L$ of all functions $\rho \in C_+(I^2,\mathbb{R})$ such that there is no L-bi-Lipschitz map $\phi : I^2 \to \mathbb{R}^2$ satisfying $Jac(\phi) = \rho$ a.e. Then $\mathcal{C}_L$ is an open dense subset of $C_+(I^2,\mathbb{R})$.

Theorem 2.2 is a direct consequence of this, since

$$\bigcap_{n\in\mathbb{N}} \mathcal{C}_n = \mathcal{C}.$$

To prove Proposition 2.3, we first establish that $\mathcal{C}_L$ is an open subset of $C_+(I^2,\mathbb{R})$ for each $L > 0$. Let $X := C_+(I^2,\mathbb{R}) \setminus \mathcal{C}_L$, and consider a sequence $(\rho_k)_{k\in\mathbb{N}} \subset X$ such that $\rho_k \xrightarrow[k\to\infty]{} \rho$. We need to show that $\rho \notin \mathcal{C}_L$. To do this, let $\phi_k : I^2 \to \mathbb{R}^2$ be a sequence of L-bi-Lipschitz maps such that $Jac(\phi_k) = \rho_k$ a.e. By the Arzelà-Ascoli theorem, $(\phi_k)_{k\in\mathbb{N}}$ has a subsequence that converges to an L-bi-Lipschitz map $\phi : I^2 \to \mathbb{R}^2$. To simplify notation, we assume without loss of generality that $\phi_k \xrightarrow[k\to\infty]{} \phi$. Our goal is to show that $Jac(\phi) = \rho$, and hence $\rho \notin \mathcal{C}_L$.

Let $U \subset I^2$ be a closed ball and denote by λ the Lebesgue measure in $\mathbb{R}^2$. Then, by the Euclidean Area Formula and as $\rho_k \to \rho$,

$$\begin{aligned} Area(\phi_k(U)) &= \int_U Jac(\phi_k) d\lambda \\ &= \int_U \rho_k d\lambda \\ &\xrightarrow[k\to\infty]{} \int_U \rho d\lambda. \end{aligned}$$

Below we show that, also, $Area(\phi_k(U)) \to Area(\phi(U))$, and hence $Jac(\phi) = \rho$, as announced. To do this, given $\varepsilon > 0$, consider the sets (see Figure 2.1)

$$V_\varepsilon^{ext}(\phi(U)) := \{x \in \mathbb{R}^2 \colon \, d(x, \phi(U)) < \varepsilon\}$$
$$V_\varepsilon^{int}(\phi(U)) := \{x \in \phi(U) \colon \, d(x, \partial\phi(U)) > \varepsilon\}.$$

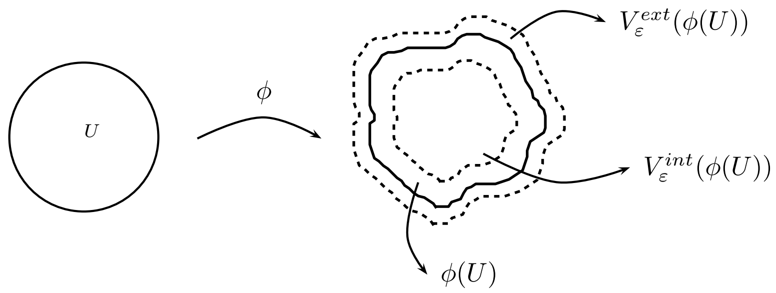

Figure 2.1: The sets $\phi(U)$, $V_\varepsilon^{ext}(\phi(U))$ and $V_\varepsilon^{int}(\phi(U))$.

Then the desired convergence

$$Area(\phi_k(U)) \to Area(\phi(U))$$

obviously follows from the next (see Figure 2.2).

Lemma 2.4:
Given $\varepsilon > 0$, there exists a positive integer k_0 such that if $k \geq k_0$, then

i) $\phi_k(U) \subset V_\varepsilon^{ext}(\phi(U))$,

ii) $V_\varepsilon^{int}(\phi(U)) \subset \phi_k(U)$.

Proof. Assume i) does not hold. Then for each $k \in \mathbb{N}$ there exists $y_k \in \phi_k(U) \setminus V_\varepsilon^{ext}(\phi(U))$. Let $x_k \in U$ be such that $y_k = \phi_k(x_k)$. Since U is closed, there exists a sequence $(x_{k_l})_{l\in\mathbb{N}}$ that converges to a certain $x \in U$. By the equicontinuity of the sequence $(\phi_k)_{k\in\mathbb{N}}$ and the convergences $x_{k_l} \xrightarrow[l\to\infty]{} x$ and $\phi_k \to \phi$, there exists $l_0 \in \mathbb{N}$ such that if $l \geq l_0$, then $|\phi_{k_l}(x_{k_l}) - \phi_{k_l}(x)| < \frac{\varepsilon}{2}$ and $|\phi_{k_l}(x) - \phi(x)| < \frac{\varepsilon}{2}$. Thus, for $l \geq l_0$,

$$|\phi_{k_l}(x_{k_l}) - \phi(x)| \leq |\phi_{k_l}(x_{k_l}) - \phi_{k_l}(x)| + |\phi_{k_l}(x) - \phi(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $y_{k_l} = \phi_{k_l}(x_{k_l}) \xrightarrow[l\to\infty]{} \phi(x) \in \phi(U) \subset V_\varepsilon^{ext}(\phi(U))$. However, this is impossible, since (y_k) is a sequence in the closed subset $\mathbb{R}^2 \setminus V_\varepsilon^{ext}(\phi(U))$, and then all its accumulation points lie in $\mathbb{R}^2 \setminus V_\varepsilon^{ext}(\phi(U))$.

To prove ii), let $x, y \in \partial U$ be such that the arc between x and y has length less than $\frac{\varepsilon}{2L}$. Let

$k_0 \in \mathbb{N}$ be such that $|\phi_k(x) - \phi(x)| < \varepsilon/4$ and $|\phi_k(y) - \phi(y)| < \varepsilon/4$ for all $k \geq k_0$. Since ϕ_k is L-bi-Lipschitz, the curve in the boundary of $\phi_k(U)$ joining $\phi_k(x)$ and $\phi_k(y)$ has a length $< \varepsilon/2$. Therefore, $\partial(\phi_k(U)) \cap V_\varepsilon^{int}(\phi(U)) = \emptyset$, and thus $V_\varepsilon^{int}(\phi(U)) \subset \phi_k(U)$, for $k \geq k_0$. $\square$

Remark 2.5:

Define $\mathcal{C}_{L,\infty}$ as being the set of all positive functions $\rho \in L_+^\infty(I^2, \mathbb{R})$ such that there is no L-bi-Lipschitz map $f : I^2 \to \mathbb{R}^2$ satisfying $Jac(f) = \rho$ a.e. Then the same argument above shows that $\mathcal{C}_{L,\infty}$ is an open subset of $L_+^\infty(I^2, \mathbb{R})$.

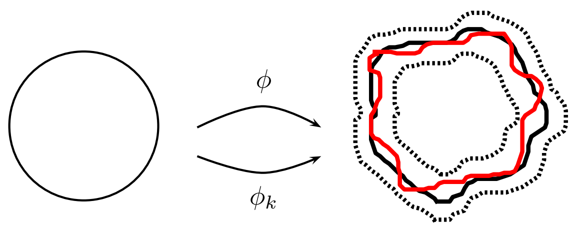

Figure 2.2: The sets $\phi(U)$ (black) and $\phi_k(U)$ (red) for $k \geq k_0$.

We next establish that $\mathcal{C}_L$ is a dense subset of $C_+(I^2, \mathbb{R})$ for each $L > 0$. Given $\varphi \in C_+(I^2, \mathbb{R})$ with image $[a, b] := \varphi(I^2) \subset (0, \infty)$ and given $\varepsilon > 0$ with $\varepsilon << b - a$, we want to construct a continuous function $\rho \in \mathcal{C}_L$ such that $||\rho - \varphi||_0 < \varepsilon$. To do this, consider the interval $I_\varepsilon := (a, a + \varepsilon) \subset [a, b]$. Let $C_\varepsilon \subset I^2$ be a connected component of $\varphi^{-1}(I_\varepsilon)$ and S_ε a sufficiently small square contained in C_ε . Let $\rho_\varepsilon : S_\varepsilon \to I_\varepsilon$ be a continuous density that is not realizable as the Jacobian of an L-bi-Lipschitz homeomorphism (see Figure 2.3). By the Tietze extension theorem, there exists a continuous function $\hat{\rho}_\varepsilon : C_\varepsilon \to [a, a + \varepsilon]$ such that $\hat{\rho}_\varepsilon|_{S_\varepsilon} = \rho_\varepsilon$ and

$$\sup_{x \in C_\varepsilon} \hat{\rho}_\varepsilon(x) = \sup_{x \in S_\varepsilon} \rho_\varepsilon(x).$$

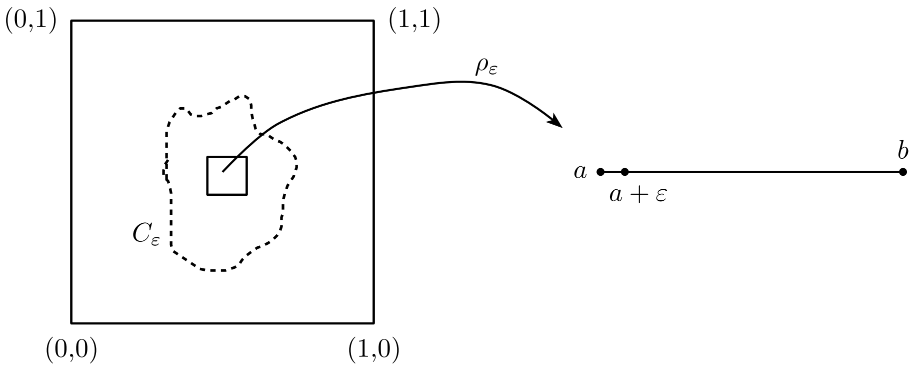

Figure 2.3: Construction of the function $\rho_\varepsilon : S_\varepsilon \longrightarrow [a, a + \varepsilon]$.

Lemma 2.6:
There exists a continuous function $\rho : I^2 \to [a,b]$ such that $\rho|_{I^2 \setminus C_\varepsilon} = \varphi$, $\rho|_{S_\varepsilon} = \rho_\varepsilon$, and $||\rho - \varphi||_0 < \varepsilon$.

Proof. Let $\{\varrho_1, \varrho_2\}$ be a partition of unity subordinate to the open cover $\{I^2 \setminus S_\varepsilon, C_\varepsilon\}$ of I^2. Define $\rho : I^2 \to [a,b]$ by letting $\rho(x) := \varphi\varrho_1 + \hat{\rho}_\varepsilon \varrho_2$.

By definition, it readily follows that $\rho|_{I^2 \setminus C_\varepsilon} = \varphi$ and $\rho|_{S_\varepsilon} = \rho_\varepsilon$. In addition, since $\{\varrho_1, \varrho_2\}$ is a partition of unity, we have

$$\begin{aligned} ||\rho - \varphi||_0 &= ||\varphi\varrho_1 + \hat{\rho}_\varepsilon\varrho_2 - \varphi||_0 \\ &= ||\varphi\varrho_1 + \hat{\rho}_\varepsilon\varrho_2 - \varphi(\varrho_1 + \varrho_2)||_0 \\ &= ||(\hat{\rho}_\varepsilon - \varphi)\varrho_2||_0. \end{aligned}$$

Now, since $\hat{\rho}_\varepsilon(U_\varepsilon) \subset [a, a+\varepsilon]$, we have $\sup\limits_{x \in C_\varepsilon \setminus S_\varepsilon} |(\hat{\rho}_\varepsilon(x) - \varphi(x))\varphi_2(x)| < \varepsilon$. Therefore, $||\rho - \varphi||_0 < \varepsilon$, as desired. □

2.2 The L^∞ case

We consider the set $L^\infty_+(I^2, \mathbb{R}) := \{\rho \in L^\infty(I^2, \mathbb{R}) : \rho(x) > 0 \text{ a.e.}\}$. Our goal is to prove the next

Theorem 2.7:
Let $\mathcal{C}_\infty$ be the set of all positive measurable functions $\rho \in L^\infty_+(I^2, \mathbb{R})$ such that there is no bi-Lipschitz map $f : I^2 \to \mathbb{R}^2$ satisfying $Jac(f) = \rho$ a.e. Then $\mathcal{C}_\infty$ is a thick subset of $L^\infty_+(I^2, \mathbb{R})$.

As in the continuous case, this theorem is a consequence of the next

Proposition 2.8:
Given $L > 1$, let $\mathcal{C}_{L,\infty}$ be the set of all positive functions $\rho \in L^\infty_+(I^2, \mathbb{R})$ such that there is no L-bi-Lipschitz map $f : I^2 \to \mathbb{R}^2$ satisfying $Jac(f) = \rho$ a.e. Then $\mathcal{C}_{L,\infty}$ is an open dense subset of $L^\infty_+(I^2, \mathbb{R})$.

Proof. From Remark 2.5, we know that $\mathcal{C}_{L,\infty}$ is open. Let us show that it is a dense subset of $L^\infty_+(I^2, \mathbb{R})$. Given $\varphi \in L^\infty_+(I^2, \mathbb{R})$, denote $b := \sup \varphi(x)$. Fix $\varepsilon > 0$, and define φ_ε as

$$\varphi_\varepsilon(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \geq \varepsilon, \\ \\ \varepsilon & \text{if } \varphi(x) < \varepsilon. \end{cases}$$

Let $a \in [\varepsilon, b]$ be such that $\varphi_\varepsilon^{-1}([a, a+\varepsilon])$ has positive measure. Let y be a Lebesgue density point of $\varphi_\varepsilon^{-1}([a, a+\varepsilon])$, which means that

$$\lim_{\delta \to 0^+} \frac{\lambda(\varphi_\varepsilon^{-1}([a, a+\varepsilon]) \cap S_\delta(y))}{\lambda(S_\delta(y))} = 1,$$

where $S_\delta(y)$ denotes the square centered at y and having sidelength δ. An easy application of Lemma 1.19 then shows the following: For a small-enough $\delta > 0$, there is a function $\rho_\varepsilon : S_\delta(y) \to [\varepsilon, b]$ that takes values in $[a, a+\varepsilon]$ for points in $\varphi_\varepsilon^{-1}([a, a+\varepsilon]) \cap S_\delta(y)$ and coincides with φ_ε on the complement of $\varphi_\varepsilon^{-1}([a, a+\varepsilon]) \cap S_\delta(y)$ such that no L-bi-Lipchitz map $f : S_\delta(y) \to \mathbb{R}^2$ can have a Jacobian equal to ρ_ε. Let ρ be defined by letting

$$\rho(x) = \begin{cases} \rho_\varepsilon(x) & \text{if } x \in S_\delta(y), \\ \\ \varphi_\varepsilon & \text{if } x \notin S_\delta(y). \end{cases}$$

By construction, ρ belongs to $\mathcal{C}_{L,\infty}$, and $\|\rho - \varphi\|_\infty \leq 2\varepsilon$. As this construction can be performed for any $\varepsilon > 0$, this shows that $\mathcal{C}_{L,\infty}$ is a dense subset of $L^\infty_+(I^2, \mathbb{R})$. $\square$

Chapter 3

N^2-point sets and Lipschitz Regular maps.

In this chapter we are devoted to study Lipschitz Regular maps and its decomposition in bi-Lipschitz pieces as a tool to deal with a question posed by Feige in [20], who asked whether every finite set of points in the plane with integer coordinates can be mapped onto the standard grid by bijections that stretch the pairwise distances in a controlled manner.

Lipschitz regular mappings were originally defined in [8], and they are vastly studied, for instance, in [9] and [15]. The section 3.1 can be seen as a basic background on Lipschitz regular maps and section 3.2 deals with its bi-Lipschitz decomposition. In Section 3.3 we explain how Feige's question can be answered negatively as a consequence of the existence of a type of *non-realizable densities* $\rho : I^2 \to \mathbb{R}$, for which the equation $f_{\#}(\rho\lambda) = \lambda|_{f(I^2)}$ does not have Lipschitz regular solutions (in a similar way than the Burago-Kleiner' construction). This Chapter is mainly based on the work of Dymond, Kaluža and Kopecká in [15].

3.1 Some basics on Lipschitz regular maps.

Lipschitz regular mappings can be understood as an intermediate class between Lipschitz and bi-Lipschitz maps. While bi-Lipschitz maps are too rigid, Lipschitz regular maps are more flexible, even they can collapse distances. But, unlike Lipschitz maps, Lipschitz Regular maps have a behaviour similar to bi-Lipschitz maps, for instance, they map connected components in an appropriate way.

Definition 3.1:
Let X and Y be two metric spaces. We say that a Lipschitz mapping $f : X \to Y$ is ***Lipschitz regular*** *if there is a constant $C \in \mathbb{N}$ such that for every $r > 0$ and every ball $B \subset Y$ of radius r, the set $f^{-1}(B)$ can be covered by at most C balls of radius Cr. The smallest such C is called the regularity constant of f, and is denoted $Reg(f)$.*

Remark 3.2:
We can think on bi-Lipschitz maps as follows: a map $f : X \to Y$ is bi-Lipschitz if it is Lipschitz

and there exists a constant $C > 0$ such that for every open ball B of radius r in Y, the set $f^{-1}(B)$ is contained in a single ball of radius Cr. Thus, bi-Lipschitz maps are obviously Lipschitz Regular.

A useful equivalent interpretation of Definition 3.1 is provided by the next lemma.

Lemma 3.3:
A Lipschitz map $f : X \to Y$ is Lipschitz regular if and only if there exists $C > 0$ such that for every $y \in Y$ and every $r > 0$, the set $f^{-1}(B_Y(y, r))$ does not contain a $2Cr$-separated set with more than C elements.

Proof. Suppose that f is Lipschitz regular with $Reg(f) = C$, and let r be a positive real number and $y \in Y$. Consider $\{x_1, \ldots, x_n\}$ be a maximal $2Cr$-separated set contained in $f^{-1}(B(y, r))$. We must to show that $n \leq C$. Notice that the family $(B(x_i, 2Cr))_{i=1}^n$ covers the set $f^{-1}(B(y, r))$; otherwise, if $x \in f^{-1}(B(y, r)) \setminus \bigcup_{i=1}^n B(x_i, 2Cr)$ we would have that $\{x_1, \ldots, x_n, x\}$ is a $2Cr$-separated set, contradicting the maximality. Since f is Lipschitz regular, we conclude that $n \geq C$.

Analogously, given $r > 0$ and $y \in Y$, if $\{x_1, \ldots, x_n\}$ is a maximal $2Cr$-separated set contained in $f^{-1}(B(y, r))$, with $n \leq C$, we have that the family $(B(x_i, 2Cr))_{i=1}^n$ is a covering of $f^{-1}(B(x_i, 2Cr))$. Since $n \leq C$, we have that f is Lipschitz regular. $\square$

From now on, we say that a Lipschitz regular mapping is (L, C)-*regular* if $Lip(f) \leq L$ and $Reg(f) \leq C$. Lemma 3.4 and Corollary 3.5 are elementary properties whose proofs can be found in [9].

Lemma 3.4:
Let X and Y be two metric spaces. If $f : X \to Y$ is Lipschitz regular and $d > 0$ then there is $C = C(Lip(f), Reg(f), d)$ such that

$$C^{-1}\mathcal{H}^d(f^{-1}(E)) \leq \mathcal{H}^d(E) \leq C\mathcal{H}^d(f^{-1}(E))$$

for every $E \subset Y$.

As a corollary of Lemma 3.4, Lipschitz Regular mappings satisfy the following Luzin's properties.

Corollary 3.5:
Let $f : X \to Y$ be Lipschitz regular. Then

1. *for every $E \subset X$ such that $\mathcal{H}^d(E) = 0$ we have that $\mathcal{H}^d(f(E)) = 0$, and*
2. *for every $F \subset Y$ such that $\mathcal{H}^d(F) = 0$ we have that $\mathcal{H}^d(f^{-1}(F)) = 0$.*

The next Proposition will be key to deal with Feige's question below. It will be settled for

Euclidean spaces, but it also holds for a more general class of metric spaces, namely, the so-called Ahlfors-regular spaces (see Lemma 12.6 in [9]).

Proposition 3.6:
Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz map and suppose that there is $C > 0$ satisfying

$$\lambda(f^{-1}(B(y,r))) \leq Cr^d$$

for every $y \in Y$ and every $r > 0$. Then f is Lipschitz regular.

The regularity property ensures us that the cardinality of the pre-image of any given point is uniformly controlled. This property is shared with bi-Lipschitz maps.

Lemma 3.7 (Observation 2.5 in [15])**:**
Let $f : X \to Y$ be a Lipschitz regular and $y \in Y$. Then we have that

$$|f^{-1}(\{y\})| \leq Reg(f).$$

3.2 Bi-Lipschitz decomposition of Lipschitz regular mappings.

In this section we deal with the bi-Lipschitz decomposition of Lipschitz Regular maps in Euclidean spaces. In [15] it is proved that a Lipschitz regular map defined in a bounded region of $\mathbb{R}^n$ can be "densely decomposed" into bi-Lipschitz pieces.

Theorem 3.8 (Theorem 2.10 in [15])**:**
Let $U \subset \mathbb{R}^d$ be open and $f : \overline{U} \to \mathbb{R}^d$ be a Lipschitz regular mapping. Then, there exist pairwise disjoint open sets $(A_n)_{n\in\mathbb{N}}$ in $\overline{U}$ such that $\bigcup_{n\in\mathbb{N}} A_n$ is dense in $\overline{U}$ and, for each $n \in \mathbb{N}$, the map $f|_{A_n}$ is bi-Lipschitz with lower bi-Lipschitz constant $b = b(Reg(f))$.

The proof of Theorem 3.8 relies on the following three steps:

1. For any given open set in the domain, there is an open subset such that the given map is almost injective (see definition 2.15 below).
2. A Lipschitz regular, almost injective map on an open set is injective.
3. A Lipschitz regular, injective map on an open set with convex image is bi-Lipschitz.

We proceed to precise the three steps above to obtain the bi-Lipschitz decomposition. We begin with the definition of almost injectivity.

Definition 3.9:
A map $h : A \subset \mathbb{R}^d \to \mathbb{R}^d$ is said to be almost injective if there is a set $B \subset A$ such that $\lambda(A \backslash B) = 0$ and $h|_B$ is injective.

Lemma 3.10 (Lemma 2.12 in [15]):
Let $U \subset \mathbb{R}^d$ be a non-empty open subset, and let $f : U \to \mathbb{R}^d$ be Lipschitz regular. Then, there exists a non-empty open set $V \subset U$ such that $f|_V$ is almost injective and $f(V)$ is an open ball.

The next step is to prove that an almost injective Lipschitz regular mapping on an open set U is actually injective on U.

Lemma 3.11 (Lemma 2.13 in [15]):
Let $U \subset \mathbb{R}^d$ be an open set, $f : \overline{U} \to \mathbb{R}^d$ be a Lipschitz regular, almost injective mapping. Then $f|_U$ is injective.

The third step is to prove that an injective, Lipschitz regular mapping with convex image is bi-Lipschitz.

Lemma 3.12 (Lemma 2.14 in [15]):
Let $U \subset \mathbb{R}^d$ be an open set and let $f : U \to \mathbb{R}^d$ be an injective, Lipschitz regular mapping such that $f(U)$ is convex. Then, f is bi-Lipschitz with lower bi-Lipschitz constant at most $\frac{1}{2Reg(f)^2}$.

As it is pointed out in [15], Theorem 3.8 can be obtained by a consecutive application of Lemmas 3.10, 3.11 and 3.12. As a consequence, we have that a Lipschitz regular mapping on an open set can be expressed, on some open subset of the image, as a "sum" of bi-Lipschitz homeomorphisms.

Proposition 3.13 (Proposition 2.15 in [15]):
Let $U \subset \mathbb{R}^d$ be a non-empty open set and $f : \overline{U} \to \mathbb{R}^d$ be a Lipschitz regular mapping. Then there exist a non-empty open set $T \subset f(\overline{U})$, $N \in \{1, \ldots, Reg(f)\}$ and pairwise disjoint open sets $W_1, \ldots W_N \subset \overline{U}$ such that $f^{-1}(T) = \bigcup_{i=1}^{N} W_i$ and $f|_{W_i} \to T$ is a bi-Lipschitz homeomorphism for each i, with lower bi-Lipschitz constant $b = b(Reg(f))$.

3.3 Feige's question

Given an undirected connected graph $G = (V, E)$ with set of vertices $V = \{v_1, \ldots, v_n\}$, we want to map V by an one-to-one mapping onto the set $\{1, \ldots, n\}$ with *bandwidth* as small as possible. More precisely, we want to find a bijective map $f : V \to \{1, \ldots, n\}$ so that $\max\{|f(v_i) - f(v_j)| : (v_i v_j) \in E\}$ is minimized (for a wide discussion to bandwidth problem, we refer to [12] and the references therein). Thus, Feige in [20] posed the following question (see Figure 2.1 below).

Question 3.14 (Feige, 2002):
Is there a constant $L > 0$ such that for every $n \in \mathbb{N}$ and every set $S \subset \mathbb{Z}^d$ with cardinality $|S| = n^d$, there exists a bijection $f : S \to \{1, \ldots, n\}^d$ with Lipschitz constant satisfying $Lip(f) \leq L$?

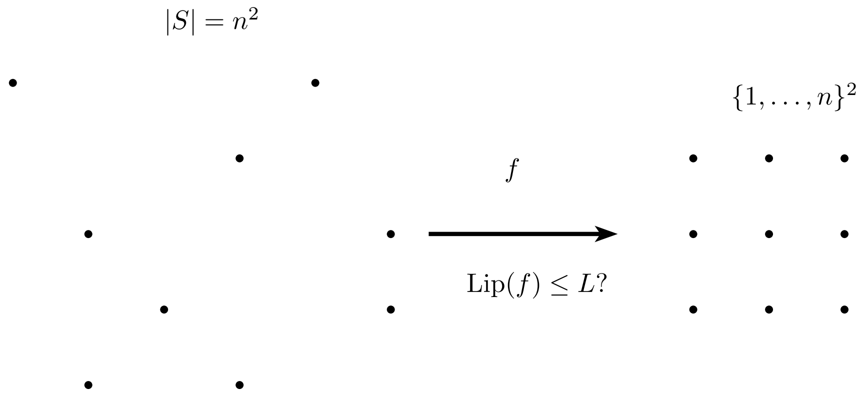

Figure 3.1: Feige's question.

Remark 3.15:
As in Remark 1.9, Feige's question has trivially a positive answer for $d = 1$.

By highly non-trivial modifications on the Burago-Kleiner' construction (see Chapter 1, Section 1.2), in [15] it is shown the following theorem which answer in the negative Feige's Question for every dimension $d \geq 2$.

Theorem 3.16 (Theorem 1.2 in [15])**:**
Let $\mathcal{F}_n$ *be the collection of all subsets* $S \subset \mathbb{Z}^d$, $d \geq 2$, *with* $|S| = n^d$ *and, for each* $S \in \mathcal{F}_n$, *define*

$$L_S := \inf\{Lip(f) :\ f : S \to \{1,2,\ldots,n\}^d \text{ is a bijection}\}.$$

Then the sequence

$$C_n := \sup\{L_S :\ S \in \mathcal{F}_n\}, \qquad n \in \mathbb{N}$$

is unbounded.

Following the Burago-Kleiner' proof on the existence of non-rectifiable Delone sets, Theorem 3.16 is showed as a consequence of the non-solvability of an extension of the equation (1.2.1), but involving Lipschitz regular maps instead of bi-Lipschitz maps. Given a function $\rho : I^2 \to \mathbb{R}$ with $0 < \inf \rho < \sup \rho < +\infty$, let $(l_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ be two sequences of positive integers such that l_i, m_i and l_i/m_i tend to $+\infty$ when i goes to $+\infty$. Let $(C_i)_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint squares with side-length equals to l_i and such that for each $i \in \mathbb{N}$, the square C_i is decomposed into m_i^2 squares $(T_{i,k})_{k=1}^{m_i^2}$ of side-length equal to l_i/m_i (as in Figure 1.3 above). Moreover, let $\varphi_i : I^2 \to S_i$ be an affine homeomorphism with scalar linear part, mapping I^2 onto C_i. As a next step, implant the function ρ in each square $(T_{i,k})_{k=1}^{m_i^d}$ by choosing a positive integer $n_{i,k} \in \left\{ \left\lfloor \int_{T_{i,k}} \rho \circ \varphi^{-1} d\lambda \right\rfloor, \left\lfloor \int_{T_{i,k}} \rho \circ \varphi^{-1} d\lambda \right\rfloor + 1 \right\}$ and putting $n_{i,k}$ points with integer coordinates in each square $T_{i,k}$. If the $n_{i,k}$ points are choosed such that $\sum_{k=1}^{m_i^2} n_{i,k}$ is of the form n_i^2, we define the set S_i as this collection of n_i^2 points.

Assume that there exists a sequence of L-Lipschitz bijections $f_i : S_i \to \{1, \ldots, n_i\}^d$. Define X_i by $\varphi^{-1}(S_i)$, and the "normalized" maps $g_i : X_i \to \mathbb{R}^d$ given by $g_i(x) := \frac{1}{n_i}(f_i \circ \varphi_i(x))$. By choosing n_i and l_i such that $n_i/l_i \longrightarrow 1$, we have that for any $L' > L$ it is possible to find a positive integer i_0 from which the mappings g_i are L' Lipschitz. Without lost of generality, we can assume that all the mappings g_i are L'-Lipschitz. By Kirzbraun-Valentine's extension theorem (see Theorem 1.1), we can extend each g_i to a map $\overline{g_i} : I^d \to \mathbb{R}^d$ by preserving the Lipschitz constant L'. By Arzelá-Ascoli theorem, there exists a subsequence of $(\overline{g_i})_{i\in\mathbb{N}}$ that converges uniformly to an L'-Lipschitz map f. Again, without lost of generality, we can assume that $\overline{g_i}$ converges to f.

For every $i \in \mathbb{N}$, define a (normalized) counting measure in X_i by

$$\mu_i(A) := \frac{1}{n_i^2}|A \cap X_i|.$$

By Lemma $A.2$ we have that μ_i converges weakly to $\rho\lambda$ and by Lemma $A.4$ and the uniform convergence, we have that $(g_i)_\#\mu_i$ converges weakly to $f_\#(\rho\lambda)$. Besides, it is proved that $(g_i)_\#\mu_i$ converges weakly to $\rho\lambda|_{I^2} = \rho\lambda|_{f(I^2)}$ (see Claim 5.3.3 in [15]). Thereby it is shown that for any $U \subset I^2$, the L'-Lipschitz map f satisfies the equality,

$$\lambda(f(U)) = \int_U \rho d\lambda.$$

Hence, as a direct consequence we have that for every measurable set $U \subset f(I^2)$ that

$$\lambda(U) = \int_{f^{-1}(U)} \rho d\lambda \geq \lambda(f^{-1}(U)) \inf \rho,$$

which implies that $\lambda(f^{-1}(U)) \leq \frac{\lambda(U)}{\inf \rho}$. By Proposition 3.6, the mapping f must be Lipschitz regular.

Therefore, the above argument shows that to answer negatively Feige's question, it is suffices to find a positive function $\rho : I^d \to \mathbb{R}$ such that $0 < \inf \rho < \sup \rho < +\infty$ for which there is no Lipschitz regular solutions of

$$f_\#(\rho\lambda) = \lambda|_{f([0,1]^2)}. \tag{3.3.1}$$

We finish this section with the following property: for a generic bounded, positive, continuous functions from to square I^2 to $\mathbb{R}$, the equation (3.3.1) has not Lipschitz regular solutions. In particular this provides a negative answer to Feige's question. In [15] the authors prove that the set of positive continuous functions which has Lipschitz regular solutions of (3.3.1) is σ-porous, which is a stronger property than to be of first Category (in the sense of Baire).

Definition 3.17:

Let $(X, ||\cdot||))$ be a Banach space.

1. *A set $P \subset X$ is called **porous** at a point $x \in X$ if there exist $\varepsilon_0 > 0$ and $\alpha \in (0,1)$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists $y \in X$ such that:*

$$||y - x|| \leq \varepsilon \quad \text{and} \quad B(y, \alpha\varepsilon) \cap P = \emptyset.$$

2. *A set $P \subset X$ is called **porous** if P is porous in every $x \in P$.*

3. *A set $E \subset X$ is called **σ-porous** if E may be expressed as a countable union of porous subsets of X.*

The next theorem provides a non-realizable density ρ for which 3.3.1 has not Lipschitz regular solutions (compare with Main Theorem in Chapter 2)

Theorem 3.18 (Theorem 4.1 in [15])**:**
Let $\Psi := \{\rho \in C(I^2, \mathbb{R}) :$ (3.3.1) has Lipschitz Regular solution $f : I^2 \to \mathbb{R}^2\}$. Then Ψ is a σ-porous subset of $C(I^2, \mathbb{R})$.

Theorem 3.18 is a consequence of the lower bi-Lipschitz constant given by Proposition 3.8 and an adequate manage of the bi-Lipschitz decomposition given by Proposition 3.13. We point out that, though the above theorem ensures the existence of a continuous function which is not Lipschitz regular realizable, it does not provided any explicit example of such a function.

We finish this chapter with some observations about Theorem 3.18. Notice that if (3.3.1) holds, then for the open subsets $T \subset f(I^2)$ and $W_1, \ldots, W_N$ given by the Proposition 3.13, we have that

$$\rho(y) = |Jac(f_1)(y)| - \sum_{i=2}^{N} \rho(f_i^{-1} \circ f_1(y))|Jac(f_i^{-1} \circ f_1)(y)| \qquad \text{for a.e } y \in W_1, \tag{3.3.2}$$

where $f_i : W_i \to T$ is bi-Lipschitz, with lower bi-Lipschitz constant only depending on the regularity constant. Observe that the equation (3.3.2) says that the density map ρ is a "linear combination" of a controlled number of Jacobians. Naively we could define the density ρ by putting a Burago-Kleiner' non-realizable density (those densities studied in Chapters 1 and 2) in each of the W_i's and then try to argue similarly to the work of Burago-Kleiner or McMullen. The problem here is that any bad-behavioured density map in a bi-Lipschitz piece can be compensated by the bad-behavioured density map in the other bi-Lipschitz pieces. For this reason the ideas in [3] cannot be used directly to construct a density map for which (3.3.1) has not Lipschitz regular solutions.

Remark 3.19:
It is an open problem to extend Theorem 3.18 to the set of positive L^∞ functions.

Chapter 4

On Delone sets that are not (bi-)Lipschitz equivalent to the standard lattice.

The aim of this Chapter is to deal with Delone sets and Lipschitz bijections. In particular, we are mainly concerned in the following question.

Question 4.1:
Are there repetitive Delone subsets of $\mathbb{R}^2$ *that admit no Lipshitz bijection with* $\mathbb{Z}^2$*?*

Some words about notation. Along this work we will deal with functions defined on discrete sets and extensions of these to the whole space. As in [6], for the former we use a standard notation f, whereas for the corresponding extension we will use the notation $\hat{f}$. As usual, for a real number A, we denote its integer part by $[A]$. Given two real numbers $A \leq B$, we denote $[\![A, B]\!]$ the set of integers n such that $A \leq n \leq B$. Given positive integers M, N, we let $R_{M,N}$ be the rectangle $[\![0, 2MN]\!] \times [\![0, M]\!]$. Given $k \in [\![1, 2N]\!]$ and a positive integer P dividing M, let S_k^P be the subset of $R_{M,N}$ formed by the points of the form

$$x_{i,j}^k := \left((k-1)M + i\frac{M}{P}, j\frac{M}{P}\right), \tag{4.0.1}$$

where i, j lie in $[\![0, P]\!]$. By some abuse of notation, the notation $x_{i,j}^k$ will still be used for $i = P+1$ (yet $x_{P+1,j}^k$ does not belong to S_k^P). The points $x_{i,j}^k$ will be called *marked points.* Notice that S_k^P also depends on M and N, but this dependence (which will be clear in each context) is suppressed just to avoid an overload of notation. For a square S of the form $S_k^P \subset R_{M,N}$, where M is a multiple of P, define $\hat{S} := \{((k-1)M + i, j) : \ i, j = 0, 1, \ldots, M-1\}$. We call $\hat{S}$ the *lower-left corner* of the corresponding square S.

To simplify, we will only work with Delone subsets $\mathcal{D}$ of $\mathbb{Z}^2$ satisfying what we call the $2\mathbb{Z}$-*property*: all points (m, n) with an even m do belong to $\mathcal{D}$. In particular, we will consider domino tilings of the plane made only of the pieces 1-1 and 1-0. More generally, we say that a subset $\mathcal{D} \subset [\![A, B]\!] \times [\![A', B']\!]$ satisfies the $2\mathbb{Z}$-*property* if all points $(m, n) \in [\![A, B]\!] \times [\![A', B']\!]$ with an even m do belong to $\mathcal{D}$.

There is a little technical problem that arises when considering maps defined on strict subsets of either $\mathbb{Z}^2$ or $R_{M,N}$. To solve this, we introduce a general construction. Namely, given either a Delone set $\mathcal{D} \subset \mathbb{Z}^2$ or a subset $\mathcal{D} \subset R_{M,N}$ satisfying the $2\mathbb{Z}$-property in each case, for every function $f\colon \mathcal{D} \to \mathbb{Z}^2$ we define its extension $\hat{f}$ to either $\mathbb{Z}^2$ or $R_{M,N}$ taking values in $\frac{1}{2}\mathbb{Z}^2$ by letting

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{D}, \\ f\big(x + (1,0)\big) - (\frac{1}{2}, 0) & \text{if } x \notin \mathcal{D}. \end{cases}$$

The proof of the next lemma is straightforward and we leave it to the reader.

Lemma 4.2:
If $f : \mathcal{D} \to \mathbb{Z}^2$ is L-bi-Lipschitz, then $\hat{f}$ is a $6L$-bi-Lipschitz map.

4.1 Some key ideas for the strategy

In order to deal with Lipschitz bijections, we need to perform slight modifications for some Lemmas and Propositions of [6]: the bi-Lipschitz condition on $\mathcal{D}$ will be replaced by a Lipschitz condition together with the existence of a lower bi-Lipschitz constant on an appropriate subset of $\mathcal{D}$ (namely, on a suitable set of marked points $x^k_{i,j}$). The proofs of these modifications follow similar arguments to those of [6].

The crucial idea of the construction in [6] is that, given an L-bi-Lipschitz map and a positive integer P, if a non-expansiveness condition holds for all the pairs $(x^k_{i+1,j}, x^k_{i,j})$, then there must exist a square S^P_k for which the extension map $\hat{f}$ is coarsely-differentiable at every point of S^P_k. This is the content of Proposition 4 of [6]. Below we state a slight variation of it, the proof of which follows the very same lines. The key point here is that we are asking for the map to be only Lipschitz yet bi-Lipschitz over a small subset of the involved Delone set.

Proposition 4.3:
Given $L \geq 1$, a positive $\varepsilon < 1$ and a positive integer P, there exist $\lambda > 0$ and positive integers M_0, N_0 such that the following holds: Given a subset $\mathcal{D} \subset R_{M,N}$ satisfying the $2\mathbb{Z}$-property, with $N \geq N_0$ and $M \geq M_0$ a multiple of $2P$, let $f : \mathcal{D} \to \mathbb{Z}^2$ be an injective L-Lipschitz map that is L-bi-Lipschitz when restricted to the set of points of the form $x^k_{i,j}$. Assume that, for all these points,

$$\frac{||f(x^k_{i+1,j}) - f(x^k_{i,j})||}{M/P} \leq (1+\lambda)\frac{||v^f_{M,N}||}{2MN}.$$

Then there is a subset

$$S = S^P_k := \left\{ \left((k-1)M + i\frac{M}{P}, j\frac{M}{P} \right) : \ 0 \leq i \leq P, 0 \leq j \leq P \right\}$$

such that every $x \in S$ satisfies

$$\left|\left|\frac{f(x+(M,0))-f(x)}{M}-\frac{f(2MN,0)-f(0,0)}{2MN}\right|\right| \leq \varepsilon. \tag{4.1.1}$$

Assume that $f : \mathcal{D} \to \mathbb{Z}^2$ is an L-Lipschitz bijection defined on a Delone set $\mathcal{D} \subset \mathbb{Z}^2$ satisfying the $2\mathbb{Z}$-property that is bi-Lipschitz on a set $S = S_k^P \subset R_{M,N}$ for a positive integer P that divides M. Let γ^* be the curve obtained by connecting points in $\hat{f}(\partial S)$ (using line segments) coming from consecutive points in ∂S. The curve γ^* is closed though not necessarily simple. However, it contains the simple curve $\gamma = \gamma_S$ obtained by "deleting short loops". We denote by $int(\gamma)$ (resp. $ext(\gamma)$) the closed, bounded (resp. unbounded) region of the plane determined by γ.

Given a sequence of (translated) squares $S_m \subset \mathcal{D}$ of the form S_k^P, we consider the map $f|_{S_m}$ defined from S_m into $\mathbb{Z}^2$. By normalizing and passing to the limit, we obtain a Lipschitz map $F\colon I^2 \to \mathbb{R}^2$. The next lemma can be seen as a slightly generalized version of Lemma 6 in [6].

Lemma 4.4:
Let $\mathcal{D} \subset \mathbb{Z}^2$ be a Delone set satisfying the $2\mathbb{Z}$-property. Given $L \geq 1$, $\varepsilon > 0$ and a positive integer P, let $S_m \subset \mathcal{D}$ be a sequence of (translated) squares of the form S_k^P, with M is a multiple of $2P$. Then there exists m_0 such that the following holds: if $m \geq m_0$ and $f\colon \mathcal{D} \to \mathbb{Z}^2$ is an L-Lipschitz bijection that is L-bi-Lipschitz on S_m and for which the limit map F above is an homeomorphism, then for $S = S_m$ we have that

i) *no point of $\hat{f}(\mathbb{Z}^2 \setminus (1+\varepsilon)\hat{S})$ lies in $int(\gamma)$;*

ii) *all points in $\hat{f}((1-\varepsilon)\hat{S})$ are contained in $int(\gamma)$.*

Proof. By contradiction, after renormalizing and passing to the limit, we obtain:

- In case i), a point in the exterior of $(1+\varepsilon)I^2$ which is mapped by F inside $F(I^2)$;
- In case ii), a point in $(1-\varepsilon)I^2$ which is mapped by F into a point outside $F(I^2)$.

This is impossible in each case, since F is an homeomorphism. □

The next step in the construction relies on that the coarse-differentiability provided by Proposition 4.3 implies that there are two neighboring squares S_k^P, S_{k+1}^P whose images under f have very close densities. This is reflected by the next claim, which corresponds to a extended version of Proposition 9 in [6].

Proposition 4.5:
Given $L \geq 1$ and $1 \geq d > d' > 0$, there exist a positive $\varepsilon < 1$ and integers P_1, M_1 such that

the following holds: Let $\mathcal{D}$ be a Delone set satisfying the $2\mathbb{Z}$-property, and let $f : \mathcal{D} \to \mathbb{Z}^2$ be a bijection. Assume that for $P \geq P_1, N \geq 1$ and $M \geq M_1$ being a multiple of $2P$, some square $S := S_k^P \subset R_{M,N}$, with $1 \leq k < 2M$, is such that every $x \in S$ satisfies (4.1.1). Denote $S' := S_{k+1}^P$. If $\hat{S} \cap \mathcal{D}$ contains $\geq dM^2$ (resp. $\leq d'M^2$) points and $\hat{S}' \cap \mathcal{D}$ contains $\leq d'M^2$ (resp. $\geq dM^2$ points), then f cannot be L-Lipschitz and, simultaneously, L-bi-Lipschitz when restricted to the set of points of the form $x_{i,j}^k$.

Propositions 4.3 and 4.5 can be put together into the next claim, which is a slight extension of Proposition 10 in [6].

Proposition 4.6:
Given $L \geq 1$ and $1 \geq d > d' > 0$, there exist a positive $\varepsilon < 1$ and integers P_*, M_*, N_* such that the following holds: let $\mathcal{D}$ be a Delone set satisfying the $2\mathbb{Z}$-property, and let $f : \mathcal{D} \to \mathbb{Z}^2$ be an L-Lipschitz bijection. Moreover, for $P \geq P_*$, $N \geq N_*$ and $M \geq M_*$ being a multiple of $2P$ assume that the map f satisfies the following:

i) for every $x_{i,j}^k, x_{s,t}^l \in R_{M,N}$, we have that $||f(x_{i,j}^k) - f(x_{s,t}^l)|| \geq \frac{1}{L}||x_{i,j}^k - x_{s,t}^l||$,

ii) f satisfies the conclusions of Lemma 4.4 for S.

With the same notation of the Proposition 4.5, suppose that $\hat{S} \cap \mathcal{D}$ contains $\geq dM^2$ (resp. $\leq d'M^2$) points and $\hat{S}' \cap \mathcal{D}$ contains $\leq d'M^2$ (resp. $\geq dM^2$ points). Then there must exist a point $x \in \mathcal{D} \cap R_{M,N}$ of the form $x_{i,j}^k$ such that either

$$\frac{||f(x + (M/P, 0)) - f(x)||}{M/P} > (1+\lambda)\frac{||f(2MN, 0) - f(0,0)||}{2MN}$$

if $x + (M/P, 0)$ belongs to $\mathcal{D}$, or

$$\frac{||f(x + (1 + M/P, 0)) - f(x)||}{1 + M/P} > (1+\lambda)\frac{||f(2MN, 0) - f(0,0)||}{2MN}$$

otherwise.

Roughly speaking, Proposition 4.6 tell us that if a Delone set $\mathcal{D}$ satisfies the $2\mathbb{Z}$-property and is mapped onto $\mathbb{Z}^2$ by an L-Lipschitz bijection that is L-bi-Lipschitz on some marked points, then variations of the local density of $\mathcal{D}$ force to stretch the distances of certain points when passing from a certain scale to a smaller one. By an inductive process, we eventually overcome any prescribed Lipschitz constant of f for a convenient construction of $\mathcal{D}$.

4.2 On Cortez-Navas' examples

We describe how Cortez and Navas produce a non bi-Lipschitz rectifiable Delone set. Fix a constant $L \geq 1$ (which will be the Lipschitz constant to discard). Consider two square patches $\mathcal{P}_1, \mathcal{P}_2$ with equal and even side-length so that they contain a different number of points. We let d_i be the density of points in the lower-left corner of $\mathcal{P}_i$ (hence $d_2 > d_1$). Additionally, assume that both patches are centered at the origin, satisfy the $2\mathbb{Z}$-property, and contain all boundary integer points.

Given these data, fix d_1', d_2' such that $d_1 < d_1' < d_2' < d_2$, and let λ, M_*, N_*, P_* be the constants provided in the Proposition 4.6 for L, $d := d_2'$ and $d' := d_1'$. Fix an integer $l \geq 1$ such that

$$\frac{(1+\lambda)^l}{L} > L. \tag{4.2.1}$$

Let $2M$ be the side-length of the patches $\mathcal{P}_i := \mathcal{P}_i^0 \subset [[-M, M]]^2$, $i = 1, 2$. The construction of new patches $\mathcal{P}_1^1$ and $\mathcal{P}_2^1$ can be sketched as follows:

Step 1: Fix an odd positive integer m so that $2mP_*M \geq M_*$ and construct a square centered at the origin made of $(mP_*)^2$ copies of $\mathcal{P}_1$ matching left sides to right sides and lower sides to upper sides (see Figure 1).

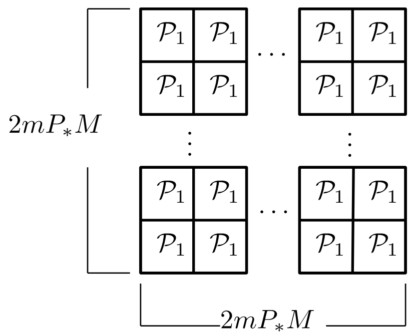

Figure 4.1: Description of Step 1.

Step 2: Match to the right a square block consisting of $(mP_*)^2$ copies of $\mathcal{P}_2$. After this, match to the right a square block consisting of $(mP_*)^2$ copies of $\mathcal{P}_1$. Proceed similarly up to having matched N blocks made of pieces $\mathcal{P}_1$ and $\mathcal{P}_2$ in an alternate way, where the integer $N \geq N_*$ is to be fixed below (see Figure 2).

Step 3: Proceed similarly to the left of the centered-at-the-origin block made of pieces $\mathcal{P}_1$. In this way, we form a rectangle of sides $2mP_*(2N+1)$ and $2mP_*M$ filled by alternate blocks of copies of $\mathcal{P}_1$ and $\mathcal{P}_2$.

Step 4: To complete $\mathcal{P}_1^1$, fill up the whole square of side $2mP_*M(2N+1)$ centered at the origin

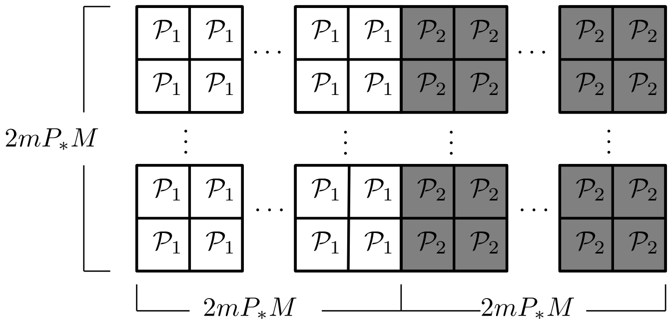

Figure 4.2: Gluing two blocks of different densities.

by matching copies of $\mathcal{P}_1$ at all places, except for those in the lower rectangle constructed above.

Step 5: Finally, to construct $\mathcal{P}_2^1$, proceed similarly as for $\mathcal{P}_1^1$ switching the roles of $\mathcal{P}_1$ and $\mathcal{P}_2$ (see Figure 3).

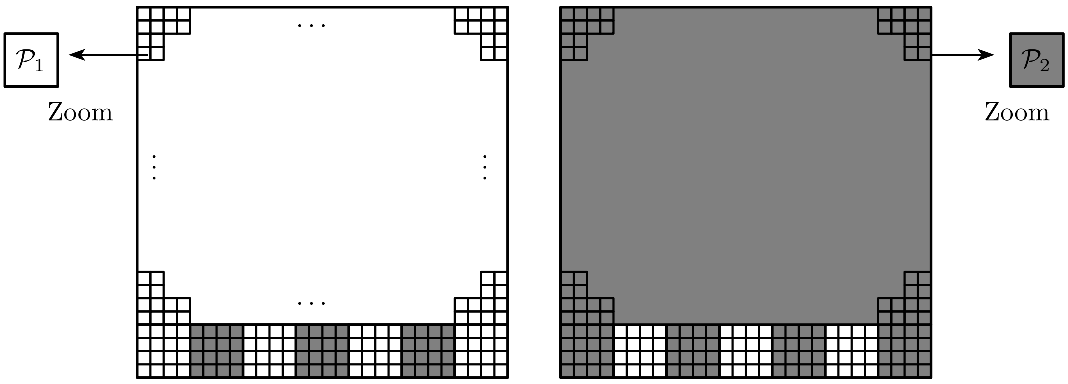

Figure 4.3: Patches $\mathcal{P}_1^1$ (left) and $\mathcal{P}_2^1$ (right).

The integer $N \geq N_*$ is taken in such a way that the density of points in the lower-left corner of $\mathcal{P}_1^1$ (respectively $\mathcal{P}_2^1$) is less than d_1' (respectively greater than d_2'). Now, by starting with the patches $\mathcal{P}_1^1$ and $\mathcal{P}_2^1$ and the same constants L, d_1', d_2', construct with the same procedure new patches $\mathcal{P}_1^2$ and $\mathcal{P}_2^2$ with densities $< d_1'$ and $> d_2'$ respectively. By repeating this procedure l times (where l is taken as in (1.1)), we obtain new patches denoted by $\mathcal{P}_1^{new}$ and $\mathcal{P}_2^{new}$. For these patches, by applying l times Proposition 4.6, one obtains the following (see Lemma 13 in [6] for the details).

Proposition 4.7:
If $\mathcal{D}$ is a Delone set that satisfies the $2\mathbb{Z}$-property and contains translated copies of either $\mathcal{P}_1^{new}$ or $\mathcal{P}_2^{new}$, then $\mathcal{D}$ cannot be mapped onto $\mathbb{Z}^2$ by an L-bi-Lipschitz map.

By the above modifications, we have the following reformulation of the Proposition 4.7 concerning to Delone sets containing copies of $\mathcal{P}_1^{new}$ and $\mathcal{P}_2^{new}$.

Proposition 4.8:
Let $\mathcal{D}$ be a Delone set satisfying the $2\mathbb{Z}$-property, which contains translated copies of either $\mathcal{P}_1^{new}$ or $\mathcal{P}_2^{new}$ as building blocks, where the parameters M and P of these patches are as in the Proposition 4.6. Then $\mathcal{D}$ cannot be mapped onto $\mathbb{Z}^2$ by an L-Lipschitz bijection which is L-bi-Lipschitz restricted to the marked points $x_{i,j}^k$.

Now, let $(L_n)_{n\in\mathbb{N}}$ be an increasing sequence of real numbers greater than 1, and start with patches $\mathcal{Q}_{1,1}$ and $Q_{1,2}$ as in Figure 4. Assuming that the patches $\mathcal{Q}_{n,1}$, $\mathcal{Q}_{n,2}$ are given, define two new patches $\mathcal{Q}_1^{new} := \mathcal{Q}_{n+1,1}$ and $\mathcal{Q}_2^{new} := \mathcal{Q}_{n+1,2}$ as in the above procedure. Hence we obtain a sequences of patches $\mathcal{Q}_{n,1}$ and $\mathcal{Q}_{n,2}$ contained in $\mathcal{D}$ as in the previous procedure. Thence, by the Proposition 4.7, the Delone set $\mathcal{D}$ cannot be L_n-bi-Lipschitz equivalent to $\mathbb{Z}^2$. Since $L_n \to +\infty$, $\mathcal{D}$ is a repetitive Delone set which cannot be rectifiable.

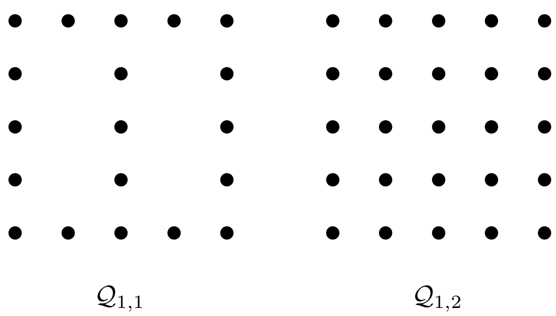

Figure 4.4: Initial patches $\mathcal{Q}_{1,1}$ and $\mathcal{Q}_{1,2}$ to construct a repetitive, non-rectifiable Delone set $\mathcal{D}$.

Remark 4.9:
For more details about the repetitiveness of $\mathcal{D}$, see Lemma 12 in [6].

4.3 Rescaling up to the limit and Lipschitz regularity

In this section we are interested in some implications of having Lipschitz maps defined on Delone sets satisfying the $2\mathbb{Z}$-property, with values in $\mathbb{Z}^2$. More precisely, we will show, after renormalize and passing to the limit, that such a function induces a Lipschitz regular map from the standard square to $\mathbb{R}^2$. Moreover, by pursuing the bi-Lipschitz decomposition of Lipschitz regular maps recently introduced by Dymond, Kaluža and Kopecká in [15] (see Chapter 3, section 3.2), we show that the preceding Lipschitz map is actually bi-Lipschitz in the marked points belonging to some suitable set.

The following result deals with maps defined on discrete sets of points. It asserts that every Lipschitz bijection defined from a Delone set that satisfies the $2\mathbb{Z}$-property onto the integer lattice must be Lipschitz regular. Actually, the next Proposition follows from the property that densities of points in any Delone set are everywhere bounded from below (away from zero); we use the $2\mathbb{Z}$-property to simplify computations. Given $x \in \mathbb{R}^2$, $r > 0$ and a subset $\mathcal{L}$ of $\mathbb{R}^2$, we will denote the set $B(x,r) \cap \mathcal{L}$ by $B_{\mathcal{L}}(x,r)$.

Proposition 4.10:
Let $\mathcal{D} \subset \mathbb{Z}^2$ be a Delone set satisfying the $2\mathbb{Z}$-property. If $f\colon \mathcal{D} \to \mathbb{Z}^2$ is an L-Lipschitz bijection, then f is Lipschitz regular, with $Reg(f) \leq C\,(L+1)^2$ for some universal constant $C > 0$.

Proof. Let $y \in \mathbb{Z}^2$ and $r > 0$. Consider $\Gamma \subset f^{-1}(B(y,r))$ a maximal r-separated set, and write $\Gamma := \{x_1, \dots x_{|\Gamma|}\}$. Then, by the L-Lipschitz condition we have that

$$f\left(\bigcup_{i=1}^{|\Gamma|} B_{\mathcal{D}}(x_i,r)\right) \subset B_{\mathbb{Z}^2}(y, r+rL).$$

Observe that, for $i = 1,\dots,|\Gamma|$, the open balls $B(x_i,r)$ are pairwise disjoint. Since f is a bijection we obtain that, for a certain constant $C_1 \geq 1$,

$$\sum_{i=1}^{|\Gamma|} |B_{\mathcal{D}}(x_i,r)| = \left| f\left(\bigcup_{i=1}^{|\Gamma|} B_{\mathcal{D}}(x_i,r)\right)\right| \leq \big|B_{\mathbb{Z}^2}(y,r+rL)\big| \leq C_1 r^2 (L+1)^2. \tag{4.3.1}$$

Now, by the $2\mathbb{Z}$-property, the cardinality $|B_{\mathcal{D}}(x_i,r)|$ is at least $C_2 r^2$ for another universal constant $C_2 > 0$. (The value of C_2 can be taken as $1/2 - \varepsilon$ provided r is large enough.) Thus, by (4.3.1),

$$C_2 r^2 |\Gamma| \leq C_1 r^2 (L+1)^2.$$

We hence conclude that $|\Gamma| \leq C_1(L+1)^2/C_2$. Therefore, by Lemma 3.3, f is Lipschitz regular with regularity constant at most $C_1(L+1)^2/C_2$. $\square$

From now on, let $(\mathcal{P}_{n,1})_{n\geq 1}$ and $(\mathcal{P}_{n,2})_{n\geq 1}$ be the two sequences of building-blocks given in the Cortez-Navas' construction from §4.2. Denote $\mathcal{P}_n := \mathcal{P}_{n,1}$, let $S_n \subset \mathbb{R}^2$ be the square (patch) that contains $\mathcal{P}_n$, and let $2M_n$ be the side-length of S_n. We choose the sequence $(M_n)_{n\geq 1}$ in such a way that M_n is a multiple of $2M_{n-1}$ and M_{n+1}/M_n is a multiple of M_n/M_{n-1} for every $n \in \mathbb{N}$. Let $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ be the homothety that maps the square S_n into I^2. In addition, define $\mathcal{R}_n := \phi_n(\mathcal{P}_n)$.

Assume there is an L-Lipschitz bijection $f : \mathcal{D} \to \mathbb{Z}^2$. As in [3], we proceed to normalize f to each building-block $\mathcal{P}_n$, that is, to consider the map $f_n : \mathcal{R}_n \to \frac{1}{2M_n}\mathbb{Z}^2$ defined by

$$f_n(x) := \frac{1}{2M_n}(f \circ \phi_n^{-1})(x).$$

Notice that $Lip(f_n) \leq L$ for all $n \geq 1$. By Kirszbraun's extension theorem[(1)], each function f_n can be extended to an L-Lipschitz map $\hat{f}_n : I^2 \to \mathbb{R}^2$. By the Arzelá-Ascoli's theorem, there exists a subsequence of $(\hat{f}_{n_k})_{k\in\mathbb{N}}$ of $(\hat{f}_n)_{n\geq 1}$, converging to an L-Lipschitz map $F : I^2 \to \mathbb{R}^2$.

Proposition 4.11:
The map $F : I^2 \to \mathbb{R}^2$ built above is Lipschitz regular, with $Reg(F) \leq 34\ Reg(f)$.

Proof. Let $y \in F(I^2)$ and $r > 0$. Consider a maximal r-separated set $\Gamma = \{x_1, \ldots, x_{|\Gamma|}\}$ contained in $F^{-1}(B(y, r))$. Given

$$0 < \varepsilon < \min\left\{\frac{(\sqrt{2+\sqrt{2}}-1)\, r}{\sqrt{2+\sqrt{2}}}, dist(\Gamma, \partial F^{-1}(B(y,r)))\right\},$$

by the convergence of $\hat{f}_n$ to F, there is a positive integer $n_0 = n_0(\varepsilon)$ such that, for every $n \geq n_0$, there exist $p_1, \ldots, p_{|\Gamma|} \in \mathcal{D}_n$ for which the following hold:

- for every $i = 1, \ldots, |\Gamma|$, we have that $||p_i - x_i|| < \varepsilon/2$,
- the set $\Gamma_n := \{p_1, \ldots, p_{|\Gamma|}\}$ is contained in $F^{-1}(B(y, r))$ and,
- $f_n(\Gamma_n) \subset B(y, r)$.

Observe that Γ_n is $(r - \varepsilon)$-separated, since Γ is r-separated.

We will delete some points in Γ_n in an appropriate way in order to obtain a set $\Gamma'_n \subset f_n^{-1}(B(y,r))$ that is r-separated and such that $|\Gamma'_n| \geq |\Gamma|/17$. By Lemma 3.3, this will imply that

$$|\Gamma| \leq 17\, |\Gamma'_n| \leq 34\, Reg(f),$$

hence F is Lipschitz-regular with $Reg(F) \leq 34\, Reg(f)$.

To build the set Γ'_n, we consider the angle

$$\alpha = \arccos\left(\frac{(r)^2 + (r-\varepsilon)^2 - (r-\varepsilon)^2}{2(r)(r-\varepsilon)}\right) = \arccos\left(\frac{r}{2(r-\varepsilon)}\right) \geq \arccos\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right) = \frac{\pi}{8},$$

where the inequality follows from the condition

$$\varepsilon < \frac{(\sqrt{2+\sqrt{2}}-1)\, r}{\sqrt{2+\sqrt{2}}}.$$

This is the angle that appears in the picture below. In the area depicted in black, no pair of points is at distance $> r - \varepsilon$. The same happens in a similar region with angle $\pi/8$. Since 16 of these

[(1)] Actually, we do not really need to keep the same Lipschitz constant L for the extension map, but just another (larger) constant that depends only on L, and a weaker form of Kirszbraum's theorem proving this is much easier to establish.

regions cover exactly the anular region between a circle of radius $r-\varepsilon$ and another of radius r (with the same center), we deduce -by the pigeonhole principle- that no more than 16 points in this anular region can be $(r-\epsilon)$-separated.

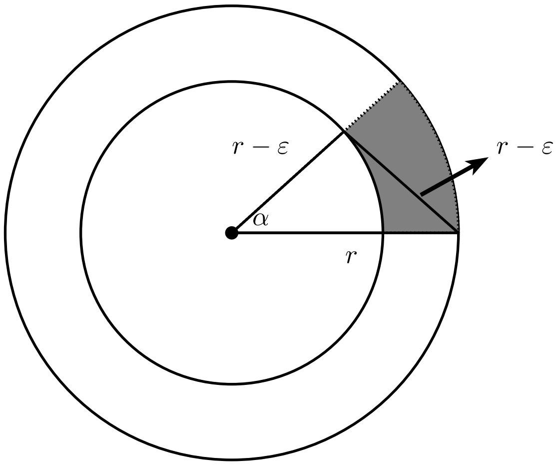

Figure 4.5:

Now, for each $i \in \{1, \ldots, |\Gamma|\}$, let Γ_n^i be the set of all points $p \in \Gamma_n$ such that $r-\varepsilon \leq ||p_i - p|| \leq r$. We have shown that this set contains at most 16 points. We erase those corresponding to p_1, then those corresponding to the p_i with minimal index that survive after the first delection ($i \geq 2$), and so on. At the end, we get the subset Γ_n' with the desired properties. $\square$

Since $\hat{f}_n$ converges to F and each set $\mathcal{R}_n$ is finite, we may choose a subsequence $\hat{f}_{n_k}$ such that, for every $x \in \bigcup_{n\geq 1} \mathcal{R}_n$, the sequence $(f_{n_k}(x))_{k\geq 1}$ converges to $F(x)$ with any prescribed rate of convergence. The next lemma (whose proof is straightforward) provides us with the necessary rate for our purposes.

Lemma 4.12:
There exists a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ such that, for every $x \in \bigcup_{n\geq 1} \mathcal{R}_n$, we have

$$\left\| f_{n_k}(x) - F(x) \right\| \leq \frac{1}{8 \cdot 34^2\, Reg(f)^2\, M_{n_{k-1}}} \tag{4.3.2}$$

From now, the subsequence $(f_{n_k})_{k\geq 1}$ will be just denoted $(f_n)_{n\geq 1}$. Since $F : I^2 \to \mathbb{R}^2$ is Lipschitz regular, Proposition 3.13 implies that there exist a non-empty open set $W \subset F(I^2)$ and open disjoint subsets $V_1, \ldots, V_N \subset I^2$, where $N \leq Reg(F)$, so that $\bigcup_{i=1}^N V_i = F^{-1}(W)$ and, for every $i \in [[1, N]]$, the map $F|_{V_i} : V_i \to W$ is bi-Lipschitz, with lower bi-Lipschitz constant $b = \frac{1}{2Reg(F)^2}$. Using this, we next prove that the L-Lipschitz map f is actually bi-Lipschitz when restricted to an appropriate subset of marked points lying in $\mathcal{D} \cap \phi_n^{-1}(V_1)$, namely, the points in $\phi_n^{-1}(\mathcal{R}_{n-1} \cap V_1)$ (these can be seen as marked points for $M = M_n$ and $P = M_{n-1}$).

Lemma 4.13:

There exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, the map $f|_{\phi_n^{-1}(\mathcal{R}_{n-1} \cap V_1)}$ is bi-Lipschitz, with lower bi-Lipschitz constant smaller than or equal to $1/4Reg(F)^2$.

Proof. By Proposition 3.13, for every x, y in V_1,

$$||F(x) - F(y)|| \geq \frac{||x-y||}{2Reg(F)^2}.$$

By the triangle inequality, for every x, y in $\mathcal{R}_{n-1} \cap V_1$,

$$||f_n(x) - f_n(y)|| \geq \frac{||x-y||}{2Reg(F)^2} - ||f_n(x) - F(x)|| - ||f_n(y) - F(y)||.$$

By Lemma 4.12 and the estimation of $Reg(F)$ in Proposition 4.11 we have that

$$\begin{aligned} ||f_n(x) - f_n(y)|| &\geq \frac{||x-y||}{2Reg(F)^2} - \frac{1}{4 \cdot 34^2\, Reg(f)^2 \cdot 2M_{n-1}} \\ &\geq \frac{||x-y||}{2Reg(F)^2} - \frac{||x-y||}{4Reg(F)^2} \\ &= \frac{1}{4Reg(F)^2} ||x-y|| \end{aligned}$$

Therefore, after rescaling, we obtain that for every z, w in $\phi_n^{-1}(\mathcal{R}_{n-1} \cap V_1)$,

$$||f(z) - f(w)|| \geq \frac{||z-w||}{4Reg(F)^2},$$

as we wanted to show. □

4.4 Some remarks on Question 4.1

In this section we discuss the problem that appears when we try to apply directly the Cortez-Navas' examples in the Lipschitz setting. We consider the function $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ that "stretches and folds" $\mathbb{Z}^2$ as is depicted in the figure below.

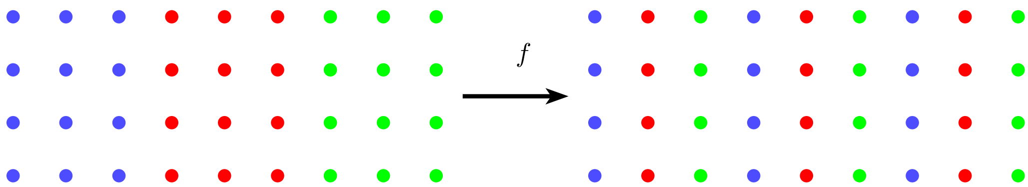

Figure 4.6: stretching and folding $\mathbb{Z}^2$ horizontally

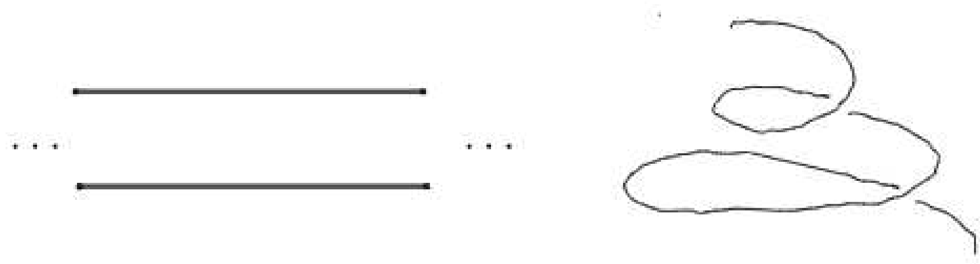

Figure 4.7: The limit map F that stretches and folds (from left to right).

This map is 3-Lipschitz and the pre-image of any ball in $\mathbb{R}^2$ under F is composed by 3 “pieces” (it is not necessary to perform precise computations to describe the phenomenon). This situation imposes some issues to the Cortez-Navas’ examples for Lipschitz maps, where variations of densities can be compensated from points coming from different pieces. Nevertheless, we think that such a situation can be avoided by “discretizing” the non-realizable densities obtained in [15] (see Theorem 3.18) and by using the results obtained in the preceding sections of this Chapter.

Chapter 5

Conclusions

Along this thesis we obtain some original results on non-rectifiable Delone sets. On the one hand, it was demonstrated that "almost all" bounded, positive continuous functions defined on the unit cube $[0,1]^d$ can be used to produce Delone sets $\mathcal{D}_d \subset \mathbb{R}^d$, $d \geq 2$, for which there are no bi-Lipschitz bijections from $\mathcal{D}_d$ to $\mathbb{Z}^d$. For this purpose, it was key the ideas introduced by Burago and Kleiner in [3]. On the other hand, in Chapter 4 we contribute some advances on the resolution of Question 4.1, where was key to develop extensions for the results obtained by Cortez and Navas in [6]. Moreover, for some Propositions in Chapter 4 it was necessary the bi-Lipschitz decomposition of Lipschitz regular maps, obtained recently by Dymond, Kaluža and Kopecká in [15].

5.1 Open Problems

On Lipschitz-rectifiability. From the work developed in Chapter 4 on Lipschitz-rectifiability, the following question arises naturally.

Question 5.1:
Does there exists a non-rectifiable Delone set in $\mathbb{R}^d$, $d \geq 2$, which can be mapped onto $\mathbb{Z}^d$ by a Lipschitz bijection?

A related problem to Question 5.1 is motivated from the work of Dymond, Kaluža and Kopecká, which fits in the "discrete-continuous" relation of non-rectifiable Delone sets and non-realizable densities.

Problem 5.2:
Find an explicit example of a bounded positive continuous (or measurable) function $\rho : [0,1]^d \to \mathbb{R}$ for which there are no Lipschitz solutions of the equation

$$f_{\#}(\rho\lambda) = \lambda|_{f(I^d)}. \tag{5.1.1}$$

We think that a solution of Problem 5.2 could be the non-realizable functions constructed by Burago and Kleiner in [3], or the McMullen's densitty constructed in [21]. These functions are bad-behaved (non-realizable) in any small scale in I^2.

On Feige's Question. We can ask for solutions to Feige's question in non-euclidean spaces. For instance, let $\mathbb{F}_2$ be the free group with two generators equipped with the word metric. Denote by $B_2(n)$ be the ball in $\mathbb{F}_2$ centered in e with radius $n \in \mathbb{N}$. It is known that

$$\begin{aligned} |B_2(n)| &= 4(1+3+9+\ldots+3^{n-1})+1 \\ &= 4\left(\frac{3^n-1}{2}\right)+1 \\ &= 2(3^n-1)+1 = 2\cdot 3^n - 1. \end{aligned}$$

Thus, in this context we pose the following non-Euclidean (Feige's) question.

Question 5.3 (Feige's question in $\mathbb{F}_2$)**:**
Is there a constant $L \geq 1$ such that for every $n \in \mathbb{N}$ and every set $S \subset \mathbb{F}_2$, with cardinality $|S| = 2\cdot 3^n - 1$, there exists a bijection $f : S \to B_2(n) \subset \mathbb{F}_2$ with $Lip(f) \leq L$?

Appendix

The aim of this appendix is to provide some basics on Lipschitz maps (section A.1) and Measure Theory (section A.2) which are used along this thesis. The results of section A.1 are rather classical and can be found, for instance, in [11], while the results in section A.2 are discussed in the appendix D of [15].

A.1 Lipschitz maps

Given two metric spaces (X, d_X) and (Y, d_Y) and a constant $L > 0$, we say that a map $f : X \to Y$ is L***-Lipschitz*** if for every $x, y \in X$:

$$d_Y(f(x), f(y)) \leq L d_X(x, y)$$

A map $f : X \to Y$ is called ***Lipschitz*** if there exists $L > 0$ such that f is L-Lipschitz. Metric spaces are plenty of Lipschitz maps; for instance, given a metric space (X, d) and a fixed point $x_0 \in X$, the map $f(x) = d(x, x_0)$ is 1-Lipschitz, because of the triangle inequality. Geometrically, a Lipschitz map does not stretch the distances more than a factor L, but the distances can be shrinked by f too much (even f can collapse several points into a single point).

A kind of mappings that do not distort too much the distances are the so-called bi-Lipschitz mappings. Given two metric spaces (X, d_X), (Y, d_Y) and two positive real numbers $\alpha < \beta$, we say that a map $f : X \to Y$ is (α, β)***-bi-Lipschitz*** if for every $x, y \in X$:

$$\alpha d_X(x, y) \leq d_Y(f(x), f(y)) \leq \beta d_X(x, y).$$

A map $f : X \to Y$ is called ***bi-Lipschitz*** if there exist positive numbers $\alpha < \beta$ such that f is (α, β)-bi-Lipschitz. Note that a bi-Lipschitz map is injective but not necessarily surjective. We say that (X, d_X) and (Y, d_Y) are ***bi-Lipschitz equivalent (BL)*** if there exists a bi-Lipschitz *bijection* from X to Y. By an L***-bi-Lipschitz*** map we mean an $\left(\frac{1}{L}, L\right)$-bi-Lipschitz map. For instance, the spaces $\mathbb{Z}$ and $2\mathbb{Z}$ with the induced metric in $\mathbb{R}$ are bi-Lipschitz equivalents via the map given by multiplication by 2. Informally speaking, a bi-Lipschitz map stretches and shrinks all the distances at most by a constant factor.

The definition of Lipschitz and bi-Lipschitz maps depends obviously in the metrics defined on the domain and in the range of a given map. Since we will work mainly in $\mathbb{R}^d$, we use in the whole thesis the Euclidean distance in $\mathbb{R}^d$.

We continue this section with three classical results about Lipschitz Analysis. The first result is the so-called Kirzbraun-Valentine's extension theorem. This theorem is an important tool since it can be used to extend Lipschitz maps defined on subsets of $\mathbb{R}^d$ (see, for instance [11], Theorem 2.10.43).

Theorem A.1 (Kirszbraun)
Let $f : A \subset \mathbb{R}^d \to \mathbb{R}^n$ be an L-Lipschitz map. Then f extends to an L-Lipschitz map $\hat{f} : \mathbb{R}^d \to \mathbb{R}^n$, i.e, $\hat{f}|_A = f$.

We point out that Kirszbraun's extension theorem does not necessarily hold if we modify the ambient and the target space where the Lipschitz map is defined. For instance, let X be the *tripod* graph as in Figure 1.1, with set of vertices being $\{u_0, u_1, u_2, u_3\}$ equipped with the shortest-path metric. The map $f : \{u_1, u_2, u_3\} \to \Delta(\omega_1, \omega_2, \omega_3)$ given by $f(u_i) = \omega_i$ for every $i = 1, 2, 3$, where $\Delta(\omega_1, \omega_2, \omega_3) \subset \mathbb{R}^2$ is an equilateral triangle with side-length equals to 2, is an 1-Lipschitz map (actually an isometry). Observe that this map cannot be extended to a 1-Lipschitz map $\hat{f}$ defined on X onto $\Delta(\omega_1, \omega_2, \omega_3)$ (see Figure 6.1).

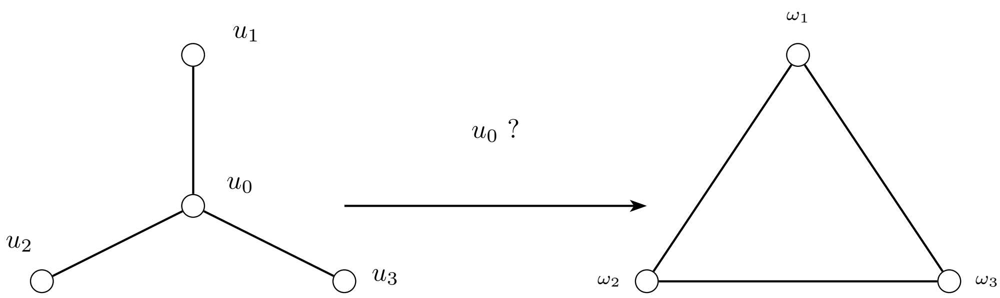

Figure 5.1: mapping a tripod onto an equilateral triangle does not admit Lipschitz extension.

Along this thesis, we work with the Jacobian of a Lipschitz map from $\mathbb{R}^d$ to itself. Differentiability makes sense for Lipschitz maps in $\mathbb{R}^d$ because the following well-known theorem, due to Rademacher. This classical result may be found in an Appendix of [14].

Theorem A.2 (Rademacher)
Every Lipschitz function on an open set in $\mathbb{R}^n$ is differentiable almost everywhere.

In geometric measure theory, the following Euclidean Area formula appears as an important tool, which relates the measure of the image of a measurable set under a Lipschitz map between Euclidean spaces and its Jacobian (see TTheorem 3.2.3 in [11]).

Theorem A.3 (Euclidean Area Formula)
Let $A \subset \mathbb{R}^d$ be a measurable set, let $f : A \to \mathbb{R}^n$ be a Lipschitz map, with $n \geq d$. Then:

$$\int_A |Jac(f)(x)| d\lambda^d(x) = \int_{\mathbb{R}^n} N(f, A, y) d\lambda^n(y),$$

where $Jac(f)$ is the Jacobian of f, λ^d is the d-dimensional Lebesgue measure and $N(f, A, y) := |\{x \in A : \ f(x) = y\}|$.

A.2 Some results in Measure Theory

In this section we enunciate some results on Measure Theory which are required to understand the convergence of some counting measure in Chapters 1 and 3. The proof of the following lemmas can be found in the Appendix D of [15].

Definition A.4:
Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of finite Borel measures in X. We say that $(\mu_k)_{k\in\mathbb{N}}$ weakly converges to a measure μ if for every $\varphi \in C_b(K)$ we have that

$$\lim_{k\to+\infty} \int_X \varphi d\mu_k = \int_X \varphi d\mu.$$

Lemma A.5 (Lemma 5.5 in [15]):
Let μ and $(\mu_k)_{k\in\mathbb{N}}$ be finite Borel measures on a compact metric space K. Suppose that there exists, for each $k \in \mathbb{N}$, a finite collection $\mathcal{Q}_k$ of Borel subsets of K such that cover μ-almost all of K and satisfying the following properties:

i) $\displaystyle\sum_{Q\in\mathcal{Q}_k} \mu(Q) = \mu(K),$

ii) $\displaystyle\lim_{k\to+\infty} \max_{Q\in\mathcal{Q}_k} diam(\mathcal{Q}) = 0,$

iii) $\displaystyle\max_{Q\in\mathcal{Q}_k} |\mu_k(Q) - \mu(Q)| = o\left(\frac{1}{|\mathcal{Q}_k|}\right).$

Then μ_k weakly converges to μ.

Given a measurable space (X, μ), we can "push" the measure μ into a measurable space Y by a measurable function from X to Y as follows.

Definition A.6:
Let μ be a measure in a space X, and $g : X \to Y$ me a measurable map. We define the pushforward

measure $g_\#\mu$ *by letting*

$$g_\#\mu(A) := \mu(g^{-1}(A)), \qquad \textit{for every } A \in Y.$$

Lemma A.7 (Lemma 5.6 in [15]):
Let K *be a compact space and* $(\mu_k)_{k\in\mathbb{N}}$ *be a sequence of finite, Borel measures on* K *weakly converging to a finite Borel measure* μ*. Let* X *be a metric space and* $h_k : K \to X$ *be a sequence of continuous mappings converging uniformly to* h*. Then* $(h_k)_\#(\mu_k)$ *weakly converges to* $h_\#(\mu)$*.*

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