UNIVERSIDAD DE SANTIAGO DE CHILE
FACULTAD DE CIENCIA
Departamento de Matemática y Ciencia de la Computación

Towards the Duflo Isomorphism in Pre-Lie Algebras and Construction of Rota^{m}-Algebras and Ballot^{m}-Algebras from Associative Algebras with a Rota-Baxter morphism and a Rota-Baxter Operator of Weights Three and Two

Wilson Arley Martinez Flor

Profesor Guía:
Enrique Reyes (USACH)
Maria Ronco (UTALCA)

Trabajo de graduación presentado a la Facultad de Ciencia en cumplimiento parcial de los requisitos exigidos para optar al Grado de Doctor en Ciencia con mención en Matemática.

SANTIAGO DE CHILE, CHILE
2015

TOWARDS THE DUFLO ISOMORPHISM IN PRE-LIE ALGEBRAS AND CONSTRUCTION OF ROTA^{M}-ALGEBRAS AND BALLOT^{M}-ALGEBRAS FROM ASSOCIATIVE ALGEBRAS WITH A ROTA-BAXTER MORPHISM AND A ROTA-BAXTER OPERATOR OF WEIGHTS THREE AND TWO

Este trabajo de Graduación fue elaborado bajo la supervición del profesor gua Dr. Enrique Reyes García de la Universidad de Santiago de Chile y la profesora Dra. María Ronco de la Universidad de Talca. Ha sido aprobada por los miembros de la Comisión Calificadora, compuesta por

Dr. Enrique Reyes García
Profesor Guía

Dra. María Ofelia Ronco
Profesor Guía

Dra. Alicia Labra Jeldres
Profesor informate

Dr. David Grimm
Profesor informate

Dra. María Isabel Cortez Muñoz
Directora Programa.

Mg. Pedro Marín Alvarez
Director Departamento.

Resumen

El teorema de Duflo es la composición del isomorfismo de Poincaré- Birkhoff-Witt (el cual es un isomorfismo a nivel de espacios vectoriales) con un automorfismo de el álgebra simétrica $S(\mathfrak{g})$ (el cual desciende al subespacio de invariantes $S(\mathfrak{g})^{\mathfrak{g}}$). Esto es, el espacio de invariantes, $S(\mathfrak{g})^{\mathfrak{g}}$ y $U(\mathfrak{g})^{\mathfrak{g}}$, **son de hecho canónicamente isomorfos como álgebras**.

En la primera parte de este trabajo generalizamos algunos conceptos del teorema de Duflo a álgebras Pre-Lie y álgebras dendriformes, en particular presentamos la noción de acciones sobre álgebras dendriformes.

La combinatoria de los árboles planares binarios es conocida por tener interesantes propiedades algebraicas, Loday y Ronco son los primeros en introducir el álgebra de Hopf de árboles planares binarios [25]. Esta álgebra de Hopf es la álgebra dendriforme libre sobre un generador.

Damos una generalización de Operadores Rota-Baxter e introducimos la noción de un Ballot^m-algebra. Un álgebra de Rota-Baxter libre sobre un conjunto pueden ser construida desde un subconjunto de bosques planares enraizados con decoraciones en los ángulos [3, 14]. Presentamos construcciones similares para obtener una álgebra asociativa en términos de árboles planares binarios con un Operador de Rota-Baxter modificado, y así construimos un Ballot^m-algebra.

Introducimos los conceptos de un Morfismo de Rota-Baxter, Dyck^m-algebra y Rota^m-algebra. Un elemento u es idempotente con respecto al producto $\cdot$ en el álgebra si: $u \cdot u = u$, y este es una identidad izquierda si $x \cdot u = x$ para todo elemento x en el álgebra. Álgebras asociativas con una identidad izquierda que simultáneamente es un elemento idempotente, nos permite presentar ejemplos de un Morfismo de Rota-Baxter y así podemos construir un Rota^m-algebra.

Enfatizamos que la construcción de Ballot^m-algebras y Rota^m-algebras a partir de álgebras asociativas con una generalización de Operadores de Rota-Baxter, son algunos de los principales resultados de este trabajo; también presentamos ejemplos interesantes.

Abstract

The Duflo theorem is the composition of the Poincaré- Birkhoff-Witt isomorphism (which is only an isomorphism at the level of vector spaces) with an automorphism of the symmetric algebra $S(\mathfrak{g})$ (which descends to subspace of invariants $S(\mathfrak{g})^{\mathfrak{g}}$). That is, the vector spaces of invariants, $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$, **are indeed canonically isomorphic as algebras**.

In the first part of this work we generalize some concepts the Duflo theorem to Pre-Lie algebras and dendriform algebras, In particular we present the notion of actions on dendriform algebras.

The combinatorics of planar binary trees is known to have very interesting algebraic properties, Loday and Ronco first introduced the Hopf Algebra of planar binary trees [25]. This Hopf algebra is the free dendriform algebra on one generator.

We give a generalization of Rota-Baxter Operators and introduce the notion of a Ballot^m-algebra. Free Rota-Baxter algebras on a set can be constructed from a subset of planar rooted forests with decorations on the angles [3, 14]. We give similar constructions for obtaining an associative algebra in terms of planar binary trees with a modified Rota-Baxter Operator, and so we construct a Ballot^m-algebra.

We introduce the concepts of a Rota-Baxter Morphism, Dyck^m-algebra and Rota^m-algebra. An element u is said to be idempotent with respect to product $\cdot$ in the algebra if: $u \cdot u = u$, and it is a left identity if $x \cdot u = x$ for all element x in the algebra. Associative algebras with a left identity that simultaneously is a element idempotent, permit us to present examples of a Rota-Baxter Morphism and so we can construct a Rota^m-algebra.

We stress that the construction of Ballot^m-algebras and Rota^m-algebras from associative algebras with a generalitation of Rota-Baxter Operators are some of the main results of this work; we also present interesting examples.

Dedicatoria

A Dios por mostrarme el camino al lado de las Matemáticas, a mi Esposa y a mi Hija por ser la motivación, a mi Madre por su darme su amor y a mi Padre por su compresión , a mis abuelos por acogernos en su hogar, a mis Hermanos y Sobrinos por ser incondicionales, a mis Profesores y Amigos por cada una de sus enseñanzas.

Agradecimientos

Gracias a Dios, por todas las bendiciones en mi formación académica y en especial por la familia que me ha dado.

Mi agradecimiento más profundo y sincero: A mi esposa Samin y mi hija Sara María por su cariño y motivación; a mis padres, Sigifredo e Ines, por su afecto y apoyo; a mis hermanos Ember, Albeiro y Yeny por ser incondicionales desde Colombia; y a mis sobrinos William y José por la felicidad que me trasmiten.

Me gustaria agradecer al profesor Dr. Enrique Reyes, mi director, por sus sugerencias y soporte durante esta investigación, de igual manera agradezco sinceramente a la profesora Dra. María Ronco, mi directora, por brindarme su ayuda en este proyecto y resolver mis dudas.

Al profesor Victor Guiñez por su apoyo, durante su dirección del programa de doctorado en matemática, a la profesora María Isabel Cortez directora actual y a los profesores del programa les manifiesto mis agradecimientos.

Debería tambien mencionar a mis compañeros, Samin, Boris, Alvaro,Inge, Patricio, Alan y María José, les agradezco por los buenos tiempos que tuvimos en su compañia.

A Evelyn Aguilar por su disposición y colaboración que mantuvo durante estos años en la secretaria de posgrado.

Finalmente, manifiesto mis agradecimiento a aquellas entidades que han hecho posible el desarrollo de mi formación doctoral:

Universidad del Cauca, Icetex, Laspau y en especial la Convocatoria 497 del Programa de Becas Francisco Jos de Caldas, modalidad en el exterior, Colciencias, Gobierno de Colombia, por patrocinar mi formación doctoral durante cuatro años.

A la Universidad de Santiago de Chile, quien financió en parte mi último año de estudio doctoral, através del Proyecto Basal por Desempeño USA1298: “Hacia una Cultura de Indicadores de Desempeño en la Educación Superior ”.

A cada uno de ellos, Gracias!

Septiembre 22, 2015.

Table of Contents

Introduction

Let $S(\mathfrak{g})$ be the symmetric algebra of a finite-dimensional Lie algebra $\mathfrak{g}$, and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. $S(\mathfrak{g})$ is a commutative algebra and $U(\mathfrak{g})$ is an associative algebra; moreover, both spaces, $S(\mathfrak{g})$ and $U(\mathfrak{g})$, are $\mathfrak{g}$-modules with respect to the adjoint action [33].

A fundamental result in the theory of enveloping algebras is the Poincaré- Birkhoff-Witt Theorem [34, 13, 10], which (in one of its incarnations) states that a natural map from the symmetric algebra $S(\mathfrak{g})$ into $U(\mathfrak{g})$ is in fact an isomorphism of $\mathfrak{g}$-modules.

Unfortunately, this map restricted to the subspaces of invariants $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$ **is not** an algebra homomorphism. Recall that $S(\mathfrak{g})^{\mathfrak{g}}$, the set of elements of $S(\mathfrak{g})$, which are annihilated under the representation induced by the adjoint representation of $\mathfrak{g}$ in $S(\mathfrak{g})$, is a graded subalgebra of $S(\mathfrak{g})$. Similarly $U(\mathfrak{g})^{\mathfrak{g}}$ is a graded subalgebra of $U(\mathfrak{g})$. However, the Duflo theorem states that the vector spaces of invariants, $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$, **are indeed canonically isomorphic as algebras**.

This isomorphism is called the Duflo isomorphism and it happens to be a composition of the Poincaré- Birkhoff-Witt isomorphism (which is only an isomorphism at the level of vector spaces) with an automorphism of the symmetric algebra $S(\mathfrak{g})$ (which descends to subspace of invariants $S(\mathfrak{g})^{\mathfrak{g}}$). The definition of this automorphism involves a formal power series on $\mathfrak{g}$, $J(x) = \det\left(\frac{1-e^{-ad_x}}{ad_x}\right)$, called Duflo element [9], where $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$.

One of the goals of this project is to study analogues when $\mathfrak{g}$ is not a Lie algebra, see for instance [28]. Although these extensions and analogues have never been studied in detail, there exists for instance, a Poincare-Birkhoff-Witt Theorem for Lie Superalgebras [29]. There is also an analogue of the classical Duflo formula in the case of L_∞-algebras and a conjecture in the case of an arbitrary Q-manifold [32]. There is a result of Ronco that says dendriform Hopf algebra is isomorphic to the enveloping algebra of its brace algebra of primitive elements [31]. We focus on classy of non-associative structures which are more general than Lie algebras, pre-Lie algebras, dendriform algebras etc.

In particular, we ask: Would an analogue of the Poincaré-Birkhoff-Witt theorem for Left and Right-Symmetric Algebras hold? If so, Can we prove an analogue of the Duflo Theorem in this case?

Right-symmetric algebras [8, 11], or RSAs in short, are also called Gerstenhaber algebras, or pre-Lie algebras. Left-symmetric algebras, or LSAs, arise in many areas of mathematics and physics and are known under many different names. LSAs are also called Vinberg algebras, Koszul algebras or quasi-associative algebras.

Chapter 1

The Poincaré-Birkhoff-Witt Theorem

Let $S(\mathfrak{g})$ be the symmetric algebra of a finnite-dimensional Lie algebra $\mathfrak{g}$, and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Both spaces, $S(\mathfrak{g})$ and $U(\mathfrak{g})$, are $\mathfrak{g}$-modules with respect to the adjoint action [33]. A fundamental result in the theory of enveloping algebras is the Poincaré-Birkhoff-Witt Theorem [34, 13, 10], which (in one of its incarnations) states that a natural map from the symmetric algebra $S(\mathfrak{g})$ into $U(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$-modules. Unfortunately this map restricted to the subspaces of invariants $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$ **is not** an algebra homomorphism. Throughout this document $\mathbb{K}$ will denote a field of characteristic 0.

1.0.1 The Symmetric Algebra

Let S to be the set of all monomials $\left\{x_1^{n_1} \dots x_k^{n_k} : (n_1, \dots n_k) \in \mathbb{N}^k\right\}$ in k different variables $x_1, \dots, x_k$. A polynomial in k variables with coefficients in $\mathbb{K}$ is a function from S to $\mathbb{K}$

$$\begin{array}{rlcl} p(x_1, \dots, x_k): & S & \to & \mathbb{K} \\ & x_1^{n_1} \dots x_k^{n_k} & \to & a_{n_1, \dots, n_k} \end{array}$$

null except for a finite number of indices in $\mathbb{N}^k$. We shall denote the set of all polynomials by $\mathbb{K}[x_1, \dots x_k]$, and we will write them as finite sums of the form

$$p = \sum_{(n_1, \dots, n_k) \in \mathbb{N}^k} a_{n_1, \dots, n_k} x_1^{n_1} \dots x_k^{n_k}.$$

If $n_1 + \cdots + n_k = n$ then $a_{n_1, \dots, n_k} x_1^{n_1} \dots x_k^{n_k}$ is called a monomial term of degree n. A polynomial is called homogeneous if all its monomial terms have the same degree. The degree of a nonzero polynomial is the largest degree of any of its monomial terms. A homogeneous polynomial of degree zero is a constant polynomial or simply a constant.

If p is a nonzero polynomial in k variables, the sum of all the monomial terms in p of degree n is called the homogeneous component of p of degree n. If p has degree d then p may be written uniquely as the sum $p_0 + p_1 + \cdots + p_d$ where p_i is the homogeneous component of p of degree i, for $0 \le i \le d$ (where some p_i may be zero).

Definition 1.0.1. We define **the ring of formal power series** over $\mathbb{K}$ in n variables as the set $\mathbb{K}[[x_1, \dots, x_n]]$ of all functions from S a $\mathbb{K}$

$$\begin{array}{lcl} S & \to & \mathbb{K} \\ x^\alpha & \to & a_\alpha \qquad (\text{ where } \alpha = (\alpha_1, \dots, \alpha_n) \text{ and } x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}). \end{array}$$

We shall indicate elements of the ring as (possibly infinite) formal sums

$$\sum_{x^\alpha \in S} a_\alpha x^\alpha,$$

where the function corresponding to this formal sum maps x^α to a_α for all $x^\alpha \in S$. In $\mathbb{K}[[x_1, \ldots, x_n]]$ we define addition and multiplication by

$$\sum_{x^\alpha \in S} a_\alpha x^\alpha + \sum_{x^\alpha \in S} b_\alpha x^\alpha = \sum_{x^\alpha \in S} (a_\alpha + b_\alpha) x^\alpha$$

$$\left(\sum_{x^\delta \in S} a_\delta x^\delta \right) \left(\sum_{x^\beta \in S} b_\beta x^\beta \right) = \sum_{x^\alpha \in S} \left(\sum_{\delta + \beta = \alpha} a_\delta b_\beta \right) x^\alpha$$

Remark 1.0.1. For all $x^\alpha \in S$, $\left\{(x^\delta, x^\beta) \in S \times S : x^\delta x^\beta = x^\alpha\right\}$ is finite, since each element of S has only finitely many factorizations as a product of two elements.

Let $S_d = \{x^\alpha \in S : \alpha_1 + \cdots + \alpha_n = d\}$ the formal sum $f_d = \sum_{x^\alpha \in S_d} a_\alpha x^\alpha$ of all the terms in $f = \sum_{x^\alpha \in S} a_\alpha x^\alpha$ of degree d is called the homogeneous componente of f of degree d. Then f may be written uniquely as the sum $f_0 + f_1 + f_2 + \cdots$ (where some f_i may be zero).

Definition 1.0.2. An **algebra over a field** $\mathbb{K}$ is a vector space A over $\mathbb{K}$ endowed with a multiplication $A \times A \to A$ which is bilinear and satisfies the following equalities:

1. $a(b + c) = ab + ac,\ (a + b)c = ac + bc$ (Distributivity)
2. $(ra)b = a(rb) = r(ab)$ (Compatibility)

for all $a, b, c \in A$ and $r \in \mathbb{K}$.

Definition 1.0.3. Let A be an algebra over $\mathbb{K}$. A is called an **associative algebra** if the multiplication in A satisfies $a(bc) = (ab)c$, for all $a, b, c \in A$. If there is an element 1 in A such that $1a = a1 = a$ for all $a \in A$, then this element is called **identity** and the algebra is called **algebra with identity**.

Definition 1.0.4. Let A be an associative algebra over $\mathbb{K}$ with identity 1. A **grading** of A is a sequence of linear subspaces A_i of A such that $A = \bigoplus_{i=0}^{\infty} A_i$, which is compatible with the ring structure of A, i.e.: $1 \in A_0$ and $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. An algebra A with a grading $\{A_i\}_{i \geq 0}$ is called a **graded algebra**, and the elements of A_i are called **homogeneous elements of degree** i.

Definition 1.0.5. Let V be a vector space. The **tensor algebra** of V is the external direct sum

$$T(V) := \bigoplus_{n=0}^{\infty} T^n(V), \text{ such that } \quad T^0(V) := \mathbb{K}, \quad T^n(V) := \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}, \quad \text{or } V^{\bigotimes n}$$

and with product given by

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) := v_1 \otimes \cdots \otimes v_n \otimes w_1 \cdots \otimes w_m,$$

for any pair of elements $v_1 \otimes \cdots \otimes v_n \in T^n(V)$ and $w_1 \otimes \cdots \otimes w_m \in T^m(V)$, and extended by linearity to $T(V)$.

Definition 1.0.6. Let V be a vector space. The **symmetric algebra** of V is the quotient

$$S(V) = T(V)/I$$

of the tensor algebra $T(V)$, by its two-side ideal I, generated by homogeneous elements of degree 2 of the form

$$x \otimes y - y \otimes x, \quad x, y \in V.$$

The projection $\pi : T(V) \to S(V)$, $t \mapsto \bar{t}$ induces an algebra operation on $S(V)$,

$$\overline{u} \bullet \overline{v} = \overline{u \cdot v} \text{ for all } u, v \in T(V)$$

Remark 1.0.2. First note that the elements of the ideal I are linear combinations of tensors of the form

$$u \cdot (x \otimes y - y \otimes x) \cdot u' \quad \text{with } \; u, u' \in T(V) \text{ and } x, y \in V.$$

Hence I is spanned by tensors of the form

$$u_1 \otimes \cdots \otimes u_n \otimes (x \otimes y - y \otimes x) \otimes u_{n+1} \otimes \cdots \otimes u_{n+m} \text{ where } u_i \in V, \text{ for each } 1 \leq i \leq n+m.$$

The map $\pi|_{\mathbb{K} \oplus V} : \mathbb{K} \oplus V \to S(V)$ is injective since $I \subseteq \bigoplus_{n \geq 2} T^n(V)$; thus we may identity $\mathbb{K}$ and V with their images in $S(V)$. Moreover, the set $\{\overline{1_{\mathbb{K}}}, \overline{v} : v \in V\}$ is a set of generators for $S(V)$, since $\{1_{\mathbb{K}}\} \cup V$ is a set of algebra generators for $T(V)$ and π is a morphism of algebras. Also, we note that the symmetric algebra $S(V)$ is an commutative algebra.

Proposition 1.0.1. *Let V be a vector space and let $S(V)$ be the symmetric algebra. Then $S(V)$ is a graded associative algebra,*

$$S(V) = \bigoplus_{n \geq 0} S^n(V), \quad \big(S^n(V) = T^n(V)/I_n\big)$$

where $I_n = I \cap T^n(V)$ is the subspace spanned by all elements of the form

$$x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

for all $x_i \in V$ and all permutation $\sigma \in \mathcal{S}_n$, where $\mathcal{S}_n$ is the group of permutations of $\{1, 2, \ldots, n\}$.

Proof. Since I is generated by homogeneous tensors, I admits the grading $I = \bigoplus_{n \geq 2} I_n$ where $I_n = T^n(V) \bigcap I$ and $I_n \cdot I_m \subseteq I_{n+m}$. We have that the elements of the form

$$v_1 \otimes \cdots \otimes (v_k \otimes v_{k+1} - v_{k+1} \otimes v_k) \otimes \cdots \otimes v_n$$

determine a set of generators for I. In turn, the latter can be rewritten as

$$v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \in I$$

for the set of the transpositions that exchange two consecutive integers, that we can replace by the group of permutations of n elements. The fact that $I_n \subset T^n(V)$ for every $n \geq 0$ implies I_n is the subspace spanned by all elements of the form

$$x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

for all $x_i \in V$ and all permutation $\sigma \in S_n$. Now, we claim that

$$S^n(V) = T^n(V)/I \simeq T^n(V)/I_n, \text{ for all } n \geq 0.$$

Indeed, the linear map $\bar{t} \mapsto t + I_n$ is well defined and it is an isomorphism of vector spaces: If $t, t' \in T^n(V)$ and $\bar{t} = \overline{t'}$ then $t - t' \in T^n(V) \cap I$ so that $t + I_n = t' + I_n$. Also, the map is injective and onto, if $t + I_n = t' + I_n$ then $t - t' \in I_n \subset I$, so that $\bar{t} = \overline{t'}$. □

1.0.2 Universal Enveloping Algebra

In what follows, we work in the non-graded case, that is the elements of vector spaces have degree 0 as well as the operations. This convention simplifies the lecture, for the graded case, it suffices to add the Koszul sign $(-1)^{|x||y|}$ whenever we exchange elements in the formula.

Definition 1.0.7. Let $\mathfrak{g}$ be a vector space over $\mathbb{K}$. A bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \to [x, y]$, is a **Lie bracket** on $\mathfrak{g}$ if for all $x, y, z \in \mathfrak{g}$, we have

- $[x, y] = -[y, x]$ (antisymmetry),
- $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi identity).

A vector space $\mathfrak{g}$ equipped with a Lie bracket is called a **Lie algebra**.

Example 1.0.1. If $(\mathcal{A}, *)$ is an associative algebra over $\mathbb{K}$ and we define a map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by

$$[x, y]_* := x * y - y * x, \text{ for all } x, y \in \mathcal{A}, \tag{1.0.1}$$

then $(\mathcal{A}, [\cdot, \cdot]_*)$ is a Lie algebra, called the **Lie algebra related to** $\mathcal{A}$. For example, let V be a vector space and $End(V)$ the algebra de endomorphisms of V with multiplication give by composition $\circ$, then $(End(V), [\cdot, \cdot]_\circ)$ is a Lie algebra. The Lie bracket defined in (1.0.1) will be referred to as the **commutator related to** $\mathcal{A}$

Definition 1.0.8. Let $\mathfrak{g}$ be a Lie algebra and $T(\mathfrak{g})$ the tensor algebra of $\mathfrak{g}$. Denote by J the bilateral ideal of $T(\mathfrak{g})$ generated by the elements

$$x \otimes y - y \otimes x - [x, y]$$

for $x, y \in \mathfrak{g}$.
Note that the associative product of $T(\mathfrak{g})$ is compatible with the quotient by J, because J is an algebra ideal, so the quotient $U(\mathfrak{g}) := T(\mathfrak{g})/J$ inherits the associative algebra structure over $\mathbb{K}$. The algebra $U(\mathfrak{g})$ is called the **universal enveloping algebra** of $\mathfrak{g}$.

We denote by $\Pi : T(\mathfrak{g}) \to U(\mathfrak{g})$, $t \mapsto t + J$ the projection. The natural operation on $U(\mathfrak{g})$ which equips $U(\mathfrak{g})$ with the structure of a unital associative algebra is given by $\Pi(t)\Pi(t') = \Pi(t.t')$ for all t, t' in $T(\mathfrak{g})$.
If $U(\mathfrak{g})$ is considered a Lie algebra, it is understood that it is equipped with the associated commutator, which we denote by $[\cdot,\cdot]_U$. The underlying Lie algebra structure of $U(\mathfrak{g})$, with the Lie bracket $[\cdot,\cdot]_U$, is easy to describe because $[\cdot,\cdot]_U$ is a derivation for the product. For instance, we get:

$$[x + J, y + J]_U = [x, y]_{\mathfrak{g}} + J,$$

$$[x \otimes y + J, z + J]_U = [x, z]_{\mathfrak{g}} \otimes y + x \otimes [y, z]_{\mathfrak{g}} + J,$$

and so on.

The map Π is a morphism of unital associative algebras, whence it is a Lie algebra morphism, when $T(\mathfrak{g})$ and $U(\mathfrak{g})$ are equipped with their respective associated commutator. The natural projection Π induces a Lie algebra morphism

$$\Gamma : \mathfrak{g} \to U(\mathfrak{g}),$$

that is, $\Gamma := \Pi|_{\mathfrak{g}}$.

Definition 1.0.9. Let A be an associative algebra over $\mathbb{K}$ with identity 1. A **filtration** of A is a chain of linear subspaces

$$A_0 \subseteq A_1 \subseteq A_2 \cdots$$

such that $A = \bigcup_{i=0}^{\infty} A_i$ which is compatible with the ring structure of A, i.e.: $1 \in A_0$ and $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. An algebra A with a filtration $\{A_i\}_{i \geq 0}$ is called a **filtered algebra**.

Definition 1.0.10. Given a filtered algebra A with filtration $\{A_i\}_{i \geq 0}$, we obtain a graded algebra as follows. We let $Gr(A)$ be the vector space

$$Gr(A) = \bigoplus_{i \geq 0} G^i(A) \text{ where } G^i(A) = A_i/A_{i-1},\ G^0(A) = A_0.$$

Elements of any factor $G^i(A)$ of the direct sum are called homogeneous elements of degree i. We define a multiplication on $Gr(A)$. It is sufficient to define a map of $G^i(A) \times G^j(A)$ into $G^{i+j}(A)$ for each $i, j \geq 0$, and to extend by linearity. Thus we let

$$(x + A_{i-1})(y + A_{j-1}) = xy + A_{i+j-1} \text{ for } x \in A_i,\ y \in A_j.$$

$Gr(A)$ is called the **associated graded algebra** of the filtered algebra A.

We use $Gr(A)$ to compare $U(\mathfrak{g})$ and $S(\mathfrak{g})$, see proposition 1.0.4 below. We denote the canonical image of $\mathbb{K} \oplus \mathfrak{g} \oplus \mathfrak{g}^{2\otimes} \oplus \cdots \oplus \mathfrak{g}^{m\otimes}$ in $U(\mathfrak{g})$ by $U_m(\mathfrak{g})$.

Lemma 1.0.2. *Let $x_1, \ldots, x_m \in \mathfrak{g}$, Γ the canonical mapping of $\mathfrak{g}$ into $U(\mathfrak{g})$, and let τ be a permutation of $\{1, \ldots, m\}$. Then*

$$\Gamma(x_1) \cdots \Gamma(x_m) - \Gamma(x_{\tau(1)}) \cdots \Gamma(x_{\tau(m)}) \in U_{m-1}(\mathfrak{g})$$

Proof. It is sufficient to prove this when τ is the transposition of j and $j+1$.

$$\Gamma(x_1) \cdots \Gamma(x_j)\sigma(x_{j+1}) \cdots \Gamma(x_m) - \Gamma(x_1) \cdots \Gamma(x_{j+1})\Gamma(x_j) \cdots \Gamma(x_m)$$
$$= \Gamma(x_1) \cdots \Gamma(x_{j-1})\Big(\Gamma(x_j)\Gamma(x_{j+1}) - \sigma(x_{j+1})\Gamma(x_j)\Big)\Gamma(x_{j+2}) \cdots \Gamma(x_m)$$
$$= \Gamma(x_1) \cdots \Gamma(x_{j-1})\,[\Gamma(x_j), \Gamma(x_{j+1})]\,\Gamma(x_{j+2}) \cdots \Gamma(x_m)$$
$$= \Gamma(x_1) \cdots \Gamma(x_{j-1})\Gamma([x_j, x_{j+1}])\Gamma(x_{j+2}) \cdots \Gamma(x_m) \in U_{m-1}(\mathfrak{g})$$

□

In the following lemma we denote $\Gamma(x_1) \cdots \Gamma(x_m)$ by $x_1 \cdots x_m$.

For integers $n, k \geq 1$, a *weak composition* of n in k parts is an ordered sequence $(n_1, \ldots, n_k)$ of nonnegative integers such that $\sum_{i=1}^{k} n_i = n$.

Lemma 1.0.3. *Let $v_1, \ldots, v_k$ be a basis of $\mathfrak{g}$. The set of elements*

$$v_1^{n_1} \ldots v_k^{n_k} \text{ for } n_1 + \cdots + n_k \leq n$$

form a basis of $U_n(\mathfrak{g})$. Moreover the elements

$$v_1^{n_1} \ldots v_k^{n_k} + U_{n-1}(\mathfrak{g}) \text{ for all weak composition of } n \text{ in } k \text{ parts}$$

form a basis for $G^n(\mathfrak{g}) = U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$.

Proof. We fix a basis $v_1, \ldots, v_k$ for $\mathfrak{g}$. The vector space $U_n(\mathfrak{g})$ is generated by $v_{i_1} \cdots v_{i_n}$ for every finite sequence $(i_1, \ldots, i_n)$ of integer between 1 and k. It is then sufficient to apply lemma 1.0.2 .

□

Proposition 1.0.4. *The graded algebra $Gr(U(\mathfrak{g}))$ is isomorphic to $S(\mathfrak{g})$.*

Proof. The natural map $\mathfrak{g} \to U_1(\mathfrak{g})/U_0(\mathfrak{g}),\ x \to x + J + U_0(\mathfrak{g})$ is an isomorphism of vector spaces. Now, any two elements of $U_1(\mathfrak{g})/U_0(\mathfrak{g})$ commute in $Gr(U(\mathfrak{g}))$: For elements $x, y \in \mathfrak{g}$ we have

$$\begin{aligned}
(x + J + U_0(\mathfrak{g}))\,(y + J + U_0(\mathfrak{g})) &= x \otimes y + J + U_1(\mathfrak{g}) \\
&= y \otimes x + [x, y] + J + U_1(\mathfrak{g}) \\
&= y \otimes x + J + U_1(\mathfrak{g}) \\
&= (y + J + U_0(\mathfrak{g}))\,(x + J + U_0(\mathfrak{g}))
\end{aligned}$$

where the product lies in $U_2(\mathfrak{g})/U_1(\mathfrak{g})$. Let $x_1,\ldots,x_k$ be a basis of $\mathfrak{g}$, the map defined by

$$\begin{array}{rccl}\varphi_n: & S^n(\mathfrak{g}) & \to & G^n(\mathfrak{g}) \\ & x_1^{n_1}\ldots x_k^{n_k} & \to & x_1^{n_1}\ldots x_k^{n_k}+U_{n-1}(\mathfrak{g})\end{array}$$

where $\sum n_j = n$, is linear. Indeed, we have

$$x_1^{n_1}\ldots x_k^{n_k}x_1^{m_1}\ldots x_k^{m_k}+U_{n+m-1}(\mathfrak{g}) \quad = \quad x_1^{n_1+m_1}\ldots x_k^{n_k+m_k}+U_{n+m-1}(\mathfrak{g}), \ ,$$

since multiplication in $Gr(U(\mathfrak{g}))$ is commutative. Thus

$$\begin{aligned}\varphi_n(x_1^{n_1}\ldots x_k^{n_k})\varphi_n(x_1^{m_1}\ldots x_k^{m_k}) &= x_1^{n_1+m_1}\ldots x_k^{n_k+m_k}+U_{n+m-1}(\mathfrak{g}) \\ &= \varphi_n(x_1^{n_1+m_1}\ldots x_k^{n_k+m_k})\end{aligned}$$

and φ_n can be extended by linearity to give an isomorphism of algebras

$$\varphi: S(\mathfrak{g}) \to Gr(U(\mathfrak{g})).$$

Therefore the associated graded algebra of $U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$. □

1.0.3 The Poincaré-Birkhoff-Witt theorem

Proposition 1.0.5. *Let $\tau \in S_n$ be a permutation of n elements.*

1. *The linear transformation $P_\tau : T^n(\mathfrak{g}) \to T^n(\mathfrak{g})$ such that*

   $$P_\tau(x_1 \otimes \cdots \otimes x_n) := x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)} \quad (x_i \in \mathfrak{g}.);$$

   moreover, P_τ is a linear isomorphism.

2. *For $\tau, \eta \in S_n,\ P_\tau P_\eta = P_{\tau\eta}$*

Proof. For $n \geq 2$ and $\tau \in S_n$, define a n-multilinear mapping $\varphi : \mathfrak{g} \times \cdots \times \mathfrak{g} \to T^n(\mathfrak{g})$

$$(x_1,\ldots,x_n) \to x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}.$$

By the Universal Property of the Tensor Product [7] , φ induces a unique linear mapping

$$\begin{array}{rccl}P_\tau: & T^n(\mathfrak{g}) & \to & T^n(\mathfrak{g}) \\ & x_1 \otimes \cdots \otimes x_n & \to & x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}\end{array}$$

Since P_τ induce a permutation of the standard basis elements, it is a linear isomorphism. Clearly $P_\tau P_\eta = P_{\tau\eta}$. □

Definition 1.0.11. Let $t \in T^n(\mathfrak{g})$. If $P_\tau(t) = t$, for all $\tau \in S_n$, then t is called a **symmetric tensor of order n**. The set of symmetric tensors is a vector subspace of $T^n(\mathfrak{g})$ denoted by $T^n_{sym}(\mathfrak{g})$; we have $T^0_{sym}(\mathfrak{g}) = \mathbb{K}$, $T^1_{sym}(\mathfrak{g}) = \mathfrak{g}$. We shall put $T_{sym}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n_{sym}(\mathfrak{g})$; this is a graded vector subspace of $T^n(\mathfrak{g})$.

Example 1.0.2. $x_1 \otimes x_2 + x_2 \otimes x_1$, is symmetric, but $x_1 \otimes x_2 - x_2 \otimes x_1$ is not.

Example 1.0.3. Let $x \in \mathfrak{g}$ and $n \in \mathbb{N}$, the tensor product of x with itself $x^{\otimes n} := x \otimes x \otimes \cdots \otimes x$ is a symmetric tensor of order n.

Suppose now that the characteristic of field $\mathbb{K}$ is 0.

Proposition 1.0.6. *The space of symmetric tensors $T^n_{sym}(\mathfrak{g})$ is the image of the linear mapping $s : T^n(\mathfrak{g}) \to T^n(\mathfrak{g})$, defined by $w \to \frac{1}{n!} \sum\limits_{\tau \in S_n} P_\tau(w)$ we call $s(w)$ the symmetrization of w.*

Proof. Note that $s(w) \in T^n_{sym}(\mathfrak{g})$, for $w \in T^n(\mathfrak{g})$. Indeed,

$$P_\tau\left(s(w)\right) \quad = \quad \frac{1}{n!} \sum_{\eta \in S_n} P_{\tau\eta}(w) \quad = \quad \frac{1}{n!} \sum_{\eta' \in S_n} P_{\eta'}(w) \quad = \quad s(w).$$

On the other hand, let $w \in T^n_{sym}(\mathfrak{g})$ be a symmetric tensor, then $w \in T^n(\mathfrak{g})$ and

$$s(w) = \frac{1}{n!} \sum_{\eta \in S_n} P_\eta(w) = \frac{1}{n!} \sum_{\eta \in S_n} w = w.$$

□

The canonical mapping $\psi : T(\mathfrak{g}) \to U(\mathfrak{g})$, given by the projection of $T(\mathfrak{g})$ onto $U(\mathfrak{g})$, induces a map $\psi_n : T^n(\mathfrak{g}) \to U_n(\mathfrak{g})$. Let $U^n(\mathfrak{g}) \subset U(\mathfrak{g})$ be the image under ψ_n of the space of symmetric tensors $T^n_{sym}(\mathfrak{g}) \subset T^n(\mathfrak{g})$. An element of $U^n(\mathfrak{g})$ is said to be **symmetric and homogeneous of degree** n.

Remark 1.0.3. Let $x_1^2 x_2 \in U_3(\mathfrak{g})$ and let $\tau \cdot \left(x_1^2 x_2\right)$ be the element obtained from $x_1^2 x_2$ by permuting the factors by the permutation $\tau = (2,3) \in S_3$. Namely,

$$\begin{aligned}
\tau \cdot \left(x_1^2 x_2\right) &= x_1 \otimes x_2 \otimes x_1 + J \\
&= \left(x_1 + J\right)\left(x_2 \otimes x_1 + J\right) \\
&= \left(x_1 + J\right)\left(x_1 \otimes x_2 + [x_2, x_1] + J\right) \\
&= x_1^2 x_2 + x_1[x_2, x_1], \text{ where } w = x_1[x_2, x_1] \in U_2(\mathfrak{g})
\end{aligned}$$

We recall that $\psi(x) := x + J$.

Proposition 1.0.7. *Let $\mathfrak{g}$ be a Lie algebra and let U_n be defined as above lemma 1.0.2. Then*

$$U_n(\mathfrak{g}) = U_{n-1}(\mathfrak{g}) \bigoplus U^n(\mathfrak{g})$$

Proof. Let $x_1^{n_1} \dots x_k^{n_k}$ be a basis element of $U_n(\mathfrak{g})$ with $n_1 + \cdots + n_k = n$. For each $\tau \in S_n$ we define $\tau \cdot (x_1^{n_1} \dots x_k^{n_k})$ to be the element obtained from $x_1^{n_1} \dots x_k^{n_k}$ by permuting the factors by the permutation τ. Since multiplication in the graded algebra of $U_n(\mathfrak{g})$ is not commutative we have

$$x_1^{n_1} \dots x_k^{n_k} = w + \frac{1}{n!} \sum_{\tau \in S_n} \tau.(x_1^{n_1} \dots x_k^{n_k})$$

where $w \in U_{n-1}(\mathfrak{g})$. Since the sum lies in $U^n(\mathfrak{g})$ we have

$$U_n(\mathfrak{g}) = U_{n-1}(\mathfrak{g}) + U^n(\mathfrak{g})$$

We next show that $U_{n-1}(\mathfrak{g}) \cap U^n(\mathfrak{g}) = 0$. Any element of $U^n(\mathfrak{g})$ has the form

$$\sum_{\substack{n_1,\dots,n_k \\ n_1+\cdots+n_k=n}} \lambda_{n_1,\dots,n_k} \sum_{\tau \in S_n} \tau.(x_1^{n_1} \dots x_k^{n_k}).$$

We express this element as a linear combination of basis elements of $U(\mathfrak{g})$. We obtain

$$\sum_{\substack{n_1,\dots,n_k \\ n_1+\cdots+n_k=n}} \lambda_{n_1,\dots,n_k} \sum_{\tau \in S_n} \tau.(x_1^{n_1} \dots x_k^{n_k}) = u + \left(n! \sum_{\substack{n_1,\dots,n_k \\ n_1+\cdots+n_k=n}} \lambda_{n_1,\dots,n_k} x_1^{n_1} \dots x_k^{n_k} \right)$$

where $u \in U_{n-1}(\mathfrak{g})$, since multiplication in the graded algebra of $U(\mathfrak{g})$ is not commutative. This element can only lie in $U_{n-1}(\mathfrak{g})$ if each $\lambda_{n_1,\dots,n_k}$ is 0. Thus

$$U_{n-1}(\mathfrak{g}) \cap U^n(\mathfrak{g}) = 0.$$

Hence we have

$$U_n(\mathfrak{g}) = U_{n-1}(\mathfrak{g}) \bigoplus U^n(\mathfrak{g}).$$

□

Proposition 1.0.8. *The following diagram of vector space isomorphisms is commutative*

$$\begin{array}{ccc} T^n_{sym}(\mathfrak{g}) & \xrightarrow{\psi_n} & U^n(\mathfrak{g}) \\ \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \theta_n} \\ S^n(\mathfrak{g}) & \xrightarrow[\varphi_n]{} & G^n(\mathfrak{g}) \end{array} \tag{1.0.2}$$

In (1.0.2), the function ψ_n is induced by the projection map $T(\mathfrak{g}) \to U(\mathfrak{g})$; τ_n is induced by the projection $T(\mathfrak{g}) \to S(\mathfrak{g})$; θ_n is induced by $U_n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ and φ_n by the isomorphism $S(\mathfrak{g}) \cong Gr\,(U(\mathfrak{g}))$.

Proof. Let $t = \sum_{\sigma \in S_n} P_\sigma(w) \in T^n_{sym}(\mathfrak{g})$, where $w \in T^n(\mathfrak{g})$. Note that in order to prove the proposition it usffices to consider the case when $w = x_1 \otimes \cdots \otimes x_n$, with $x_k \in \mathfrak{g}$ for $1 \leq k \leq n$, is an homogeneous monomial.
We have that:

$$(\varphi_n \circ \tau_n)(t) = n!\, x_1 \otimes \cdots \otimes x_n + J + U_{n-1}(\mathfrak{g})\ ,$$
$$(\theta_n \circ \psi_n)(t) = \sum_{\sigma \in S_n} P_\sigma(w) + J + U_{n-1}(\mathfrak{g})$$

Now, by Lemma 1.0.2 we have $P_\sigma(w) - x_1 \otimes \cdots \otimes x_n + J \in U_{n-1}(\mathfrak{g})$, and thus

$$\sum_{\sigma \in S_n} (P_\sigma(w) - x_1 \otimes \cdots \otimes x_n) + J + U_{n-1}(\mathfrak{g}) = U_{n-1}(\mathfrak{g}).$$

Hence $(\theta_n \circ \psi_n)(t) = (\varphi_n \circ \tau_n)(t)$. □

Example 1.0.4.

$$\begin{array}{ccc} x_1 \otimes x_2 + x_2 \otimes x_1 & \xrightarrow{\psi_n} & x_1 \otimes x_2 + x_2 \otimes x_1 + J \\ \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \theta_n} \\ 2x_2 \otimes x_1 + I & \xrightarrow[\varphi_n]{} & \frac{x_1 \otimes x_2 + x_2 \otimes x_1 + J + U_1(\mathfrak{g})}{2x_2 \otimes x_1 + J + U_1(\mathfrak{g})} \end{array}$$

Since $x_1 \otimes x_2 + x_2 \otimes x_1 - 2x_2 \otimes x_1 + J = x_1 \otimes x_2 - x_2 \otimes x_1 + J = [x_1, x_2] + J \in U_1(\mathfrak{g})$. Hence $x_1 \otimes x_2 + x_2 \otimes x_1 + J + U_1(\mathfrak{g}) = 2x_2 \otimes x_1 + J + U_1(\mathfrak{g})$

Definition 1.0.12. Let I_n be the map defined by $I_n = \theta_n^{-1} \circ \varphi_n$, and let us extend this map by linearity to give

$$I_{PBW} : S(\mathfrak{g}) \to U(\mathfrak{g}); \quad I_{PBW}(x_1 \otimes \cdots \otimes x_n + I) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} + J.$$

I_{PBW} is called **the symmetrization map**.

Remark 1.0.4. I_{PBW} is an isomorphism of vector spaces. The symmetrization map $I_{PBW} : S(\mathfrak{g}) \to U(\mathfrak{g})$ restricts to an linear isomorphism on invariants, $(S(\mathfrak{g}))^{\mathfrak{g}} \to (U(\mathfrak{g}))^{\mathfrak{g}} = Z(U(\mathfrak{g}))$ (see Definition 2.0.19). Unfortunately, this restricted map is not an algebra homomorphism.

Definition 1.0.13. A **representation** of a Lie algebra $\mathfrak{g}$ on a vector space V is a morphism of Lie algebras

$$\rho : \mathfrak{g} \to End(V), \quad \rho([x, y]) = [\rho(x), \rho(y)].$$

For $x, y \in \mathfrak{g}$, $v \in V$, we denote $\rho(x)(v)$ by $x.v$ and we say that V is a $\mathfrak{g}$-**module**, or that $\mathfrak{g}$ **acts** on V under the condition:

$$[x, y].v = x.(y.v) - y.(x.v).$$

The importance of the enveloping algebra $U(\mathfrak{g})$ is that it has the same representation theory as $\mathfrak{g}$: Indeed, given any Lie algebra homomorphism $\theta : \mathfrak{g} \to End(V)$, it can be extended to an associative algebra homomorphism $\theta' : T(\mathfrak{g}) \to End(V)$ in a natural way: $\theta'(x_1 \otimes \cdots \otimes x_n) = \theta(x_1) \circ \cdots \circ \theta(x_n)$. Let $x, y \in \mathfrak{g}$. Then we have

$$\begin{aligned} \theta'(x \otimes y - y \otimes x - [x, y]) &= \theta(x) \circ \theta(y) - \theta(y) \circ \theta(x) - \theta([x, y]) \\ &= [\theta(x), \theta(y)]_{\circ} - \theta([x, y]) \\ &= 0 \end{aligned}$$

Since θ, is a Lie algebra homomorphism.

Thus all the generators of the 2-sided ideal J of $T(\mathfrak{g})$ lie in the kernel of θ'. This shows there is an induced homomorphism $\phi : U(\mathfrak{g}) \to End(V)$.

Conversely, let ϕ be an associative algebra homomorphism, $\phi : U(\mathfrak{g}) \to End(V)$. We have a linear map $\Gamma : \mathfrak{g} \to U(\mathfrak{g})$ such that $\Gamma([x, y]) = [\Gamma(x), \Gamma(y)]$ for all $x, y \in \mathfrak{g}$, and so Γ is a Lie algebra homomorphism. We now define $\theta : \mathfrak{g} \to End(V)$ by $\theta := \phi \circ \Gamma$. Then θ is a Lie homomorphism of the required type.

Remark 1.0.5. The notation $x_1 \cdots x_n$ for $x_1 \otimes \cdots \otimes x_n + J$ will sometimes apply.

Example 1.0.5. Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra. $\mathfrak{g}$ acts on $S(\mathfrak{g})$ under the action

$$x.(x_1 \cdots x_n) = \sum_{i=1}^{n} x_1 \cdots [x, x_i] \cdots x_n,$$

where $x \in \mathfrak{g}$ and $x_1 \cdots x_n \in S(\mathfrak{g})$. That is, $S(\mathfrak{g})$ is a $\mathfrak{g}$-**module**.

Definition 1.0.14. Let V and W be $\mathfrak{g}$-modules which are isomorphic as vector spaces. We say that $\psi : V \to W$, a linear isomorphism of V onto W, is an **isomorphism of $\mathfrak{g}$-modules** if

$$\psi(x.v) = x.\psi(v)$$

for all $x \in \mathfrak{g}$ and for all $v \in V$. In other words, ψ is an **isomorphism of $\mathfrak{g}$-modules** , if ψ is equivariant with respect to the actions of $\mathfrak{g}$ on V and W, that is, if the following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{\rho_1(x)} & V \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ W & \xrightarrow[\rho_2(x)]{} & W \end{array}$$

Proposition 1.0.9. $I_{PBW} : S(\mathfrak{g}) \to U(\mathfrak{g})$ *is a isomorphism of* $\mathfrak{g}$*-modules.*

Proof. It suffices to show that $x.I_{PBW}(p) = I_{PBW}(x.p)$, for $p = x_1 \cdots x_n \in S(\mathfrak{g})$, for $x_i \in \mathfrak{g}$ and $x \in \mathfrak{g}$. Indeed,

$$x.I_{PBW}(p) = x \cdot I_{PBW}(p) - I_{PBW}(p) \cdot x = \frac{1}{n!} \sum_{\sigma \in S_n} \left(x \cdot x_{\sigma(1)} \cdots x_{\sigma(n)} - x_{\sigma(1)} \cdots x_{\sigma(n)} \cdot x \right).$$

On the other hand,

$$\begin{aligned} I_{PBW}\,(x.p) &= I_{PBW} \left(\sum_{i=1}^{n} x_1 \cdots [x, x_i] \cdots x_n \right) \\ &= \sum_{i=1}^{n} I_{PBW}\,(x_1 \cdots [x, x_i] \cdots x_n) \\ &= \sum_{i=1}^{n} \tfrac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots [x, x_{\sigma(i)}] \cdots x_{\sigma(n)} \\ &= \tfrac{1}{n!} \sum_{\sigma \in S_n} \sum_{i=1}^{n} x_{\sigma(1)} \cdots [x, x_{\sigma(i)}] \cdots x_{\sigma(n)} \\ &= \tfrac{1}{n!} \sum_{\sigma \in S_n} \sum_{i=1}^{n} x_{\sigma(1)} \cdots \left(x \cdot x_{\sigma(i)} - x_{\sigma(i)} \cdot x \right) \cdots x_{\sigma(n)} \\ &= \tfrac{1}{n!} \sum_{\sigma \in S_n} \sum_{i=1}^{n} x_{\sigma(1)} \cdots x \cdot x_{\sigma(i)} \cdots x_{\sigma(n)} - x_{\sigma(1)} \cdots x_{\sigma(i)} \cdot x \cdots x_{\sigma(n)} \\ &= \tfrac{1}{n!} \sum_{\sigma \in S_n} \left(x \cdot x_{\sigma(1)} \cdots x_{\sigma(n)} - x_{\sigma(1)} \cdots x_{\sigma(n)} \cdot x \right). \end{aligned}$$

Thus $x.I_{PBW}(p) = I_{PBW}(x.p)$, and hence $I_{PBW} : S(\mathfrak{g}) \to U(\mathfrak{g})$ is a isomorphism of $\mathfrak{g}$-modules □

Chapter 2

Lie Algebras

Let $S(\mathfrak{g})^{\mathfrak{g}}$ be the subspace of all the elements of the symmetric algebra $S(\mathfrak{g})$, which are annihilated under the representation of $\mathfrak{g}$ on $S(\mathfrak{g})$ induced by the adjoint representation. The vector space $S(\mathfrak{g})^{\mathfrak{g}}$ is a graded subalgebra of $S(\mathfrak{g})$. Similarly, the subspace $U(\mathfrak{g})^{\mathfrak{g}}$ is a graded subalgebra of $U(\mathfrak{g})$. Neither the Poincaré- Birkhoff-Witt map from $S(\mathfrak{g})$ to $U(\mathfrak{g})$, nor its restriction to the subspaces of invariants $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$ **are** algebra homomorphisms. However, the Duflo Theorem states that the vector spaces of invariants, $S(\mathfrak{g})^{\mathfrak{g}}$ and $U(\mathfrak{g})^{\mathfrak{g}}$, **are canonically isomorphic as algebras**. This isomorphism is called the Duflo isomorphism and it happens to be a composition of the Poincaré- Birkhoff-Witt isomorphism (which is only an isomorphism at the level of vector spaces) with an automorphism of the symmetric algebra $S(\mathfrak{g})$ (which descends to subspace of invariants $S(\mathfrak{g})^{\mathfrak{g}}$). The definition of this automorphism involves the formal power series on $\mathfrak{g}$: $J(x) = \det\left(\frac{1-e^{-ad_x}}{ad_x}\right)$, called Duflo element [9], where $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$.

2.0.4 Derivations of Lie Algebras and Representations

Definition 2.0.15. Let $\mathfrak{g}$ be a Lie algebra. A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called a **derivation** of $\mathfrak{g}$ if for all $x, y \in \mathfrak{g}$, we have

$$D\left([x,y]\right) = [D(x), y] + [x, D(y)]$$

The set of derivations of $\mathfrak{g}$ is a Lie subalgebra of $End(\mathfrak{g})$.

Definition 2.0.16. Let $\mathfrak{g}$ be a Lie algebra. For $x \in \mathfrak{g}$, the linear map of $\mathfrak{g}$ into $\mathfrak{g}$ defined by

$$\begin{array}{rccc} ad(x): & \mathfrak{g} & \to & \mathfrak{g} \\ & y & \to & [x,y] \end{array}$$

is a derivation of g, called an **inner derivation** of $\mathfrak{g}$.

Definition 2.0.17. If $(\mathfrak{g}, [\cdot,\cdot])$ and $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a **morphism of Lie algebras** if $\phi([x,y]) = [\phi(x), \phi(y)]_{\mathfrak{h}}$, for all $x, y \in \mathfrak{g}$.

Proposition 2.0.10. *Let $\mathfrak{g}$ be a Lie algebra, and consider the Lie algebra $(\, End(\mathfrak{g}), [\ ,\ ]_{\circ}\,)$. We have that,*

$$ad([x,y]) = ad(x) \circ ad(y) - ad(y) \circ ad(x) = [ad(x), ad(y)]_{\circ};$$

That is, $ad : \mathfrak{g} \to End(\mathfrak{g})$, $x \to ad(x)$ is a morphism of Lie algebras.

Proof. Observe that $ad([x,y])(z) = [[x,y],z]$, whereas the Jacobi identity

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0,$$

implies that

$$ad([x,y])(z) = [x,[y,z]] - [y,[x,z]] = ad(x) \circ ad(y)(z) - ad(y) \circ ad(x)(z).$$

□

Example 2.0.6. By Proposition 2.0.10 the mapping $ad : \mathfrak{g} \to End(\mathfrak{g})$, $x \to ad(x)$ is a representation of $\mathfrak{g}$ in $\mathfrak{g}$ called the **adjoint representation**.

Definition 2.0.18. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to End(V)$ a representation of $\mathfrak{g}$. A vector subspace J of V is said to be **stable under** ρ if $\rho(x)(J) \subseteq J$, for all $x \in \mathfrak{g}$. In this case, the mapping $x \mapsto \rho(x)|_J$ is a representation of $\mathfrak{g}$ in J, called a **subrepresentation of** $\mathfrak{g}$, and the $\mathfrak{g}$-module J is called a **sub**-$\mathfrak{g}$-**module**. Moreover, the quotient V/J is a $\mathfrak{g}$-module, called **quotient**-$\mathfrak{g}$-**module**.

The adjoint representation of $\mathfrak{g}$ extends naturally to a representation on $T(\mathfrak{g})$ as follows:

$$x \cdot (x_1 \otimes \cdots \otimes x_n + I) = \sum_{j=1}^{n} x_1 \otimes \cdots \otimes x_{j-1} \otimes [x, x_j] \otimes x_{j+1} \otimes \cdots \otimes x_n + I.$$

Remark 2.0.6. The above definition is valid if we considered the degree of the elements of $\mathfrak{g}$ equal to zero, otherwise you have to apply the convention the Koszul sign.

Now, we note that the ideal I of $T(\mathfrak{g})$ (see definition 1.0.6) is also a $\mathfrak{g}$-module (sub-$\mathfrak{g}$-module of $T(\mathfrak{g})$) since, for $x_1, x_2 \in \mathfrak{g}$, we have

$$\begin{aligned} x \cdot (x_1 \otimes x_2 - x_2 \otimes x_1) &= [x, x_1] \otimes x_2 + x_1 \otimes [x, x_2] - [x, x_2] \otimes x_1 - x_2 \otimes [x, x_1] \\ &= ([x, x_1] \otimes x_2 - x_2 \otimes [x, x_1]) + (x_1 \otimes [x, x_2] - [x, x_2] \otimes x_1) \in I, \end{aligned}$$

and so, $S(\mathfrak{g})$ has also a natural structure of $\mathfrak{g}$-module .

Example 2.0.7. Let $\mathfrak{g}$ be a Lie algebra. The mapping $\mathfrak{g} \to End(S(\mathfrak{g}))$ defined by

$$x \cdot (u + I) := \sum_{i=1}^{n} u_1 \otimes \cdots \otimes [x, u_i] \otimes \cdots \otimes u_n + I,$$

for $x, u_i \in \mathfrak{g}$ and $u + I = u_1 \otimes \cdots \otimes u_n + I \in S^n(\mathfrak{g})$, is a representation of $\mathfrak{g}$ in $S(\mathfrak{g})$ induced by the adjoint representation

$$ad : \mathfrak{g} \to End(\mathfrak{g}), \qquad x \to ad(x).$$

Definition 2.0.19. Let V be a $\mathfrak{g}$-module. An element v of V is an **invariant of the** $\mathfrak{g}$-**module** V, or an **invariant of the representation of** $\mathfrak{g}$ **in** V, if

$$\rho(x)(v) = x \cdot v = 0, \text{ for all } x \in \mathfrak{g}.$$

We denote the set of invariants of V by $V^{\mathfrak{g}}$.

2.0.5 Universal Property of $S(V)$

We want to see that for any vector space V the commutative algebra $S(V)$ is free over V.

Theorem 2.0.11. *(**Universal Property of** $S(V)$). Let A be an associative algebra, not necessarily commutative, with unit element 1_A, and let φ be a linear mapping from the vector space V into A,*

$$\varphi : V \to A$$

such that

$$\varphi(x) \cdot \varphi(y) - \varphi(y) \cdot \varphi(x) = 0 \ \text{ for all } x, y \in V,$$

then φ is uniquely extended to an algebra homomorphism,

$$\theta : S(V) \to A$$

such that $\theta(1) = 1_A$ and $\theta \circ i = \varphi$, where i is the inclusion mapping of V into $S(V)$.

Proof. For every $n \geq 2$, define a n-multilinear mapping $\varphi_n : V \times \cdots \times V \to A$

$$(x_1, \ldots, x_n) \to \varphi(x_1) \cdot \ldots \cdot \varphi(x_n),$$

where $\cdot$ is the product of A.
By the Universal Property of the Tensor Product [7], φ_n induces a linear mapping

$$\begin{array}{llll} \phi_n : & T^n(V) & \to & A \\ & x_1 \otimes \cdots \otimes x_n & \to & \varphi(x_1) \cdot \ldots \cdot \varphi(x_n) \end{array}$$

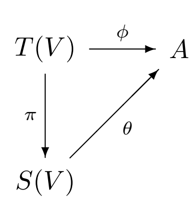
Defining $\phi_1 = \varphi$ and $\phi_0(\alpha) = \alpha 1_A$ ($\alpha \in \mathbb{K}$), we extend by linearity and get an associative algebra homomorphism ϕ from $T(V)$ to A.
By construction, ϕ is an associative algebra homomorphism. Now, let $x, y \in V$, then

$$\phi(x \otimes y - y \otimes x) = \phi(x).\phi(y) - \phi(y).\phi(x) = 0$$

Since

$$\varphi(x) \cdot \varphi(y) - \varphi(y) \cdot \varphi(x) = 0 \ \text{ for all } x, y \in V,$$

all the generators of the two-side ideal I of $T(V)$ lie in the kernel of ϕ, and so I lies in the kernel of ϕ, and therefore induces an algebra homomorphism θ of $S(V)$ into A such that $\theta \circ \pi = \phi$



□

Now we will consider dual representations. We denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ equipped with its standard vector space structure. Henceforth, $v = f_1 \cdots f_n$ in $S(\mathfrak{g}^*)$ denotes the element $f_1 \otimes \cdots \otimes f_n + I$.

Example 2.0.8. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$ and let φ be the linear map

$$\begin{array}{rccl} \varphi: & \mathfrak{g}^* & \to & End(S(\mathfrak{g})) \\ & f & \to & \varphi(f) \end{array}$$

defined by

$$\begin{array}{rlcl} \varphi(f): & S(\mathfrak{g}) & \to & S(\mathfrak{g}) \\ & x_1 \otimes \cdots \otimes x_n + I & \mapsto & \sum_{i=1}^{n} f(x_i) x_1 \otimes \cdots \otimes \widehat{x_i} \otimes \cdots \otimes x_n + I \end{array}$$

where $\varphi(f)(1) = 0$, and the sign $\widehat{\ }$ above a letter indicates that it should be omitted.
We have

$$\varphi(f) \circ \varphi(g) - \varphi(g) \circ \varphi(f) = 0,$$

Indeed,

$$\begin{aligned} \varphi(f) \circ \varphi(g)(x_1 \otimes \cdots \otimes x_n + I) &= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} g(x_i) f(x_j) x_1 \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes \widehat{x_i} \otimes \cdots \otimes x_n + I \\ &= \varphi(g) \circ \varphi(f)(x_1 \otimes \cdots \otimes x_n + I) \end{aligned}$$

Hence the map φ extends, by the the universal property (Theorem 2.0.11) to an associative algebra homomorphism θ defined by:

$$\begin{array}{rlcl} \theta: & S(\mathfrak{g}^*) & \to & End(S(\mathfrak{g})) \\ & f_1 \otimes \cdots \otimes f_k + I & \mapsto & \varphi(f_1) \circ \cdots \circ \varphi(f_k) \,. \end{array}$$

Moreover, $\varphi(f)$ is a derivation of $S(\mathfrak{g})$ and it sends $S^n(\mathfrak{g})$ to $S^{n-1}(\mathfrak{g})$. If $v \in S^k(\mathfrak{g}^*)$, then $\theta(v)$ is a **differential operator de order** k, that is, it annihilates every element of $S(\mathfrak{g})$ of degree $< k$, and it sends $S^n(\mathfrak{g})$ to $S^{n-k}(\mathfrak{g})$.

Theorem 2.0.12. *If $\mathfrak{g}^*$ is a finite dimension $\mathbb{K}$-vector space of dimension n, then $S(\mathfrak{g}^*)$ is isomorphic to the polynomial algebra in n variables over $\mathbb{K}$, $\mathbb{K}[x_1, \ldots, x_n]$.*

Proof. Take a basis $\xi_1^*, \ldots, \xi_n^*$ of $\mathfrak{g}^*$. Let φ be the linear mapping of $\mathfrak{g}^*$ into $\mathbb{K}[x_1, \ldots, x_n]$ which assigns x_i to ξ_i^*, this is,

$$\begin{array}{rccl} \varphi: & \mathfrak{g}^* & \to & \mathbb{K}[x_1, \ldots, x_n] \\ & \xi_i^* & \mapsto & x_i \,. \end{array}$$

By Theorem 2.0.11, φ induces an associative algebra homomorphism ϕ from $S(\mathfrak{g}^*)$ into $\mathbb{K}[x_1, \ldots, x_n]$, given by

$$\begin{array}{llll} \phi : & S(\mathfrak{g}^*) & \to & \mathbb{K}[x_1, \dots, x_n] \\ & \xi_{i_1}^* \cdots \xi_{i_k}^* & \mapsto & x_{i_1} \cdots x_{i_k} . \end{array}$$

For any polynomial $P(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$, by construction of ϕ, we get that

$$\phi(P(\xi_1^*, \dots, \xi_n^*)) = P(x_1, \dots, x_n),$$

which implies that ϕ is onto, and that $P(\xi_1^*, \dots, \xi_n^*) \neq 0$,for $P(x_1, \dots, x_n) \neq 0$.
All elements of $S(\mathfrak{g}^*)$ can be expressed in the form $P(\xi_1^*, \dots, \xi_n^*)$ for certain $P(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$. Thus we have $Ker\theta = \{0\}$, and ϕ is an isomorphism. □

Definition 2.0.20. Let $\rho : \mathfrak{g} \to End(V)$ be a representation of the Lie algebra $\mathfrak{g}$ and let V^* be the dual space of V. The map $\rho' : \mathfrak{g} \to End(V^*)$ defined by

$$x \cdot f := -f \circ \rho(x), \text{ for all } x \in \mathfrak{g} \text{ and } f \in V^*,$$

is a morphism of Lie algebras. That is, V^* is a $\mathfrak{g}$**-module** called **dual** $\mathfrak{g}$**-module** and ρ' is **the dual representation of** ρ.

Example 2.0.9. Let $\mathfrak{g}$ be a Lie algebra. The mapping $\mathfrak{g} \to End(S(\mathfrak{g}^*))$ defined by

$$x \cdot v = \sum_{i=1}^{n} f_1 \otimes \cdots (-f_i \circ ad(x)) \otimes \cdots \otimes f_n + I,$$

for $x \in \mathfrak{g}$ and $v = f_1 \otimes \cdots \otimes f_n + I \in S^n(\mathfrak{g}^*)$ with $f_i \in \mathfrak{g}^*$, for $1 \leq i \leq n$, is a representation of $\mathfrak{g}$ in $S(\mathfrak{g}^*)$ induced by the coadjoint representation:

$$ad' : \mathfrak{g} \to End(\mathfrak{g}^*), \quad x \mapsto ad'(x),$$

where

$$\begin{array}{llll} ad'(x) : & \mathfrak{g}^* & \to & \mathfrak{g}^* \\ & f & \mapsto & (-f \circ ad(x)) . \end{array}$$

Lemma 2.0.13. *Let $\mathfrak{g}$ be a Lie algebra. The algebra homomorphism θ defined in Example 2.0.8*

$$\theta : S(\mathfrak{g}^*) \to End(S(\mathfrak{g})) \quad f_1 \otimes \cdots \otimes f_k + I^* \longmapsto \varphi(f_1) \circ \cdots \circ \varphi(f_k)$$

satisfies the equality:

$$x \cdot (\theta(v)(u_1 \otimes \cdots \otimes u_n + I)) = \theta(x \cdot v)(u_1 \otimes \cdots \otimes u_n + I) + \theta(v)(x \cdot (u_1 \otimes \cdots \otimes u_n + I)),$$

for all elements $x \in \mathfrak{g}$ and $u_1 \otimes \cdots \otimes u_n + I \in S(\mathfrak{g})$, where $x \cdot v$ is the coadjoint representation:

$$x \cdot v = x \cdot (f_1 \otimes \cdots \otimes f_k + I^*) = \sum_{i=1}^{k} f_1 \otimes \cdots \otimes (-f_i \circ ad(x)) \otimes \cdots \otimes f_k + I^*,$$

and

$$x \cdot (u_1 \cdots u_n) = \sum_{i=1}^{n} u_1 \otimes \cdots \otimes [x, u_i] \otimes \cdots \otimes u_n + I$$

is the adjoint representation.

Proof. It suffices to prove the lemma when v is of the form $f^k := f \otimes \cdots \otimes f$, with $f \in \mathfrak{g}^*$ and $k \in \mathbb{N}$. For $k = 1$, and $u^n := u \otimes \cdots \otimes u + I$ where $u \in \mathfrak{g}$ and $n \in \mathbb{N}$, we have that:

$$x \cdot (\theta(f)(u^n)) = x \cdot \left(nf(u)u^{n-1}\right) = n(n-1)f(u)[x,u] \cdot u^{n-2},$$

On the other hand,

$$\begin{aligned}
\theta(x \cdot f)(u^n) + \theta(f)(x \cdot u^n) = \;& \theta(-f \circ ad(x))(u^n) + \theta(f)\left(n[x,u] \cdot u^{n-1}\right) \\
= \;& n(-f \circ ad(x))(u)\left(u^{n-1}\right) + \\
& nf([x,u])u^{n-1} + n(n-1)f(u)[x,u] \cdot u^{n-2} \\
= \;& n(n-1)f(u)[x,u] \cdot u^{n-1}.
\end{aligned}$$

Thus,

$$x \cdot (\theta(f)(u^n)) = \theta(x \cdot f)(u^n) + \theta(f)(x \cdot u^n).$$

We now argue by induction on k. It is assumed that $k > 1$ and the formula is true for all v of the form f^j with $j < k$. Let $q = f^{k-1}$, then

$$\begin{aligned}
x \cdot \left(\theta\left(f^k\right)(u)\right) = \;& x \cdot (\theta(f)(\theta(q)(u))) \\
= \;& \theta(x \cdot f)(\theta(q)(u)) + \theta(f)(x \cdot (\theta(q)(u))) \\
= \;& \theta(x \cdot f)(\theta(q)(u)) + \theta(f)(\theta(x \cdot q)(u) + \theta(q)(x \cdot u)) \\
= \;& \theta(x \cdot f)(\theta(q)(u)) + \theta(f)(\theta(x \cdot q)(u)) + \theta(f)(\theta(q)(x \cdot u)) \\
= \;& \theta((x \cdot f) \cdot q)(u) + \theta(f \cdot (x \cdot q))(u) + \theta(f) \circ \theta(q)(x \cdot u) \\
= \;& \theta((x \cdot f) \cdot q + f \cdot (x \cdot q))(u) + \theta(f \cdot q)(x \cdot u) \\
= \;& \theta(x \cdot (f \cdot q))(u) + \theta\left(f^k\right)(x \cdot u) \\
= \;& \theta\left(x \cdot f^k\right)(u) + \theta\left(f^k\right)(x \cdot u),
\end{aligned}$$

which completes the proof. □

Corollary 2.0.14. *The restriction of θ to the sets of invariants of $S(\mathfrak{g}^*)$ and $S(\mathfrak{g})$,*

$$\begin{array}{rccc}
\theta : & S(\mathfrak{g}^*)^{\mathfrak{g}} & \to & End(S(\mathfrak{g})^{\mathfrak{g}}) \\
& v & \mapsto & \theta(v)
\end{array}$$

is well defined. That is, $S(\mathfrak{g}^*)^{\mathfrak{g}}$ acts on $S(\mathfrak{g})^{\mathfrak{g}}$.

Proof. Let $v \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ and $u \in S(\mathfrak{g})^{\mathfrak{g}}$, then

$$x.\big(\theta\,(v)\,(u)\big) = \theta\,(x.v)\,(u) + \theta\,(v)\,(x.u) = 0$$

for all $x \in \mathfrak{g}$. Thus, the element $\theta\,(v)$ belongs to $End(S(\mathfrak{g})^{\mathfrak{g}})$. □

Remark 2.0.7. Let Γ be the canonical mapping, $x \to x + J$ of $\mathfrak{g}$ into $U(\mathfrak{g})$ considered in section 1.0.2. For all values of $x, y \in \mathfrak{g}$ we have

$$\begin{aligned}
\Gamma([x,y]) &= [x,y] + J \\
&= (x \otimes y - y \otimes x) + J \\
&= (x+J)(y+J) - (y+J)(x+J) \\
&= \Gamma(x)\Gamma(y) - \Gamma(y)\Gamma(x) \\
&= [\Gamma(x), \Gamma(y)]_U.
\end{aligned}$$

So, $\Gamma : \mathfrak{g} \to U(\mathfrak{g})$ is a morphism of Lie algebras.

The ideal J of $T(\mathfrak{g})$ is also a $\mathfrak{g}$-submodule since, for $a, b \in \mathfrak{g}$, we have that

$$x \cdot (a \otimes b - b \otimes a - [a,b]) = [x,a] \otimes b + a \otimes [x,b] - [x,b] \otimes a - b \otimes [x,a] - [x,[a,b]] \in J\,,$$

and

$$[x,[a,b]] = [[x,a],b] + [a,[x,b]]\,;$$

thus, $U(\mathfrak{g})$ can be given the structure of a $\mathfrak{g}$-module (quotient-$\mathfrak{g}$-module).

Definition 2.0.21. There is a $\mathfrak{g}$-action on $U(\mathfrak{g})$, $\mathfrak{g} \to End(U(\mathfrak{g}))$, defined by:

$$x \cdot u := \sum_{i=1}^{n} u_1 \otimes \cdots \otimes [x, u_i] \otimes \cdots \otimes u_n + J := x{\bullet}u - u{\bullet}x,$$

for any $x \in \mathfrak{g}$ and $u = u_1 \otimes \cdots \otimes u_n + J \in U(\mathfrak{g})$, with $u_i \in \mathfrak{g}$ for $1 \le i \le n$. The action is obtained from the adjoint action:

$$ad : \mathfrak{g} \to End(\mathfrak{g}), \qquad x \to ad(x)\,.$$

Remark 2.0.8. Note that

$$\begin{aligned}
U(\mathfrak{g})^{\mathfrak{g}} &= \{u \in U(\mathfrak{g}) : x.u = 0 \text{ for all } x \in \mathfrak{g}\} \\
&= \{u \in U(\mathfrak{g}) : x{\bullet}u - u{\bullet}x = 0 \text{ for all } x \in \mathfrak{g}\} \\
&= \{u \in U(\mathfrak{g}) : x{\bullet}u = u{\bullet}x \text{ for all } x \in \mathfrak{g}\}
\end{aligned}$$

An element u lie in the center of $U(\mathfrak{g})$, denoted $Z(U(\mathfrak{g}))$, if, and only if, it commutes with all generators $x \in \mathfrak{g}$ subject to relation $x \cdot y - y \cdot x = [x,y]$. Elements of the center are also called Casimir elements.

Therefore, $U(\mathfrak{g})^{\mathfrak{g}} = Z(U(\mathfrak{g}))$.

2.0.6 Formal Power Series and the Exponential Series

Definition 2.0.22. Let $A = \bigoplus_{j\geq 0} A^j$ be an associative graded algebra with unit element. Then

$$\hat{A} := \left\{(a_0, a_1, \ldots) : a_j \in A^j \text{ for } j = 0, 1, \ldots\right\},$$

is the set of **formal power series on** A. On $\hat{A}$ we consider a product defined by

$$(a_0, a_1, \ldots)\hat{\cdot}\big(b_0, b_1, \ldots\big) = \Big(a_0{\cdot}b_0,\;\; a_1{\cdot}b_0 + a_0{\cdot}b_1,\;\; \ldots,\;\; \sum_{k=0}^{j} a_{j-k}{\cdot}b_k\;\; ,\ldots\Big)$$

The space $\hat{A}$ with the product defined above, is a algebra called the algebra of **formal power series on** A. The notation $\sum_{j\geq 0} a_j$ for $(a_j)_{j\geq 0}$ will sometimes apply.

Lemma 2.0.15. *The algebra $\widehat{S(\mathfrak{g}^*)}$ of formal power series of $S(\mathfrak{g}^*)$ is a $\mathfrak{g}$-module with the action given by:*

$$x \cdot (a_j)_{j\geq 0} := (x \cdot a_j)_{j\geq 0}\,, \text{ for all } x \in \mathfrak{g}\,,$$

where

$$x \cdot a_j := \sum_{k=1}^{n} u_1 \otimes \cdots \otimes (-u_k \circ ad(x)) \otimes \cdots \otimes u_n + I$$

is the action of the representation

$$\mathfrak{g} \to End\,(S(\mathfrak{g}^*))\,.$$

Proof. Let $x, y \in \mathfrak{g}$ and $(a_j)_{j\geq 0} \in \widehat{S(\mathfrak{g}^*)}$. We have that:

$$\begin{aligned}
x.\cdot(y\cdot(a_j)_{j\geq 0}) - y\cdot(x\cdot(a_j)_{j\geq 0}) &= \big(x\cdot\big(y\cdot a_j\big)\big)_{j\geq 0} - \big(y\cdot(x\cdot a_j)\big)_{j\geq 0}\\
&= \Big(x\cdot(y\cdot a_j) - y\cdot(x\cdot a_j)\Big)_{j\geq 0}\\
&= \big([x,y]\cdot a_j\big)_{j\geq 0}\\
&= [x,y]\cdot\big(a_j\big)_{j\geq 0}\,.
\end{aligned}$$

□

Remark 2.0.9*.* We have $(a_j)_{j\geq 0} \in \widehat{S(\mathfrak{g}^*)^{\mathfrak{g}}}$ if, and only if, $a_j \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ for every $j \geq 0$.

Lemma 2.0.16. *The algebra homomorphism*

$$\begin{aligned}
\theta :\;\; & S(\mathfrak{g}^*) && \to\;\; End(S(\mathfrak{g}))\\
& f_1 \otimes \cdots \otimes f_k + I && \mapsto\;\; \varphi(f_1) \circ \cdots \circ \varphi(f_k)
\end{aligned}$$

extends to an algebra homomorphism

$$\begin{aligned}
\widehat{\theta} :\;\; & \widehat{S(\mathfrak{g}^*)} && \to\;\; End(S(\mathfrak{g}))\\
& (a_j)_j && \mapsto\;\; \widehat{\theta}((a_j)_j)
\end{aligned}$$

given by

$$\begin{array}{rcl} \widehat{\theta}((a_j)_{j\geq 0}): \quad S(\mathfrak{g}) & \to & S(\mathfrak{g}) \\ x_1\otimes\cdots\otimes x_n + I & \mapsto & \sum_{j=0}^{n}\theta(a_j)(x_1\otimes\cdots\otimes x_n)+I\,. \end{array}$$

Proof. Let $x^n \in S^n(\mathfrak{g})$, $(a_j)_{j\geq 0}$ and $(b_j)_{j\geq 0}$ in $\widehat{S(\mathfrak{g}^*)}$. We get that:

$$\begin{aligned} \widehat{\theta}\Big((a_j)_j\cdot(b_j)_j\Big)(x^n) &= \widehat{\theta}\Big(\big(\sum_{k=0}^{j} a_{j-k}\cdot b_k\big)_j\Big)(x^n) \\ &= \sum_{j=0}^{n}\theta\Big(\sum_{k=0}^{j} a_{j-k}\cdot b_k\Big)(x^n) \\ &= \sum_{j=0}^{n}\Big(\sum_{k=0}^{j}\theta(a_{j-k})\circ\theta(b_k)(x^n)\Big)\,. \end{aligned}$$

On the other hand, we have that:

$$\begin{aligned} \widehat{\theta}\Big((a_j)_j\Big)\circ\widehat{\theta}\Big((b_j)_j\Big)(x^n) &= \widehat{\theta}\Big((a_j)_j\Big)\big(\sum_{k=0}^{n}\theta(b_k)(x^n)\big) \\ &= \sum_{j=0}^{n-k}\theta(a_j)\Big(\sum_{k=0}^{n}\theta(b_k)(x^n)\Big) \\ &= \sum_{j=0}^{n-k}\Big(\sum_{k=0}^{n}\theta(a_j)\circ\theta(b_k)(x^n)\Big) \\ &= \sum_{k=0}^{n}\left(\sum_{j=0}^{n-k}\theta(a_j)\circ\theta(b_k)(x^n)\right) \\ &= \sum_{k=0}^{n}\left(\sum_{j=k}^{n}\theta(a_{j-k})\circ\theta(b_k)(x^n)\right), \end{aligned}$$

and thus we get

$$\widehat{\theta}\Big((a_j)_j\cdot(b_j)_j\Big)(x^n)=\widehat{\theta}\Big((a_j)_j\Big)\circ\widehat{\theta}\Big((b_j)_j\Big)(x^n)\,.$$

Therefore, $\widehat{\theta}$ is an algebra homomorphism.

□

Corollary 2.0.17. *The restriction of $\widehat{\theta}$ to the set of invariants of $\widehat{S(\mathfrak{g}^*)}$,*

$$\begin{array}{rccl} \widehat{\theta}: & \widehat{S(\mathfrak{g}^*)}^{\mathfrak{g}} & \to & End(S(\mathfrak{g})^{\mathfrak{g}}) \\ & (a_j)_j & \mapsto & \widehat{\theta}((a_j)_j) \end{array}$$

is well defined. That is, $\widehat{S(\mathfrak{g}^*)}^{\mathfrak{g}}$ acts on $S(\mathfrak{g})^{\mathfrak{g}}$.

Proof. Let $u \in S(\mathfrak{g})^{\mathfrak{g}}$, we have that:

$$\begin{aligned} x\cdot\big(\widehat{\theta}\,((a_j)_j)\,(u)\big) &= x.\big(\sum_{j=0}^{\infty}\theta\,(a_j)\,(u)\big) \\ &= \sum_{j=0}^{\infty}\theta\,(x.a_j)\,(u)+\theta\,(a_j)\,(x.u) = \quad 0, \end{aligned}$$

for all $x \in \mathfrak{g}$. Therefore, we get that $\widehat{\theta}\,((a_j)_j)\,(u) \in S(\mathfrak{g}^*)^{\mathfrak{g}}$. □

Definition 2.0.23. Let $A = \bigoplus_{j\geq 0} A^j$ be an associative graded algebra with unit element $1_{\hat{A}}$, satisfying that $A^0 = \mathbb{K}$. Define $A_0 = \Big\{(a_n)_n \in \hat{A} : a_0 = 0, a_n \in A^n,$ for all $n \in \mathbb{N}\Big\}$. We set

$$Exp : A_0 \to 1_{\hat{A}} + A_0, \quad Exp(a) := \sum_{k=0}^{\infty} \tfrac{1}{k!} a^k,$$
$$Log : 1_{\hat{A}} + A_0 \to A_0, \quad Log(1_{\hat{A}} + a) := \sum_{k=1}^{\infty} \tfrac{(-1)^{k+1}}{k} a^k,$$

where $a^0 := 1_{\hat{A}} = (1, 0, 0, \ldots)$ and $a^k := \underbrace{a \hat{\cdot} \cdots \hat{\cdot} a}_{k\text{-times}}$. The set

$$1_{\hat{A}} + A_0 := \Big\{(1, a_1, a_2, \ldots) \in \hat{A} : 1 \in \mathbb{K} \text{ and } a_j \in A^j, \text{ for every } j \geq 1\Big\}$$

is a subgroup of $\hat{A}$ called the **Magnus group**.

Remark 2.0.10. With all the above notation, if $a \in A_0$, then we have $a^n = (\underbrace{0, \ldots, 0}_{n-times}, u_n, u_{n+1}, \ldots)$ for every $n \in \mathbb{N}$.

Example 2.0.10. Applying the explicit definition of Exp for every element

$$u \quad = \ (0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6, \ldots) \in A_0,$$

we have that:

$$Exp(u) = \Big(1,\ a_1,\ a_2 + \frac{1}{2!}b_2,\ a_3 + \frac{1}{2!}b_3 + \frac{1}{3!}c_3,\ a_4 + \frac{1}{2!}b_4 + \frac{1}{3!}c_4 + \frac{1}{4!}d_4,\ \ldots\Big),$$

where

$$\tfrac{1}{2!}u^2 = \tfrac{1}{2!}(0,\ 0,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6, \ldots) \quad b_{j_2} = \sum_{j_1=1}^{j_2-1} a_{j_2-j_1} * a_{j_1}$$

$$\tfrac{1}{3!}u^3 = \tfrac{1}{3!}(0,\ 0,\ 0,\ c_3,\ c_4,\ c_5,\ c_6, \ldots) \quad c_{j_3} = \sum_{j_2=2}^{j_3-1} a_{j_3-j_2} * b_{j_2}$$

$$\vdots$$

Let us give the explicit description of some terms in the following sums,

$$b_{j_2} = \sum_{j_1=1}^{j_2-1} a_{j_2-j_1} * a_{j_1} \text{ for } j_2 \geq 2, \quad c_{j_3} = \sum_{j_2=2}^{j_3-1} a_{j_3-j_2} * b_{j_2} \text{ for } j_3 \geq 3 \text{ and } d_{j_4} = \sum_{j_3=3}^{j_4-1} a_{j_4-j_3} * c_{j_3}, \text{ for } j_4 \geq 4\,.$$

$$b_2 = a_1 * a_1$$
$$b_3 = a_2 * a_1 + a_1 * a_2$$
$$b_4 = a_3 * a_1 + a_2 * a_2 + a_1 * a_3$$
$$b_5 = a_4 * a_1 + a_3 * a_2 + a_2 * a_3 + a_1 * a_4$$
$$\vdots$$

$c_3 = a_1 * b_2$
$c_4 = a_2 * b_2 + a_1 * b_3$
$c_5 = a_3 * b_2 + a_2 * b_3 + a_1 * b_4$
$c_6 = a_4 * b_2 + a_3 * b_3 + a_2 * b_4 + a_1 * b_5$
$\vdots$
$d_4 = a_1 * c_3$
$d_5 = a_2 * c_3 + a_1 * c_4$
$d_6 = a_3 * c_3 + a_2 * c_4 + a_1 * c_5$
$d_7 = a_4 * c_3 + a_3 * c_4 + a_2 * c_5 + a_1 * c_6$
$\vdots$

2.0.7 The Duflo Isomorphism

Definition 2.0.24. Let $\mathfrak{g}$ be a finite-dimensional vector space over a field $\mathbb{K}$. A map

$$f : \mathfrak{g} \to \mathbb{K}$$

is called a **polynomial function on** $\mathfrak{g}$ if for any basis $\xi_1, \ldots, \xi_m$ of $\mathfrak{g}$, there exists some polynomial

$$p \in \mathbb{K}[x_1, \ldots, x_m]$$

such that

$$f(x) = p(\alpha_1, \ldots, \alpha_m)$$

for all $x = \alpha_1\xi_1 + \cdots + \alpha_m\xi_m \in \mathfrak{g}$. We denote by $\mathbb{K}[\mathfrak{g}]$ the algebra of polynomial functions on $\mathfrak{g}$.

Example 2.0.11. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $x \in \mathfrak{g}$ be the element $x = \alpha_1\xi_1 + \cdots + \alpha_m\xi_m,\ \alpha_i \in \mathbb{K}$. We have that:

$$(ad(x))^2 (y) = (ad(x) \circ ad(x)) (y) = [x, [x, y]].$$

$$\begin{aligned} (ad(x))^2 (y) = & \left[\alpha_1\xi_1 + \cdots + \alpha_m\xi_m, [\alpha_1\xi_1 + \cdots + \alpha_m\xi_m, y]\right] \\ = & \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i\alpha_j[\xi_i, [\xi_j, y]] \end{aligned}$$

That is, $(ad(x))^2$ is the function

$$(ad(x))^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i\alpha_j \left(ad(\xi_i) \circ ad(\xi_j)\right) \text{ from } \mathfrak{g} \text{ to } \mathfrak{g}.$$

The function $c_2 : \mathfrak{g} \to \mathbb{K}$ given by

$$c_2(x) := tr\left((ad(x))^2\right) = \sum_{i=1}^{m}\sum_{j=1}^{m} tr\left(ad(\xi_i) \circ ad(\xi_j)\right) \alpha_i\alpha_j,$$

for all $x \in \mathfrak{g}$, is an homogeneous polynomial function of degree 2.

Lemma 2.0.18. *Define the map $h : S(\mathfrak{g}^*) \to \mathbb{K}[\mathfrak{g}]$ as the unique linear homomorphism such that*

$$h(f_1 \otimes \cdots \otimes f_n + I)(x) := f_1(x) \cdots f_n(x),$$

for $x \in \mathfrak{g}$ and $f_1, \ldots, f_n \in \mathfrak{g}^*$. The linear map h is an algebra isomorphism

$$S(\mathfrak{g}^*) \cong \mathbb{K}[\mathfrak{g}].$$

Proof. Let $\xi_1, \ldots, \xi_m$ be a basis of the vector space $\mathfrak{g}$. For an element $x \in \mathfrak{g}$ such that $x = \alpha_1\xi_1 + \cdots + \alpha_m\xi_m$, $\alpha_i \in \mathbb{K}$, we have that:

$$\begin{aligned}
f_1(x)\ldots f_n(x) \quad &= \quad f_1\left(\sum_{i_1}^{m} \alpha_{i_1}\xi_{i_1}\right) f_2(x)\ldots f_n(x) \\
&= \quad \left[\sum_{i_1=1}^{m} \alpha_{i_1} f_1(\xi_{i_1})\right] f_2(x)\ldots f_n(x) \\
&= \quad \left[\sum_{i_1=1}^{m} \alpha_{i_1} f_1(\xi_{i_1}) f_2(x)\right] f_3(x)\ldots f_n(x) \\
&= \quad \left[\sum_{i_1=1}^{m}\sum_{i_2=1}^{m} \alpha_{i_1}\alpha_{i_2} f_1(\xi_{i_1}) f_2(\xi_{i_2})\right] f_3(x)\ldots f_n(x) \\
&\ldots \\
&= \quad \sum_{i_1,\ldots,i_n=1}^{m} f_1(\xi_{i_1})\ldots f_n(\xi_{i_n})\alpha_{i_1}\ldots\alpha_{i_n} = p(\alpha_1,\ldots,\alpha_m)\,,
\end{aligned}$$

where $p \in \mathbb{K}[x_1, \ldots, x_m]$ is homogeneous of degree n.

□

Proposition 2.0.19. *The linear map $\mu : T^n(\mathfrak{g}^*) \to (T^n(\mathfrak{g}))^*$ determined by*

$$\mu(f_1 \otimes \cdots \otimes f_n)(x_1 \otimes \cdots \otimes x_n) := f_1(x_1)f_2(x_2)\ldots f_n(x_n)$$

is an isomorphism of $\mathfrak{g}$-modules. Here $x_1, \ldots, x_n$ lie in $\mathfrak{g}$ and $f_1, \ldots, f_n$ belong to $\mathfrak{g}^*$.

Proof. The linear map μ is injective: $\mu(f_1 \otimes \cdots \otimes f_n) = 0$ if and only if $f_1 \otimes \cdots \otimes f_n = 0$. Moreover, $T^n(\mathfrak{g}^*)$ and $(T^n(\mathfrak{g}))^*$ have the same dimension, and so μ must also be surjective. Thus μ is an isomorphism of vector spaces. We must also show equivariance, that is

$$\mu(x.(f_1 \otimes \cdots \otimes f_n)) = x.\mu(f_1 \otimes \cdots \otimes f_n)$$

for all $x \in \mathfrak{g}$. Indeed, we have

$$x.(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^{n} f_1 \otimes \cdots \otimes (-f_i \circ ad_x) \otimes \cdots \otimes f_n.$$

Thus

$$\mu(x.(f_1 \otimes \cdots \otimes f_n)) = \sum_{i=1}^{n} \mu(f_1 \otimes \cdots \otimes (-f_i \circ ad_x) \otimes \cdots \otimes f_n)$$

and

$$\mu(x.(f_1 \otimes \cdots \otimes f_n))(x_1 \otimes \cdots \otimes x_n) = -\big[f_1([x, x_1]) \cdots f_n(x_n)\big] - \cdots - \big[f_1(x_1) \cdots f_n([x, x_n])\big]$$

On the other hand,

$$\begin{aligned}(x.\mu(f_1 \otimes \cdots \otimes f_n))(x_1 \otimes \cdots \otimes x_n) \quad &= -\mu(f_1 \otimes \cdots \otimes f_n)(x.(x_1 \otimes \cdots \otimes x_n)) \text{ (see Definition ???)} \\ &= -\mu(f_1 \otimes \cdots \otimes f_n)(\sum_{i=1}^{n} x_1 \otimes \cdots \otimes [x, x_i] \otimes \cdots \otimes x_n) \\ &= -\sum_{i=1}^{n} \mu(f_1 \otimes \cdots \otimes f_n)(x_1 \otimes \cdots \otimes [x, x_i] \otimes \cdots \otimes x_n) \\ &= -\sum_{i=1}^{n} f_1(x_1) \cdots f_i([x, x_i]) \cdots f_n(x_n).\end{aligned}$$

Therefore

$$(x.\mu(f_1 \otimes \cdots \otimes f_n))(x_1 \otimes \cdots \otimes x_n) = \mu(x.(f_1 \otimes \cdots \otimes f_n))(x_1 \otimes \cdots \otimes x_n)$$

So the result follows. □

Definition 2.0.25. An element $f \in (T^n(\mathfrak{g}))^*$ is called **symmetric** if

$$f(x_1 \otimes \cdots \otimes x_n) = f(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)})$$

for all $x_1, \ldots, x_n \in \mathfrak{g}$ and all $\sigma \in S_n$. The set of symmetric elements of $(T^n(\mathfrak{g}))^*$ will be denoted by $(T^n(\mathfrak{g}))^*_{sym}$.

Lemma 2.0.20. *Consider the maps* $S^n(\mathfrak{g}^*) \xrightarrow{\beta} T^n_{sym}(\mathfrak{g}^*) \xrightarrow{\mu} (T^n(\mathfrak{g}))^*_{sym}$ *given by*

$$f_1 \ldots f_n \to \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \to \frac{1}{n!} \sum_{\sigma \in S_n} \mu(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)})$$

Then, the subspaces $T^n_{sym}(\mathfrak{g}^*)$ *and* $(T^n(\mathfrak{g}))^*_{sym}$ *are* $\mathfrak{g}$*-submodules and the maps* β, μ *are isomorphisms.*

Proof. Let $f = \frac{1}{n!} \sum_{\sigma \in S_n} \mu(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)})$ and fix $\tau \in S_n$. Since $f_{\sigma(j)}(x_{\tau(j)}) = f_{\sigma(j)}(x_k)$ for $\tau(j) = k$, (i.e., $j = \tau^{-1}(k)$), then $f_{\sigma(j)}(x_{\tau(j)}) = f_{\sigma\tau^{-1}(k)}(x_k)$ and

$$\begin{aligned}f(x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}) &= \tfrac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)}(x_{\tau(1)}) \cdots f_{\sigma(n)}(x_{\tau(n)}) \\ &= \tfrac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma\tau^{-1}(1)}(x_1) \cdots f_{\sigma\tau^{-1}(n)}(x_n) \\ &= f(x_1 \otimes \cdots \otimes x_n).\end{aligned}$$

Thus $f \in (T^n(\mathfrak{g}))^*_{sym}$. Now we have

$$\beta(f_{\tau(1)} \cdots f_{\tau(n)}) = \frac{1}{n!} \sum_{\sigma \in S_n} \mu(f_{\sigma\tau(1)} \otimes \cdots \otimes f_{\sigma\tau(n)}) = \beta(f_1 \cdots f_n)$$

□

Lemma 2.0.21. *Let $\mathfrak{g}$ be a Lie algebra and $\xi_1, \ldots, \xi_m$ be a basis for $\mathfrak{g}$. The map*

$$c_n : \mathfrak{g} \mapsto \mathbb{K},\ x \mapsto tr\left((ad(x))^n\right), \quad \left(x = \alpha_1\xi_1 + \cdots + \alpha_m\xi_m\right) \quad \textit{given by}$$

$$c_n(x) = tr\left((ad(x))^n\right) = \sum_{i_1,\ldots,i_n=1}^{m} tr\left(ad(\xi_{i_1}) \circ \cdots \circ ad(\xi_{i_n})\right)\alpha_{i_1}\ldots\alpha_{i_n}.$$

*is an **invariant** polynomial function on $\mathfrak{g}$ which is homogeneous of degree n, that is $x.c_n = 0$ for all $x \in \mathfrak{g}$, and so $c_n \in S(\mathfrak{g}^*)^{\mathfrak{g}}$.*

Proof. Let us recall that ad is the linear map

$$\begin{array}{rcl} ad: \ \mathfrak{g} & \to & End(\mathfrak{g}) \\ x & \to & ad_x \end{array}$$

defined by $ad_x(y) = [x, y]$ for all $x, y \in \mathfrak{g}$. We define the following map $f : T^n(\mathfrak{g}) \to \mathbb{K}$,

$$f(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!}\sum_{\sigma \in S_n} tr(ad_{x_{\sigma(1)}} \circ \cdots \circ ad_{x_{\sigma(n)}})$$

$f(x_1 \otimes \cdots \otimes x_n) = f(x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)})$ for all $x_1, \ldots, x_n \in \mathfrak{g}$ and all $\tau \in S_n$ then $f \in (T^n(\mathfrak{g}))^*_{sym}$.
By Lemma 2.0.20 we have that f corresponds to c_n and

$$f(x \otimes \cdots \otimes x) = c_n(x) = tr\left((ad(x))^n\right).$$

We wish to show that $c_n \in S(\mathfrak{g}^*)^{\mathfrak{g}}$, that is c_n is an invariant polynomial function. We shall make use of the isomorphism $S^n(\mathfrak{g}^*) \cong (T^n(\mathfrak{g}))^*_{sym}$.
We recall that $T^n(\mathfrak{g})$ may be regarded as an $\mathfrak{g}$-module under the action

$$x.(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} x_1 \otimes \cdots \otimes [x, x_i] \otimes \cdots \otimes x_n .$$

Its dual space $(T^n(\mathfrak{g}))^*$ then becomes an $\mathfrak{g}$-module under the action

$$(x.f)(x_1 \otimes \cdots \otimes x_n) = -f(x.(x_1 \otimes \cdots \otimes x_n)).$$

We now consider $x.f$ where $f \in (T^n(\mathfrak{g}))^*_{sym}$ is the function defined above.
We have

$$\begin{aligned}
(x.f)(x_1 \otimes \cdots \otimes x_n) &= -\sum_{i=1}^{n} f(x_1 \otimes \cdots \otimes [x, x_i] \otimes \cdots \otimes x_n) \\
&= -\tfrac{1}{n!}\sum_{\sigma \in S_n}\sum_{i=1}^{n} tr(ad_{x_{\sigma(1)}} \circ \cdots \circ ad_{[x, x_{\sigma(i)}]} \circ \ldots \circ ad_{x_{\sigma(n)}}) \\
&= -\tfrac{1}{n!}\sum_{\sigma \in S_n}\sum_{i=1}^{n} tr(ad_{x_{\sigma(1)}} \circ \cdots \circ ad_x \circ ad_{x_{\sigma(i)}} \circ \cdots \circ ad_{x_{\sigma(n)}}) \\
&\quad +\tfrac{1}{n!}\sum_{\sigma \in S_n}\sum_{i=1}^{n} tr(ad_{x_{\sigma(1)}} \circ \cdots \circ ad_{x_{\sigma(i)}} \circ ad_x \circ \cdots \circ ad_{x_{\sigma(n)}}) \\
&= \tfrac{1}{n!}\sum_{\sigma \in S_n}\big(tr(ad_{x_{\sigma(1)}} \circ \cdots \circ ad_{x_{\sigma(n)}} \circ ad_x) - \\
&\qquad tr(ad_x \circ ad_{x_{\sigma(1)}} \circ \cdots \circ ad_{x_{\sigma(n)}})\big) \\
&= 0 \quad \text{since tr(AB)=tr(BA).}
\end{aligned}$$

Thus $x.f = 0$ for all $x \in \mathfrak{g}$. $\square$

Definition 2.0.26. Let us define, inductively, a sequence of rational numbers $\{B_n\}$ by the following recursion formula:

$$B_0 := 1, \qquad B_n := -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!} \qquad (n \geq 1) \ .$$

The B_n are referred to as the **Bernoulli numbers**.

Definition 2.0.27. Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$; we define an element

$$J^{1/2} \in \widehat{S(\mathfrak{g}^*)},$$

called **Duflo element** as the formal power series:

$$Exp\left((0,\ 0,\ a_2,\ 0,\ a_4,\ 0,\ a_6,\ 0,\ a_8,\ 0,\ a_{10},\ 0,\ \ldots)\right),$$

where $a_n \in S^n(\mathfrak{g}^*)$ is such that

$$a_n(x) = \frac{B_n}{(2n)(n!)} \cdot tr\Big((ad(x))^n\Big), \quad a_n \in \mathbb{K}[\mathfrak{g}]^n \,,$$

in which B_n are the Bernoulli numbers.

Remark 2.0.11. Modulo terms of orden ≥ 12, one finds that

$J^{1/2} = (1,\ 0,\ a_2,\ 0,\ a_4 + \frac{1}{2!}\left(a_2 * a_2\right),\ 0,\ a_6 + \frac{1}{2!}\left(a_4 * a_2 + a_2 * a_4\right) + \frac{1}{3!}\left(a_2 * a_2 * a_2\right),\ 0,$
$a_8 + \frac{1}{2!}\left(a_6 * a_2 + a_4 * a_4 + a_2 * a_6\right) + \frac{1}{3!}\left(a_4 * a_2 * a_2 + a_2 * a_4 * a_2 + a_2 * a_2 * a_4\right) + \frac{1}{4!}\left(a_2 * a_2 * a_2 * a_2\right),\ 0,$
$a_{10} + \frac{1}{2!}\left(a_8 * a_2 + a_6 * a_4 + a_4 * a_6 + a_2 * a_8\right) + \frac{1}{3!}\left(a_6 * a_2 * a_2 + a_2 * a_6 * a_2 + a_2 * a_2 * a_6\right) +$
$\frac{1}{4!}\left(a_4 * a_2 * a_2 * a_2 + a_2 * a_4 * a_2 * a_2 + a_2 * a_2 * a_4 * a_2 + a_2 * a_2 * a_2 * a_4\right) + \frac{1}{5!}\left(a_2 * a_2 * a_2 * a_2 * a_2\right),\ 0, \ldots)$

Theorem 2.0.22. ***(The Duflo isomorphism,*** *[26, 27]). $I_{PBW} \circ \widehat{\theta}(J^{1/2})$, defines an isomorphism of algebras*

$$S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}.$$

The proof for semisimple Lie algebra and finite-dimensional solvable Lie algebras is given in [26]. Duflo's Theorem is valid for arbitrary Lie algebras. In recent years, new proofs for the general case have been found using Kontsevich's theory of deformation quantization [17], and more recently by AlekseevTorossian [6] in their approach to the KashiwaraVergne conjecture [16]. Some of these proofs are given in [9].

Chapter 3

Left Pre-Lie and Dendriform Algebra

Right-symmetric algebras [8, 11], or RSAs in short, are also called Gerstenhaber algebras, or pre-Lie algebras. Left-symmetric algebras, or LSAs, arise in many areas of mathematics and physics and are known under many different names. LSAs are also called Vinberg algebras, Koszul algebras or quasi-associative algebras. Based on the results of the previous chapter, we give some similar conclusions for Left-symmetric algebras and dendriform algebra. This third part of the thesis contains our next main results: Theorems 3.0.23, 3.0.24, 3.0.28, 3.0.29, where, we identify properties similar to the over of chapter 2 for Dendriform algebra and Left-symmetric algebras. This means that we should be able to investigate in the context of dendriform algebra in parallel with Lie algebras. The obstacle that did not allow continue with an analogue of the Duflo theorem was that we could not give an invariant in this context.

3.0.8 Left Pre-Lie Algebra

Definition 3.0.28. An algebra A over $\mathbb{K}$ with a bilinear product $\langle\ ,\ \rangle$ is called **Left-Symmetric Algebra** or **Left Pre-Lie Algebra**, if the product satisfies the following identity:

$$\langle\langle x,y\rangle,z\rangle-\langle x,\langle y,z\rangle\rangle=\langle\langle y,x\rangle,z\rangle-\langle y,\langle x,z\rangle\rangle,$$

for all $x, y, z \in A$.

An Left-Symmetric Algebra A is endowed with a structure of Lie algebra with commutator $[z, x] = \langle z, x\rangle - \langle x, z\rangle$.

Definition 3.0.29. A vector space M is said to be a **module over the left-symmetric algebra** $(A, \langle\cdot,\cdot\rangle)$, if it is endowed with a left action $. : A \times M \longrightarrow M$ such that

$$[z,x].m = z.(x.m) - x.(z.m) \text{ , for any } z,x \in A, m \in M \text{ ,}$$

and with a right action $\circ : M \times A \longrightarrow M$ such that

$$m\circ\langle b,a\rangle = (m\circ b)\circ a + b.(m\circ a) - (b.m)\circ a$$

for any $b, a \in A, m \in M$.

Definition 3.0.30. Let M be a module over a left-symmetric algebra $(A, \langle\ ,\ \rangle)$. The subspace

$$M^{r.ass} = \{m \in M : \langle b,a\rangle.m = b.(a.m),\ \forall a,b \in A\}$$

is called the **right associative invariant subspace of** M and

$$M^{r.inv} = \{m \in M : a.m = 0,\ \forall a \in A\}$$

is called the **right invariant subspace of** M.

Example 3.0.12. Let A be a left-symmetric algebra, then $T(A) = \bigoplus\limits_{n \geq 0} A^{\otimes n}$ is an A-module with left and right actions given by:

$$\begin{aligned} a.(a_1 \otimes \cdots \otimes a_n) &= \sum_{i=1}^{n} a_1 \otimes \cdots \otimes \langle a, a_i \rangle \otimes \cdots \otimes a_n \\ (a_1 \otimes \cdots \otimes a_n) \circ a &= a_1 \otimes \cdots \otimes \langle a_n, a \rangle \end{aligned}$$

We check the, $n = 3$ case. We have,

$$\begin{aligned} y.m = y.(a \otimes b \otimes c) = \;& \langle y, a \rangle \otimes b \otimes c + a \otimes \langle y, b \rangle \otimes c + a \otimes b \otimes \langle y, c \rangle \\ x.(y.m) = \;& \langle x, \langle y, a \rangle \rangle \otimes b \otimes c + \langle y, a \rangle \otimes \langle x, b \rangle \otimes c + \langle y, a \rangle \otimes b \otimes \langle x, c \rangle \\ & + \langle x, a \rangle \otimes \langle y, b \rangle \otimes c + a \otimes \langle x, \langle y, b \rangle \rangle \otimes c + a \otimes \langle y, b \rangle \otimes \langle x, c \rangle \\ & + \langle x, a \rangle \otimes b \otimes \langle y, c \rangle + a \otimes \langle x, b \rangle \otimes \langle y, c \rangle + a \otimes b \otimes \langle x, \langle y, c \rangle \rangle \\ y.(x.m) = \;& \langle y, \langle x, a \rangle \rangle \otimes b \otimes c + \langle x, a \rangle \otimes \langle y, b \rangle \otimes c + \langle x, a \rangle \otimes b \otimes \langle y, c \rangle \\ & + \langle y, a \rangle \otimes \langle x, b \rangle \otimes c + a \otimes \langle y, \langle x, b \rangle \rangle \otimes c + a \otimes \langle x, b \rangle \otimes \langle y, c \rangle \\ & + \langle y, a \rangle \otimes b \otimes \langle x, c \rangle + a \otimes \langle y, b \rangle \otimes \langle x, c \rangle + a \otimes b \otimes \langle y, \langle x, c \rangle \rangle \end{aligned}$$

Thus

$$\begin{aligned} x.(y.m) - y.(x.m) = \;& (\langle x, \langle y, a \rangle \rangle - \langle y, \langle x, a \rangle \rangle) \otimes b \otimes c + a \otimes (\langle x, \langle y, b \rangle \rangle - \langle y, \langle x, b \rangle \rangle) \otimes c \\ & + a \otimes b \otimes (\langle x, \langle y, c \rangle \rangle - \langle y, \langle x, c \rangle \rangle) \\ = \;& \langle [x, y], a \rangle \otimes b \otimes c + a \otimes \langle [x, y], b \rangle \otimes c + a \otimes b \otimes \langle [x, y], c \rangle \\ = \;& [x, y].m \end{aligned}$$

since

$$\langle x, \langle y, a \rangle \rangle - \langle y, \langle x, a \rangle \rangle = \langle \langle x, y \rangle, a \rangle - \langle \langle y, x \rangle, a \rangle = \langle [x, y], a \rangle .$$

On the other hand,

$$
\begin{aligned}
(m\circ x)\circ y + x.(m\circ y) - (x.m)\circ y &= a\otimes b\otimes\langle\langle c,x\rangle,y\rangle + \langle x,a\rangle\otimes b\otimes\langle c,y\rangle +\\
&\quad a\otimes\langle x,b\rangle\otimes\langle c,y\rangle + a\otimes b\otimes\langle x,\langle c,y\rangle\rangle +\\
&\quad -\left(\langle x,a\rangle\otimes b\otimes c + a\otimes\langle x,b\rangle\otimes c + a\otimes b\otimes\langle x,c\rangle\right)\circ y\\
&= a\otimes b\otimes\big(\langle\langle c,x\rangle,y\rangle + \langle x,\langle c,y\rangle\rangle - \langle\langle x,c\rangle,y\rangle\big)\\
&= a\otimes b\otimes\langle c,\langle x,y\rangle\rangle\\
&= (a\otimes b\otimes c)\circ\langle x,y\rangle\\
&= m\circ\langle x,y\rangle\ .
\end{aligned}
$$

In the lemma below we will use the following construction:

Let A^* be the dual space of A and let φ be the linear map

$$
\begin{array}{rccl}
\varphi: & A^* & \to & End(T(A))\\
 & f & \to & \varphi(f)
\end{array}
$$

defined by

$$
\begin{array}{rccl}
\varphi(f): & T(A) & \to & T(A)\\
 & x_1\otimes\cdots\otimes x_n & \to & \sum_{i=1}^{n} f(x_i)x_1\otimes\cdots\otimes\widehat{x_i}\otimes\cdots\otimes x_n
\end{array}
$$

where $\varphi(f)(1) := 0$, $\varphi(f)(x) := f(x)$ for all $x\in A$, and the sign $\widehat{\ }$ above a letter indicates that it has been omitted.

The map φ extends, by the the universal property to an associative algebra homomorphism θ defined by

$$
\begin{array}{rccl}
\theta: & T(A^*) & \to & End(T(A))\\
 & f_1\otimes\cdots\otimes f_k & \to & \varphi(f_1)\circ\cdots\circ\varphi(f_k)\,.
\end{array}
$$

Moreover, $\varphi(f)$ sends $T^n(A)$ to $T^{n-1}(A)$. If $v\in T^k(A^*)$, then $\theta(v)$ is a differential operator de order k, that is, annihilates every element of $T(A)$ of degree $< k$, and it sends $T^n(A)$ to $T^{n-k}(A)$.

Lemma 3.0.23. *Let $(A,\langle\ ,\ \rangle)$ be a Pre-Lie algebra. Then, for the algebra homomorphism*

$$
\theta: T(A^*)\to End(T(A))
$$

defined above, we have

$$
x.\big(\theta(v)(u_1\otimes\cdots\otimes u_n)\big) = \theta(x.v)(u_1\otimes\cdots\otimes u_n) + \theta(v)(x.(u_1\otimes\cdots\otimes u_n))
$$

for all $x\in A$, where $x\cdot v$ is the action:

$$
x.v = x\cdot(f_1\otimes\cdots\otimes f_k) = \sum_{i=1}^{k} f_1\otimes\cdots\otimes(-f_i\circ L(x))\otimes\cdots\otimes f_k
$$

and

$$x.(u_1 \otimes \cdots \otimes u_n) = \sum_{i=1}^{n} u_1 \otimes \cdots \otimes \langle x, u_i \rangle \cdots \otimes u_n$$

is the left action of the left-symmetric algebra A on $T(A)$, where $L(x) : A \to A,$ is the linear map defined by $L(x)(y) := \langle x, y \rangle$.

Proof. It suffices to prove the lemma when $v = f_1 \otimes \cdots \otimes f_k$, with $f_i \in A^*$ and $u_1 \otimes \cdots \otimes u_n \in T(A)$ where $u \in A$ and $n, k \in \mathbb{N}$. For $k = 1$, we have

$$x.\big(\theta(f)(u_1 \otimes \cdots \otimes u_n)\big) = x.\left(\sum_{i=1}^{n} f(u_i) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n\right).$$

We define $w_{i,k} := u_k,$ if $1 \leq k \leq i-1$ and $w_{i,k} := u_{k+1},$ if $i \leq k \leq n-1$. For each $1 \leq i \leq n$, then we have

$$\begin{aligned} x.\big(\theta(f)(u_1 \otimes \cdots \otimes u_n)\big) &= x.\left(\sum_{i=1}^{n} f(u_i) w_{i,1} \otimes \cdots \otimes w_{i,n-1}\right) \\ &= \sum_{i=1}^{n} f(u_i)\left(\sum_{j=1}^{n-1} w_{i,1} \otimes \cdots \otimes \langle x, w_{i,j} \rangle \otimes \cdots \otimes w_{i,n-1}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \theta(x.f)(u_1 \otimes \cdots \otimes u_n) + \theta(f)(x.(u_1 \otimes \cdots \otimes u_n)) &= \theta(-f \circ L(x))(u_1 \otimes \cdots \otimes u_n) + \\ &\quad \theta(f)\left(\sum_{i=1}^{n}(u_1 \otimes \cdots \otimes \langle x, u_i \rangle \otimes \cdots \otimes u_n)\right) \end{aligned}$$

$$\begin{aligned} &= -\sum_{i=1}^{n} f(\langle x, u_i \rangle) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n + \theta(f)\left(\sum_{i=1}^{n}(u_1 \otimes \cdots \otimes \langle x, u_i \rangle \otimes \cdots \otimes u_n)\right) \\ &= -\sum_{i=1}^{n} f(\langle x, u_i \rangle) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n + \sum_{i=1}^{n} f(\langle x, u_i \rangle) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n + \\ &\quad \sum_{i=1}^{n} f(u_i)\left(\sum_{j=1}^{n-1} w_{i,1} \otimes \cdots \otimes \langle x, w_{i,j} \rangle \otimes \cdots \otimes w_{i,n-1}\right) \\ &= \sum_{i=1}^{n} f(u_i)\left(\sum_{j=1}^{n-1} w_{i,1} \otimes \cdots \otimes \langle x, w_{i,j} \rangle \otimes \cdots \otimes w_{i,n-1}\right) \end{aligned}$$

Thus, $x.\big(\theta(f)(u_1 \otimes \cdots \otimes u_n)\big) = \theta(x.f)(u_1 \otimes \cdots \otimes u_n) + \theta(f)(x.(u_1 \otimes \cdots \otimes u_n))$

We now argue by induction on k. It is assumed that $k > 1$ and that the formula

$$x.\big(\theta(v)(u_1 \otimes \cdots \otimes u_n)\big) = \theta(x.v)(u_1 \otimes \cdots \otimes u_n) + \theta(v)(x.(u_1 \otimes \cdots \otimes u_n))$$

is true for all v of the form $f_1 \otimes \cdots \otimes f_{k-1}$. Let $q = f_1 \otimes \cdots \otimes f_k$, then

$$
\begin{aligned}
x.\big(\theta(q)(u_1\otimes\cdots\otimes u_n)\big) &= x.\Big(\theta(v)\circ\theta(f_k)(u_1\otimes\cdots\otimes u_n)\Big)\\
&= x.\Big(\theta(v)\left(\theta(f_k)(u_1\otimes\cdots\otimes u_n)\right)\Big)\\
&= \theta(x.v)\Big(\theta(f_k)(u_1\otimes\cdots\otimes u_n)\Big)+\theta(v)\Big(x.\big(\theta(f_k)(u_1\otimes\cdots\otimes u_n)\big)\Big)\\
&= \theta(x.v)\Big(\theta(f_k)(u_1\otimes\cdots\otimes u_n)\Big)+\\
&\quad\ \theta(v)\Big(\theta(x.f_k)(u_1\otimes\cdots\otimes u_n)+\theta(f_k)(x.(u_1\otimes\cdots\otimes u_n))\Big)\\
&= \theta(x.v)\Big(\theta(f_k)(u_1\otimes\cdots\otimes u_n)\Big)+\theta(v)\Big(\theta(x.f_k)(u_1\otimes\cdots\otimes u_n)\Big)+\\
&\quad\ \theta(v)\Big(\theta(f_k)(x.(u_1\otimes\cdots\otimes u_n))\Big)\\
&= \theta(x.v)\circ\theta(f_k)(u_1\otimes\cdots\otimes u_n)+\theta(v)\circ\theta(x.f_k)(u_1\otimes\cdots\otimes u_n)\ +\\
&\quad\ \theta(q)(x.(u_1\otimes\cdots\otimes u_n))\\
&= \theta(x.q)(u_1\otimes\cdots\otimes u_n)+\theta(q)(x.(u_1\otimes\cdots\otimes u_n))\ .
\end{aligned}
$$

This complete the proof. □

Example 3.0.13. Let A be a left-symmetric algebra; then $T(A)$ is an A-module:

$$
\begin{aligned}
a.(a_1\otimes\cdots\otimes a_n) &= a_1\otimes\cdots\otimes\langle a,a_n\rangle+\sum_{i=1}^{n-1}a_1\otimes\cdots\otimes[a,a_i]\otimes\cdots\otimes a_n\\
(a_1\otimes\cdots\otimes a_n)\circ a &= a_1\otimes\cdots\otimes\langle a_n,a\rangle
\end{aligned}
$$

where $[a,a_i]=\langle a,a_i\rangle-\langle a_i,a\rangle$.

Indeed, we check the $n=3$ case:

$$
\begin{aligned}
z.m=z.(a\otimes b\otimes y)= &\ [z,a]\otimes b\otimes y+a\otimes[z,b]\otimes y+a\otimes b\otimes\langle z,y\rangle\\
w.(z.m)= &\ [w,[z,a]]\otimes b\otimes y+[z,a]\otimes[w,b]\otimes y+[z,a]\otimes b\otimes\langle w,y\rangle\\
&+[w,a]\otimes[z,b]\otimes y+a\otimes[w,[z,b]]\otimes y+\ a\otimes[z,b]\otimes\langle w,y\rangle\\
&+[w,a]\otimes b\otimes\langle z,y\rangle+a\otimes[w,b]\otimes\langle z,y\rangle+a\otimes b\otimes\langle w,\langle z,y\rangle\rangle\\
z.(w.m)= &\ [z,[w,a]]\otimes b\otimes y+[w,a]\otimes[z,b]\otimes y+[w,a]\otimes b\otimes\langle z,y\rangle\\
&+[z,a]\otimes[w,b]\otimes y+a\otimes[z,[w,b]]\otimes y+a\otimes[w,b]\otimes\langle z,y\rangle\\
&+[z,a]\otimes b\otimes\langle w,y\rangle+a\otimes[z,b]\otimes\langle w,y\rangle+a\otimes b\otimes\langle z,\langle w,y\rangle\rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
w.(z.m)-z.(w.m)= &\ ([w,[z,a]]-[z,[w,a]])\otimes b\otimes y+a\otimes([w,[z,b]]-[z,[w,b]])\otimes y\\
&+a\otimes b\otimes(\langle w,\langle z,y\rangle\rangle-\langle z,\langle w,y\rangle\rangle)\\
= &\ [[w,z],a]\otimes b\otimes y+a\otimes[[w,z],b]\otimes y+a\otimes b\otimes(\langle\langle w,z\rangle,y\rangle-\langle\langle z,w\rangle,y\rangle)\\
= &\ \langle w,z\rangle.m-\langle z,w\rangle.m
\end{aligned}
$$

since

$$[w,[z,x]]-[z,[w,x]]=[[w,z],x] \text{ and } \langle w,\langle z,a\rangle\rangle-\langle z,\langle w,a\rangle\rangle=\langle\langle w,z\rangle,a\rangle-\langle\langle z,w\rangle,a\rangle$$

Also, we have

$$\begin{aligned}(m\circ x)\circ w+x.(m\circ w)-(x.m)\circ w &= a\otimes b\otimes\langle\langle y,x\rangle,w\rangle+[x,a]\otimes b\otimes\langle y,w\rangle+\\ &\quad a\otimes[x,b]\otimes\langle y,w\rangle+a\otimes b\otimes\langle x,\langle y,w\rangle\rangle+\\ &\quad -([x,a]\otimes b\otimes y+a\otimes[x,b]\otimes y+a\otimes b\otimes\langle x,y\rangle)\circ w\\ &= a\otimes b\otimes\big(\langle\langle y,x\rangle,w\rangle+\langle x,\langle y,w\rangle\rangle-\langle\langle x,y\rangle,w\rangle\big)\\ &= a\otimes b\otimes\langle y,\langle x,w\rangle\rangle\\ &= (a\otimes b\otimes y)\circ\langle x,w\rangle\\ &= m\circ\langle x,w\rangle.\end{aligned}$$

Example 3.0.14. Let A^* be the dual space of A and let $\varphi^{'}$ the linear map

$$\begin{array}{rccl}\varphi^{'}: & A^* & \to & End(T(A))\\ & f & \to & \varphi(f)\end{array}$$

defined by

$$\begin{array}{rlcl}\varphi^{'}(f): & T(A) & \to & T(A)\\ & x_1\otimes\cdots\otimes x_n & \to & \sum_{i=1}^{n-1} f(x_i)x_1\otimes\cdots\otimes\widehat{x_i}\otimes\cdots\otimes x_n\end{array}$$

where $\varphi^{'}(f)(1):=0$, $\varphi^{'}(f)(x):=f(x)$ for all $x\in A$, and the sign $\widehat{}$ above a letter indicates that it has been omitted.

The map $\varphi^{'}$ extends, by the the universal property to an associative algebra homomorphism θ defined by:

$$\begin{array}{rlcl}\theta: & T(A^*) & \to & End(T(A))\\ & f_1\otimes\cdots\otimes f_k & \to & \varphi^{'}(f_1)\circ\cdots\circ\varphi^{'}(f_k)\end{array}.$$

Moreover, $\varphi^{'}(f)$ sends $T^n(A)$ to $T^{n-1}(A)$. If $v\in T^k(A^*)$, then $\theta_1(v)$ is a differential operator de order k, that is, annihilates every element of $T(A)$ of degree $<k$, and it sends $T^n(A)$ to $T^{n-k}(A)$.

Lemma 3.0.24. *Let $(A,\langle\ ,\ \rangle)$ be a Pre-Lie algebra, then for the algebra homomorphism*

$$\theta: T(A^*)\to End(T(A))$$

defined above, we have

$$x.\big(\theta(v)(u_1\otimes\cdots\otimes u_n)\big)=\theta(x.v)(u_1\otimes\cdots\otimes u_n)+\theta(v)(x.(u_1\otimes\cdots\otimes u_n))$$

for all $x \in A$, where $x \cdot v$ is the coadjoint representation,

$$x.v = x \cdot (f_1 \otimes \cdots \otimes f_k) = \sum_{i=1}^{k} f_1 \otimes \cdots \otimes (-f_i \circ ad(x)) \otimes \cdots \otimes f_k \,,$$

and

$$x.(u_1 \otimes \cdots \otimes u_n) = u_1 \otimes \cdots \otimes \langle x, u_n \rangle + \sum_{i=1}^{n-1} u_1 \otimes \cdots \otimes [x, u_i] \cdots \otimes u_n$$

is the left action of the left-symmetric algebra A on $T(A)$.

Proof. It suffices to prove this when $v = f_1 \otimes \cdots \otimes f_k$, with $f_i \in A^*$ and $u_1 \otimes \cdots \otimes u_n \in T(A)$ where $u_i \in A$ and $n, k \in \mathbb{N}$. For $k = 1$, we have

$$x.\big(\theta\,(f)\,(u_1 \otimes \cdots \otimes u_n)\big) \quad = \quad x.\left(\sum_{i=1}^{n-1} f(u_i) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n\right)$$

We define $w_{i,k} := u_k,$ if $1 \leq k \leq i-1$ and $w_{i,k} := u_{k+1},$ if $i \leq k \leq n-2,$ for each $1 \leq i \leq n-1,$ then

$$\begin{aligned} x.\big(\theta\,(f)\,(u_1 \otimes \cdots \otimes u_n)\big) &= x.\left(\sum_{i=1}^{n-1} f(u_i) w_{i,1} \otimes \cdots \otimes w_{i,n-2} \otimes u_n\right) \\ &= \sum_{i=1}^{n-1} f(u_i) w_{i,1} \otimes \cdots \otimes w_{i,n-2} \otimes \langle x, u_n \rangle + \\ &\quad \sum_{i=1}^{n-1} f(u_i) \left(\sum_{j=1}^{n-1} w_{i,1} \otimes \cdots \otimes [x, w_{i,j}] \otimes \cdots \otimes w_{i,n-2} \otimes u_n\right) . \end{aligned}$$

On the other hand,

$$\theta\,(x.f)\,(u_1 \otimes \cdots \otimes u_n) + \theta\,(f)\,(x.\,(u_1 \otimes \cdots \otimes u_n)) \quad = \quad \theta\,(-f \circ ad(x))\,(u_1 \otimes \cdots \otimes u_n) \; +$$

$$\theta\,(f) \left(u_1 \otimes \cdots \otimes \langle x, u_n \rangle + \sum_{i=1}^{n-1} (u_1 \otimes \cdots \otimes [x, u_i] \otimes \cdots \otimes u_n)\right)$$

$$\begin{aligned} &= -\sum_{i=1}^{n-1} f\left([x, u_i]\right) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n + \\ &\quad \theta\,(f) \left(u_1 \otimes \cdots \otimes \langle x, u_n \rangle + \sum_{i=1}^{n-1} (u_1 \otimes \cdots \otimes [x, u_i] \otimes \cdots \otimes u_n)\right) \\ &= -\sum_{i=1}^{n-1} f\left([x, u_i]\right) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n + \sum_{i=1}^{n-1} f\left(u_i\right) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes \langle x, u_n \rangle + \\ &\quad \sum_{i=1}^{n-1} f\left([x, u_i]\right) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_n + \sum_{i=1}^{n-1} f(u_i) \left(\sum_{j=1}^{n-1} w_{i,1} \otimes \cdots \otimes [x, w_{i,j}] \otimes \cdots \otimes w_{i,n-2} \otimes u_n\right) \\ &= \sum_{i=1}^{n-1} f\left(u_i\right) u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes \langle x, u_n \rangle + \sum_{i=1}^{n-1} f(u_i) \left(\sum_{j=1}^{n-1} w_{i,1} \otimes \cdots \otimes [x, w_{i,j}] \otimes \cdots \otimes w_{i,n-2} \otimes u_n\right) \end{aligned}$$

Thus,

$x.\big(\theta(f)(u_1 \otimes \cdots \otimes u_n)\big) = \theta(x.f)(u_1 \otimes \cdots \otimes u_n) + \theta(f)(x.(u_1 \otimes \cdots \otimes u_n))$

We now aim to argue by induction on k. It is assumed that $k > 1$ and the formula

$$x.\big(\theta(v)(u_1 \otimes \cdots \otimes u_n)\big) = \theta(x.v)(u_1 \otimes \cdots \otimes u_n) + \theta(v)(x.(u_1 \otimes \cdots \otimes u_n))$$

is true for all v of the form $f_1 \otimes \cdots \otimes f_{k-1}$. Let $q = f_1 \otimes \cdots \otimes f_k$, then

$$\begin{aligned}
x.\big(\theta(q)(u_1 \otimes \cdots \otimes u_n)\big) &= x.\Big(\theta(v) \circ \theta(f_k)(u_1 \otimes \cdots \otimes u_n)\Big) \\
&= x.\Big(\theta(v)(\theta(f_k)(u_1 \otimes \cdots \otimes u_n))\Big) \\
&= \theta(x.v)\Big(\theta(f_k)(u_1 \otimes \cdots \otimes u_n)\Big) + \theta(v)\Big(x.\big(\theta(f_k)(u_1 \otimes \cdots \otimes u_n)\big)\Big) \\
&= \theta(x.v)\Big(\theta(f_k)(u_1 \otimes \cdots \otimes u_n)\Big) + \\
&\quad \theta(v)\Big(\theta(x.f_k)(u_1 \otimes \cdots \otimes u_n) + \theta(f_k)(x.(u_1 \otimes \cdots \otimes u_n))\Big) \\
&= \theta(x.v)\Big(\theta(f_k)(u_1 \otimes \cdots \otimes u_n)\Big) + \theta(v)\Big(\theta(x.f_k)(u_1 \otimes \cdots \otimes u_n)\Big) + \\
&\quad \theta(v)\Big(\theta(f_k)(x.(u_1 \otimes \cdots \otimes u_n))\Big) \\
&= \theta(x.v) \circ \theta(f_k)(u_1 \otimes \cdots \otimes u_n) + \theta(v) \circ \theta(x.f_k)(u_1 \otimes \cdots \otimes u_n) + \\
&\quad \theta(q)(x.(u_1 \otimes \cdots \otimes u_n)) \\
&= \theta(x.q)(u_1 \otimes \cdots \otimes u_n) + \theta(q)(x.(u_1 \otimes \cdots \otimes u_n))
\end{aligned}$$

This complete the proof. □

3.0.9 Dendriform Algebra

Definition 3.0.31. A **dendriform algebra** over $\mathbb{K}$ is a $\mathbb{K}$-vector space H endowed with two binary operations $\succ, \prec$: $H \otimes H \to H$ satisfying the following properties:

1. $(x \prec y) \prec z = x \prec (y * z)$
2. $(x \succ y) \prec z = x \succ (y \prec z)$
3. $(x * y) \succ z = x \succ (y \succ z)$

for any elements x, y and z in H; here $x * y := x \succ y + x \prec y$, for any $x, y \in H$.

Remark 3.0.12. If $(H, \succ, \prec)$ is a dendriform algebra, then $(H, *)$ is an associative algebra. Dendriform algebras were introduced by Loday [20] in 1995 with motivation from algebraic K-theory, and have been further studied with connections to several areas in mathematics and physics, including operads [22] homology [1, 2] Hopf algebras [12, 15, 31, 23] , Lie and Leibniz algebras, combinatorics [18, 24, 5, 4] arithmetic [21] and quantum field theory [18].

Theorem 3.0.25. *Let $(D, \succ, \prec)$ be a dendriform algebra. Then D is equipped with a left pre-lie algebra operation $\langle\ ,\ \rangle$ given by*

$$\langle x, y\rangle := x \succ y - y \prec x$$

for all $x, y \in D$.

Proof. We have

$$\begin{aligned}
\langle\langle x, y\rangle, z\rangle - \langle\langle y, x\rangle, z\rangle &= \langle [x, y], z\rangle \\
&= [x, y] \succ z - z \prec [x, y] \\
&= (x * y - y * x) \succ z - z \prec (x * y - y * x)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle x, \langle y, z\rangle\rangle - \langle y, \langle x, z\rangle\rangle &= x \succ \langle y, z\rangle - \langle y, z\rangle \prec x - y \succ \langle x, z\rangle + \langle x, z\rangle \prec y \\
&= x \succ (y \succ z - z \prec y) - (y \succ z - z \prec y) \prec x \\
&\quad - y \succ (x \succ z - z \prec x) + (x \succ z - z \prec x) \prec y \\
&= x \succ (y \succ z) + (z \prec y) \prec x - y \succ (x \succ z) - (z \prec x) \prec y \\
&= (x * y) \succ z + z \prec (y * x) - (y * x) \succ z - z \prec (x * y) \\
&= (x * y - y * x) \succ z - z \prec (x * y - y * x)\,.
\end{aligned}$$

Thus

$$\langle\langle x, y\rangle, z\rangle - \langle\langle y, x\rangle, z\rangle = \langle x, \langle y, z\rangle\rangle - \langle y, \langle x, z\rangle\rangle\,.$$

Therefore $(D, \langle\ ,\ \rangle)$ is a left-symmetric algebra.

□

Example 3.0.15. The graded vector space $\overline{T}(B) = \bigoplus_{n\geq 1} B^{\otimes n}$ endowed with the products $\succ$ and $\prec$ defined as follows:

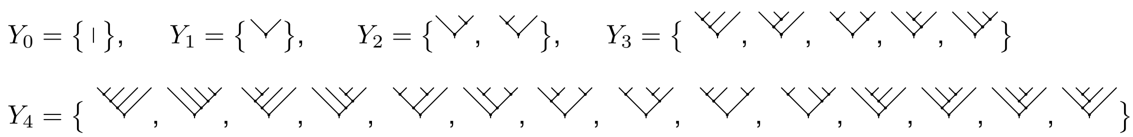
1. $(v_1 \otimes \cdots \otimes v_n) \succ (v_{n+1} \otimes \cdots \otimes v_{n+m}) := \Big((v_1 \otimes \cdots \otimes v_n) * (v_{n+1} \otimes \cdots \otimes v_{n+m-1})\Big) \otimes v_{n+m}.$
2. $(v_1 \otimes \cdots \otimes v_n) \prec (v_{n+1} \otimes \cdots \otimes v_{n+m}) := \Big((v_1 \otimes \cdots \otimes v_{n-1}) * (v_{n+1} \otimes \cdots \otimes v_{n+m})\Big) \otimes v_n.$
3. $* = \succ + \prec$. ($*$ is called **shuffle product**)

is a dendriform algebra.

Definition 3.0.32. Let B be a $\mathbb{K}$-vector space. **The cotensor algebra** $\overline{T}(B)$ is the $\mathbb{K}$-vector space $\overline{T}(B) = \bigoplus_{n\geq 1} B^{\otimes n}$ equipped with the shuffle product. The shuffle product is associative and commutative.

Now we concentrate on a special type of dendriform algebras which we will use in the following chapters.

Definition 3.0.33. A **planar binary tree** is an oriented planar graph drawn in the plane with one root, $n+1$ leaves and n interior vertices, and such that each vertex is connected by three edges, two incoming edges and one outgoing edge.
An edge can be internal (connecting two vertices) or external (with one loose end). The external incoming edges are the leaves. The root is the unique edge not ending in a vertex.
The integer n is the number of internal vertices and is called the **degree of the tree**. We denote by Y_n the set of trees of degree n. For example,

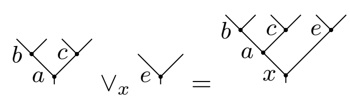


One can show that the cardinality of Y_n is given by the n-th Catalan number,

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

Definition 3.0.34. Let B be a vector space and X be a basis of B. We denote by $Y_{n,X}$ the set of all planar binary trees with $n+1$ leaves whose vertices are labeled by the elements of X. The $\mathbb{K}$-vector space generated by $Y_{n,X}$ is denoted $\mathbb{K}[Y_{n,X}]$.

Definition 3.0.35. Let $t \in Y_{n,X}$ and $w \in Y_{m,X}$ be labeled trees, and let x be an element of X. The tree $t \vee_x w \in Y_{n+m+1,X}$ is called the **grafting** of t and w over x, and it is obtained by joining the roots of t and w in a new vertices, which is labeled with x.



Remark 3.0.13. For any tree $t \in Y_{n,X}$ there exist unique elements $t_1 \in Y_{n-k,X}$, $t_2 \in Y_{k-1,X}$ and $x \in X$, such that

$$t = t_1 \vee_x t_2$$

Definition 3.0.36. We define recursively operations $\succ$ and $\prec$ over the graded space

$$Dend(B) := \bigoplus_{n\geq 1} \mathbb{K}[Y_{n,X}]$$

as follows:

1. $t \succ | := 0$, and $| \prec t := 0$ for $t \neq |$
2. $| \succ t := t$, and $t \prec | := t$ for $t \neq |$
3. If $t = t_1 \vee_x t_2$ and $w = w_1 \vee_y w_2$, then

$$t \prec w := t_1 \vee_x (t_2 * w) \quad \text{and} \quad t \succ w := (t * w_1) \vee_y w_2$$

4. $t * w := t \succ w + t \prec w$

The following result is due to J-L Loday [19]

Theorem 3.0.26. *For any finite set X, the space $Dend(B)$, with the products $\prec$ and $\succ$ defined above, is the free dendriform algebra on B spanned by X.*

Definition 3.0.37. Let $(D, \succ, \prec)$ be a dendriform algebra; for any $n \geq 1$ define the operations $w^n_\succ$, $w^n_\prec$ from $D^{\otimes n}$ into D as follows: $w^1_\succ(x) = w^n_\prec(x) := x$,

$$w^n_\succ(x_1, \ldots, x_n) := ((\cdots((x_1 \succ x_2) \succ x_3) \succ \cdots) \succ x_{n-1}) \succ x_n,$$

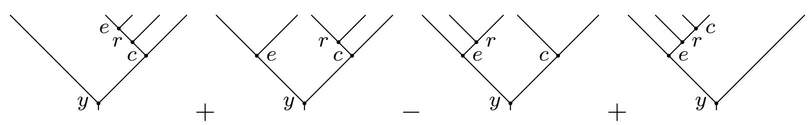
$$w^n_\prec(x_1, \ldots, x_n) := x_1 \prec (x_2 \prec \cdots (x_{n-2} \prec (x_{n-1} \prec x_n)) \cdots),$$

and define also $\langle x_1, \ldots, x_n, x \rangle_D \in D$ by the formula

$$\langle x_1, \ldots, x_n, x \rangle_D := \sum_{i=0}^{n} (-1)^{n-i} w^i_\prec(x_1, \ldots, x_i) \succ x \prec w^{n-i}_\succ(x_{i+1}, \ldots, x_n)$$

for any family of elements $x_1, \ldots, x_n, x$ of D.

Example 3.0.16.

$\langle e, r, c, y \rangle_{Dend(B)} := -$ 

$$\langle e, r, c,\ y \rangle_{Dend(B)} := -y \prec ((e \succ r) \succ c) + e \succ y \prec (r \succ c) - (e \prec r) \succ y \prec c + (e \prec (r \prec c)) \succ y$$

Definition 3.0.38. Let $(B, \succ, \prec)$ be a dendriform algebra and let

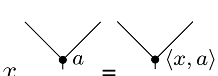
$$Dend(B) := \bigoplus_{n \geq 1} \mathbb{K}[Y_{n,X}]$$

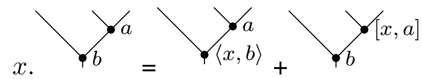
be the free dendriform algebra on B. Denote by I_B the two-sided ideal of $Dend(B)$ spanned by the elements

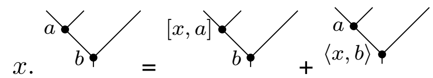
$$\langle x_1, \ldots, x_n, y \rangle_B - \langle x_1, \ldots, x_n, y \rangle_{Dend(B)}, \text{ for all } x, y \in B.$$

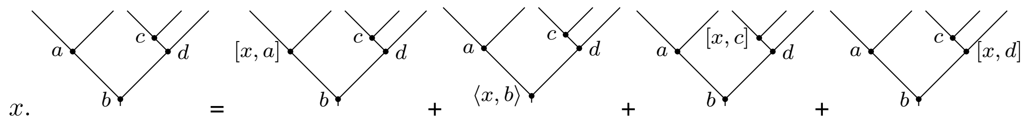
The quotient $Dend(B)/I_B$ is a dendriform algebra called the **enveloping algebra** of B; we denote it by $U_{dend}(B)$.

Let $(B, \langle\ ,\ \rangle)$ be a left-symmetric algebra; then $Dend(B)$ is an B-module, that is, there is a B-action on $Dend(B)$ defined in each tree as in the following examples, for all $x \in B$:

$x.$ 







Let J the two-side ideal of $Dend(B)$ spanned by the elements

$$\langle x,y\rangle - \langle x,y\rangle_{Dend(B)}, \text{ for all } x,y \in B.$$

This action satisfies $z.J \subseteq J$.

The following result is due to M.Ronco [30]

Theorem 3.0.27. *Let $(D,\succ,\prec)$ be a dendriform algebra. If $x_1,\dots,x_n,z_{n+1},\dots,z_{n+m}$ are elements of D then:*

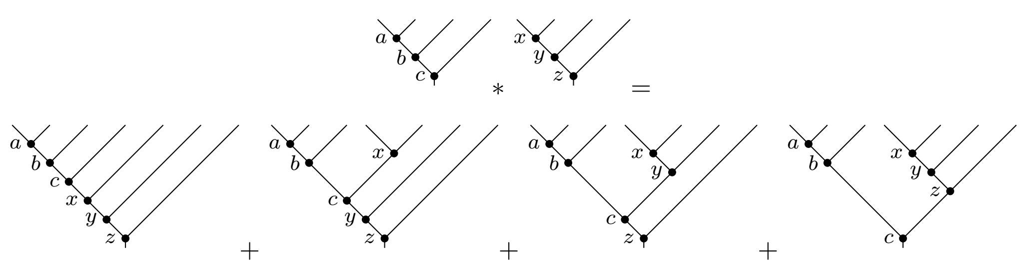
$$w_{\succ}(x_1,\dots,x_n) * w_{\succ}(z_{n+1},\dots,z_{n+m}) = \sum_{j=0}^{m} w_{\succ}(x_1,\dots,x_n \prec w_{\succ}(z_{n+1},\dots,z_{n+j}),\dots,z_{n+m})$$

Example 3.0.17. we have, for example,

$$w_{\succ}(a,b,c) * w_{\succ}(x,y,z) =$$

$$w_{\succ}(a,b,c,x,y,z) + w_{\succ}(a,b,c \prec x,y,z) + w_{\succ}(a,b,c \prec w_{\succ}(x,y),z) + w_{\succ}(a,b,c \prec w_{\succ}(x,y,z))\,.$$

In the case of $D = Dend(B)$ we have:



Theorem 3.0.28. *We can express the shuffle product as a recursive procedure as follows:*

$$\begin{aligned}(x_1 \otimes \cdots \otimes x_n) * (z_{n+1} \otimes \cdots \otimes z_{n+m}) \quad = \quad & (x_1 \otimes \cdots \otimes x_{n-1} * z_{n+1} \otimes \cdots \otimes z_{n+m}) \otimes x_n \quad + \\ & (x_1 \otimes \cdots \otimes x_{n-1} * z_{n+1} \otimes \cdots \otimes z_{n+m-1}) \otimes x_n \otimes z_{n+m} \quad + \\ & (x_1 \otimes \cdots \otimes x_{n-1} * z_{n+1} \otimes \cdots \otimes z_{n+m-2}) \otimes x_n \otimes z_{n+m-1} \otimes z_{n+m} \quad + \\ & \vdots \\ & (x_1 \otimes \cdots \otimes x_{n-1} * z_{n+1}) \otimes x_n \otimes z_{n+2} \otimes \cdots \otimes z_{n+m-1} \otimes z_{n+m} \quad + \\ & x_1 \otimes \cdots \otimes x_n \otimes z_{n+1} \otimes \cdots \otimes z_{n+m}\end{aligned}$$

Proof. If we compare the calculations of the example 3.0.15, the result is obtained. □

Example 3.0.18. Let $\overline{T}(V)$ be the cotensor algebra, Definition 3.0.32 . If $a \otimes b \otimes c$ and $x \otimes y \otimes z$ are elements in $T(V)$ then:

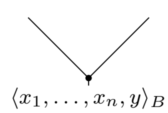
$$\begin{aligned}(a \otimes b \otimes c) * (x \otimes y \otimes z) \quad = \quad & ((a \otimes b) * (x \otimes y \otimes z)) \otimes c \quad + \\ & ((a \otimes b) * (x \otimes y)) \otimes c \otimes z \quad + \\ & ((a \otimes b) * x) \otimes c \otimes y \otimes z \quad + \\ & a \otimes b \otimes c \otimes x \otimes y \otimes z\end{aligned}$$

Theorem 3.0.29. *Let* $(B, \succ, \prec)$ *be a dendriform algebra. Then every element in the enveloping algebra of* B*,* $U_{dend}(B)$ *can be written as a finite sum of elements of the form*

$$w_{\succ}(x_1, \ldots, x_n)$$

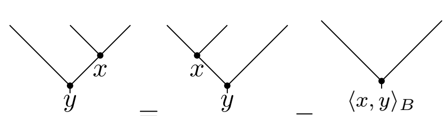
where $x_i \in B_m = span\left\{\langle y_1, \ldots, y_m\rangle : y_1, \ldots, y_m \in \bigcup_{i=1}^{m-1} B_i\right\}$ *and* $B_1 = B$.

Proof. We use the following equalities in $U_{dend}(B)$, Definition 3.0.37 and Example 3.0.16.

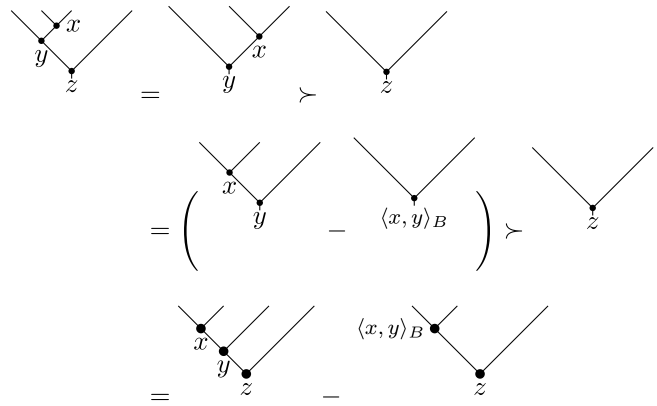


$$= \langle x_1, \ldots, x_n, y\rangle_{Dend(B)}$$

For instance,

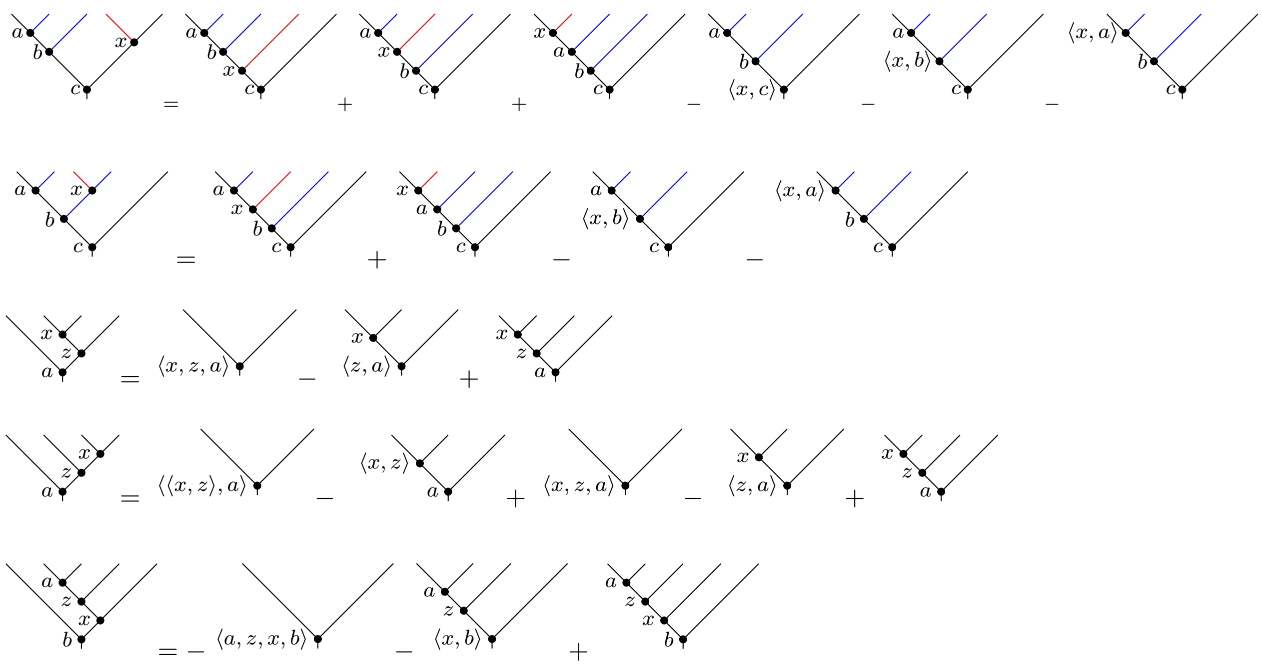


and therefore



□

Example 3.0.19. The following equalities are satisfied in $D = U_{dend}(B)$



Chapter 4

Construction of Ballot^m-Algebras from Associative Algebras with a Rota-Baxter Operator of Weight Three and Two

In this chapter we give a generalitation of Rota-Baxter Operators and introduce the notion of a Ballot^m-algebra. Combinatorial objects such as rooted trees that carry a recursive structure have found important applications recently in mathematics. This is the case for free Rota-Baxter algebras. It was shown in [3, 14] that free Rota-Baxter algebras on a set can be constructed from a subset of planar rooted forests with decorations on the angles. We now give similar constructions for obtain an associative algebra in terms of planar binary trees with a modified Rota-Baxter Operator, and so we construct a Ballot^m-algebra.

Definition 4.0.39. A **Ballot^m-algebra** over $\mathbb{K}$ is a $\mathbb{K}$-vector space H endowed with $m+1$ binary operations $\{*_j\}_{0\leq j\leq m}$ satisfying the following properties:

1. $x *_i (y *_j z) = (x *_i y) *_j z,$ for $0 \leq i < j \leq m,$
2. $x *_0 (y *_0 z) = (x *_0 y + \cdots + x *_m y) *_0 z,$
3. $x *_i (y *_0 z + \cdots + y *_{i-1} z) = (x *_{i+1} y + \cdots + x *_m y) *_i z,$ for $1 \leq i \leq m-1,$
4. $x *_m (y *_0 z + \cdots + y *_m z) = (x *_m y) *_m z,$

for all x, y and z in H.

Example 4.0.20. The properties of a Ballot^2-algebra are

$$\begin{aligned}
x *_0 (y *_1 z) &= (x *_0 y) *_1 z,\\
x *_0 (y *_2 z) &= (x *_0 y) *_2 z,\\
x *_1 (y *_2 z) &= (x *_1 y) *_2 z,\\
x *_2 (y *_0 z + y *_1 z + y *_2 z) &= (x *_2 y) *_2 z,\\
x *_1 (y *_0 z) &= (x *_2 y) *_1 z,\\
x *_0 (y *_0 z) &= (x *_0 y + x *_1 y + x *_2 y) *_0 z,
\end{aligned}$$

Remark 4.0.14. If $1 \leq j \leq m-1$, then, the operations $\succ^j := *_0 + *_1 + \cdots + *_j$ and $\prec^j := *_{j+1} + \cdots + *_m$, equips to a Ballot^m-algebra A of $m-1$ dendriform algebra structures given by: $(A, \succ^j, \prec^j)$.

4.0.10 Rota-Baxter Operator of Weight λ

Definition 4.0.40. Let $(A, \cdot)$ be an associative algebra. A linear map $R : A \to A$ is called a **Rota-Baxter operador** of weight λ on A if R satisfies

$$R(x) \cdot R(y) = R\left(R(x) \cdot y + x \cdot R(y) + \lambda\, x \cdot y\right),$$

for all $x, y \in A$. A **Rota-Baxter algebra** (also known as a **Baxter algebra**) is an associative algebra A with a **Rota-Baxter operador**.

Example 4.0.21. Let A be a $\mathbb{K}$-algebra. For a given $\lambda \in \mathbb{K}$, define $R_\lambda : A \to A$, $x \to -\lambda x$, for all $x \in A$. Then (A, R_λ) is a Rota-Baxter algebra of weight λ. In particular, id is a Rota-Baxter operator of weight -1.

Example 4.0.22. The Rota-Baxter relation finds its most prominent example in the integration by parts rule for the Riemann integral on a suitable function space, denoted by $cont(\mathbb{R})$ the ring of continuous function on $cont(\mathbb{R})$ with the standard pointwise multiplication. For a given $f \in cont(\mathbb{R})$, define $R(f) \in cont(\mathbb{R})$ by $R(f)(x) = \int\limits_0^x f(t)dt$, for all $x \in \mathbb{R}$. Then $(cont(\mathbb{R}), R)$ is a Rota-Baxter algebra of weight zero. Indeed, for $f, g \in cont(\mathbb{R})$, let $F(x) := \int\limits_0^x f(t)dt = R(f)(x)$ and $G(x) := \int\limits_0^x g(t)dt = R(g)(x)$. Then $\frac{d}{dx}F(x) := F'(x) = f(x)$ and $G'(x) = g(x)$. Integration by parts gives

$$\begin{aligned}
R(f \cdot R(g))(x) &= \int\limits_0^x f(t) \cdot G(t)dt \\
&= F(t) \cdot G(t) \mid_0^x - \int\limits_0^x F(t) \cdot g(t)dt \\
&= F(x) \cdot G(x) - \int\limits_0^x F(t) \cdot g(t)dt \\
&= (R(f) \cdot R(g))(x) - R(R(f) \cdot g)(x)
\end{aligned}$$

Thus, $R(f \cdot R(g)) = R(f) \cdot R(g) - R(R(f) \cdot g)$, which is just the Rota-Baxter relation for weight 0.

The following result provides the construction of dendriform algebras from associative algebras with a Rota-Baxter operator of weight zero.

Proposition 4.0.30. *Let A be an associative $\mathbb{K}$-algebra and $R : A \to A$ be a Rota-Baxter operator of weight zero on A. Define new operations on A by*

$$x \succ y := R(x) \cdot y \text{ and } x \prec y := x \cdot R(y).$$

for all $x, y \in A$. Then $(A, \succ, \prec)$ is a dendriform algebra.

Proof. We verify the last axiom in 3.0.31; the others are similar. We have

$$
\begin{aligned}
x \succ (y \succ z) &= R(x) \cdot (y \succ z) \\
&= R(x) \cdot (R(y) \cdot z) \\
&= (R(x) \cdot R(y)) \cdot z \\
&= R(R(x) \cdot y + x \cdot R(y))z \\
&= R(x \succ y + x \prec y) \cdot z \\
&= (x \succ y + x \prec y) \succ z
\end{aligned}
$$

□

Definition 4.0.41. If a linear map $D : A \longrightarrow A$ satisfies

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y) \text{ for all } x, y \in A,$$

then D is called a derivation on A. More generally, a linear map $d : A \longrightarrow A$ is called a derivation of weight λ and β if

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y) + \lambda d(x) \cdot d(y) + \beta d\big(d(x) \cdot d(y)\big), \text{ for all } x, y \in A.$$

Take in the definition 4.0.41 a derivation of weight λ and β, that is

$$d(x \cdot y) = d(x) \cdot y + x \cdot d(y) + \lambda d(x) \cdot d(y) + \beta d\big(d(x) \cdot d(y)\big).$$

We can use the definition 4.0.41 recursively in the last term and thus we have

$$d(x \cdot y) = \sum_{n=1}^{\infty} \beta^{n-1} \bigg(d^n(x) \cdot d^{n-1}(y) + d^{n-1}(x) \cdot d^n(y) + \lambda\, d^n(x) \cdot d^n(y) \bigg).$$

Define the operator

$$S(x, y) = d(x) \cdot y + x \cdot d(y) + \lambda\, d(x) \cdot d(y),$$

then

$$d(x \cdot y) = \sum_{n=1}^{\infty} \beta^{n-1} S(d^{n-1}(x), d^{n-1}(y)).$$

Denote $S_n(x, y) = S(d^{n-1}(x), d^{n-1}(y))$ for each $n \geq 1$. If we defined $x * y = x \cdot y + \beta\, d(x \cdot y)$, then

$$x * y = x \cdot y + \sum_{n=1}^{\infty} \beta^n S_n(x, y).$$

Therefore the definition 4.0.41 give a deformation of the product in A, with deformation parameter β.

Example 4.0.23. Let A be an commutative algebra with unit 1, and $D : A \longrightarrow A$ be a derivation on A . Let ξ be a formal variable not in A. The algebra of pseudo-differential symbols over A is the vector space

$$\Psi(A) = \left\{ \sum_{-\infty}^{n} a_\nu \xi^\nu : a_\nu \in A,\ \nu \in \mathbb{Z} \right\}$$

with the relation

$$\xi^\nu \circ a = \sum_{j=0}^{\infty} \frac{1}{j!} D^j(a) \frac{\partial^j}{\partial \xi^j}(\xi^\nu)$$

For all $a \in A$ and $\nu \in \mathbb{Z}$, for instance

$$\begin{aligned} \xi^{-3} \circ a &= \cdots + 15\, D^4(a)\xi^{-7} - 10\, D^3(a)\xi^{-6} + 6\, D^2(a)\xi^{-5} - 3\, D(a)\xi^{-4} + a\xi^{-3} \\ \xi^4 \circ a &= \cdots + 0 + 0 + D^4(a) + 4\, D^3(a)\xi^1 + 6\, D^2(a)\xi^2 + 4\, D(a)\xi^3 + a\xi^4 \end{aligned}$$

We extend $\circ$ to the associative multiplication on $\Psi(A)$ given by

$$\Big(\sum_{-\infty}^{n} b_\nu \xi^\nu\Big) \circ \Big(\sum_{-\infty}^{m} a_k \xi^k\Big) = \sum_{j=0}^{\infty} \frac{1}{j!} \Big(\sum_{-\infty}^{m} D^j(a_k)\xi^k\Big)\Big(\sum_{-\infty}^{n} b_\nu \frac{\partial^j}{\partial \xi^j}(\xi^\nu)\Big)$$

Where $D^0(a) = a,\ \frac{\partial^0}{\partial \xi^0}(\xi^\nu) = \xi^\nu$ and if we denote $\nu(\nu-1)\cdots(\nu-j+1)$ by $\binom{\nu}{j}$, then

$$\Big(\sum_{-\infty}^{m} D^j(a_k)\xi^k\Big)\Big(\sum_{-\infty}^{n} b_\nu \frac{\partial^j}{\partial \xi^j}(\xi^\nu)\Big) = \sum_{\mu} \left(\sum_{k+\nu=\mu} D^j(a_k) b_\nu \binom{\nu}{j} \right) \xi^{\mu-j}.$$

The derivation D on A extends to a derivation $\mathcal{D}$ on $\Psi(A)$ via

$$\mathcal{D}(\sum_{-\infty}^{n} a_\nu \xi^\nu) = \sum_{-\infty}^{n} D(a_\nu)\xi^\nu$$

If D is invertible, then $\mathcal{D}^{-1}$ a Rota-Baxter operator on $\Psi(A)$.

For the following proposition see definition 4.0.42.

Proposition 4.0.31. *Let* $(A, \cdot)$ *be an algebra over* $\mathbb{K}$. *An invertible linear mapping* $P : A \to A$ *is a Rota-Baxter operator of weight* λ *and* β *on* A *if and only if* P^{-1} *is a derivation of weight* λ *and* β *on* A.

Proof. If an invertible linear mapping $P : A \longrightarrow A$ is a Rota-Baxter Operator of weight λ, and β then

$$P(x) \cdot P(y) = P(P(x) \cdot y + x \cdot P(y) + \lambda x \cdot y) + \beta x \cdot y \text{ for all } x, y \in A$$

Set $z = P(x), w = P(y)$. Then by last equation we have

$$P^{-1}(z \cdot w) = z \cdot P^{-1}(w) + P^{-1}(z) \cdot w + \lambda P^{-1}(z) \cdot P^{-1}(w) + \beta P^{-1}\big(P^{-1}(z) \cdot P^{-1}(y)\big).$$

Therefore P^{-1} is a derivation of weight λ and β.
Conversely, let P be a derivation of weight λ, and β then

$$P(P^{-1}(z) \cdot P^{-1}(w)) = z \cdot P^{-1}(w) + P^{-1}(z) \cdot w + \lambda z \cdot w + P\big(z \cdot w\big) \text{ for all } z, w \in A$$

Therefore

$$P^{-1}(z) \cdot P^{-1}(w) = P^{-1}(z \cdot P^{-1}(w) + P^{-1}(z) \cdot w + \lambda z \cdot w) + \beta z \cdot w \text{ for all } z, w \in A$$

This proves the result. □

Proposition 4.0.32. *Let $(A, \cdot)$ be an associative algebra and $R : A \to A$ be a Rota-Baxter operator of weight one on A. That is,*

$$R(x) \cdot R(y) = R\left(R(x) \cdot y + x \cdot R(y) + x \cdot y\right),$$

for all $x, y \in A$. Define three new operations on A by

$$x *_0 y := R(x) \cdot y, \quad x *_1 y := x \cdot y \quad \text{and} \quad x *_2 y := x \cdot R(y),$$

for all $x, y \in A$. Then, $(A, *_0, *_1, *_2)$ is a Ballot^2-algebra.

Proof. We have for all $x, y, z \in A$

$$\begin{aligned} x *_0 (y *_1 z) &= R(x) \cdot (y \cdot z) \\ &= (R(x) \cdot y) \cdot z \\ &= (x *_0 y) *_1 z \end{aligned}$$

$$\begin{aligned} x *_1 (y *_0 z) &= x \cdot (R(y) \cdot z) \\ &= (x \cdot R(y)) \cdot z \\ &= (x *_2 y) *_1 z \end{aligned}$$

$$\begin{aligned} x *_0 (y *_2 z) &= R(x) \cdot (y \cdot R(z)) \\ &= (R(x) \cdot y) \cdot R(z) \\ &= (x *_0 y) *_2 z \end{aligned}$$

$$\begin{aligned} x *_0 (y *_0 z) &= R(x) \cdot (R(y) \cdot z) \\ &= (R(x) \cdot R(y)) \cdot z \\ &= R(R(x) \cdot y + x \cdot y + x \cdot R(y)) \cdot z \\ &= (x *_0 y + x *_1 y + x *_2 y) *_0 z \end{aligned}$$

$$\begin{aligned} x *_1 (y *_2 z) &= x \cdot (y \cdot R(z)) \\ &= (x \cdot y) \cdot R(z) \\ &= (x *_1 y) *_2 z \end{aligned}$$

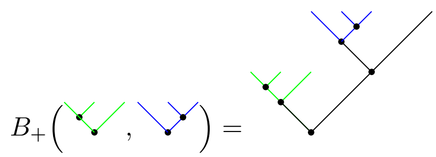
$$\begin{aligned} x *_2 (y *_0 z + y *_1 z + y *_2 z) &= x \cdot R(R(y) \cdot z + y \cdot z + y \cdot R(z)) \\ &= x \cdot (R(y) \cdot R(z)) \\ &= (x \cdot R(y)) \cdot R(z) \\ &= (x *_2 y) *_2 z \end{aligned}$$

□

Example 4.0.24. Let $\mathbb{K}[Y_n]$ be the vector space generated by the set Y_n of all planar binary trees with n + 1 leaves. For any $t \in Y_k$ and $w \in Y_m$ we define recursively the following product on $\bigoplus_{n \geq 0} \mathbb{K}[Y_n]$ extending it by linearity:

$$t \cdot w := \begin{cases} t, & \text{if} \quad w = |, \\ w, & \text{if} \quad t = |, \\ B_+(t_1, \dots, t_n \cdot w) + B_+(t \cdot w_1, \dots, w_m) & \\ \quad + B_+(t_1, \dots, t_n \cdot w_1, \dots, w_m), & \text{if} \quad t = B_+(t_1, \dots, t_n) \text{ and } w = B_+(w_1, \dots, w_m), \end{cases}$$

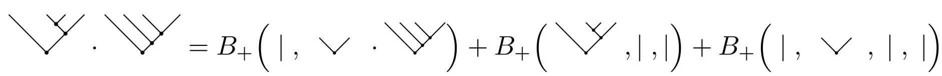
where B_+ associates to the planar binary trees $t_1, \dots, t_n$, the tree obtained by the joining of the root of each t_j, $1 \le j \le n$ with a new vertex on a unique edge, drawing a new internal edge for each j, this is.



Observe that any planar binary tree of degree bigger than one can be written in a unique way as

$$t = B_+(t_1, \dots, t_n).$$

As an example we have



We define the operator

$$R(t) = B_+(t), \text{ for all } t \in \bigoplus_{n \ge 0} \mathbb{K}[Y_n].$$

Then, R is a Rota-Baxter operator of weight one on $\bigoplus_{n \ge 0} \mathbb{K}[Y_n]$.

Proposition 4.0.33. *Let* $t = B_+(t_1, \dots, t_n)$*,* $w = B_+(w_1, \dots, w_m)$ *and* $z = B_+(z_1, \dots, z_k)$ *be planar binary trees in the algebra* $\big(\bigoplus_{n \ge 0} \mathbb{K}[Y_n], \cdot\big)$*. Suppose that*

$$\begin{aligned} t \cdot (w \cdot z_1) &= (t \cdot w) \cdot z_1 \\ t_n \cdot (w \cdot z_1) &= (t_n \cdot w) \cdot z_1 \\ t_n \cdot (w \cdot z) &= (t_n \cdot w) \cdot z \end{aligned}$$

Then,

$$t \cdot (w \cdot z) = (t \cdot w) \cdot z.$$

Proof. $(t.w).z = (B_+(t_1, \dots, t_n \cdot w) + B_+(t \cdot w_1, \dots, w_m) + B_+(t_1, \dots, t_n \cdot w_1, \dots, w_m)) \cdot z$

$$\begin{aligned} (B_+(t_1, \dots, t_n \cdot w)) \cdot z = \;& B_+(t_1, \dots, (t_n \cdot w) \cdot z) + \\ & B_+(B_+(t_1, \dots, t_n \cdot w) \cdot z_1, \dots, z_k) + \\ & B_+(t_1, \dots, (t_n \cdot w) \cdot z_1, \dots, z_k) \;. \end{aligned}$$

$$\begin{aligned} (B_+(t \cdot w_1, \dots, w_m)) \cdot z = \;& B_+(t \cdot w_1, \dots, w_m \cdot z) + \\ & B_+(B_+(t \cdot w_1, \dots, w_m) \cdot z_1, \dots, z_k) + \\ & B_+(t \cdot w_1, \dots, w_m \cdot z_1, \dots, z_k) \;. \end{aligned}$$

$$
\begin{aligned}
(B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m))\cdot z = \quad & B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z) + \\
& B_+(B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m)\cdot z_1,\ldots,z_k) + \\
& B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)\ .
\end{aligned}
$$

We can simplify the second term of the previous development by $B_+((t\cdot w)\cdot z_1,\ldots,z_k)$, since

$$
\begin{aligned}
B_+((t\cdot w)\cdot z_1,\ldots,z_k) = \quad & B_+(B_+(t\cdot w_1,\ldots,w_m)\cdot z_1,\ldots,z_k) + \\
& B_+(B_+(t_1,\ldots,t_n\cdot w)\cdot z_1,\ldots,z_k) + \\
& B_+(B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m)\cdot z_1,\ldots,z_k)\ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(t\cdot w)\cdot z = \quad & B_+(t_1,\ldots,(t_n\cdot w)\cdot z) + \\
& B_+(t_1,\ldots,(t_n\cdot w)\cdot z_1,\ldots,z_k) + \\
& B_+(t\cdot w_1,\ldots,w_m\cdot z) + \\
& B_+(t\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k) + \\
& B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z) + \\
& B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k) + \\
& B_+((t\cdot w)\cdot z_1,\ldots,z_k)\ .
\end{aligned}
$$

On the other hand, we have

$t.(w.z) = t\cdot(B_+(w_1,\ldots,w_m\cdot z) + B_+(w\cdot z_1,\ldots,z_k) + B_+(w_1,\ldots,w_m\cdot z_1,\ldots,z_k))$, and

$$
\begin{aligned}
t\cdot(B_+(w_1,\ldots,w_m\cdot z)) = \quad & B_+(t\cdot w_1,\ldots,w_m\cdot z) + \\
& B_+(t_1\ldots,t_n\cdot B_+(w_1,\ldots,w_m\cdot z)) + \\
& B_+(t_1\ldots,t_n\cdot w_1,\ldots,w_m\cdot z)\ . \\
t\cdot(B_+(w\cdot z_1,\ldots,z_k)) = \quad & B_+(t\cdot(w\cdot z_1),\ldots,z_k) + \\
& B_+(t_1,\ldots,t_n\cdot B_+(w\cdot z_1,\ldots,z_k)) + \\
& B_+(t_1,\ldots,t_n\cdot(w\cdot z_1),\ldots,z_k)\ . \\
t\cdot(B_+(w_1,\ldots,w_m\cdot z_1,\ldots,z_k)) = \quad & B_+(t\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k) + \\
& B_+(t_1,\ldots,t_n\cdot B_+(w_1,\ldots,w_m\cdot z_1,\ldots,z_k)) + \\
& B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)\ .
\end{aligned}
$$

We can simplify the second term of the above development by $B_+(t_1,\ldots,t_n\cdot(w\cdot z))$, since

$$
\begin{aligned}
B_+(t_1,\ldots,t_n\cdot(w\cdot z)) = \quad & B_+(t_1\ldots,t_n\cdot B_+(w_1,\ldots,w_m\cdot z)) + \\
& B_+(t_1,\ldots,t_n\cdot B_+(w\cdot z_1,\ldots,z_k)) + \\
& B_+(t_1,\ldots,t_n\cdot B_+(w_1,\ldots,w_m\cdot z_1,\ldots,z_k)).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
t\cdot(w\cdot z) = \quad & B_+(t\cdot w_1,\ldots,w_m\cdot z) + \\
& B_+(t_1\ldots,t_n\cdot w_1,\ldots,w_m\cdot z) + \\
& B_+(t\cdot(w\cdot z_1),\ldots,z_k) + \\
& B_+(t_1,\ldots,t_n\cdot(w\cdot z_1),\ldots,z_k) + \\
& B_+(t\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k) + \\
& B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k) + \\
& B_+(t_1,\ldots,t_n\cdot(w\cdot z))
\end{aligned}
$$

Finally, if we compare the calculations of the two parts, the result is obtained. □

Corollary 4.0.34. *Let* $t = B_+(t_1, \ldots, t_n)$*,* $w = B_+(w_1, \ldots, w_m)$ *and* $z = B_+(z_1, \ldots, z_k)$ *be planar binary trees in the algebra* $\big(\bigoplus_{n\geq 0} \mathbb{K}[Y_n],\ \cdot\ \big)$*. Suppose that*

$$t_n = | = z_1 \,.$$

Then,

$$t \cdot (w \cdot z) = (t \cdot w) \cdot z \,.$$

Proof. We have that by proposition 4.0.33 with $t_n = |$ and $z_1 = |$, the corollary is satisfied.

□

Corollary 4.0.35. *Let* $t = B_+(t_1, \ldots, t_n)$*,* $w = B_+(w_1, \ldots, w_m)$ *and* $z = B_+(z_1, \ldots, z_k)$ *be planar binary trees in the algebra* $\big(\bigoplus_{n\geq 0} \mathbb{K}[Y_n],\ \cdot\ \big)$*. Suppose that*

$$t_n = B_+(t_{n1}, \ldots, t_{nn'}) \text{ with } t_{nn'} = | \text{ and that}$$

$$z_1 = B_+(z_{11}, \ldots, z_{1k'}) \text{ with } z_{11} = | \,.$$

Then,

$$t \cdot (w \cdot z) = (t \cdot w) \cdot z.$$

Proof. We have that by corollary 4.0.34 with $t_{nn'} = |$ and $z_{11} = |$, the equality

$$t_n \cdot (w \cdot z_1) = (t_n \cdot w) \cdot z_1$$

is satisfied.
Thus we have that the equations

$$\begin{aligned} t \cdot (w \cdot z_{11}) &= (t \cdot w) \cdot z_{11} \\ t_n \cdot (w \cdot z_{11}) &= (t_n \cdot w) \cdot z_{11} \\ t_n \cdot (w \cdot z_1) &= (t_n \cdot w) \cdot z_1 \end{aligned}$$

are true, and therefore by proposition 4.0.33 we have

$$t \cdot (w \cdot z_1) = (t \cdot w) \cdot z_1.$$

In the same manner, we have that equations

$$\begin{aligned} t_n \cdot (w \cdot z_1) &= (t_n \cdot w) \cdot z_1 \\ t_{nn'} \cdot (w \cdot z_1) &= (t_{nn'} \cdot w) \cdot z_1 \\ t_{nn'} \cdot (w \cdot z) &= (t_{nn'} \cdot w) \cdot z \end{aligned}$$

are true, and so by proposition 4.0.33 we have

$$t_n \cdot (w \cdot z) = (t_n \cdot w) \cdot z.$$

Finally by proposition 4.0.33, the equalities

$$t \cdot (w \cdot z_1) = (t \cdot w) \cdot z_1.$$

$$t_n \cdot (w \cdot z_1) = t_n \cdot (w \cdot z_1)$$

$$t_n \cdot (w \cdot z) = (t_n \cdot w) \cdot z.$$

imply that

$$t \cdot (w \cdot z) = (t \cdot w) \cdot z.$$

□

Remark 4.0.15. All planar binary trees of degree n can be expressed in terms of

$$|,\ B_+(|),\ B_+(|,|), \ldots, B_+(\underbrace{|, \ldots, |}_{n-1}),$$

For instance, The set planar binary trees of degree 4 is:

$$Y_4 = \Big\{ B_+(\,|,|,|,|\,),\quad B_+(\,B_+(\,|,|,|\,)\,),\quad B_+(\,B_+(\,B_+(\,|,|\,)\,)\,),\quad B_+(\,B_+(\,B_+(\,B_+(\,|\,)\,)\,)\,),$$

$$B_+(\,B_+(\,B_+(\,|\,),|\,)\,),\quad B_+(B_+(\,|,B_+(\,|\,)\,)\,),\quad B_+(\,B_+(\,|\,),|,|\,),\quad B_+(\,|,B_+(\,|\,),|\,),\quad B_+(\,|,|,B_+(\,|\,)\,),$$

$$B_+(\,B_+(\,|,|\,),|\,),\ \ B_+(\,|,B_+(\,|,|\,)\,),\ \ B_+(\,|,B_+(\,B_+(\,|\,)\,)\,),\ \ B_+(\,B_+(\,B_+(\,|\,)\,),|\,),\ \ B_+(\,B_+(\,|\,),B_+(\,|\,)\,)\Big\}$$

Corollary 4.0.36. *The algebra* $\big(\bigoplus_{n \geq 0} \mathbb{K}[Y_n],\ \cdot\,\big)$ *is associative.*

Proof. We have that by corollaries 4.0.34 and 4.0.35 and by remark 4.0.15, this corollary is satisfied. □

4.0.11 Rota-Baxter Operator of Weight λ and β

Definition 4.0.42. Let $(A, \cdot)$ be an associative algebra. A linear map $R : A \to A$ is called a **Rota-Baxter operador** of weight λ and β on A if R satisfies

$$R(x) \cdot R(y) = R\left(R(x) \cdot y + x \cdot R(y) + \lambda\, x \cdot y\right) + \beta\, x \cdot y,$$

for all $x, y \in A$.

Example 4.0.25. Let Y be the set of planar binary trees and let $F = \{(t_1, \ldots, t_n) : t_1, \ldots, t_n \in Y, n \in \mathbb{N}\}$; each element of F is called a forest. We are going to define a product $\cdot$ on the vector space generated by the set F, $\mathbb{K}[F]$. We define $\cdot$ by given a set map

$$\cdot \;: Y \times Y \to \mathbb{K}[F],$$

and then extending it bilinearly. First assume that t and w are planar binary trees. Then we note that a tree is either $|$ or is of the form $B_+(t_1, \ldots, t_n)$ for unique trees $t_1, \ldots, t_n$. Thus we can define

$$t \cdot w := \begin{cases} t, & \text{if} \quad w = |, \\ w, & \text{if} \quad t = |, \\ B_+(t_1, \ldots, t_n \cdot w) + B_+(t \cdot w_1, \ldots, w_m) & \\ \quad + 3\, B_+(t_1, \ldots, t_n \cdot w_1, \ldots, w_m) + 2(t_1, \ldots, t_n \cdot w_1, \ldots, w_m), & \text{if} \quad t = B_+(t_1, \ldots, t_n) \text{ and} \\ & \qquad w = B_+(w_1, \ldots, w_m), \end{cases}$$

Now, consider arbitrary forests $(t_1, \ldots, t_n)$ and $(w_1, \ldots, w_m)$ we define

$$\begin{aligned} (t_1, \ldots, t_n) \cdot (w_1, \ldots, w_m) &:= (t_1, \ldots, t_n \cdot (w_1, \ldots, w_m)) \\ &:= (t_1, \ldots, (t_n \cdot w_1, \ldots, w_m)) \\ &:= ((t_1, \ldots, t_n) \cdot w_1, \ldots, w_m) \\ &:= ((t_1, \ldots, t_n \cdot w_1), \ldots, w_m) \\ &:= (t_1, \ldots, t_n \cdot w_1, \ldots, w_m). \end{aligned}$$

Extending $\cdot$ bilinearly, we obtain a binary operation

$$\cdot \;: \mathbb{K}[F] \times \mathbb{K}[F] \to \mathbb{K}[F].$$

Define

$$R((t_1, \ldots, t_n)) = B_+(t_1, \ldots, t_n), \text{ for all } (t_1, \ldots, t_n) \in \mathbb{K}[F].$$

Then, R is a Rota-Baxter operator of weight three and two on $\mathbb{K}[F]$. We will check this fact in several steps. First of all, we need to check that $(\mathbb{K}[F]\,.)$ is really an associative algebra:

For $m \neq 1$ the associativity

$$((t_1, \ldots, t_n) \cdot (w_1, \ldots, w_m)) \cdot (z_1, \ldots, z_k) = (t_1, \ldots, t_n) \cdot ((w_1, \ldots, w_m) \cdot (z_1, \ldots, z_k))$$

is satisfied in the algebra $(\mathbb{K}[F], \cdot)$. We have to verify associativity for planar binary trees.

Proposition 4.0.37. *Let* $t = B_+(t_1, \ldots, t_n)$, $w = B_+(w_1, \ldots, w_m)$ *and* $z = B_+(z_1, \ldots, z_k)$ *be planar binary trees in the algebra* $(\mathbb{K}[F], \cdot)$. *Suppose that*

$$\begin{aligned} t \cdot (w \cdot z_1) &= (t \cdot w) \cdot z_1 \\ t_n \cdot (w \cdot z_1) &= (t_n \cdot w) \cdot z_1 \\ t_n \cdot (w \cdot z) &= (t_n \cdot w) \cdot z \end{aligned}$$

Then,

$$t \cdot (w \cdot z) = (t \cdot w) \cdot z.$$

Proof.

$$\begin{aligned}(t.w).z \quad = \quad & \big(B_+(t_1,\ldots,t_n\cdot w)+B_+(t\cdot w_1,\ldots,w_m)+3\,B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m)+\\ & 2\,(t_1,\ldots,t_n\cdot w_1,\ldots,w_m)\big)\cdot z\end{aligned}$$

$$\begin{aligned}(B_+(t_1,\ldots,t_n\cdot w))\cdot z = \quad & B_+(t_1,\ldots,(t_n\cdot w)\cdot z)+\\ & B_+(B_+(t_1,\ldots,t_n\cdot w)\cdot z_1,\ldots,z_k)+\\ & 3\,B_+(t_1,\ldots,(t_n\cdot w)\cdot z_1,\ldots,z_k)+\\ & 2\,(t_1,\ldots,(t_n\cdot w)\cdot z_1,\ldots,z_k)\,.\\ (B_+(t\cdot w_1,\ldots,w_m))\cdot z = \quad & B_+(t\cdot w_1,\ldots,w_m\cdot z)+\\ & B_+(B_+(t\cdot w_1,\ldots,w_m)\cdot z_1,\ldots,z_k)+\\ & 3\,B_+(t\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)+\\ & 2\,(t\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)\\ (3\,B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m))\cdot z = \quad & 3\,B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z)+\\ & 3\,B_+(B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m)\cdot z_1,\ldots,z_k)+\\ & 9\,B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)+\\ & 6\,(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)\,.\\ 2\,\,(t_1,\ldots,t_n\cdot w_1,\ldots,w_m)\cdot z = \quad & 2\,\,(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z)\end{aligned}$$

We simplify some terms of the previous development by $B_+((t\cdot w)\cdot z_1,\ldots,z_k)$, since,

$$\begin{aligned}B_+((t\cdot w)\cdot z_1,\ldots,z_k) = \quad & B_+(B_+(t_1,\ldots,t_n\cdot w)\cdot z_1,\ldots,z_k)+\\ & B_+(B_+(t\cdot w_1,\ldots,w_m)\cdot z_1,\ldots,z_k)+\\ & 3\,B_+(B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m)\cdot z_1,\ldots,z_k)\,.\\ & 2\,B_+\,(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)\,\,.\end{aligned}$$

Thus,

$$\begin{aligned}(t\cdot w)\cdot z = \quad & B_+(t_1,\ldots,(t_n\cdot w)\cdot z)+\\ & 3\,B_+(t_1,\ldots,(t_n\cdot w)\cdot z_1,\ldots,z_k)+\\ & 2(t_1,\ldots,(t_n\cdot w)\cdot z_1,\ldots,z_k)+\\ & B_+(t\cdot w_1,\ldots,w_m\cdot z)+\\ & 3\,B_+(t\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)+\\ & 2\,(t\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)+\\ & 3\,B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z)+\\ & 7\,B_+(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)+\\ & 6\,(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z_1,\ldots,z_k)+\\ & 6\,(t_1,\ldots,t_n\cdot w_1,\ldots,w_m\cdot z)+\\ & B_+((t\cdot w)\cdot z_1,\ldots,z_k)\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}t.(w.z) \quad = \quad & t\cdot\big(B_+(w_1,\ldots,w_m\cdot z)+B_+(w\cdot z_1,\ldots,z_k)+3\,B_+(w_1,\ldots,w_m\cdot z_1,\ldots,z_k)+\\ & 2\,(w_1,\ldots,w_m\cdot z_1,\ldots,z_k)\big)\,,\text{ and we compute each term in turn:}\end{aligned}$$

$$
\begin{aligned}
t \cdot (B_+(w_1, \ldots, w_m \cdot z)) = \quad & B_+(t \cdot w_1, \ldots, w_m \cdot z) + \\
& B_+(t_1 \ldots, t_n \cdot B_+(w_1, \ldots, w_m \cdot z)) + \\
& 3B_+(t_1 \ldots, t_n \cdot w_1, \ldots, w_m \cdot z) + \\
& 2\,(t_1 \ldots, t_n \cdot w_1, \ldots, w_m \cdot z)\,.
\end{aligned}
$$

$$
\begin{aligned}
t \cdot (B_+(w \cdot z_1, \ldots, z_k)) = \quad & B_+(t \cdot (w \cdot z_1), \ldots, z_k) + \\
& B_+(t_1, \ldots, t_n \cdot B_+(w \cdot z_1, \ldots, z_k)) + \\
& 3\,B_+(t_1, \ldots, t_n \cdot (w \cdot z_1), \ldots, z_k) + \\
& 2\,(t_1, \ldots, t_n \cdot (w \cdot z_1), \ldots, z_k)\,.
\end{aligned}
$$

$$
\begin{aligned}
3\,t \cdot (B_+(w_1, \ldots, w_m \cdot z_1, \ldots, z_k)) = \quad & 3\,B_+(t \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k) + \\
& 3\,B_+(t_1, \ldots, t_n \cdot B_+(w_1, \ldots, w_m \cdot z_1, \ldots, z_k)) + \\
& 9\,B_+(t_1, \ldots, t_n \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k) + \\
& 6\,(t_1, \ldots, t_n \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k)\,.
\end{aligned}
$$

$$
2\,t \cdot (w_1, \ldots, w_m \cdot z_1, \ldots, z_k) = \quad 2\,(t \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k)\,.
$$

We simplify some terms of the above development by $B_+(t_1, \ldots, t_n \cdot (w \cdot z))$, since

$$
\begin{aligned}
B_+(t_1, \ldots, t_n \cdot (w \cdot z)) = \quad & B_+(t_1 \ldots, t_n \cdot B_+(w_1, \ldots, w_m \cdot z)) + \\
& B_+(t_1, \ldots, t_n \cdot B_+(w \cdot z_1, \ldots, z_k)) + \\
& 3\,B_+(t_1, \ldots, t_n \cdot B_+(w_1, \ldots, w_m \cdot z_1, \ldots, z_k)) + \\
& 2\,B_+(t_1, \ldots, t_n \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k)\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
t \cdot (w \cdot z) = \quad & B_+(t \cdot w_1, \ldots, w_m \cdot z) + \\
& 3 \cdot B_+(t_1 \ldots, t_n \cdot w_1, \ldots, w_m \cdot z) + \\
& 2\,(t_1 \ldots, t_n \cdot w_1, \ldots, w_m \cdot z) + \\
& B_+(t \cdot (w \cdot z_1), \ldots, z_k) + \\
& 3\,B_+(t_1, \ldots, t_n \cdot (w \cdot z_1), \ldots, z_k) + \\
& 2\,(t_1, \ldots, t_n \cdot (w \cdot z_1), \ldots, z_k) + \\
& 3\,B_+(t \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k) + \\
& 7\,B_+(t_1, \ldots, t_n \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k) + \\
& 6\,(t_1, \ldots, t_n \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k) + \\
& 2\,(t \cdot w_1, \ldots, w_m \cdot z_1, \ldots, z_k) + \\
& B_+(t_1, \ldots, t_n \cdot (w \cdot z))\,.
\end{aligned}
$$

Finally, if we compare the calculations of the two parts and suppose that

$$
\begin{aligned}
t \cdot (w \cdot z_1) &= (t \cdot w) \cdot z_1 \\
t_n \cdot (w \cdot z_1) &= (t_n \cdot w) \cdot z_1 \\
t_n \cdot (w \cdot z) &= (t_n \cdot w) \cdot z,
\end{aligned}
$$

the result is obtained. □

Corollary 4.0.38. *Let* $t = B_+(t_1, \ldots, t_n)$, $w = B_+(w_1, \ldots, w_m)$ *and* $z = B_+(z_1, \ldots, z_k)$ *be planar binary trees in the algebra* $(\,\mathbb{K}[F],\,\cdot\,)$. *Suppose that*

$$t_n = | = z_1$$

Then,

$$t \cdot (w \cdot z) = (t \cdot w) \cdot z.$$

Proof. We have that by proposition 4.0.37 with $t_n = |$ and $z_1 = |$, the corollary is satisfied.

□

Corollary 4.0.39. *Let* $t = B_+(t_1, \dots, t_n)$*,* $w = B_+(w_1, \dots, w_m)$ *and* $z = B_+(z_1, \dots, z_k)$ *be planar binary trees in the algebra* $\left(\mathbb{K}[F], \cdot \right)$*. Suppose that*

$$t_n = B_+(t_{n1}, \dots, t_{nn'}) \text{ with } t_{nn'} = |$$

$$z_1 = B_+(z_{11}, \dots, z_{1k'}) \text{ with } z_{11} = |$$

Then,

$$t \cdot (w \cdot z) = (t \cdot w) \cdot z.$$

Proof. We have that by corollary 4.0.38 with $t_{nn'} = |$ and $z_{11} = |$, the equality is satisfied.

$$t_n \cdot (w \cdot z_1) = t_n \cdot (w \cdot z_1)$$

and the proof continues in the same way as in the corollary 4.0.35, using in this case the corollary 4.0.38 and proposition 4.0.37.

□

Corollary 4.0.40. *The algebra* $\left(\mathbb{K}[F], \cdot \right)$ *is associative.*

Proof. We have that by corollaries 4.0.38 and 4.0.39 and by remark 4.0.15, this corollary is satisfied □

Proposition 4.0.41. *The linear map* $R : \mathbb{K}[F] \to \mathbb{K}[F]$*,*

$$R((t_1, \dots, t_n)) = B_+(t_1, \dots, t_n), \text{ for all } (t_1, \dots, t_n) \in \mathbb{K}[F].$$

is a Rota-Baxter operator of weight three and two on $\mathbb{K}[F]$*.*

Proof. We just need to prove that

$$R((t_1, \dots, t_n)) = B_+(t_1, \dots, t_n), \text{ for all } (t_1, \dots, t_n) \in \mathbb{K}[F].$$

is a Rota-Baxter operator of weight three and two on $\mathbb{K}[F]$. This is immediate from equation

$$t \cdot w := \begin{cases} t, & \text{if} \quad w = |, \\ w, & \text{if} \quad t = |, \\ B_+(t_1, \ldots, t_n \cdot w) + B_+(t \cdot w_1, \ldots, w_m) & \\ \quad + 3\, B_+(t_1, \ldots, t_n \cdot w_1, \ldots, w_m) + 2(t_1, \ldots, t_n \cdot w_1, \ldots, w_m), & \text{if} \quad t = B_+(t_1, \ldots, t_n) \text{ and} \\ & \quad\;\; w = B_+(w_1, \ldots, w_m), \end{cases}$$

□

Proposition 4.0.42. *Let $(A, \cdot)$ be an associative algebra and $R : A \to A$ be a Rota-Baxter operator of weight three and two on A. That is,*

$$R(x) \cdot R(y) = R\left(R(x) \cdot y + x \cdot R(y) + 3x \cdot y\right) + 2x \cdot y,$$

for all $x, y \in A$. Define four new operations on A by

$$x *_0 y := R(x) \cdot y + x \cdot y, \quad x *_1 y := x \cdot y, \quad x *_2 y := -x \cdot y \quad \text{and} \quad x *_3 y := x \cdot R(y) + 2x \cdot y,$$

for all $x, y \in A$. Then, $(A, *_0, *_1, *_2, *_3)$ is a Ballot^3-algebra.

Proof. We have for all $x, y, z \in A$

$$\begin{aligned} x *_0 (y *_3 z) &= R(x) \cdot (y \cdot R(z) + 2y \cdot z) + x \cdot (y \cdot R(z) + 2y \cdot z) \\ &= R(x) \cdot (y \cdot R(z)) + x \cdot (y \cdot R(z)) + R(x) \cdot (2y \cdot z) + x \cdot (2y \cdot z) \\ &= (R(x) \cdot y) \cdot R(z) + (x \cdot y) \cdot R(z) + (R(x) \cdot 2y) \cdot z + (x \cdot 2y) \cdot z \\ &= (R(x) \cdot y + x \cdot y) \cdot R(z) + 2(R(x) \cdot y + x \cdot y) \cdot z \\ &= (x *_0 y) *_3 z \end{aligned}$$

$$\begin{aligned} x *_0 (y *_1 z) &= R(x) \cdot (y \cdot z) + x \cdot (y \cdot z) \\ &= (R(x) \cdot y) \cdot z + (x \cdot y) \cdot z \\ &= (R(x) \cdot y + x \cdot y) \cdot z \\ &= (x *_0 y) *_1 z \end{aligned}$$

$$\begin{aligned} x *_1 (y *_2 z) &= x \cdot (-y \cdot z) \\ &= -(x \cdot y) \cdot z \\ &= (x *_1 y) *_2 z \end{aligned}$$

$$\begin{aligned} x *_1 (y *_3 z) &= x \cdot (y \cdot R(z) + 2y \cdot z) \\ &= (x \cdot y) \cdot R(z) + 2(x \cdot y) \cdot z \\ &= (x *_1 y) *_3 z \end{aligned}$$

$$\begin{aligned} x *_0 (y *_2 z) &= R(x) \cdot (-y \cdot z) + x \cdot (-y \cdot z) \\ &= -(R(x) \cdot y) \cdot z - (x \cdot y) \cdot z \\ &= -(R(x) \cdot y + x \cdot y) \cdot z \\ &= (x *_0 y) *_2 z \end{aligned}$$

$$\begin{aligned} x *_2 (y *_3 z) &= -x \cdot (y \cdot R(z) + 2y \cdot z) \\ &= -x \cdot (y \cdot R(z)) - x \cdot (2y \cdot z) \\ &= -(x \cdot y) \cdot R(z) - 2(x \cdot y) \cdot z \\ &= (x *_2 y) *_3 z \end{aligned}$$

$$\begin{aligned}
x *_3 (y *_0 z + y *_1 z + y *_2 z + y *_3 z) &= x \cdot R\big(R(y) \cdot z + y \cdot z + y \cdot z - y \cdot z + y \cdot R(z) + 2y \cdot z\big) \\
&\quad + 2x \cdot \big(R(y) \cdot z + y \cdot z + y \cdot z - y \cdot z + y \cdot R(z) + 2y \cdot z\big) \\
&= x \cdot R\big(R(y) \cdot z + y \cdot R(z) + 3y \cdot z\big) \\
&\quad +2x \cdot \big(y \cdot R(z) + R(y) \cdot z + 3y \cdot z\big) \\
&= x \cdot \big(R(R(y) \cdot z + y \cdot R(z) + 3y \cdot z) + 2y \cdot z\big) \\
&\quad + 2x \cdot (y \cdot R(z)) + 2x \cdot (R(y) \cdot z) + 4x \cdot (y \cdot z) \\
&= x \cdot (R(y) \cdot R(z)) + 2x \cdot (y \cdot R(z)) + 2x \cdot (R(y) \cdot z) + 4x \cdot (y \cdot z) \\
&= (x \cdot R(y)) \cdot R(z) + (2x \cdot y) \cdot R(z) + 2x \cdot (R(y) \cdot z) + 4x \cdot (y \cdot z) \\
&= (x \cdot R(y) + 2x \cdot y) \cdot R(z) + 2(x \cdot R(y) + 2x \cdot y) \cdot z \\
&= (x *_3 y) *_3 z
\end{aligned}$$

$$\begin{aligned}
x *_2 (y *_0 z + y *_1 z) &= -x \cdot (R(y) \cdot z + y \cdot z + y \cdot z) \\
&= -x \cdot (R(y) \cdot z + 2y \cdot z) \\
&= -x \cdot (R(y) + 2y) \cdot z \\
&= -(x \cdot R(y) + 2x \cdot y) \cdot z \\
&= (x *_3 y) *_2 z
\end{aligned}$$

$$\begin{aligned}
x *_1 (y *_0 z) &= x \cdot (R(y) \cdot z + y \cdot z) \\
&= (x \cdot R(y) + x \cdot y) \cdot z \\
&= (-x \cdot y + x \cdot R(y) + 2x \cdot y) \cdot z \\
&= (x *_2 y + x *_3 y) *_1 z
\end{aligned}$$

$$\begin{aligned}
x *_0 (y *_0 z) &= R(x) \cdot (R(y) \cdot z + y \cdot z) + x \cdot (R(y) \cdot z + y \cdot z) \\
&= R(x) \cdot (R(y) \cdot z) + R(x) \cdot (y \cdot z) + x \cdot (R(y) \cdot z + y \cdot z) \\
&= (R(x) \cdot R(y)) \cdot z + (R(x) \cdot y) \cdot z + (x \cdot R(y) + x \cdot y) \cdot z \\
&= (R(R(x) \cdot y + x \cdot R(y) + 3x \cdot y) + 2x \cdot y) \cdot z + (R(x) \cdot y + x \cdot R(y) + x \cdot y) \cdot z \\
&= R(R(x) \cdot y + x \cdot R(y) + 3x \cdot y) \cdot z + (R(x) \cdot y + x \cdot R(y) + 3x \cdot y) \cdot z \\
&= (R(x) \cdot y + x \cdot R(y) + 3x \cdot y) *_0 z \\
&= (R(x) \cdot y + x \cdot y + x \cdot y - x \cdot y + x \cdot R(y) + 2x \cdot y) *_0 z \\
&= (x *_0 y + x *_1 y + x *_2 y + x *_3 y) *_0 z
\end{aligned}$$

□

Theorem 4.0.43. *Let $(A, \cdot)$ be an associative algebra and let $R : A \to A$ be a Rota-Baxter operator of weight three and two on A. That is, let us assume that*

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y) + 3x \cdot y) + 2x \cdot y,$$

for all $x, y \in A$. *If* m *is odd, we define* $m+1$ *new operations on* A *by*

$$x *_0 y := R(x) \cdot y + x \cdot y,$$
$$x *_j y := (-1)^{j+1} x \cdot y, \quad \textit{for} \quad 1 \leq j \leq m-1 \textit{ and}$$
$$x *_m y := x \cdot R(y) + 2x \cdot y,$$

for all $x, y \in A$. *Then,* $(A, \{*_i\}_{1 \leq i \leq m})$ *is a Ballot*^{m}*-algebra.*

Proof. We need to prove $x *_i (y *_j z) = (x *_i y) *_j z$, for $0 \leq i < j \leq m$.

We have for all $x, y, z \in A$

$$\begin{aligned} x *_0 (y *_m z) &= R(x) \cdot (y \cdot R(z) + 2y \cdot z) + x \cdot (y \cdot R(z) + 2y \cdot z) \\ &= R(x) \cdot (y \cdot R(z)) + x \cdot (y \cdot R(z)) + R(x) \cdot (2y \cdot z) + x \cdot (2y \cdot z) \\ &= (R(x) \cdot y) \cdot R(z) + (x \cdot y) \cdot R(z) + (R(x) \cdot 2y) \cdot z + (x \cdot 2y) \cdot z \\ &= (R(x) \cdot y + x \cdot y) \cdot R(z) + 2(R(x) \cdot y + x \cdot y) \cdot z \\ &= (x *_0 y) *_m z \end{aligned}$$

Let $1 \leq j \leq m$, then

$$\begin{aligned} x *_0 (y *_j z) &= R(x) \cdot ((-1)^{j+1} y \cdot z) + x \cdot ((-1)^{j+1} y \cdot z) \\ &= (-1)^{j+1} (R(x) \cdot y) \cdot z + (-1)^{j+1} (x \cdot y) \cdot z \\ &= (-1)^{j+1} (R(x) \cdot y + x \cdot y) \cdot z \\ &= (x *_0 y) *_j z \end{aligned}$$

Let $1 \leq i < j \leq m-1$, then

$$\begin{aligned} x *_i (y *_j z) &= (-1)^{i+1} x \cdot ((-1)^{j+1} y \cdot z) \\ &= (-1)^{j+1} ((-1)^{i+1} x \cdot y) \cdot z \\ &= (x *_i y) *_j z \end{aligned}$$

Let $1 \leq j \leq m-1$, then

$$\begin{aligned} x *_j (y *_m z) &= (-1)^{j+1} x \cdot (y \cdot R(z) + 2y \cdot z) \\ &= (-1)^{j+1} x \cdot (y \cdot R(z)) + (-1)^{j+1} x \cdot (2y \cdot z) \\ &= ((-1)^{j+1} x \cdot y) \cdot R(z) + 2((-1)^{j+1} x \cdot y) \cdot z \\ &= (x *_j y) \cdot R(z) + 2(x *_j y) \cdot z \\ &= (x *_j y) *_m z \end{aligned}$$

Now, we shall prove $x *_i (y *_0 z + \cdots + y *_i z) = (x *_i y + \cdots + x *_m y) *_i z$, for $i = m$ and $i = 0$

For $i = m$.

As m odd, we have $y *_1 z + \cdots + y *_{m-1} z = 0$ and,

$$
\begin{aligned}
x *_m (y *_0 z + y *_1 z + \ldots + y *_{m-1} z + y *_m z) &= x \cdot R\big(R(y) \cdot z + y \cdot z + y \cdot R(z) + 2y \cdot z\big) \\
&\quad + 2x \cdot \big(R(y) \cdot z + y \cdot z + y \cdot R(z) + 2y \cdot z\big) \\
&= x \cdot R\big(R(y) \cdot z + y \cdot R(z) + 3y \cdot z\big) \\
&\quad + 2x \cdot \big(y \cdot R(z) + R(y) \cdot z + 3y \cdot z\big) \\
&= x \cdot \big(R(R(y) \cdot z + y \cdot R(z) + 3y \cdot z) + 2y \cdot z\big) \\
&\quad + 2x \cdot (y \cdot R(z)) + 2x \cdot (R(y) \cdot z) + 4x \cdot (y \cdot z) \\
&= x \cdot (R(y) \cdot R(z)) + 2x \cdot (y \cdot R(z)) \\
&\quad + 2x \cdot (R(y) \cdot z) + 4x \cdot (y \cdot z) \\
&= (x \cdot R(y)) \cdot R(z) + (2x \cdot y) \cdot R(z) \\
&\quad + 2x \cdot (R(y) \cdot z) + 4x \cdot (y \cdot z) \\
&= (x \cdot R(y) + 2x \cdot y) \cdot R(z) + 2(x \cdot R(y) + 2x \cdot y) \cdot z \\
&= (x *_m y) *_m z
\end{aligned}
$$

For $i = 0$.

We have $x *_1 y + \cdots + x *_{m-1} y = 0$, since m is odd,

$$
\begin{aligned}
x *_0 (y *_0 z) &= R(x) \cdot (R(y) \cdot z + y \cdot z) + x \cdot (R(y) \cdot z + y \cdot z) \\
&= R(x) \cdot (R(y) \cdot z) + R(x) \cdot (y \cdot z) + x \cdot (R(y) \cdot z + y \cdot z) \\
&= (R(x) \cdot R(y)) \cdot z + (R(x) \cdot y) \cdot z + (x \cdot R(y) + x \cdot y) \cdot z \\
&= (R(R(x) \cdot y + x \cdot R(y) + 3x \cdot y) + 2x \cdot y) \cdot z + (R(x) \cdot y + x \cdot R(y) + x \cdot y) \cdot z \\
&= R(R(x) \cdot y + x \cdot R(y) + 3x \cdot y) \cdot z + (R(x) \cdot y + x \cdot R(y) + 3x \cdot y) \cdot z \\
&= (R(x) \cdot y + x \cdot R(y) + 3x \cdot y) *_0 z \\
&= (R(x) \cdot y + x \cdot y + x \cdot R(y) + 2x \cdot y) *_0 z \\
&= (x *_0 y + x *_1 y + \cdots + x *_{m-1} y + x *_m y) *_0 z
\end{aligned}
$$

Now, we shall prove $x *_i (y *_0 z + \cdots + y *_{i-1} z) = (x *_{i+1} y + \cdots + x *_m y) *_i z$, for $1 \leq i \leq m-1$:

If i is even, then $y *_1 z + \cdots + y *_{i-1} z = y \cdot z$ and $x *_{i+1} y + \cdots + x *_{m-1} y = 0$. Thus

$$
\begin{aligned}
x *_i (y *_0 z + \cdots + y *_{i-1} z) &= -x \cdot (R(y) \cdot z + y \cdot z + y \cdot z) \\
&= -x \cdot (R(y) + 2y) \cdot z \\
&= -(x \cdot R(y) + 2x \cdot y) \cdot z \\
&= (x *_m y) *_i z \\
&= (x *_{i+1} y + \cdots + x *_m y) *_i z
\end{aligned}
$$

If i is odd, then $y *_1 z + \cdots + y *_{i-1} z = 0$ and $x *_{i+1} y + \cdots + x *_{m-1} y = -x \cdot y$. Thus

$$
\begin{aligned}
x *_i (y *_0 z + \cdots + y *_{i-1} z) &= x \cdot (R(y) \cdot z + y \cdot z) \\
&= (x \cdot R(y) + x \cdot y) \cdot z \\
&= (-x \cdot y + x \cdot R(y) + 2x \cdot y) \cdot z \\
&= (x *_{i+1} y + \cdots + x *_{m-1} y + x *_m y) *_i z
\end{aligned}
$$

Therefore, $(A, \{*_i\}_{1\leq i\leq m})$ is a Ballot^m-algebra.

$\square$

The following results can be similarly demonstrated.

Proposition 4.0.44. *Let $(A, \cdot)$ be an associative algebra and $R : A \to A$ be a Rota-Baxter operator of weight three and two on A. That is, assume that*

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y) + 3x \cdot y) + 2x \cdot y,$$

for all $x, y \in A$. Define five new operations on A by

$$x *_0 y := R(x)\cdot y + x\cdot y, \quad x *_1 y := x\cdot y, \quad x *_2 y := -x\cdot y, \quad x *_3 y := x\cdot y \quad \textit{and} \quad x *_4 y := x\cdot R(y) + x\cdot y,$$

for all $x, y \in A$. Then, $(A, *_0, *_1, *_2, *_3, *_4)$ is a Ballot^4-algebra.

Proposition 4.0.45. *Let $(A, \cdot)$ be an associative algebra and $R : A \to A$ be a Rota-Baxter operator of weight three and two on A. That is, assume that*

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y) + 3x \cdot y) + 2x \cdot y,$$

for all $x, y \in A$. If m is even, define $m + 1$ new operations on A by

$$\begin{aligned} &x *_0 y := R(x) \cdot y + x \cdot y, \\ &x *_j y := (-1)^{j+1} x \cdot y, \quad \textit{for} \quad 1 \leq j \leq m - 1 \ \textit{and} \\ &x *_m y := x \cdot R(y) + x \cdot y, \end{aligned}$$

for all $x, y \in A$. Then, $(A, \{*_i\}_{1\leq i\leq m})$ is a Ballot^m-algebra.

Chapter 5

Construction of Rota^m-Algebras from Associative Algebras with a Rota-Baxter Morphism

In this chapter we introduce the concepts of a Rota-Baxter Morphism and a Dyck^m-algebra. We present examples of Rota-Baxter Morphism from associative algebras with a left identity that simultaneously is a element idempotent. An element u is said to be idempotent with respect to product $\cdot$ in the algebra if: $x \cdot x = x$ and is left identity if $x \cdot u = x$ for all element x in the algebra. Finally, we give a constructions with a Rota-Baxter Morphism, and so we construct a Rota^m-algebra.

5.0.12 Dyck^m-Algebras

Definition 5.0.43. A **Dyck^m-algebra** over $\mathbb{K}$ is a $\mathbb{K}$-vector space H endowed with $m+1$ binary operations $\{*_j\}_{0\leq j\leq m}$ satisfying the following properties:

1. $x *_i (y *_j z) = (x *_i y) *_j z,$ for $0 \leq i < j \leq m,$
2. $x *_i (y *_0 z + y *_1 z + \cdots + y *_i z) = (x *_i y + x *_{i+1} y + \cdots + x *_m y) *_i z,$ for $0 \leq i \leq m,$

for all x, y and z in H.

Remark 5.0.16. Observes that the products $*_0, *_1, ..., *_m$ are not associative and if $1 \leq j \leq m-1,$ then, the operations $\succ^j := *_0 + *_1 + \cdots + *_j$ and $\prec^j := *_{j+1} + \cdots + *_m$, equips to a Dyck^m-algebra A of $m-1$ dendriform algebra structures given by: $(A, \succ^j, \prec^j)$.

Example 5.0.26. The properties of a Dyck^2-algebra are

$$\begin{aligned}
x *_0 (y *_1 z) &= (x *_0 y) *_1 z,\\
x *_0 (y *_2 z) &= (x *_0 y) *_2 z,\\
x *_1 (y *_2 z) &= (x *_1 y) *_2 z,\\
x *_2 (y *_0 z + y *_1 z + y *_2 z) &= (x *_2 y) *_2 z,\\
x *_1 (y *_0 z + y *_1 z) &= (x *_1 y + x *_2 y) *_1 z,\\
x *_0 (y *_0 z) &= (x *_0 y + x *_1 y + x *_2 y) *_0 z,
\end{aligned}$$

Definition 5.0.44. A **Rota^m-algebra** over $\mathbb{K}$ is a $\mathbb{K}$-vector space H endowed with $m+1$ binary operations $\{*_j\}_{0\leq j\leq m}$ satisfying the following properties:

1. $x *_i (y *_j z) = (x *_i y) *_j z,$ for $0 \leq i < j \leq m,$ and $0 \leq i = j \leq m-1$
2. $0 = (x *_1 y + \cdots + x *_m y) *_0 z,$
3. $x *_i (y *_0 z + \cdots + y *_{i-1} z) = (x *_{i+1} y + \cdots + x *_m y) *_i z,$ for $1 \leq i \leq m-1,$

4. $x *_m (y *_0 z + \cdots + y *_m z) = (x *_m y) *_m z\,,$

for all x, y and z in H.

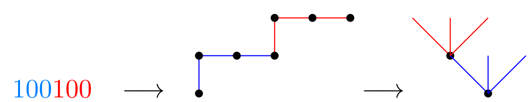
Example 5.0.27. The properties of a Rota^2-algebra are

$$\begin{aligned}
x *_0 (y *_1 z) &= (x *_0 y) *_1 z,\\
x *_0 (y *_2 z) &= (x *_0 y) *_2 z,\\
x *_1 (y *_2 z) &= (x *_1 y) *_2 z,\\
x *_0 (y *_0 z) &= (x *_0 y) *_0 z,\\
x *_1 (y *_1 z) &= (x *_1 y) *_1 z,\\
x *_2 (y *_0 z + y *_1 z + y *_2 z) &= (x *_2 y) *_2 z\,,\\
x *_1 (y *_0 z) &= (x *_2 y) *_1 z,\\
0 &= (x *_0 y + x *_1 y + x *_2 y) *_0 z,
\end{aligned}$$

Remark 5.0.17. If $1 \leq j \leq m-1$, then, the operations $\succ^j := *_0 + *_1 + \cdots + *_j$ and $\prec^j := *_{j+1} + \cdots + *_m$, equips to a Rota^m-algebra A of $m-1$ dendriform algebra structures given by: $(A, \succ^j, \prec^j)$.

Definition 5.0.45. For $m, n \geq 1$, a m**-Ballot path of height** n is a path on the real plane $\mathbb{R}^2$ from $(0,0)$ to (nm, n) made from vertical steps $(0,1)$ and horizontal steps $(1,0)$ which always stays above the line $my = x$. Note that the initial and terminal points of each step lie on $\mathbb{N} \times \mathbb{N}$.

m-Ballot path can be also interpreted as words on a binary alphabet by replacing vertical steps by the letter 1 and horizontal steps by 0.

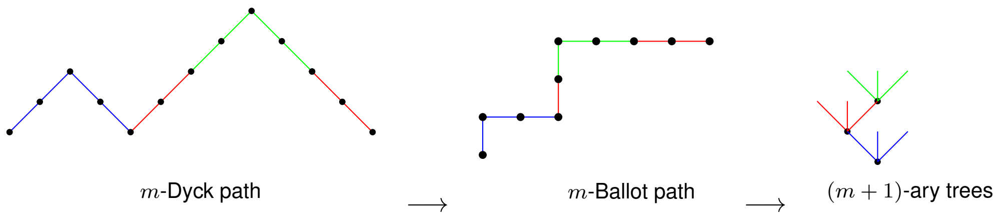


An oriented planar graph drawn in the plane with one root, $n \cdot (m-1) + 1$ leaves and n interior vertices is called an m**-ary tree** if every internal vertex has exactly m incoming edges and one outgoing edge.

Also, there is a bijection between $(m+1)$-ary trees with n interior vertices and m-ballot paths of height n. The number of m-Ballot paths of height n is given by the Fuss-Catalan number:

$$C(m,n) = \frac{1}{mn+1}\binom{mn+n}{n}$$

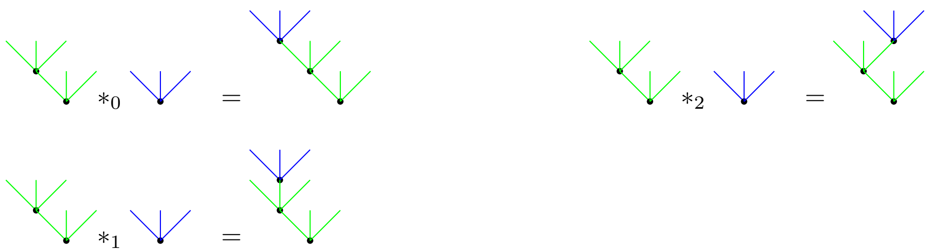
Definition 5.0.46. For $m, n \geq 1$, **an** m**-Dyck path of size** n is a path on the real plane $\mathbb{R}^2$, starting at $(0,0)$ and ending at $(nm, 0)$, consisting on up steps (m, m) and down steps $(1, 1)$, which never goes below the line x-axis. Note that the initial and terminal points of each step lie on $\mathbb{N}^2$.



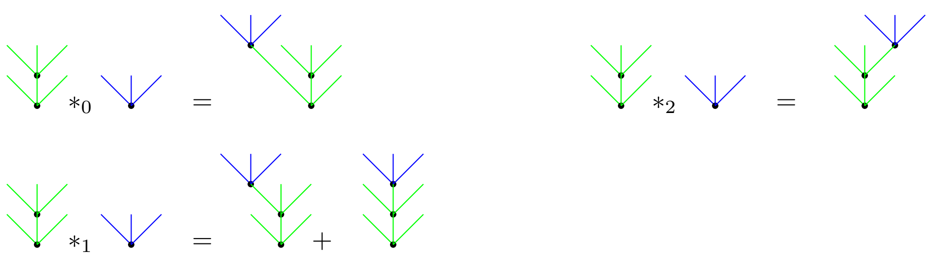
If D is a m-Dyck path, we can obtain the m-Ballot path B by replacing each sequence of m consecutive up steps by one vertical step and each down step by an horizontal step. We wish to describe in the examples below, operations $*_0, *_1, *_2$ on 3-ary trees; these operations were defined by Ronco on m-Dyck paths, which are in bijection with m-Ballot paths.

Now we describe operations $*_0, *_1, *_2$:

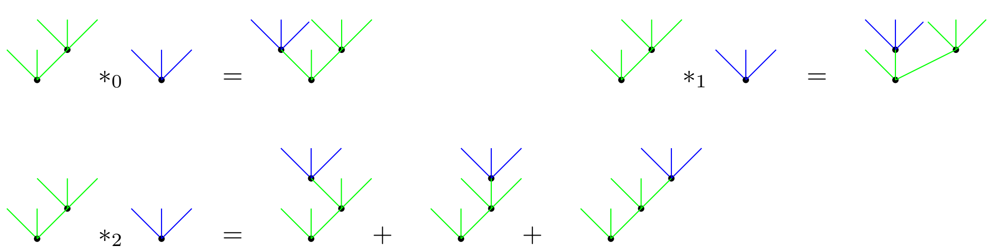
Example 5.0.28.



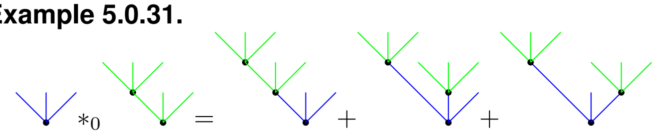
Example 5.0.29.

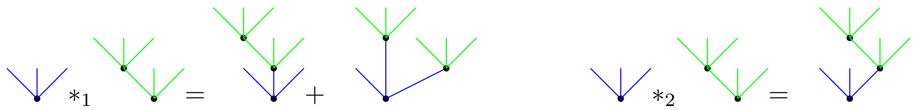


Example 5.0.30.



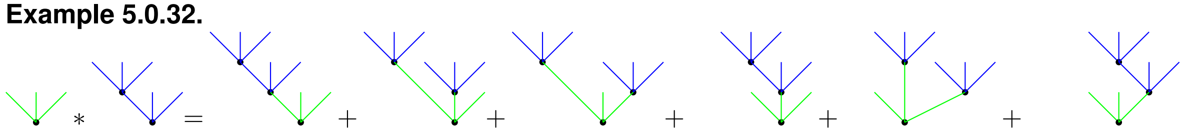
Example 5.0.31.



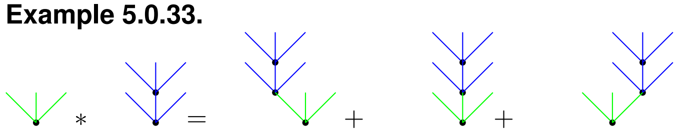


Now we describe the operation $* = *_0 + *_1 + *_2$ on 3-ary trees.

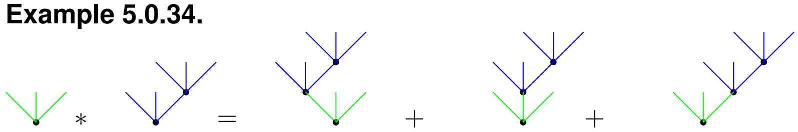
Example 5.0.32.



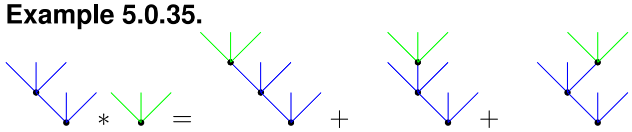
Example 5.0.33.



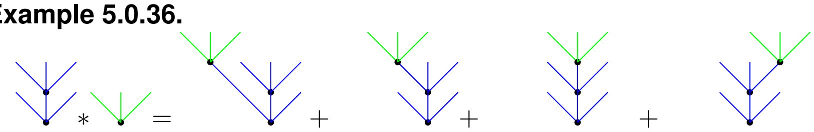
Example 5.0.34.



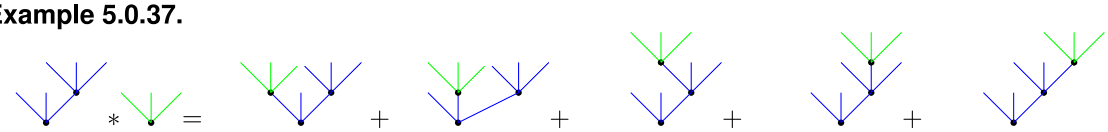
Example 5.0.35.



Example 5.0.36.

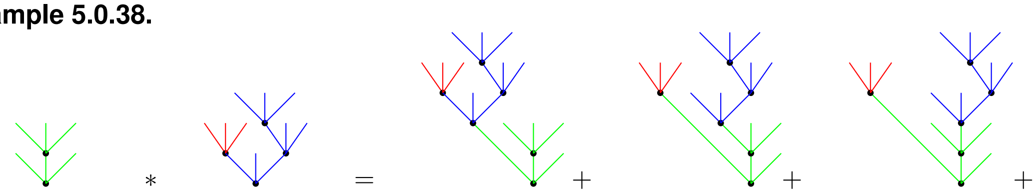


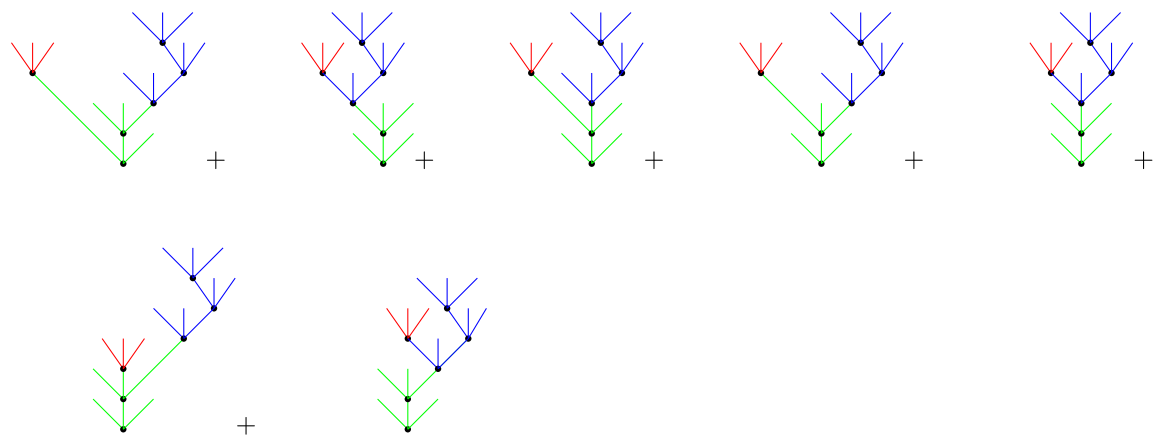
Example 5.0.37.



finally,

Example 5.0.38.





Example 5.0.39. The operations $*_0, *_1, *_2$ described in the above examples determines a Dyck^2-algebra on 3-ary trees.

5.0.13 Rota-Baxter Morphisms

Definition 5.0.47. Let $(A, \cdot)$ be an associative algebra. A linear map $R : A \to A$ is called a **Rota-Baxter Morphism** on A if R satisfies

$$R(x) \cdot R(y) = R\left(R(x) \cdot y\right) = R(x \cdot y),$$

for all $x, y \in A$.

The following lemma gives an elementary example of a Rota-Baxter morphism.

Lemma 5.0.46. *Let $(A, \cdot)$ be the associative algebra defined by $x \cdot y = y$ for all $x, y \in A$, and $R : A \longrightarrow A$ any linear map, then R satisfies*

$$R(x) \cdot R(y) = R\left(R(x) \cdot y\right) = R(x \cdot y),$$

for all $x, y \in A$.

Proof.

$$R(y) = R(x) \cdot R(y) = R\left(R(x) \cdot y\right) = R(x \cdot y),$$

for all $x, y \in A$. ☐

Less trivial constructions are given in the following propositions.

Proposition 5.0.47. *Let $(\, A, \, \cdot \,)$ be an associative subalgebra of an algebra B, and suppose that there exists $u \in B$ such that $x \cdot u \in A$ and $u \cdot x = x$ for all $x \in A$. Then the linear map $R : A \longrightarrow A$ defined by $R(x) = x \cdot u$ satisfies*

$$R(a \cdot b) = R\left(\, R(a) \cdot b\right) = R(a) \cdot R(b) \text{ for all } a, b \in A$$

Proof. Let $x, y \in A$, then

$$\begin{aligned} R(x) \cdot R(y) &= (x \cdot u) \cdot (y \cdot u) \\ &= (x \cdot (u \cdot y)) \cdot u \\ &= (x \cdot y) \cdot u \\ &= R(x \cdot y) \end{aligned}$$

On the other hand,

$$\begin{aligned} R(R(x) \cdot y) &= R((x \cdot u) \cdot y) \\ &= R(x \cdot (u \cdot y)) \\ &= R(x \cdot y) \end{aligned}$$

Therefore $R(x \cdot y) = R(R(x) \cdot y) = R(x) \cdot R(y)$ for all $x, y \in A$.

□

Example 5.0.40. The element $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $u \cdot x = x$ for all x in the algebra of matrices

$$A = \left\{ \begin{pmatrix} b & a \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

considered as a subalgebra of $B = M_{2\times 2}$ under the usual matrix multiplication. Indeed,

$$u \cdot x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a \\ b & a \end{pmatrix} = \begin{pmatrix} b & a \\ b & a \end{pmatrix},$$

If we define the map $R : A \longrightarrow A$ by

$$R(\begin{pmatrix} b & a \\ b & a \end{pmatrix}) = \begin{pmatrix} b & a \\ b & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

then R satisfies $R(x) \cdot R(y) = R(R(x) \cdot y) = R(x \cdot y)$; for all $x, y \in A$.

Proposition 5.0.48. *Let* $(A, \cdot)$ *be an associative algebra and suppose that there exists* $u \in A$ *such that* $u^2 = u$ *and* $x \cdot u = x$ *for all* $x \in A$. *Then the linear map* $R : A \longrightarrow A$ *defined by* $R(x) = u \cdot x$ *satisfies*

$$R(a \cdot b) = R(R(a) \cdot b) = R(a) \cdot R(b) \text{ for all } a, b \in A$$

Proof. Let $x, y \in A$, then

$$\begin{aligned} R(x) \cdot R(y) &= (u \cdot x) \cdot (u \cdot y) \\ &= u \cdot ((x \cdot u) \cdot y) \\ &= u \cdot (x \cdot y) \\ &= R(x \cdot y) \end{aligned}$$

On the other hand,

$$\begin{aligned} R(R(x)\cdot y) &= R((u\cdot x)\cdot y) \\ &= u\cdot((u\cdot x)\cdot y) \\ &= (u\cdot(u\cdot x))\cdot y \\ &= (u^2\cdot x)\cdot y \\ &= u\cdot(x\cdot y) \\ &= R(x\cdot y) \end{aligned}$$

Therefore

$$R(x\cdot y) = R\,(\,R(x)\cdot y) = R(x)\cdot R(y) \text{ for all } x, y \in A$$

□

Example 5.0.41. The element $u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is an idempotent in the algebra of matrices

$$A = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

under the usual matrix multiplication, that is $u^2 = u$. Observe that $x \cdot u = x$ for all $x \in A$, and if $x \in A$ with $a \neq 0$, then $u \cdot x \neq x$:

$$x\cdot u = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \qquad u\cdot x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

If we define the map $R : A \longrightarrow A$ by

$$R(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

then R satisfies $R(x) \cdot R(y) = R(R(x) \cdot y) = R(x \cdot y)$; for all $x, y \in A$.

Example 5.0.42. The element $v = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ is an idempotent in the algebra of matrices

$$A = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

under the usual matrix multiplication, that is $v^2 = v$. Observe that $x.v = x$ for all $x \in A$ and if $x \in A$ with $a \neq b$, then $v.x \neq x$:

$$x.v = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \qquad v.x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix}$$

If we define the map $S : A \longrightarrow A$ by

$$S(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix}$$

then S satisfies $S(x) \cdot S(y) = S(S(x) \cdot y) = S(x \cdot y)$; for all $x, y \in A$.

Remark 5.0.18. If we consider the last two examples, then we have the following relation:

$S(x) \cdot T(y) = S(T(x) \cdot y) = S(x \cdot y),$ for all $x, y \in A$. This is a general fact:

Proposition 5.0.49. *Let $(A, \cdot)$ be an associative algebra and suppose that there exists $u, w \in A$ such that $x \cdot u = x$ and $x \cdot w = x$ for all $x \in A$. Then the linear maps $T,\ S : A \longrightarrow A$ defined by $T(x) = u \cdot x$ and $S(x) = w \cdot x$ satisfy*

$$T(a \cdot b) = T\,(\,S(a) \cdot b) = T(a) \cdot S(b) \text{ for all } a, b \in A\,.$$

Proof. Let $x, y \in A$, then

$$\begin{aligned} T(x) \cdot S(y) &= (u \cdot x) \cdot (w \cdot y) \\ &= u \cdot ((x \cdot w) \cdot y) \\ &= u \cdot (x \cdot y) \\ &= T(x \cdot y) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(S(x) \cdot y) &= T((w \cdot x) \cdot y) \\ &= u \cdot ((w \cdot x) \cdot y) \\ &= (u \cdot (w \cdot x)) \cdot y \\ &= ((u \cdot w) \cdot x) \cdot y \\ &= u \cdot (x \cdot y) \\ &= T(x \cdot y) \end{aligned}$$

Therefore

$$T(x \cdot y) = T\,(\,S(x) \cdot y) = T(x) \cdot S(y) \text{ for all } x, y \in A$$

□

Proposition 5.0.50. *Let A be an associative algebra and let $R : A \longrightarrow A$ a linear map such that*

$$R(a \cdot b) = R\,(\,R(a) \cdot b) = R(a) \cdot R(b) \text{ for all } a, b \in A\,.$$

Then we can define a dendriform algebra structures on A given by

$$a \prec b := a \cdot b - R(a) \cdot b, \qquad a \succ b := R(a) \cdot b$$

Proof.

$$\begin{aligned}(a \prec b) \prec c &= (a \cdot b - R(a) \cdot b) \prec c\\ &= (a \cdot b - R(a) \cdot b) \cdot c - R(a \cdot b - R(a) \cdot b) \cdot c\\ &= (a \cdot b) \cdot c - (R(a) \cdot b) \cdot c - R(a \cdot b - R(a) \cdot b) \cdot c\end{aligned}$$

$$\begin{aligned}a \prec (b \prec c + b \succ c) &= a \prec (b \cdot c - R(b) \cdot c + R(b) \cdot c)\\ &= a \prec (b \cdot c)\\ &= a \cdot (b \cdot c) - R(a) \cdot (b \cdot c)\end{aligned}$$

$R(a \cdot b - R(a) \cdot b\,) = 0$, then $(a \prec b) \prec c = a \prec (b \prec c + b \succ c)$

$$\begin{aligned}a \succ (b \prec c) &= a \succ (b \cdot c - R(b) \cdot c)\\ &= R(a) \cdot (b \cdot c - R(b) \cdot c)\\ &= R(a) \cdot (b \cdot c) - R(a) \cdot (R(b) \cdot c)\end{aligned}$$

$$\begin{aligned}(a \succ b) \prec c &= (R(a) \cdot b) \prec c\\ &= (R(a) \cdot b) \cdot c - R(R(a) \cdot b) \cdot c\end{aligned}$$

$R(R(a) \cdot b) = R(a) \cdot R(b)$, then $a \succ (b \prec c) = (a \succ b) \prec c$

$$a \succ (b \succ c) = R(a)(R(b) \cdot c)$$

$$\begin{aligned}(a \succ b + a \prec b) \succ c &= (R(a) \cdot b + a \cdot b - R(a) \cdot b) \succ c\\ &= (a \cdot b) \succ c\\ &= R(a \cdot b) \cdot c\end{aligned}$$

$R(a \cdot b) = R(a) \cdot R(b)$, then $a \succ (b \succ c) = (a \succ b + a \prec b) \succ c\,.$

□

Remark 5.0.19. Note that of the definition $a \succ b := R(a) \cdot b$, we have $(a \succ b) \succ c = a \succ (b \succ c)$.

Proposition 5.0.51. *Let $(A, \cdot)$ be an associative algebra and let $T : A \longrightarrow A$ and $S : A \longrightarrow A$ be linear maps such that*

$$T(a \cdot b) = T(T(a) \cdot b) = T(a) \cdot T(b) \text{ for all } a, b \in A\,,$$

$$S(a \cdot b) = S(S(a) \cdot b) = S(a) \cdot S(b) \text{ for all } a, b \in A\,,$$

$$S(a) \cdot T(b) = T(S(a) \cdot b) \text{ for all } a, b \in A\,,$$

$$T(a) \cdot S(b) = S(T(a) \cdot b) \text{ for all } a, b \in A\,.$$

Define three new operations on A by

$$\begin{aligned} x *_0 y &:= T(x) \cdot y, \\ x *_1 y &:= S(x) \cdot y - T(x) \cdot y, \\ x *_2 y &:= T(x) \cdot y - S(x) \cdot y \quad \text{and} \\ x *_3 y &:= x \cdot y - T(x) \cdot y \end{aligned}$$

for all $x, y \in A$. Then, $(A, *_0, *_1, *_2, *_3)$ is a Rota[3]-algebra.

Proof.

$$\begin{aligned} a *_0 (b *_1 c) &= a *_0 (S(b) \cdot c - T(b) \cdot c) \\ &= T(a) \cdot (S(b) \cdot c - T(b) \cdot c) \\ &= T(a) \cdot (S(b) \cdot c) - T(a) \cdot (T(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_0 b) *_1 c &= (T(a) \cdot b) *_1 c \\ &= S(T(a) \cdot b) \cdot c - T(T(a) \cdot b) \cdot c \end{aligned}$$

$T(a) \cdot S(b) - T(a) \cdot T(b) = S(T(a) \cdot b) - T(T(a) \cdot b)$, then $a *_0 (b *_1 c) = (a *_0 b) *_1 c$

$$\begin{aligned} a *_0 (b *_2 c) &= a *_0 (T(b) \cdot c - S(b) \cdot c) \\ &= T(a) \cdot (T(b) \cdot c - S(b) \cdot c) \\ &= T(a) \cdot (T(b) \cdot c) - T(a) \cdot (S(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_0 b) *_2 c &= (T(a) \cdot b) *_2 c \\ &= T(T(a) \cdot b) \cdot c - S(T(a) \cdot b) \cdot c \end{aligned}$$

$T(a) \cdot T(b) - T(a) \cdot S(b) = T(T(a) \cdot b) - S(T(a) \cdot b)$, then $a *_0 (b *_2 c) = (a *_0 b) *_2 c$

$$\begin{aligned} a *_0 (b *_3 c) &= a *_0 (b \cdot c - T(b) \cdot c) \\ &= T(a) \cdot (b \cdot c - T(b) \cdot c) \\ &= T(a) \cdot (b \cdot c) - T(a) \cdot (T(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_0 b) *_3 c &= (T(a) \cdot b) *_3 c \\ &= (T(a) \cdot b) \cdot c - T(T(a) \cdot b) \cdot c \end{aligned}$$

$T(a) \cdot T(b) = T(T(a) \cdot b)$, then $a *_0 (b *_3 c) = (a *_0 b) *_3 c$

$$\begin{aligned} a *_1 (b *_2 c) &= a *_1 (T(b) \cdot c - S(b) \cdot c) \\ &= S(a) \cdot (T(b) \cdot c - S(b) \cdot c) - T(a) \cdot (T(b) \cdot c - S(b) \cdot c) \\ &= S(a) \cdot (T(b) \cdot c) - S(a) \cdot (S(b) \cdot c) - T(a) \cdot (T(b) \cdot c) + T(a) \cdot (S(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_1 b) *_2 c &= (S(a) \cdot b - T(a) \cdot b) *_2 c \\ &= T(S(a) \cdot b - T(a) \cdot b) \cdot c - S(S(a) \cdot b - T(a) \cdot b) \cdot c \end{aligned}$$

$S(a) \cdot T(b) - S(a) \cdot S(b) - T(a) \cdot T(b) + T(a) \cdot S(b) = T(S(a) \cdot b - T(a) \cdot b) - S(S(a) \cdot b - T(a) \cdot b)$, then $a *_1 (b *_2 c) = (a *_1 b) *_2 c$

$$\begin{aligned} a *_1 (b *_3 c) &= a *_1 (b \cdot c - T(b) \cdot c) \\ &= S(a) \cdot (b \cdot c - T(b) \cdot c) - T(a) \cdot (b \cdot c - T(b) \cdot c) \\ &= S(a) \cdot (b \cdot c) - S(a) \cdot (T(b) \cdot c) - T(a) \cdot (b \cdot c) + T(a) \cdot (T(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_1 b) *_3 c &= (S(a) \cdot b - T(a) \cdot b) *_3 c \\ &= (S(a) \cdot b - T(a) \cdot b) \cdot c - T(S(a) \cdot b - T(a) \cdot b) \cdot c \end{aligned}$$

$S(a) \cdot T(b) - T(a) \cdot T(b) = T(S(a) \cdot b - T(a) \cdot b)$, then $a *_1 (b *_3 c) = (a *_1 b) *_3 c$

$$\begin{aligned} a *_2 (b *_3 c) &= a *_2 (b \cdot c - T(b) \cdot c) \\ &= T(a) \cdot (b \cdot c - T(b) \cdot c) - S(a) \cdot (b \cdot c - T(b) \cdot c) \\ &= T(a) \cdot (b \cdot c) - T(a) \cdot (T(b) \cdot c) - S(a) \cdot (b \cdot c) - S(a) \cdot (T(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_2 b) *_3 c &= (T(a) \cdot b - S(a) \cdot b) *_3 c \\ &= (T(a) \cdot b - S(a) \cdot b) \cdot c - T(T(a) \cdot b - S(a) \cdot b) \cdot c \end{aligned}$$

$T(a) \cdot T(b) - S(a) \cdot T(b) = T(T(a) \cdot b - S(a) \cdot b)$, then $a *_2 (b *_3 c) = (a *_2 b) *_3 c$

$$\begin{aligned} a *_0 (b *_0 c) &= a *_0 (T(b) \cdot c) \\ &= T(a) \cdot (T(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_0 b) *_0 c &= (T(a) \cdot b) *_0 c \\ &= T(T(a) \cdot b) \cdot c \end{aligned}$$

$T(a) \cdot T(b) = T(T(a) \cdot b)$, then $a *_0 (b *_0 c) = (a *_0 b) *_0 c$

$$
\begin{aligned}
a *_1 (b *_1 c) &= a *_1 (S(b) \cdot c - T(b) \cdot c) \\
&= S(a) \cdot (S(b) \cdot c - T(b) \cdot c) - T(a) \cdot (S(b) \cdot c - T(b) \cdot c) \\
&= (S(a) \cdot S(b) - S(a) \cdot T(b) - T(a) \cdot S(b) + T(a) \cdot T(b)) \cdot c
\end{aligned}
$$

$$
\begin{aligned}
(a *_1 b) *_1 c &= (S(a) \cdot b - T(a) \cdot b) *_1 c \\
&= S(S(a) \cdot b - T(a) \cdot b) \cdot c - T(S(a) \cdot b - T(a) \cdot b) \cdot c \\
&= (S(S(a) \cdot b - T(a) \cdot b) - T(S(a) \cdot b - T(a) \cdot b)) \cdot c
\end{aligned}
$$

$S(a) \cdot S(b) - S(a) \cdot T(b) - T(a) \cdot S(b) + T(a) \cdot T(b) = S(S(a) \cdot b - T(a) \cdot b) - T(S(a) \cdot b - T(a) \cdot b)$, then $a *_1 (b *_1 c) = (a *_1 b) *_1 c$

$$
\begin{aligned}
a *_2 (b *_2 c) &= a *_2 (T(b) \cdot c - S(b) \cdot c) \\
&= T(a) \cdot (T(b) \cdot c - S(b) \cdot c) - S(a) \cdot (T(b) \cdot c - S(b) \cdot c) \\
&= (T(a) \cdot T(b) - T(a) \cdot S(b) - S(a) \cdot T(b) + S(a) \cdot S(b)) \cdot c
\end{aligned}
$$

$$
\begin{aligned}
(a *_2 b) *_2 c &= (T(a) \cdot b - S(a) \cdot b) *_2 c \\
&= T(T(a) \cdot b - S(a) \cdot b) \cdot c - S(T(a) \cdot b - S(a) \cdot b) \cdot c \\
&= (T(T(a) \cdot b - S(a) \cdot b) - S(T(a) \cdot b - S(a) \cdot b)) \cdot c
\end{aligned}
$$

$T(a) \cdot T(b) - T(a) \cdot S(b) - S(a) \cdot T(b) + S(a) \cdot S(b) = T(T(a) \cdot b - S(a) \cdot b) - S(T(a) \cdot b - S(a) \cdot b)$, then $a *_2 (b *_2 c) = (a *_2 b) *_2 c$

$$
\begin{aligned}
a *_3 (b *_0 c + b *_1 c + b *_2 c + b *_3 c) &= a *_3 (b \cdot c) \\
&= a \cdot (b \cdot c) - T(a) \cdot (b \cdot c) \\
\\
(a *_3 b) *_3 c &= (a \cdot b - T(a) \cdot b) *_3 c \\
&= (a \cdot b - T(a) \cdot b) \cdot c - T(a \cdot b - T(a) \cdot b) \cdot c
\end{aligned}
$$

$T(a \cdot b - T(a) \cdot b) = 0$, then $a *_3 (b *_0 c + b *_1 c + b *_2 c + b *_3 c) = (a *_3 b) *_3 c$

$$
\begin{aligned}
a *_2 (b *_0 c + b *_1 c) &= a *_2 (S(b) \cdot c) \\
&= T(a) \cdot (S(b) \cdot c) - S(a) \cdot (S(b) \cdot c) \\
\\
(a *_3 b) *_2 c &= (a \cdot b - T(a) \cdot b) *_2 c \\
&= T(a \cdot b - T(a) \cdot b) \cdot c - S(a \cdot b - T(a) \cdot b) \cdot c
\end{aligned}
$$

$T(a) \cdot S(b) - S(a) \cdot S(b) = T(a \cdot b - T(a) \cdot b) - S(a \cdot b - T(a) \cdot b)$, then

$$
a *_2 (b *_0 c + b *_1 c) = (a *_3 b) *_2 c
$$

$$\begin{aligned} a *_1 (b *_0 c) &= a *_1 (T(b) \cdot c) \\ &= S(a) \cdot (T(b) \cdot c) - T(a) \cdot (T(b) \cdot c) \end{aligned}$$

$$\begin{aligned} (a *_2 b + a *_3 b) *_1 c &= (a \cdot b - S(a) \cdot b) *_1 c \\ &= S(a \cdot b - S(a) \cdot b) \cdot c - T(a \cdot b - S(a) \cdot b) \cdot c \end{aligned}$$

$S(a) \cdot T(b) - T(a) \cdot T(b) = S(a \cdot b - S(a) \cdot b) - T(a \cdot b - S(a) \cdot b)$, then

$$a *_1 (b *_0 c) = (a *_2 b + a *_3 b) *_1 c$$

$$\begin{aligned} (a *_1 b + a *_2 b + a *_3 b) *_0 c &= (a \cdot b - T(a) \cdot b) *_0 c \\ &= T(a \cdot b - T(a) \cdot b) \cdot c \\ &= 0 \end{aligned}$$

$T(T(a) \cdot b) = T(a \cdot b)$, then $0 = (a *_1 b + a *_2 b + a *_3 b) *_0 c$

□

Proposition 5.0.52. *Let* $(A, \cdot)$ *be an associative subalgebra of an algebra* B*, suppose that there exist* $u, \theta \in B$ *such that* $u \cdot u = u$*,* $\theta \cdot \theta = \theta$*,* $u \cdot \theta = \theta$*,* $\theta \cdot u = u$*; furthermore that* $u \cdot x \in A$*,* $\theta \cdot x \in A$ *and* $x \cdot u = x$*,* $x \cdot \theta = x$ *for all* $x \in A$*. Then the linear maps* $T : A \longrightarrow A$ *defined by* $T(y) = u \cdot y$ *and* $S : A \longrightarrow A$ *defined by* $S(x) = \theta \cdot x$ *satisfy*

$$S(x) \cdot T(y) = T(S(x) \cdot y) \text{ for all } x, y \in A$$

$$T(x) \cdot S(y) = S(T(x) \cdot y) \text{ for all } x, y \in A$$

Proof. We observe the following equalities:

$$S(x) \cdot T(y) = (\theta \cdot x) \cdot (u \cdot y) = \theta \cdot ((x \cdot u) \cdot y) = \theta \cdot (x \cdot y),$$

$$T(S(x) \cdot y) = T((\theta \cdot x) \cdot y) = u \cdot ((\theta \cdot x) \cdot y) = (u \cdot \theta) \cdot (x \cdot y) = \theta \cdot (x \cdot y),$$

and

$$T(x) \cdot S(y) = (u \cdot x) \cdot (\theta \cdot y) = u \cdot ((x \cdot \theta) \cdot y) = u \cdot (x \cdot y),$$

$$S(T(x) \cdot y) = S((u \cdot x) \cdot y) = \theta \cdot ((u \cdot x) \cdot y) = (\theta \cdot u) \cdot (x \cdot y) = u(x \cdot y).$$

□

Example 5.0.43. The elements $u = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\theta = \begin{pmatrix} a & 0 & 1-a \\ 1-a & 1 & a-1 \\ a & 0 & 1-a \end{pmatrix}$ where $a \in \mathbb{R}$ are idempotents in the algebra of matrices $M_{3\times 3}$. We consider the subalgebra

$$A = \left\{ \begin{pmatrix} y & y & 0 \\ n & n & 0 \\ r & r & 0 \end{pmatrix} : r, n, y \in \mathbb{R} \right\}$$

under the usual matrix multiplication. In fact,

$$u^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = u,$$

and

$$\theta^2 = \begin{pmatrix} a & 0 & 1-a \\ 1-a & 1 & a-1 \\ a & 0 & 1-a \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 1-a \\ 1-a & 1 & a-1 \\ a & 0 & 1-a \end{pmatrix}$$
$$= \begin{pmatrix} a^2 + (1-a)a & 0 & a(1-a) + (1-a)^2 \\ (1-a)a + (1-a) + (a-1)a & 1 & (1-a)^2 + (a-1) + (a-1)(1-a) \\ a^2 + (1-a)a & 0 & a(1-a) + (1-a)^2 \end{pmatrix} = \theta.$$

We observe that $x \cdot u = u$ and $x \cdot \theta = x$ for all $x \in A$, since

$$x \cdot u = \begin{pmatrix} y & y & 0 \\ n & n & 0 \\ r & r & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} y & y & 0 \\ n & n & 0 \\ r & r & 0 \end{pmatrix} = x$$

$$x \cdot \theta = \begin{pmatrix} y & y & 0 \\ n & n & 0 \\ r & r & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 1-a \\ 1-a & 1 & a-1 \\ a & 0 & 1-a \end{pmatrix} = \begin{pmatrix} ya + y(1-a) & y & y(1-a) + y(a-1) \\ na + n(1-a) & n & n(1-a) + n(a-1) \\ ra + r(1-a) & r & r(1-a) + r(a-1) \end{pmatrix} = x$$

If we define the maps $T : A \longrightarrow A$ by $T(y) = u \cdot y$ and $S_a : A \longrightarrow A$ by $S_a(x) = \theta \cdot x$ for $a \in \mathbb{R}$. Then we have that:

$$T(x) = u \cdot x = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & y & 0 \\ n & n & 0 \\ r & r & 0 \end{pmatrix} = \begin{pmatrix} r & r & 0 \\ y+n-n & y+n-r & 0 \\ r & r & 0 \end{pmatrix} \in A,$$

$$S_a(x) = \theta \cdot x = \begin{pmatrix} a & 0 & 1-a \\ 1-a & 1 & a-1 \\ a & 0 & 1-a \end{pmatrix} \cdot \begin{pmatrix} y & y & 0 \\ n & n & 0 \\ r & r & 0 \end{pmatrix}$$
$$= \begin{pmatrix} ay+r-ar & ay+r-ar & 0 \\ y-ay+n+ar-r & y-ay+n+ar-r & 0 \\ ay+r-ar & ay+r-ar & 0 \end{pmatrix} \in A.$$

Moreover, the elements u and θ satisfy:

$$u\cdot\theta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 1-a \\ 1-a & 1 & a-1 \\ a & 0 & 1-a \end{pmatrix} = \begin{pmatrix} a & 0 & 1-a \\ a+(1-a)-a & 1 & 1-a+a-1-(1-a) \\ a & 0 & 1-a \end{pmatrix} = \theta$$

$$\theta \cdot u = \begin{pmatrix} a & 0 & 1-a \\ 1-a & 1 & a-1 \\ a & 0 & 1-a \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a+(1-a) \\ 1 & 1 & (1-a)-1+(a-1) \\ 0 & 0 & a+(1-a) \end{pmatrix} = u$$

Therefore, we have that A is a subalgebra of $M_{3\times 3}$ and the elements $u, \theta \in M_{3\times 3}$ satisfy the hypothesis of the last theorem.

Proposition 5.0.53. *Let $(A, \cdot)$ be an associative algebra and let $T : A \longrightarrow A$ and $S_n : A \longrightarrow A$, $1 \leq n \leq k$ linear maps such that*

$$T(a \cdot b) = T(T(a) \cdot b) = T(a) \cdot T(b) \text{ for all } a, b \in A$$

$$S_n(a \cdot b) = S_n(S_n(a) \cdot b) = S_n(a) \cdot S_n(b) \text{ for all } a, b \in A$$

$$S_n(a) \cdot T(b) = T(S_n(a) \cdot b) \text{ for all } a, b \in A$$

$$T(a) \cdot S_n(b) = S_n(T(a) \cdot b) \text{ for all } a, b \in A$$

$$S_q(S_n(a) \cdot b) = S_n(a) \cdot S_q(b) \text{ for all } 1 \leq n \leq q \leq k \text{ and } a, b \in A$$

If m is odd, $m = 2k+1$ define $m+1$ new operations on A by

$$\begin{aligned} x *_0 y &:= T(x) \cdot y, \\ x *_{2n-1} y &:= S_n(x) \cdot y - T(x) \cdot y, \text{ and} \\ x *_{2n} y &:= T(x) \cdot y - S_n(x) \cdot y \quad \text{for } 1 \leq n \leq k \\ x *_m y &:= x \cdot y - T(x) \cdot y \end{aligned}$$

for all $x, y \in A$. Then, $(A, *_0, \ldots, *_m)$ is a Rota^m-algebra.

Proof. We need to prove $x *_i (y *_j z) = (x *_i y) *_j z$, for $0 \leq i < j \leq m$ and $0 \leq i = j \leq m-1$
We have for all $x, y, z \in A$

$$\begin{aligned} x *_0 (y *_m z) &= x *_0 (y \cdot z - T(y) \cdot z) \\ &= T(x) \cdot (y \cdot z - T(y) \cdot z) \\ &= (T(x) \cdot y - T(x) \cdot T(y)) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_0 y) *_m z &= (T(x) \cdot y) *_m z \\ &= (T(x) \cdot y) \cdot z - T(T(x) \cdot y) \cdot z \\ &= (T(x) \cdot y - T(T(x) \cdot y)) \cdot z \end{aligned}$$

$T(x) \cdot T(y) = T(T(x) \cdot y)$, then $x *_0 (y *_m z) = (x *_0 y) *_m z$.

Let $1 \leq n \leq k$, then

$$\begin{aligned} x *_0 (y *_{2n-1} z) &= x *_0 (S_n(y) \cdot z - T(y) \cdot z) \\ &= T(x) \cdot (S_n(y) \cdot z - T(y) \cdot z) \\ &= (T(x) \cdot S_n(y) - T(x) \cdot T(y)) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_0 y) *_{2n-1} z &= (T(x) \cdot y) *_{2n-1} z \\ &= S_n(T(x) \cdot y) \cdot z - T(T(x) \cdot y) \cdot z \\ &= (S_n(T(x) \cdot y) - T(T(x) \cdot y)) \cdot z \end{aligned}$$

$T(x) \cdot S_n(y) - T(x) \cdot T(y) = S_n(T(x) \cdot y) - T(T(x) \cdot y)$, then $x *_0 (y *_{2n-1} z) = (x *_0 y) *_{2n-1} z$.

$$\begin{aligned} x *_0 (y *_{2n} z) &= x *_0 (T(y) \cdot z - S_n(y) \cdot z) \\ &= T(x) \cdot (T(y) \cdot z - S_n(y) \cdot z) \\ &= (T(x) \cdot T(y) - T(x) \cdot S_n(y)) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_0 y) *_{2n} z &= (T(x) \cdot y) *_{2n} z \\ &= T(T(x) \cdot y) \cdot z - S_n(T(x) \cdot y) \cdot z \\ &= (T(T(x) \cdot y) - S_n(T(x) \cdot y)) \cdot z \end{aligned}$$

$T(x) \cdot T(y) - T(x) \cdot S_n(y) = T(T(x) \cdot y) - S_n(T(x) \cdot y)$, then $x *_0 (y *_{2n} z) = (x *_0 y) *_{2n} z$.

$$\begin{aligned} x *_{2n-1} (y *_{2n} z) &= x *_{2n-1} (T(y) \cdot z - S_n(y) \cdot z) \\ &= S_n(x) \cdot (T(y) \cdot z - S_n(y) \cdot z) - T(x) \cdot (T(y) \cdot z - S_n(y) \cdot z) \\ &= (S_n(x) \cdot T(y) - S_n(x) \cdot S_n(y) - T(x) \cdot T(y) + T(x) \cdot S_n(y)) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_{2n-1} y) *_{2n} z &= (S_n(x) \cdot y - T(x) \cdot y) *_{2n} z \\ &= T(S_n(x) \cdot y - T(x) \cdot y) \cdot z - S_n(S_n(x) \cdot y - T(x) \cdot y) \cdot z \\ &= (T(S_n(x) \cdot y - T(x) \cdot y) - S_n(S_n(x) \cdot y - T(x) \cdot y)) \cdot z \end{aligned}$$

$S_n(x)\cdot T(y)-S_n(x)\cdot S_n(y)-T(x)\cdot T(y)+T(x)\cdot S_n(y)=T(S_n(x)\cdot y-T(x)\cdot y)-S_n(S_n(x)\cdot y-T(x)\cdot y)$, then $x *_{2n-1} (y *_{2n} z) = (x *_{2n-1} y) *_{2n} z$.

$$\begin{aligned} x *_{2n-1} (y *_m z) &= x *_{2n-1} (y \cdot z - T(y) \cdot z) \\ &= S_n(x) \cdot (y \cdot z - T(y) \cdot z) - T(x) \cdot (y \cdot z - T(y) \cdot z) \\ &= (S_n(x) \cdot y - S_n(x) \cdot T(y) - T(x) \cdot y + T(x) \cdot T(y)) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_{2n-1} y) *_m z &= (S_n(x) \cdot y - T(x) \cdot y) *_m z \\ &= (S_n(x) \cdot y - T(x) \cdot y) \cdot z - T(S_n(x) \cdot y - T(x) \cdot y) \cdot z \\ &= (S_n(x) \cdot y - T(x) \cdot y - T(S_n(x) \cdot y - T(x) \cdot y)) \cdot z \end{aligned}$$

$S_n(x) \cdot y - S_n(x) \cdot T(y) - T(x) \cdot y + T(x) \cdot T(y) = S_n(x) \cdot y - T(x) \cdot y - T(S_n(x) \cdot y - T(x) \cdot y)$, then $x *_{2n-1} (y *_m z) = (x *_{2n-1} y) *_m z$.

$$\begin{aligned} x *_{2n} (y *_m z) &= x *_{2n} (y \cdot z - T(y) \cdot z) \\ &= T(x) \cdot (y \cdot z - T(y) \cdot z) - S_n(x) \cdot (y \cdot z - T(y) \cdot z) \\ &= (T(x) \cdot y - T(x) \cdot T(y) - S_n(x) \cdot y + S_n(x) \cdot T(y)) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_{2n} y) *_m z &= (T(x) \cdot y - S_n(x) \cdot y) *_m z \\ &= (T(x) \cdot y - S_n(x) \cdot y) \cdot z - T(T(x) \cdot y - S_n(x) \cdot y) \cdot z \\ &= (T(x) \cdot y - S_n(x) \cdot y - T(T(x) \cdot y - S_n(x) \cdot y)) \cdot z \end{aligned}$$

$T(x) \cdot y - T(x) \cdot T(y) - S_n(x) \cdot y + S_n(x) \cdot T(y) = T(x) \cdot y - S_n(x) \cdot y - T(T(x) \cdot y - S_n(x) \cdot y)$, then $x *_{2n} (y *_m z) = (x *_{2n} y) *_m z$.

$$\begin{aligned} x *_0 (y *_0 z) &= x *_0 (T(y) \cdot z) \\ &= T(x) \cdot (T(y) \cdot z) \end{aligned}$$

$$\begin{aligned} (x *_0 y) *_0 z &= (T(x) \cdot y) *_0 z \\ &= T(T(x) \cdot y) \cdot z \end{aligned}$$

$T(x) \cdot T(y) = T(T(x) \cdot y)$, then $x *_0 (y *_0 z) = (x *_0 y) *_0 z$.

$$\begin{aligned} x *_{2n-1} (y *_{2n-1} z) &= x *_{2n-1} (S_n(y) \cdot z - T(y) \cdot z) \\ &= S_n(x) \cdot (S_n(y) \cdot z - T(y) \cdot z) - T(x) \cdot (S_n(y) \cdot z - T(y) \cdot z) \\ &= (S_n(x) \cdot S_n(y) - S_n(x) \cdot T(y) - T(x) \cdot S_n(y) + T(x) \cdot T(y)) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_{2n-1} y) *_{2n-1} z &= (S_n(x) \cdot y - T(x) \cdot y) *_{2n-1} z \\ &= S_n(S_n(x) \cdot y - T(x) \cdot y) \cdot z - T(S_n(x) \cdot y - T(x) \cdot y) \cdot z \\ &= (S_n(S_n(x) \cdot y - T(x) \cdot y) - T(S_n(x) \cdot y - T(x) \cdot y)) \cdot z \end{aligned}$$

$S_n(x)\cdot S_n(y)-S_n(x)\cdot T(y)-T(x)\cdot S_n(y)+T(x)\cdot T(y)=S_n(S_n(x)\cdot y-T(x)\cdot y)-T(S_n(x)\cdot y-T(x)\cdot y)$, then $x *_{2n-1} (y *_{2n-1} z) = (x *_{2n-1} y) *_{2n-1} z$.

$$\begin{aligned}
x *_{2n} (y *_{2n} z) &= x *_{2n} (T(y) \cdot z - S_n(y) \cdot z) \\
&= T(x) \cdot (T(y) \cdot z - S_n(y) \cdot z) - S_n(x) \cdot (T(y) \cdot z - S_n(y) \cdot z) \\
&= (T(x) \cdot T(y) - T(x) \cdot S_n(y)) \cdot z - (S_n(x) \cdot T(y) - S_n(x) \cdot S_n(y)) \cdot z \\
&= (T(x) \cdot T(y) - T(x) \cdot S_n(y) - S_n(x) \cdot T(y) + S_n(x) \cdot S_n(y)) \cdot z
\end{aligned}$$

$$\begin{aligned}
(x *_{2n} y) *_{2n} z &= (T(x) \cdot y - S_n(x) \cdot y) *_{2n} z \\
&= T(T(x) \cdot y - S_n(x) \cdot y) \cdot z - S_n(T(x) \cdot y - S_n(x) \cdot y) \cdot z \\
&= (T(T(x) \cdot y - S_n(x) \cdot y) - S_n(T(x) \cdot y - S_n(x) \cdot y)) \cdot z
\end{aligned}$$

$T(x) \cdot T(y) - T(x) \cdot S_n(y) - S_n(x) \cdot T(y) + S_n(x) \cdot S_n(y) = T(T(x) \cdot y - S_n(x) \cdot y) - S_n(T(x) \cdot y - S_n(x) \cdot y)$, then $x *_{2n} (y *_{2n} z) = (x *_{2n} y) *_{2n} z$.

Now, we shall prove $x *_i (y *_0 z + \cdots + y *_{i-1} z) = (x *_{i+1} y + \cdots + x *_m y) *_i z$, for $1 \leq i \leq m-1$:

If i is even $i = 2n \;\; 1 \leq n \leq k$, then $y *_1 z + \cdots + y *_{i-1} z = S_n(y) \cdot z$ and $x *_{i+1} y + \cdots + x *_m y = x \cdot y - T(x) \cdot y$. Thus

$$\begin{aligned}
x *_i (y *_0 z + \cdots + y *_{i-1} z) &= x *_i (S_n(y) \cdot z) \\
&= T(x) \cdot (S_n(y) \cdot z) - S_n(x) \cdot (S_n(y) \cdot z) \\
&= (T(x) \cdot S_n(y) - S_n(x) \cdot S_n(y)) \cdot z
\end{aligned}$$

$$\begin{aligned}
(y *_{i+1} z + \cdots + y *_m z) *_i z &= (x \cdot y - T(x) \cdot y) *_i z \\
&= T(x \cdot y - T(x) \cdot y) \cdot z - S_n(x \cdot y - T(x) \cdot y) \cdot z \\
&= (T(x \cdot y - T(x) \cdot y) - S_n(x \cdot y - T(x) \cdot y)) \cdot z
\end{aligned}$$

$T(x) \cdot S_n(y) - S_n(x) \cdot S_n(y) = T(x \cdot y - T(x) \cdot y) - S_n(x \cdot y - T(x) \cdot y)$, then

$$x *_i (y *_0 z + \cdots + y *_{i-1} z) = (y *_{i+1} z + \cdots + y *_m z) *_i z$$

If i is odd $i = 2n-1$, $1 \leq n \leq k$ then $y *_0 z + \cdots + y *_{i-1} z = T(y) \cdot z$ and $x *_{i+1} y + \cdots + x *_m y = x \cdot y - S_n(x) \cdot y$. Thus

$$\begin{aligned}
x *_i (y *_0 z + \cdots + y *_{i-1} z) &= x *_i (T(y) \cdot z) \\
&= S_n(x) \cdot (T(y) \cdot z) - T(x) \cdot (T(y) \cdot z) \\
&= (S_n(x) \cdot T(y) - T(x) \cdot T(y)) \cdot z
\end{aligned}$$

$$\begin{aligned}
(x *_{i+1} y + \cdots + x *_m y) *_i z &= (x \cdot y - S_n(x) \cdot y) *_i z \\
&= S_n(x \cdot y - S_n(x) \cdot y) \cdot z - T(x \cdot y - S_n(x) \cdot y) \cdot z \\
&= (S_n(x \cdot y - S_n(x) \cdot y) - T(x \cdot y - S_n(x) \cdot y)) \cdot z
\end{aligned}$$

$S_n(x \cdot y - S_n(x) \cdot y) - T(x \cdot y - S_n(x) \cdot y) = S_n(x) \cdot T(y) - T(x) \cdot T(y)$ then $x *_i (y *_0 z + \cdots + y *_{i-1} z) = (x *_{i+1} y + \cdots + x *_m y) *_i z$.

Now, we shall prove $x *_m (y *_0 z + \cdots + y *_m z) = (x *_m y) *_m z$.
We have for all $x, y, z \in A$, $y *_0 z + \cdots + y *_m z = y \cdot z$, then

$$\begin{aligned} x *_m (y *_0 z + \cdots + y *_m z) &= x *_m (y \cdot z) \\ &= x \cdot (y \cdot z) - T(x) \cdot (y \cdot z) \\ &= (x \cdot y - T(x) \cdot y) \cdot z \end{aligned}$$

$$\begin{aligned} (x *_m y) *_m z &= (x \cdot y - T(x) \cdot y) *_m z \\ &= (x \cdot y - T(x) \cdot y) \cdot z - T(x \cdot y - T(x) \cdot y) \cdot z \\ &= x \cdot y - T(x) \cdot y - T(x \cdot y - T(x) \cdot y) \cdot z \end{aligned}$$

$x \cdot y - T(x) \cdot y = x \cdot y - T(x) \cdot y - T(x \cdot y - T(x) \cdot y)$ then $x *_m (y *_0 z + \cdots + y *_m z) = (x *_m y) *_m z$.

Finally, we need to prove $(x *_1 y + \cdots + x *_m y) *_0 z = 0,$

we have, $x *_1 y + \cdots + x *_m y = x \cdot y - T(x) \cdot y,$ then

$$\begin{aligned} (x *_1 y + \cdots + x *_m y) *_0 z &= (x \cdot y - T(x) \cdot y) *_0 z \\ &= T(x \cdot y - T(x) \cdot y) \cdot z \end{aligned}$$

$T(x \cdot y) = T(T(x) \cdot y),$ then $(x *_1 y + \cdots + x *_m y) *_0 z = 0$.

Therefore, $(A, \{*_i\}_{1 \leq i \leq m})$ is a Rota^m-algebra.

□

The following result can be similarly demonstrated.

Proposition 5.0.54. *Let $(A, \cdot)$ be an associative algebra and let $T : A \longrightarrow A$ and $S_n : A \longrightarrow A$, $1 \leq n \leq k$ linear maps such that*

$$T(a \cdot b) = T(T(a) \cdot b) = T(a) \cdot T(b) \text{ for all } a, b \in A$$

$$S_n(a \cdot b) = S_n(S_n(a) \cdot b) = S_n(a) \cdot S_n(b) \text{ for all } a, b \in A$$

$$S_n(a) \cdot T(b) = T(S_n(a) \cdot b) \text{ for all } a, b \in A$$

$$T(a) \cdot S_n(b) = S_n(T(a) \cdot b) \text{ for all } a, b \in A$$

$$S_q(S_n(a) \cdot b) = S_n(a) \cdot S_q(b) \text{ for all } 1 \leq n \leq q \leq k-1 \text{ and } a, b \in A$$

If m is even, $m = 2k$ define $m+1$ new operations on A by

$$\begin{aligned} x *_0 y &:= T(x) \cdot y, \\ x *_{2n-1} y &:= S_n(x) \cdot y - T(x) \cdot y, \text{ and} \\ x *_{2n} y &:= T(x) \cdot y - S_n(x) \cdot y \quad \text{for } 1 \leq n \leq k-1 \\ x *_{2k-1} y &:= S_k(x) \cdot y - T(x) \cdot y \\ x *_m y &:= x \cdot y - S_k(x) \cdot y \end{aligned}$$

for all $x, y \in A$. Then, $(A, *_0, \ldots, *_m)$ is a Rota^m-algebra.

Bibliography

[1] A.Frabetti, *Dialgebra homology of associative algebras*, C. R. Acad. Sci. Paris **325** (1997), 135–140.

[2] ______, *Leibniz homology of dialgebras of matrices*, J. Pure Appl. Alg. **129** (1998), 123–141.

[3] M. Aguiar and W. Moreira, *Combinatorics of the free baxter algebra*, Electron. J. Combin. **13** (2006), R17. arXiv:math.CO/0510169.

[4] M. Aguiar and F. Sottile, *Cocommutative Hopf algebras of permutations and trees*, preprint arXiv:math.QA/0403101. (March 2004).

[5] ______, *Structure of the Loday-Ronco Hopf algebra of trees*, preprint (May 2004).

[6] A. Alekseev and C. Torossian, *The kashiwaravergne conjecture and drinfeld's associators*, Annals of Mathematics **175 (2)** (2012), 415–463.

[7] A. Bonfiglioli and R. Fulci, *Topics in noncommutative algebra the theorem of campbell, baker, hausdorff and dynkin*, Springer-Verlag Berlin Heidelberg, London New York, 2012.

[8] D. Burde, *Left-symmetric algebras, or pre-lie algebras in geometry and physics*, Central European Journal of Mathematics **4 (3)** (2006), 323–357.

[9] D. Calaque and C. A. Rossi, *Lectures on Duflo Isomorphisms in Lie Algebra and Complex Geometry*, European Mathematical Society. Series of Lectures in Mathematics, Zurich, 2011.

[10] R. Carter, *Lie Algebras of Finite and Affine Type*, Springer-Verlag Berlin Heidelberg, New York, 2013.

[11] P. Cartier, *Vinberg algebras and combinatorics*, Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France, (2009).

[12] F. Chapoton, *Un théorème de Cartier-Milnor-Moore-Quillen pour les bigèbres dendriformes et les algèbres braces*, J. Pure Appl. Alg **168 (1)** (2002), 1–18.

[13] J. Dixmier, *Enveloping algebras*, Grad. Stud. Math. 11, Amer. Math. Soc., Providence, RI,1996. 2.

[14] K. Ebrahimi-Fard, , and L. Guo, *Free Rota-Baxter algebras and rooted trees*, arXiv:math.RA/0510266.

[15] R. Holtkamp, *On Hopf algebra structures over operads*, preprint (July 2004), arXiv:math.RA/0407074.

[16] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff formula and invariant hyperfunctions*, Invent. Math. **47 (3)** (1978), 249 – 272.

[17] M. Kontsevich, *Deformation quantization of poisson manifolds*, Lett. Math. Phys. **66 (3)** (2003), 157–216.

[18] L.Foissy, *Les algèbres de Hopf des arbres enracinés décorés II*, Bull. Sci. Math. **126** (2002), 249–288.

[19] J.-L. Loday, *Dialgebras*, Prepublication de l'Ins.de Recherche Math. Avancée, Strasbourg, 1999.

[20] J.L. Loday, *Dialgebras, in Dialgebras and related operads, Lecture Notes in Math.*, 1763 (2001), 7–66.(preprint 2001, arXiv:math.QA/0102053).

[21] ———, *Arithmetree*, J. Algebra **258** (2002), 275–309.

[22] ———, *Scindement d'associativitè et algèbres de Hopf, Proceedings of the Conference in honor of Jean Leray, Nantes*, Sèminaire et Congrès (SMF) **9** (2004), 155–172.

[23] J.L. Loday and M. Ronco, *Algèbre de Hopf colibres*, C. R. Acad. Sci. Paris **337** (2003), 153–158.

[24] ———, *Trialgebras and families of polytopes in Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic k-theory*, Contemporary Mathematics **346** (preprint May 2002, 2004), arXiv:math.AT/0205043.

[25] J.L. Loday and M. O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139 (2)** (1998), 293–309.

[26] M.Duflo, *Caractères des groupes et des algèbres de lie résolubles*, Ann. Sci. École Norm. Sup **(4) 3** (1970), 23–74.

[27] ———, *Opérateurs différentiels bi-invariants sur un groupe de Lie*, Ann. Sci. École Norm. Sup **(4) 10** (1977), 265–288.

[28] E. Meinrenken, *Clifford Algebras and Lie Theory*, Cambridge University Press, New York, 2005.

[29] Ian M. Musson, *Lie superalgebras and Enveloping algebras*, Grad. Stud. Math. 131, Amer. Math. Soc., Providence, RI,2012.

[30] M. O. Ronco, *Primitive elements in a free dendriform algebra*, In New trends in Hopf algebra Theory, (La Falda, 1999), Contemporary Mathematics **267** (2000), 245–263.

[31] ———, *Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras*, J. Algebra **254 (1)** (2002), 152–172.

[32] B. Shoikhet, *On the Duflo formula for L_∞-algebras and Q-manifolds*, arXiv:math/9812009v2 [math.QA] (7 Dec 1998), 1–11.

[33] P. Tauvel and Rupert.W.T.Yu, *Lie Algebras and Algebraic Groups*, Springer-Verlag Berlin Heidelberg, Germany, 2005.

[34] Tuong Ton-That and Thai-Duong Tran, *Poincaré's proof of the so-called Birkhoff-Wiff theorem*, Revue d'histoire des mathématiques, Société mathématique de France **5** (1999), 249–284.