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GENERATION OF SULLIVAN DECOMPOSABLE ALGEBRAS VIA CERTAIN PDES

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Abstract

In this Thesis we investigate properties of certain commutative differential graded algebras naturally associated to some submanifolds of a infinite Jet manifold which is determined by a finite system of finite-order PDEs, particularly those inspired by the study of linear gauge complexes and by one-forms associated to equations of pseudo-spherical type. More explicitly, we identify linear gauge complexes as a particular type of certain twisted complexes and we will generate Sullivan decomposable algebras using the hierarchies of equations of pseudo-spherical type. Finally, we will relate this process with the A_{PL} functor and spatial realization functor, taking advantage of its important properties which relate Sullivan algebras to topological spaces.

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Introduction

Dennis Sullivan in [25] defined a *Sullivan Decomposable algebra* as a free commutative differential graded algebra (*cdga*) generated by a graded vector space $V = \{V^i\}_{i\geq 0}$ of the form $(\Lambda V, d)$, where the differential satisfies the condition $dV \subseteq \Lambda V^{\geq 2} V$. Sullivan also proved that this notion can be described in terms of a Lie algebra and a sequence of *twisted cohomology classes*.

Subsequently he refines this notion and defines a *Sullivan minimal algebra* free commutative differential graded algebra generated by a graded vector space V, which admits a well-ordered homogeneous basis $\{v_\alpha\}_{\alpha\in I}$ compatible with the degree [that is, if $\beta < \alpha$, then $|v_\beta| \leq |v_\alpha|$ in which $|V_*|$ denotes the degree of V_*] and such that the differential d satisfies $d(v_\alpha) \in \Lambda V_{<\alpha}$ for each $\alpha \in I$, where $\Lambda V_{<\alpha}$ denotes the subspace of V generated by $\{v_\beta\}_{\beta<\alpha}$.

Inspired by the construction of De Rham complex of differential forms on a manifold, Sullivan introduces the so-called *minimal model* of a given topological space. For this purpose, first he builds the *cdga* of polynomial differential forms $A_{PL}(X)$ of a given topological space X. Then, for any *cdga* A such that $H^0(A) = \mathbb{Q}$ Sullivan builds its *minimal model*, this is, a quasi-isomorphism $\rho\colon (\Lambda V, d) \to A$ where $(\Lambda V, d)$ is a Sullivan minimal algebra. The *minimal model of* X is, by definition the minimal model of $A_{PL}(X)$. As show in [7], $A_{PL}(X)$ encodes important (rational) homotopic information of X

The purpose of this paper is to identify properties *cdga's* generated by differential equations, particularly those related with the study of linear gauge complex (see [12] and [13]) and differential equations of pseudo-spherical type (see [17], [19] and [20]). These two theories allow us to generate Sullivan decomposable algebras, minimal Sullivan algebras and minimal models. We interpret these algebras as determining (via the functor A_{PL}) topological models for differential equations

The thesis is organised as follows:

Chapter 1.
This chapter is devoted to studying in detail some notation and results about graded algebras and algebraic systems of coefficients in a *cdga*. These structure are needed to describe the main notion of a Sullivan decomposable algebra and its relation to twisting matrix and twisting cohomology in hom version.

Chapter 2.
We introduce the definition of a Sullivan minimal algebra, and we also consider one of the most important concepts in rational homotopy theory, the notion of *minimal model of a cdga*. The existence of a minimal model is guaranteed by Sullivan's Theorem [25]. We remark that in spite of

its important a fully general proof of Sullivan's theorem is hard to find in the literature. This is why we have decided to present one in full detail. We give the definition of extensions of a *dga*, which permits us to prove Sullivan's theorem in full generality, we present a detailed proof following [25], [8], [10], and [15]. We remark that spite of its importance a fully general proof of Sullivan's theorem is hard to find in the literature. This is why we have decided to present one in full detail. We finish this chapter relating this demonstration (in the case that the homology of the *cdga* is of finite type i.e., all homology space are of finite dimension) with the twisting matrices and twisted cohomology, such a relation is possible since a Sullivan minimal algebra is a Sullivan decomposable algebra.

Chapter 3.
The third part of the thesis contains our next main result: We prove that the horizontal Gauge cohomology studied for Marvan in [12] and [13] is the twisted cohomology with coefficients in $\mathfrak{g}$ a lie algebra. For this purpose, we study basic facts about Geometry of Infinite Jet manifold and the Variational Bicomplex. Furthermore, since the linear Gauge complex use $\mathfrak{g}$-valued differential forms as elements and considers submanifolds of a infinite Jet bundle determined by PDEs, we introduce the concept of $\mathfrak{g}$-valued differential forms in the second section, and in the third section we present some basic constructions about submanifolds of infinite Jet manifolds determined by a finite system of finite-order PDEs, following the works of Anderson and Kamran in [2] and Reyes in [21].

In the last section we give an example of Marvan [12] about the nonlinear Klein-Gordon equation $\frac{\partial u}{\partial x \partial y} = g(u)$, for which he finds an $\mathfrak{sl}(2)$-valued zero curvature representation, and we generate a twisting matrix and we give a element that satisfies the Maurer-Cartan condition by the vertical differential.

Chapter 4.
We apply our results to the construction of twisting matrices, and we generate Sullivan decomposable algebras by certain forms determined by a manifold of pseudo-spherical type on a submanifold of a infinite Jet bundle. We use the Burgers' equation and Sine-Gordon equation (Examples that appear in [18]) and we interpret our results as a way to generate Sullivan decomposable algebras. We also introduce the notion of gauge transformation, we show that these transformations can be used to obtain new twisting matrices, and also that the Sullivan decomposable algebra generated are not necessarily isomorphic to the original.

Finally, we present a Sullivan decomposable algebra generated by a hierarchy of pseudo-spherical type using twisting matrices, this allows us to generalize the construction of algebra generated by a single equation. We close our work with a short section about the functors, A_{PL} and the spatial realization of a *cdga*, and we relate its important properties with our study of Sullivan descomposable algebras generated by manifolds of pseudo-spherical type.

Chapter 1

Preliminaries

In this chapter, we first include some notation and basic facts on graded algebras. This material is required to introduce the notion of Sullivan decomposable algebras, also called by Sullivan in the article [25] minimal algebras, which are determined by a Lie algebra and a sequence of twisted cohomology classes. We also consider an algebraic system of coefficients in a commutative differential graded algebra [9], and afterwards relate this notion to twisting matrices and twisting cohomology in tensor and Hom version.

1.1 Graded module over a ring k

Definition 1.1.1. *A (non-negatively)* ***graded module over a ring*** $\boldsymbol{k}$ *is a family of modules over* k *:* $M = \{M^i\}_{i\geq 0}$*, indexed by non-negative integers. Elements belonging to* M^i *are called* ***homogeneous elements of degree*** $\boldsymbol{i}$*, and if* $x \in M^i$*, we denote its degree as* $|x| = i$*. We say that* M *is* ***concentrated in degrees*** $\boldsymbol{i \in I}$ *if* $M^i = 0$*,* $i \notin I$*; in this case, we write* $M = \{M^i\}_{i\in I}$*.*

We will consider some typical constructions of graded linear algebra. We always assume that $M = \{M^i\}_{i\geq 0}$ is a graded module.

1. The ***direct sum*** of two graded modules M, N over the same ring, is the graded module $M \oplus N$ with the grading: $(M \oplus N)^r = M^r \oplus N^r$.
2. A ***graded submodule*** $B \subseteq M$ is a graded module $B = \{B^i\}_{i\geq 0}$, such that, for all $i \geq 0$, B^i is a submodule of M^i.
3. Given a graded submodule $B \subseteq M$, the ***quotient module*** of M by B is the graded module $M/B = \{M^i/B^i\}_{i\geq 0}$ with $|x + B^i| = |x|$.
4. The ***tensor product*** of two graded modules M, N over the same ring, is again a graded module whose degree r component is given by:
$$(M \otimes N)^r = \bigoplus_{p+q=r} M^p \otimes N^q.$$
5. A ***linear map of degree*** $\boldsymbol{i}$ between graded modules $f\colon M \to P$ is a family of linear maps $f^j\colon M^j \to N^{j+i}$. Also, each linear map f determines graded submodules:
$$Ker(f) \subseteq M \text{ and } Im(f) \subset N, \text{ where } (Ker(f))^j = Ker(f^j) \text{ and } (Im(f))^j = Im(f^{j-i}).$$
6. $Hom(M, N)$ is the graded module whose elements of degree i are the linear maps from M to N of degree i.

In particular, if $M(i)$ is the graded module defined by

$$M(i)^j = \begin{cases} M^i, & j = i \\ 0, & \text{otherwise}, \end{cases}$$

then, $M(i)$ is concentrated in degree i. And $M = \bigoplus_{i\geq 0} M(i)$; it is usual to write $M = \bigoplus_{i\geq 0} M^i$. It is important to note there is no addition defined for M^i and M^j if $i \neq j$; when M is seen as the direct sum $M = \bigoplus_{i\geq 0} M^i$ the addition $x + y$ is always defined as a formal sum.

If k is a field and M is a graded module over k, we say that M is a ***graded vector space***; M is said to be of ***finite type*** if each M^i is finite dimensional. A graded vector space is ***finite dimensional*** if each M^i is finite dimensional and M is concentrated in finitely many degrees. In this case $dimM = \sum_i dimM^i$.

Definition 1.1.2. *A **differential on a graded module** M is a linear map $d_M : M \to M$ of degree 1, for which $d_M^{n+1} \circ d_M^n = 0$ for any $n \geq 0$. We call (M, d_M), or simply M abusing of language, a **differential graded module** (dgm for short).*

A *dgm* is also known as a cochain complex. Any differential graded module (M, d_M) has an associated graded module $H(M, d_M)$ defined by:

$$H^n(M, d_M) = Ker(d_M^n)/Im(d_M^{n-1}), \ \text{ for each } \ n \geq 1 \ \text{ and } H^0(M, d_M) = Ker(d_M^0).$$

The elements of $Ker(d_M^n)$ are called n-***cocycles*** and the elements of $Im(d_M^n)$ are called n-***coboundaries***. The graded module $H(M, d_N)$ is called ***the cohomology*** of M.

Note that $Hom(M, N)$ has two structures of *dgm*:

$$\left.\begin{array}{r} (Hom(M,N), d_{Hom}), \ \text{ where } \ d_{Hom}(f) = d_N \circ f, \\ (Hom(M,N), \overline{d}_{Hom}), \ \text{ where } \ \overline{d}_{Hom}(f) = d_N \circ f - (-1)^{|f|} f \circ d_M. \end{array}\right\} \tag{1.1.1}$$

We will see in Example (1.3.2) that $\overline{d}_{Hom}$ has an useful property which d_{Hom} does not always satisfy.

1.2 Commutative differential graded algebras

Definition 1.2.1. *A **graded algebra** A is a graded module equipped with a linear map $A \otimes A \to A$ of degree zero, called multiplication and defined by $x \otimes y \mapsto xy$, together with an identity element $1 \in A^0$, such that for all $x, y, z \in A$:*

$$(xy)z = x(yz) \ \text{ and } \ 1x = x1 = x.$$

A graded algebra A is ***commutative*** (in the graded sense) if $yx = (-1)^{|x||y|} xy$ for all $x, y \in A$. A ***left ideal*** I of A is a graded submodule, such that if $x \in A$ and $y \in I$, then $xy \in I$; a right ideal is defined analogously, and we refer to a two sided ideal as an ***ideal***.

Examples 1.2.1.

1. *Given A and B graded algebras, the tensor product $A \otimes B$ admits a graded algebra structure. We define the multiplication as*

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb', \quad \text{for } a, b \in A \text{ and } a', b' \in B. \tag{1.2.1}$$

Also, we can define other multiplications as:

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \text{ or } (a \otimes b)(a' \otimes b') = a'a \otimes b'b.$$

The sign convention in (1.2.1) ensures that the tensor product of two commutative graded algebras is also commutative.

2. *The ring k can be regarded as a graded algebra concentrated in degree 0, in the following way: $k = \{k^n\}_{n \geq 0}$ such that:*

$$k^n = \begin{cases} k, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

3. *Let V be a graded vector space; the **tensor algebra** of V is defined by*

$$T(V) = \bigoplus_{r \geq 0} T^r(V), \text{ where } T^0(V) = k, \text{ and } T^r(V) = \underbrace{V \otimes \cdots \otimes V}_{r-times} \text{ for } r \geq 1.$$

The multiplication is given as follows: if $x \in T^r(V)$ and $y \in T^l(V)$, then $xy = x \otimes y \in T^{r+l}(V)$; the identity is $1 \in T^0(V)$, and the degree of a generic element $v_1 \otimes \cdots \otimes v_r \in T^r(V)$ of word length r is defined to be $\sum_{i=1}^{r} |v_i|$. If $(T^r(V))^n$ denotes all elements of degree n in $T^r(V)$, then the degree n component of $T(V)$ is $(T(V))^n = \bigoplus_{r \geq 0} (T^r(V))^n$.

4. *The elements $x \otimes y - (-1)^{|x||y|} y \otimes x$, with x and y homogeneous elements of V, generate an ideal I in $T(V)$. Thus, the quotient $\Lambda V = T(V)/I$ inherits a structure of graded algebra, which is commutative by construction; it is called the **free commutative graded algebra on** V. We denote its multiplication by $x \wedge y$ for $x, y \in \Lambda V$. In particular $v \wedge v = 0$ if $|v|$ is odd: in fact from the commutativity of ΛV, $v \wedge v = (-1)^{|v||v|} v \wedge v$, hence $2v \wedge v = 0$ (here we consider k a field of characteristic different from 2).*

   We may write $\Lambda V = \bigoplus_{r \geq 0} (\Lambda^r V)$ in which $\Lambda^r V = \pi(T^r(V))$ with $\pi \colon T(V) \to \Lambda V$ the canonical projection. Note that $\Lambda^r V$ stands for all elements of word length r. It will cause no confusion with the notation $(\Lambda V)^r$, which stands for all elements in ΛV of degree r. Let us denote ΛV by $\Lambda(v_1, v_2, \ldots)$ if $\{v_1, v_2, \ldots\}$ is basis of V.

5. *Let V be a graded vector space concentrated in degree 1 (or in an odd degree). Then, we can construct $\Lambda V = T(V)/I$, the free commutative graded algebra on V as above. We note that I is the ideal in $T(V)$ generated by the set of elements of the form $v \otimes v$ for $v \in V$. In fact, by construction I is generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x$ with x, y homogeneous elements in V (since V is concentrated in degree 1, it suffices to consider the elements of degree 1); in particular for $x = y$ with $|x| = 1$, we obtain $x \otimes x \in I$; now if we*

consider the ideal generated by elements of the form $v \otimes v$ for $v \in V$, then if $x, y \in V$ with $|x| = |y| = 1$, we have $(x+y) \otimes (x+y) \in I$, and therefore

$$(x+y) \otimes (x+y) - x \otimes x + y \otimes y = x \otimes y + y \otimes x \in I.$$

*So when V is a (not graded) vector space then we can consider it as being concentrated in degree 1. The corresponding ΛV is known as the **exterior graded algebra of** V, and the multiplication $\wedge$ is called the **wedge** or **exterior product**. If we consider the subspace $I_r(V) = I \cap T^r(V)$ then $\Lambda^r V \cong T^r(V)/I_r(V)$.*

We also observe, after [26], that the Universal property of ΛV establishes a natural isomorphism $(\Lambda^r V)^* \cong A_r(V)$ where $A_r(V)$ is the vector space of all alternating r-multilinear maps, that is, the space of r-multilinear maps

$$h \colon \underbrace{V \times V \times \cdots \times V}_{r-\text{copies}} \to \mathbb{R}$$

such that for all σ in the permutation group S_r: $h(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) = (sgn\sigma) h(v_1, \ldots, v_r)$, in which $sgn\sigma$ is the sign of σ.

On the other hand using a pairing we can establish an isomorphism between $\Lambda^r(V^*)$ and $\Lambda^r(V)^*$, and so we have an isomorphism $\Lambda^r(V^*) \cong A_r(V)$.

Definition 1.2.2. *A **morphism** of graded algebras $\varphi \colon A \to B$ is a linear map of degree zero such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$ and $\varphi(1) = 1$.*

Definition 1.2.3. *A **derivation of degree i** in a graded algebra A is a linear map $d \colon A \to A$ of degree i such that:*

$$d(xy) = d(x)y + (-1)^{i|x|} x\, d(y), \quad \text{for all } x, y \in A$$

The last equality when $i = 1$ is called ***the graded Leibnitz rule***.

Any linear map $f \colon V \to A$ from a graded vector space V to a graded algebra A extends to a unique morphism $\widehat{f}$ of graded algebras:

$$\begin{array}{rccc} \widehat{f}\colon & T(V) & \to & A \\ & x \otimes y & \mapsto & f(x)f(y) \end{array}$$

and, any linear map $g \colon V \to T(V)$ of degree i, extends to a unique derivation $\breve{g}$ of degree i in $T(V)$ by:

$$\begin{array}{rccc} \breve{g}\colon & T(V) & \to & T(V) \\ & x \otimes y & \to & g(x) \otimes y + (-1)^{i|x|} x \otimes g(y). \end{array} \tag{1.2.2}$$

Furthermore, let us assume that A is a commutative graded algebra and V a graded vector space: Any linear map $f \colon V \to A$ of degree 0 extends to a unique morphism $\widehat{f}$ of commutative graded algebras, $\widehat{f}\colon \Lambda V \to A$, and any linear map $g\colon V \to \Lambda V$ of degree i extends to a unique derivation of degree i in ΛV.

Definition 1.2.4. *A* ***differential graded algebra*** *(dga for short) is a graded algebra A equipped with a differential $d_A\colon A \to A$ (where A is considered a graded module) which is also a derivation of degree 1, this is d_A satisfies the graded Leibniz rule:*

$$d_A(xy) = d_A(x)\, y + (-1)^{|x|} x\, d_A(y), \quad \text{for all } \ x, y \in A \tag{1.2.3}$$

and moreover $d_A \circ d_A = 0$.

We often denote a cdga A by the pair (A, d_A). If A is commutative, we call it a ***commutative differential graded algebra*** *(cdga for short)*

A ***morphism*** of *(c)dga's* $\varphi\colon (A, d_A) \to (B, d_B)$ is a graded algebra morphism such that $fd_B = d_A f$.

Let A be a *(c)dga*; a graded submodule which is closed under the maps multiplication and differential is called a subalgebra of A. So $Ker(d_A)$ is a subalgebra of A and $Im(d_A)$ is an ideal of this subalgebra, hence $H(A, d_A) = Ker(d_A)/Im(d_A)$ is also a graded algebra with multiplication $[a][a'] = [aa']$ for $a, a' \in Ker(d_A)$. We will say that a is a cocycle if $a \in Ker(d_A)$. A morphism f of *(c)dga's* induces a homomorphism in cohomology:

$$\begin{array}{rccc} H(f): & H(A, d_A) & \to & H(B, d_B) \\ & [x] & \mapsto & [f(x)] \end{array}$$

Definition 1.2.5. *A morphism $f\colon (A, d_A) \to (B, d_B)$ between (c)dga's is called* ***quasi-isomorphism*** *if the induced homomorphism in cohomology $H(f)$ is an isomorphism.*

Examples 1.2.2.

1. *Let (A, d_A) be a dga, then $k \subset H^0(A, d_A)$. Indeed, by the graded Leibniz rule, for the identity element $1 \in A^0$ we have that $d(1) = d(1.1) = d(1) + d(1)$, therefore $d(1) = 0$ (here we consider k a field of characteristic different from 2).*

2. *Given two (c)dga's (A, d_A) and (B, d_B), then $A \otimes B$ is also a (c)dga with differential*

$$d(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b)$$

   and multiplication given in (1.2.1).

3. *The ring k is a cdga with differential the null map.*

4. *Let X be a smooth manifold; the space of differential forms, denoted by $\Omega(X) = \{\Omega^i(X)\}_{i \geq 0}$, where $\Omega^i(X)$ denote the real vector space of all smooth i-forms on X, with the exterior product $\wedge$ and exterior differential d (see [26]) is a cdga. The cohomology of this cdga is called the deRham cohomology, the i-th deRham cohomology group of X is denoted by $H^i_{DR}(X) = Ker(d^i)/Im(d^{i-1})$.*

5. *A differential on $T(V)$ is completely characterized by its values on V and the graded Leibnitz rule, since the restriction of d to V is a linear map of degree 1 and so it extends to a derivation in $T(V)$, as explained in the expression (1.2.2)*

6. *As in the previous example, a differential on ΛV is completely characterized by its values on V and the graded Leibnitz rule. We call the cdga $(\Lambda V, d)$ the **free commutative differential graded algebra (free cdga) on** V, with differential d.*

   Below, we present two properties of the differential in $(\Lambda V, d)$:

   I *If $w \in (\Lambda^r V)^n$ (recall that $\Lambda^r V$ stands for all elements in ΛV of word length r and $(\Lambda V)^n$ stands for all elements in ΛV of degree n), then $d(w) \in (\Lambda^{\geq r} V)^{n+1}$.*

   We observe that, $(\Lambda^0 V)^0 = k$ and $(\Lambda^0 V)^n = \{0\}$ if $n \geq 1$. So, for $w \in (\Lambda^0 V)^n$, then $d(w) \in (\Lambda V)^{n+1} = (\Lambda^{\geq 0} V)^{n+1}$ and for $w \in (\Lambda^1 V)^n$, then $d(w) \in (\Lambda V)^{n+1}$, as $(\Lambda^0 V)^{n+1} = \{0\}$, we have $d(w) \in (\Lambda^{\geq 1} V)^{n+1}$. Now, if $w = x \wedge y$, where $x \in (\Lambda^1 V)^p$ and $y \in (\Lambda^1 V)^q$ (this is x, y are homogeneous elements of V) then, we obtain of the above that $d(w) \in (\Lambda^{\geq 2} V)^{p+q+1}$, in fact $d(w) = d(x) \wedge y + (-1)^{|x|} x \wedge d(y)$.

   By induction on the length of the word, and equality (1.2.3) and the linearity of d, this property is obtained.

   So, we have that if $w \in (\Lambda^r V)^n$, then $d(w) = (d(w))_r + (d(w))_{r+1} + \cdots + (d(w))_m$, for some $m \in \mathbb{N}$ where $(d(w))_i \in (\Lambda^i V)^{n+1}$, for $r \leq i \leq m$.

   II *The differential d can always be written as a sum $d = d_0 + d_1 + d_2 + \cdots + d_p$ for some $p \in \mathbb{N}$, where d_i is a derivation of degree 1.*

   To prove this property, we consider $w \in (\Lambda V)^r$, according to the length of words that form w, then $w = w_0 + w_1 + \cdots + w_j$ for some $j \in \mathbb{N}$, where $w_i \in (\Lambda^i V)^r$ for $0 \leq i \leq j$. Hence, $d(w) = d(w_0) + d(w_1) + \cdots + d(w_j)$ and by the first property we have:

$$
\begin{aligned}
d(w) = d(w_0)_0 + d(w_0)_1 + \cdots + d(w_0)_{p_0} + \\
d(w_1)_1 + d(w_1)_2 + \cdots + d(w_1)_{p_1} + \cdots + \\
d(w_j)_j + d(w_1)_j + 1 + \cdots + d(w_j)_{p_j}
\end{aligned}
$$

   Let $p = \max\{p_0, \ldots, p_j\}$. We define $d(w_i)_{p_i+n} = 0$, for $p_i < p_i + n \leq p$ and according to the columns of the previous sum, we denote:

$$
\begin{aligned}
d_0(w) &= d(w_0)_0 + d(w_1)_1 + \cdots + d(w_j)_j, \\
d_1(w) &= d(w_0)_1 + d(w_1)_2 + \cdots + d(w_j)_{j+1}, \\
&\vdots \\
d_p(w) &= d(w_0)_p + d(w_1)_p + \cdots + d(w_j)_{j+p}
\end{aligned}
$$

   Then, $d(w) = d_0(w) + d_1(w) + \cdots + d_p(w)$. In particular for $w = w_t$, this is, w is formed by a element of length t, then $d_i(w) = d(w)_{i+t}$.

Furthermore, we can define the following linear maps:

$$\begin{array}{rccc} d_i: & \Lambda V & \to & \Lambda V \\ & w & \to & d_i(w) \end{array}$$

so, d_i increases the of length word by exactly i, this is, $d_i(\Lambda^r V) \subseteq \Lambda^{r+i}V$.

Also, d_i is a derivation of degree 1, since, for x, y homogeneous elements of V we have:

$$\begin{aligned} d_i(x \wedge y) &= (d(x \wedge y))_{i+2} \\ &= (d(x) \wedge y)_{i+2} + (-1)^{|x|}(x \wedge d(y))_{i+2} \\ &= (d(x))_{i+1} \wedge y + (-1)^{|x|} x \wedge (dy)_{i+1} \\ &= d_i(x) \wedge y + (-1)^{|x|} x \wedge d_i(y). \end{aligned}$$

Now, if (A, d) be a free cdga, then we call d_0 the ***linear part*** *of the differential and d_1 the* ***quadratic part*** *of the differential.*

7. *Let (A, d_A) be a dga; the* ***opposite differential graded algebra*** *(A^{opp}, d) is the dga defined by $A^{opp} = A$, $d = d_A$ and multiplication given by $a \bullet_{opp} b = (-1)^{|b||a|} ba$, for homogeneous elements $a, b \in A$. So, $d(a \bullet_{opp} b) = d(a) \bullet_{opp} b + (-1)^{|a|} a \bullet_{opp} d(b)$ and if A is commutative, then $(A^{opp}, d_A) = (A, d_A)$ as cdga's.*

1.3 Modules over a differential graded algebra

Definition 1.3.1. *Let A be a graded algebra. A (left)*A***-module*** *is a graded module M over k together with a linear map of degree zero $A \otimes M \to M$, $x \otimes m \mapsto xm$, such that $x(ym) = (xy)m$ and $1m = m$, for all $x, y \in A$ and $m \in M$.*

We can define a *(right)*A-***module*** in an analogous manner. We say that M is an A***-module*** if M is a *(left)*A*-module* and a *(right)*A*-module* such that $mx = (-1)^{|x||m|} xm$, for all $x \in A$ and $m \in M$. Also, An A*-module* M satisfies the following equalities, for all $x, a \in A$ and $m \in M$:

$$x(ma) = (-1)^{|a||m|}(xa)m = (-1)^{|a||m|}(-1)^{|xa||m|} m(xa) = (-1)^{|x||m|}(mx)a = (xm)a.$$

Now, each *(left)*A*-module* M can be seen as a *(right)*A^{opp}*-module*, with linear map

$$M \otimes A^{opp} \to M, \quad m \otimes x \mapsto mx = (-1)^{|x||m|} xm.$$

Moreover, from Example (1.2.2-7) we see that if A is a commutative graded algebra, the *(left)*A*-module* M is also a *(right)*A*-module*; in this case M is an A*-module*.

Examples 1.3.1.

1. *Let A be a commutative graded algebra. An* A***-linear map*** *$f: M \to N$ of degree i between (left)A-modules M and N, is a linear map of degree i of graded modules such that*

$$f(am) = (-1)^{|a||f|} a f(m), \quad \textit{for all } a \in A \textit{ and } m \in M.$$

These maps form a graded submodule $Hom_A(M,N) \subseteq Hom(M,N)$. Moreover $Hom_A(M,N)$ is an A-module with operation given by the linear map:

$$\begin{array}{ccccc} A \otimes Hom_A(M,N) & \to & Hom_A(M,N) & & \\ a \otimes f & \mapsto & af: & M & \to & N \\ & & & m & \mapsto & af(m). \end{array}$$

2. *Let M be a (right)A-module and let N be a (left)A-module; then we define the module $M \otimes_A N = (M \otimes N)/I$, where I is the submodule of $M \otimes N$ spanned by elements of the form $ma \otimes n - m \otimes an$ with $a \in A$. So $M \otimes_A N$ admits a structure of (left)A-module with operation given by the linear map $x(m \otimes_A n) = (-1)^{|m||x|} mx \otimes_A n = (-1)^{|m||x|} m \otimes_A xn$ for $x \in A$ and $m \otimes_A n \in M \otimes_A N$. If A is commutative we can define $x(m \otimes_A n) = xm \otimes_A n$, since $xm = (-1)^{|m||x|} mx$.*

Definition 1.3.2. *Let (A, d_A) be a dga. A (left)A-**differential graded module** ((left)A-dgm for short), is a (left)A-module M together with a differential d_M in M satisfying:*

$$d_M(am) = d_A(a)m + (-1)^{|a|} a\, d_M(m), \quad \text{for } a \in A \text{ and } m \in M. \tag{1.3.1}$$

Moreover, if M is an A-*module*, we call M an A-***differential graded module*** (A-*dgm* for short). Note that if M is a *dga*, then M is itself a *(left)M-dgm* and (1.3.1) is equivalent to the property of the differential (1.2.3).

If (A, d_A) is a *cdga*, we have:

$$\begin{aligned} d_M(ma) &= (-1)^{|a||m|}(d_A(a)\, m + (-1)^{|a|} a\, d_M(m)) \\ &= (-1)^{|a||m|}\big((-1)^{(|a|+1)|m|} m\, d_A(a) + (-1)^{|a|+|a|(|m|+1)} d_M(m) a\big) \\ &= d_M(m)\, a + (-1)^{|m|} m\, d_A(a). \end{aligned}$$

A ***morphism*** of A-*dgm's*, $f\colon (M, d_M) \to (N, d_N)$ is an A-linear map of degree 0, such that $d_M f = f d_N$.

Examples 1.3.2.

1. *Let us recall that we pointed out in (1.1.1) that $Hom(M,N)$ has two differentials, d_{Hom} and $\overline{d}_{Hom}$. We remark that the map $\overline{d}_{Hom}$ is a differential for $Hom_A(M,N)$ but d_{Hom} is not, because $d_N \circ f$ does not necessarily belong to $Hom_A(M,N)$. Indeed,*

$$d_N \circ f(am) = d_N((-1)^{|a||f|} af(m)) = (-1)^{|a||f|} d_A(a)\,(f(m)) + (-1)^{|a|(|f|+1)} a\, d_N(f(m)),$$

while, $(-1)^{|a||d_N \circ f|} a\,(d_N \circ f(m)) = (-1)^{|a|(|f|+1)} a\, d_N(f(m))$.

2. *Let A be a cdga and let V be a graded vector space over a field k. Then $A \otimes V$ is an A-dgm. It is enough to define the linear map $x(a \otimes v) = xa \otimes v$ and the differential $d(a \otimes v) = d_A(a) \otimes v$ for $x \in A$ and $a \otimes v \in A \otimes V$.*

3. *Let A be a cdga and let V be a graded vector space over a field k concentrated in degree zero. We observe that $End(V) = Hom(V,V)$ is a graded algebra concentrated in degree zero with multiplication given by composition of maps. Then $A \otimes End(V)$ is a graded algebra with component of degree n given by*

$$(A \otimes End(V))^n = A^n \otimes (End(V))^0 = A^n \otimes End(V)$$

and with multiplication defined by:

$$(a \otimes f) \circ (b \otimes g) = ab \otimes g \circ f.$$

We can define a trivial differential on $End(V)$, that is, $d_{End(V)}(f) = 0$, for all $f \in End(V)$; it follows that $A \otimes End(V)$ has differential $d(a \otimes f) = d_A(a) \otimes f$. Thus, $A \otimes End(V)$ is a A-dgm with operation given by the linear map

$$x(a \otimes f) = xa \otimes f.$$

1.4 Algebraic Systems of Coefficients

1.4.1 Tensor version of the cohomology of a module with values in an algebraic system of coefficients

Definition 1.4.1. *An* ***algebraic system of coefficients*** *in a cdga over a field k $A = \{A^i\}_{i \geq 0}$ is a pair (Θ, V) consisting of a k-vector space V (a graded vector space concentrated in degree zero) and an element Θ, of degree 1, in the A-dgm $A \otimes End(V)$ such that:*

$$d\Theta - \Theta \circ \Theta = 0. \tag{1.4.1}$$

The ***dimension*** *of the system is the dimension of the vector space V.*

Note that the element $\Theta \in A^1 \otimes (End(V))$ can be expressed as:

$$\Theta = \sum_{\alpha \in \Lambda} a_\alpha \otimes f_\alpha \tag{1.4.2}$$

where $|a_\alpha| = 1$, f_α is a linear function of V into V and Λ is a finitely indexed set.

Thus, recalling Example (1.3.2-3) we have $\Theta \circ \Theta \in A^1 \otimes (End(V))$ and $d\Theta \in A^1 \otimes (End(V))$, we have:

$$\Theta \circ \Theta = (\sum_{\alpha \in \Lambda} a_\alpha \otimes f_\alpha) \circ (\sum_{\beta \in \Lambda} a_\beta \otimes f_\beta) = \sum_{\alpha \in \Lambda} \sum_{\beta \in \Lambda} a_\alpha a_\beta \otimes f_\beta \circ f_\alpha,$$

$$d\Theta = \sum_{\alpha \in \Lambda} d_A(a_\alpha) \otimes f_\alpha.$$

Hence, if we write Θ as in the expression (1.4.2), a system (Θ, V) can be seen as a linear map of V into $A^1 \otimes V$,

$$\begin{array}{rccl} \Theta: & V & \to & A^1 \otimes V \\ & v & \mapsto & \sum_{\alpha \in \Lambda} a_\alpha \otimes f_\alpha(v), \end{array}$$

and we can also consider $\Theta \circ \Theta$ and $d\Theta$ as the linear maps:

$$\begin{array}{rcl} \Theta \circ \Theta : \quad V & \to & A^2 \otimes V \\ v & \mapsto & \sum_{\alpha \in \Lambda} \sum_{\beta \in \Lambda} a_\alpha a_\beta \otimes f_\beta \circ f_\alpha(v), \end{array}$$

$$\begin{array}{rcl} d\Theta : \quad V & \to & A^2 \otimes V \\ v & \mapsto & \sum_{\alpha \in \Lambda} d_A(a_\alpha) \otimes f_\alpha(v). \end{array}$$

Now, let (Θ, V) be a system of coefficients on A and let (Y, d_Y) be a A-*dgm*; then the module $Y \otimes V$ inherits a structure of A-*dgm* with operation and differential given by the following linear maps,

$$\left. \begin{array}{r} a(y \otimes v) = ay \otimes v, \\ d_\Theta(y \otimes v) = d_Y(y) \otimes v + (-1)^{|y|} \Psi_y \circ \Theta(v), \end{array} \right\} \tag{1.4.3}$$

where for a given element $y \in Y$, Ψ_y is the linear map of degree $|y|$ between graded modules defined by:

$$\begin{array}{rcl} \Psi_y : \quad A \otimes V & \to & Y \otimes V \\ a \otimes v & \mapsto & ya \otimes v. \end{array}$$

Moreover Ψ_y is an A-linear map of degree $|y|$, and satisfies:

$$\Psi_{y+x} = \Psi_y + \Psi_x; \text{ for } x, y \in Y,$$

$$\Psi_{\lambda y} = \lambda \Psi_y; \text{ for } \lambda \in k \text{ and } y \in Y,$$

$$\Psi_{by} = b \Psi_y; \text{ for } b \in A.$$

We write $\Psi_y(a \otimes v) = y(a \otimes v)$ and we prove that d_Θ is a differential. In fact:

$$\begin{aligned} d_\Theta \circ d_\Theta(y \otimes v) &= d_\Theta(d_Y(y) \otimes v + (-1)^{|y|} \Psi_y \circ \Theta(v)) \\ &= d_Y(d_Y(y)) \otimes v + (-1)^{|d_Y(y)|} \Psi_{d_Y(y)} \circ \Theta(v) + (-1)^{|y|} (d_\Theta(\Psi_y \circ \Theta(v))) \\ &= (-1)^{|y|+1} d_Y(y) \Big(\sum_{\alpha \in \Lambda} a_\alpha \otimes f_\alpha(v) \Big) + (-1)^{|y|} d_\Theta \Big(y \sum_{\alpha \in \Lambda} a_\alpha \otimes f_\alpha(v) \Big) \\ &= (-1)^{|y|+1} \Big(\sum_{\alpha \in \Lambda} d_Y(y) a_\alpha \otimes f_\alpha(v) \Big) + (-1)^{|y|} \Big(\sum_{\alpha \in \Lambda} \big(d_Y(y a_\alpha) \otimes f_\alpha(v) + \\ &\qquad (-1)^{|y|+1} \Psi_{y a_\alpha} \circ \Theta(f_\alpha(v)) \big) \Big) \\ &= (-1)^{|y|+1} \big(\sum_{\alpha \in \Lambda} d_Y(y) a_\alpha \otimes f_\alpha(v) \big) + (-1)^{|y|} \Big(\sum_{\alpha \in \Lambda} \big(d_Y(y) a_\alpha \otimes f_\alpha(v) + \\ &\qquad (-1)^{|y|} y d_A(a_\alpha) \otimes f_\alpha(v) + (-1)^{|y|+1} y a_\alpha \Theta(f_\alpha(v)) \big) \Big) \\ &= \sum_{\alpha \in \Lambda} y d_A(a_\alpha) \otimes f_\alpha(v) - \sum_{\alpha \in \Lambda} y a_\alpha \Theta(f_\alpha(v)) \\ &= y(d\Theta(v) - \Theta \circ \Theta(v)) \end{aligned} \tag{1.4.4}$$

The last equality holds because:

$$\begin{aligned}\sum_{\alpha\in\Lambda} a_\alpha \Theta(f_\alpha(v)) &= \sum_{\alpha\in\Lambda} a_\alpha \Big(\sum_{\beta\in\Lambda} a_\beta \otimes f_\beta(f_\alpha(v))\Big) \\ &= \sum_{\alpha\in\Lambda}(\sum_{\beta\in\Lambda} a_\alpha a_\beta \otimes f_\beta(f_\alpha(v))) \\ &= \Theta\circ\Theta(v). \end{aligned} \tag{1.4.5}$$

Thus, the equation $d_\Theta \circ d_\Theta = 0$ reduces to equation (1.4.1) above. Moreover d_Θ does satisfy the equality (1.3.1):

$$\begin{aligned} d_\Theta(a(y\otimes v)) &= d_\Theta(ay\otimes v) \\ &= d_Y(ay)\otimes v + (-1)^{|ay|}\Psi_{ay}\circ\Theta(v) \\ &= \big(d_A(a)y + (-1)^{|a|}a\,(d_Y(y)\big)\otimes v + (-1)^{|a|+|y|}\Psi_{ay}\circ\Theta(v) \\ &= d_A(a)(y\otimes v) + (-1)^{|a|}a\,d_Y(y)\otimes v + (-1)^{|y|}a\,\Psi_y\circ\Theta(v) \\ &= d_A(a)(y\otimes v) + (-1)^{|a|}a\,d_\Theta(y\otimes v). \end{aligned}$$

The cohomology of the A-*dgm* $(Y\otimes V, d_\Theta)$ is called the ***cohomology of*** (Y,d) ***with values in the system*** (Θ, V) (tensor version).

1.4.2 Hom version of the cohomology of a module with values in an algebraic system of coefficients

Instead of the differential module $(Y\otimes V, d_\Theta)$, we now consider the module $Hom(V,Y)$. It also inherits a structure of A-*dgm*:

$$\left.\begin{aligned} (ag)(v) &= ag(v), \\ d_\Theta(g) &= d_Y\circ g - \Phi_g\circ\Theta, \end{aligned}\right\} \tag{1.4.6}$$

where Φ_g is the linear map of degree $|g|$ on $A\otimes V$ given by:

$$\begin{aligned} \Phi_g: \quad A\otimes V &\to Y \\ a\otimes v &\mapsto ag(v). \end{aligned}$$

This map satisfies the following properties:

$$\Phi_{g+h} = \Phi_g + \Psi_h;\ \text{for } g,h\in Hom(V,Y),$$

$$\Phi_{\lambda g} = \lambda\Phi_g;\ \text{for } \lambda\in k \text{ and } g\in Hom(V,Y),$$

$$\Phi_{ag}(x\otimes v) = (-1)^{|x||a|}a\Phi_g(x\otimes v);\ \text{for } a\in A \text{ and } x\otimes v\in A\otimes V.$$

Furthermore:

$$
\begin{aligned}
d_\Theta \circ d_\Theta(g)(v) &= (d_\Theta(d_Y \circ g - \Phi_g \circ \Theta))(v) \\
&= d_Y \circ d_Y \circ g - \Phi_{d_Y \circ g} \circ \Theta(v) - d_Y \circ \Phi_g \circ \Theta(v) + \Phi_{\Phi_g \circ \Theta} \circ \Theta(v) \\
&= -\Phi_{d_Y \circ g}\big(\sum_{\alpha\in\Lambda} a_\alpha \otimes f_\alpha(v)\big) - d_Y \circ \Phi_g\big(\sum_{\alpha\in\Lambda} a_\alpha \otimes f_\alpha(v)\big) + \Phi_{\Phi_g \circ \Theta}\big(\sum_{\alpha\in\Lambda} a_\alpha \otimes f_\alpha(v)\big) \\
&= \sum_{\alpha\in\Lambda}\Big(-a_\alpha d_Y \circ g(f_\alpha(v)) - d_Y\big(a_\alpha g(f_\alpha(v))\big)\Big) + \Big(\sum_{\alpha\in\Lambda} a_\alpha \otimes \Phi_g \circ \Theta(f_\alpha(v))\Big) \\
&= \sum_{\alpha\in\Lambda}(-a_\alpha d_Y \circ g(f_\alpha(v)) - \sum_{\alpha\in\Lambda}\big[d_A(a_\alpha)g(f_\alpha(v)) + (-1)^{|a_\alpha|}a_\alpha d_Y \circ g(f_\alpha(v))\big] + \\
&\qquad \sum_{\alpha\in\Lambda} a_\alpha \Phi_g\Big(\sum_{\beta\in\Lambda} a_\beta \otimes f_\beta(f_\alpha(v))\Big) \\
&= -\sum_{\alpha\in\Lambda}(d_A(a_\alpha)\Phi_g(1 \otimes f_\alpha(v)) + \sum_{\alpha\in\Lambda}\sum_{\beta\in\Lambda} a_\alpha a_\beta \Phi_g(1 \otimes f_\beta \circ f_\alpha(v)) \\
&= \Phi_g(-\sum_{\alpha\in\Lambda}(d_A(a_\alpha)f_\alpha(v) + \sum_{\alpha\in\Lambda}\sum_{\beta\in\Lambda} a_\alpha a_\beta (f_\beta \circ f_\alpha(v))) \\
&= \Phi_g(-d\Theta(v) + \Theta \circ \Theta(v)) \\
&= 0.
\end{aligned}
$$

Thus, the equation $d_\Theta \circ d_\Theta = 0$ reduces to equation (1.4.1) above. Moreover, we can check that (1.3.1) also holds:

$$
\begin{aligned}
d_\Theta(ag)(v) &= (d_Y \circ ag - \Phi_{ag} \circ \Theta)(v) \\
&= d_Y(ag(v)) - \Phi_{ag}\big(\sum_{\alpha\in\Lambda} a_\alpha \otimes f_\alpha(v)\big) \\
&= d_A(a)g(v) + (-1)^{|a|} a\, d_Y(g(v)) - \sum_{\alpha\in\Lambda}(-1)^{|a_\alpha||a|} a \Phi_g(a_\alpha \otimes f_\alpha(v)) \\
&= d_A(a)g(v) + (-1)^{|a|} a\,(d_Y(g(v)) - \Phi_g \circ \Theta(v)) \\
&= (d_A(a)g + (-1)^{|a|} a\, d_\Theta g)(v),
\end{aligned}
$$

The cohomology of the A-*dgm* $(Hom(V,Y), d_\Theta)$ is called the ***cohomology of*** (Y, d_Y) ***with values in the system*** (Θ, V) (Hom version).

Remark 1.4.1. *The graded module $A \otimes End(V)$ also has a multiplication defined by:*

$$(a \otimes f)(b \otimes g) = ab \otimes f \circ g.$$

If we consider this multiplication, for Θ as in (1.4.2) we have

$$\Theta \circ \Theta = (\sum_{\beta\in\Lambda} a_\beta \otimes f_\beta) \circ (\sum_{\alpha\in\Lambda} a_\alpha \otimes f_\alpha) = \sum_{\beta\in\Lambda}\sum_{\alpha\in\Lambda} a_\beta a_\alpha \otimes f_\beta \circ f_\alpha.$$

We can define an action and a map d_Θ as in (1.4.3), but then the equality (1.4.4) becomes

$$d_\Theta \circ d_\Theta(y \otimes v) = y(d\Theta + \Theta \circ \Theta)(v),$$

because the equality (1.4.5) is $\sum_{\alpha\in\Lambda} a_\alpha \Theta(f_\alpha(v)) = -\Theta \circ \Theta$. So, we could change the Definition (1.4.1) replacing the equality (1.4.1) by $d\Theta + \Theta \circ \Theta = 0$, and we would obtain analogous cohomologies of $(Y \otimes V, d_\Theta)$ and $(Hom(V, Y), d_\Theta)$.

1.4.3 Twisting matrix and twisting cohomology

Now we consider the element Θ appearing in the Definition (1.4.1) as a linear transformation $\Theta\colon V \to A^1 \otimes V$ such that $\Theta(v_k) = \sum_{i\in I} \theta_{ki} \otimes v_i$, where $dim(V) < \infty$ with $\{v_i\}_{i\in I}$ a basis for V and $\theta_{ki} \in A^1$. In fact: if $dim(V) = n < \infty$ and if $\{v_1, v_2, ..., v_n\}$ is a basis for V, then $f_\alpha(v_k) = \sum_{i=1}^{n} \lambda^\alpha_{ki} v_i$, where $\lambda^\alpha_{ki} \in k$, and hence

$$\begin{array}{rccl} \Theta: & V & \to & A^1 \otimes V \\ & v_k & \mapsto & \sum_{i=1}^{n} \sum_{\alpha\in\Lambda} a_\alpha \lambda^\alpha_{ki} \otimes v_i \end{array}$$

for $a_\alpha \in A^1$ as in the expression (1.4.2). We denote by $\theta_{ki} = \sum_{\alpha\in\Lambda} a_\alpha \lambda^\alpha_{ki}$, so $\Theta(v_k) = \sum_{i=1}^{n} \theta_{ki} \otimes v_i$.

Now, using the equality (1.4.5):

$$\begin{aligned} \Theta \circ \Theta(v_k) &= \sum_{\alpha\in\Lambda} \sum_{\beta\in\Lambda} a_\alpha a_\beta \otimes f_\beta(f_\alpha(v_k)) = \sum_{\alpha\in\Lambda} \sum_{\beta\in\Lambda} a_\alpha a_\beta \otimes f_\beta(\sum_{i=1}^{n} \lambda^\alpha_{ki} v_i) \\ &= \sum_{\alpha\in\Lambda} \sum_{\beta\in\Lambda} a_\alpha a_\beta \otimes \sum_{i=1}^{n} \lambda^\alpha_{ki} f_\beta(v_i) = \sum_{\alpha\in\Lambda} \sum_{\beta\in\Lambda} a_\alpha a_\beta \otimes \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda^\alpha_{ki} \lambda^\beta_{ij} v_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\alpha\in\Lambda} \sum_{\beta\in\Lambda} a_\alpha a_\beta \lambda^\alpha_{ki} \lambda^\beta_{ij} \otimes v_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\alpha\in\Lambda} a_\alpha \lambda^\alpha_{ki} \sum_{\beta\in\Lambda} a_\beta \lambda^\beta_{ij} \otimes v_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ki} \theta_{ij} \otimes v_j, \end{aligned}$$

and

$$d\Theta(_k) = \sum_{\alpha\in\Lambda} d_A(a_\alpha) \otimes f_\alpha(v_k) = \sum_{\alpha\in\Lambda} d_A(a_\alpha) \otimes \sum_{i=1}^{n} \lambda^\alpha_{ki} v_i = \sum_{i=1}^{n} d_A(\sum_{\alpha\in\Lambda} \lambda^\alpha_{ki}) \otimes v_i = \sum_{i=1}^{n} d_A(\theta_{ki}) \otimes v_i.$$

We can certainly define by any linear transformation $\Theta\colon V \to A^1 \otimes V$ the following maps:

$$\begin{array}{rccl} \Theta \circ \Theta: & V & \to & A^2 \otimes V \\ & v_k & \mapsto & \sum_{i\in I} \sum_{j\in I} \theta_{ki} \theta_{ij} \otimes v_j \\ & & & = \sum_{i\in I} \theta_{ki} \Theta(v_i) \end{array} \qquad \begin{array}{rccl} d\Theta: & V & \to & A^2 \otimes V \\ & v_k & \mapsto & \sum_{i\in I} d_A(\theta_{ki}) \otimes v_i. \end{array} \tag{1.4.7}$$

This motivates the following definition:

Definition 1.4.2. *Let A be a cdga and V a vector space of finite dimension. A linear transformation $\Theta\colon V \to A^1 \otimes V$ is called a **twisting matrix** if $d\Theta - \Theta \circ \Theta = 0$.*

The following theorem states that if V is a vector space of finite dimension, an algebraic system of coefficient determines a twisting matrix and the converse is also true.

Theorem 1.4.1. *Let A be a cdga and V a graded vector space concentrated in degree zero with $dim(V) < \infty$. Then (Θ, V) is an algebraic system of coefficients in A if and only if Θ is a twisting matrix.*

Proof. We observed at the beginning of this subsection that Θ holds the maps in (1.4.7) and therefore $d\Theta - \Theta \circ \Theta = 0$. Thus, an algebraic system of coefficients determines a twisting matrix.

On the other hand, let $\Theta\colon V \to A^1 \otimes V$ a linear map such that $\Theta(v_k) = \sum_{i=1}^{n} \theta_{ki} \otimes v_i$, and assume that this map satisfies (1.4.7). We want to express Θ as in the expression (1.4.2), such that $\Theta(v_k) = \sum_{\alpha\in\Lambda} a_\alpha \otimes f_\alpha(v_k)$. Thus, we can consider $\Lambda = I \times I$, and for $l \in I$, $i \in I$ we define $a_{li} = \theta_{li}$ and the linear map given by

$$f_{li}(v_k) = \begin{cases} v_i, & \text{if} \quad k = l, \\ 0, & \text{if} \quad k \neq l. \end{cases}$$

Therefore $\Theta = \sum_{\alpha\in I\times I} a_\alpha \otimes f_\alpha$ and $\Theta(v_k) = \sum_{i=1}^{n} a_{ki} \otimes f_{ki}(v_k) = \sum_{i=1}^{n} \theta_{ki} \otimes v_i$. Thus, a twisting matrix Θ determines an algebraic system of coefficients in A, (Θ, V).

□

As in the previous section, if (Y, d_Y) is an A-*dgm*, then the module $Y \otimes V$ inherits a structure of A-*module* via the linear map of degree 0 appearing in (1.4.3). Moreover, $Y \otimes V$ is a A-*dgm* with d_Θ as in (1.4.3) by the following result:

Theorem 1.4.2. *d_Θ is a differential on $Y \otimes V$ if only if Θ is a twisting matrix.*

Proof. We note that:

$$\begin{aligned}
d_\Theta \circ d_\Theta(y \otimes v_k) &= d_\Theta(d_Y(y) \otimes v_k + (-1)^{|y|}\Psi_y(\Theta(v_k))) \\
&= (-1)^{|d_Y(y)|}\Psi_{d_Y}(y)(\Theta(v_k)) + (-1)^{|y|} d_\Theta\big(\sum_{i\in I} y\theta_{ki} \otimes v_i\big) \\
&= (-1)^{|y+1|}\sum_{i\in I} d_Y(y)\theta_{ki} \otimes v_i + (-1)^{|y|}\Big(\sum_{i\in I} d_Y(y\theta_{ki}) \otimes v_i + (-1)^{|y\theta_{ki}|}\Psi_{y\theta_{ki}}(\Theta(v_i))\Big) \\
&= y\Big(\sum_{i\in I} d_A(\theta_{ki}) \otimes v_i - \sum_{i\in I}\sum_{j\in I} \theta_{ki}\theta_{ij} \otimes v_j\Big) \\
&= y(d\Theta(v_k) - \Theta \circ \Theta(v_k)).
\end{aligned}$$

Now, if $d_\Theta \circ d_\Theta = 0$ then $y(d\Theta - \Theta \circ \Theta) = 0$ for all $y \in Y$; in particular if $y = 1 \in Y^0$, then $d\Theta - \Theta \circ \Theta = 0$ and Θ is twisting matrix. On the other hand, if Θ is twisting matrix then $d_\Theta \circ d_\Theta = 0$. Moreover d_Θ satisfies the condition $d_\Theta(a(y \otimes v)) = d_A(a)\,(y \otimes v) + (-1)^{|a|} a\, d_\Theta(y \otimes v)$ as before.

□

The cohomology of the A-*dgm* $(Y \otimes V, d_\Theta)$ is called the ***twisted cohomology of*** Y ***with coefficients in*** V(tensor version); we denote it by $H_{\otimes,\Theta}(Y;V)$.

If (Y, d_Y) is an A-*dgm* and we consider the module $Hom(V,Y) = \{g \colon V \to Y^i : g$ is a linear map $\}$ (since V is concentrated in degree zero), via the linear map of degree 0 in (1.4.6), then $Hom(V,Y)$ has a structure of A-*dgm* with d_Θ as in (1.4.6), by the following result, analogous to theorem (1.4.2):

Theorem 1.4.3. *d_Θ is a differential on $Hom(V,Y)$ if only if Θ is a twisting matrix.*

Proof. Observe that:

$$\begin{aligned}
d_\Theta \circ d_\Theta(g)(v_k) &= d_\Theta(d_Y \circ g - \Phi_y \circ \Theta)(v_k) \\
&= -\Phi_{d_Y \circ g} \circ \Theta(v_k) - d_Y \circ \Phi_g(\Theta(v_k)) + \Phi_{\Phi_g \circ \Theta}(\Theta(v_k)) \\
&= -\sum_{i \in I} \theta_{ki}(d_Y \circ g)(v_i) - d_Y\big(\sum_{i \in I} \theta_{ki} g(v_i)\big) + \sum_{i \in I} \theta_{ki}(\Phi_g \circ \Theta)(v_i) \\
&= -\sum_{i \in I} \theta_{ki} d_Y(g(v_i)) - \big(\sum_{i \in I} d_A(\theta_{ki}) g(v_i) + (-1)^{|\theta_{ki}|} \theta_{ki} d_Y(g(v_i))\big) + \sum_{i \in I} \theta_{ki} \sum_{j \in I} \theta_{ij} g(v_j) \\
&= \Phi_g(-d\Theta(v_k) + \Theta \circ \Theta(v_k)).
\end{aligned}$$

Therefore, if $d_\Theta \circ d_\Theta = 0$ then $\Phi_g(-d\Theta + \Theta \circ \Theta) = 0$ for all $g \in Hom(V,Y)$; in particular if $g = 1$ (this is $g(v) = 1 \in Y^0$ for all $v \in V$), then $d\Theta - \Theta \circ \Theta = 0$; and if Θ is twisting matrix then $d_\Theta \circ d_\Theta = 0$. Also d_Θ meets the condition $d_\Theta(ag) = d_A(a)\, g + (-1)^{|a|} a\, d_\Theta(g)$ as before.

□

The cohomology of the A-*dgm* $(Hom(V,Y), d_\Theta)$ is called the ***twisted cohomology of*** Y ***with coefficients in*** V(Hom version); we denote it by $H_{Hom,\Theta}(V;Y)$.

The next section, we use the twisting cohomology (Hom version) to determine a Sullivan Decomposable Algebra of finite type and the concept of minimal model, see Theorems (1.5.1), (1.5.4) and section 2.3.

As we shall see in the section 3.4 and the chapter 4, we will relate the notions of twisting matrix and twisting cohomology (tensor version) to linear gauge complexes and manifolds of pseudo-spherical type.

1.5 Sullivan Decomposable Algebra

To end this preliminary section we enunciate the definition of a Sullivan Decomposable algebra and its characterization as a Lie algebra and a sequence of twisted cohomology classes. We recall from Example (1.2.1-5):

Let $V = \{V^i\}_{i\geq 0}$ be a graded vector space over a field k; then $A = \Lambda V$ satisfies that:

$$A^0 = (\Lambda V)^0 = k \oplus V^0 \oplus (V^0 \wedge V^0) \oplus (V^0 \wedge V^0 \wedge V^0) \oplus \cdots \tag{1.5.1}$$

And for $n \geq 1$:

$$A^n = (\Lambda V)^n = V^n \oplus (\oplus_{i+j=n} V^i \wedge V^j) \oplus (\oplus_{i+j+h=n} V^i \wedge V^j \wedge V^h) \oplus \cdots \tag{1.5.2}$$

Moreover $(\Lambda V)^0 = k$ if and only if $V^0 = \{0\}$.

Definition 1.5.1. *A* ***Sullivan decomposable algebra*** *is a free commutative differential graded algebra $A = (\Lambda V, d)$ which is connected (this is, $A^0 = k$) and such that $dV \subseteq \Lambda^{\geq 2}V$, where $\Lambda^{\geq 2}V$ stands for all elements of word length ≥ 2.*

For instance $(\Lambda(x, y, z), d)$ with $|x| = |y| = |z| = 1$ and $dx = y \wedge z$, $dy = x \wedge z$, $dz = x \wedge y$, and $(\Lambda(x, y, z), d)$ with $|x| = |y| = 2$ and $|z| = 3$, $dx = dy = 0$ and $dz = x \wedge x - y \wedge y$ are Sullivan decomposable algebras. Observe that if A is a Sullivan decomposable algebra then, by Example (1.2.2-1) $ker(d^0) = k$ and $H^0(A) = k$.

The following theorem can be found in [25]. Here we present a detailed version of this result because its converse will be one of our main objects of study. We will relate it to manifolds of pseudo-spherical type in chapter 4.

Theorem 1.5.1. *A Sullivan decomposable algebra $A = (\Lambda V, d)$ of finite type determines a Lie algebra and a sequence of twisted cohomology classes (Hom version).*

Proof. Suppose that A is a Sullivan decomposable algebra (thus, $V^0 = \{0\}$, see (1.5.1); we denote by $A_{\leq\, k}$ the space

$$A_{\leq\, k} = \Lambda(V^1 \oplus \cdots \oplus V^k) \quad \text{for} \quad k \geq 1,$$

That is, $A_{\leq\, k}$ is the subalgebra of A generated by elements of degree $\leq k$.

The proof of Theorem (1.5.1) consists in constructing a Lie algebra by means of the subalgebra $A_{\leq\, 1}$, and in constructing a twisted cohomology class by means of the subalgebras $A_{\leq\, k}$ for $k \geq 2$.

We first construct the Lie algebra. To this end, let us consider $A_{\leq\, 1} = \Lambda(V^1)$ and let us fix a basis $\{v_\alpha\}_{\alpha \in I}$ of V^1. Then, since A is a Sullivan decomposable algebra, the differential of these elements are necessarily of the form

$$d(v_\alpha) = \sum_{i<j} \lambda_{ij}^{\alpha} v_i \wedge v_j, \tag{1.5.3}$$

where $\lambda_{ij}^{\alpha} \in k$ and $\lambda_{ij}^{\alpha} \neq 0$ for a finite number of indices i, j such that $(i,j) \in I \times I$ and $i < j$, since $v_i \wedge v_i = 0$ and $v_i \wedge v_j = -v_j \wedge v_i$. We conclude that $d(A_{\leq 1}) \subseteq A_{\leq 1} A_{\leq 1}$. We denote by L the vector space dual of V^1 with dual basis $\{v_i^*\}_{i \in I}$, and define a bracket in L by:

$$\begin{array}{rccl} [\ ,\ ]: & L \times L & \rightarrow & L \\ & (v_i^*, v_j^*) & \mapsto & \begin{cases} [v_i^*, v_j^*], & \text{if } i < j, \\ -[v_j^*, v_i^*], & \text{if } i > j, \\ 0, & \text{if } i = j. \end{cases} \end{array}$$

where

$$\begin{array}{rccl} [v_i^*, v_j^*]: & V & \rightarrow & k \\ & v_\alpha & \mapsto & \lambda_{ij}^{\alpha}. \end{array}$$

Now, according to our description of the differential d in (1.5.3), we only have coefficients λ_{ij} for $i < j$; therefore, to facilitate the calculations related to the bracket let us denote by

$$\lambda_{ji}^{\alpha} := -\lambda_{ij}^{\alpha}, \quad \text{for } i < j. \tag{1.5.4}$$

The bracket $[\ ,\ ]$ is a linear map, since if $v = \sum_{\alpha \in I} \beta^\alpha v_\alpha$, then $[v_i^*, v_j^*](v) = \sum_{\alpha \in I} \beta^\alpha \lambda_{ij}^{\alpha}$, besides, by definition the bracket $[\ ,\ ]$ satisfies the antisymmetric property. We need to verify the Jacobi identity. We first observe that $[v_i^*, v_j^*] = \sum_{\alpha \in I} \lambda_{ij}^{\alpha} v_\alpha^*$, and

$$[v_h^*, [v_i^*, v_j^*]] = [v_h^*, \sum_{\alpha \in I} \lambda_{ij}^{\alpha} v_\alpha^*] = \sum_{\alpha \in I} \lambda_{ij}^{\alpha} [v_h^*, v_\alpha^*] = \sum_{\alpha \in I} \lambda_{ij}^{\alpha} \sum_{r \in I} \lambda_{h\alpha}^{r} v_r^*. \tag{1.5.5}$$

With the notation $J := [v_i^*, [v_j^*, v_h^*]] + [v_j^*, [v_h^*, v_i^*]] + [v_h^*, [v_i^*, v_j^*]]$ and the last equality, we have that

$$J = \sum_{\alpha \in I} \sum_{r \in I} (\lambda_{jh}^{\alpha} \lambda_{i\alpha}^{r} + \lambda_{hi}^{\alpha} \lambda_{j\alpha}^{r} + \lambda_{ij}^{\alpha} \lambda_{h\alpha}^{r}) v_r^*;$$

thus

$$J(v_m) = \sum_{\alpha \in I} \lambda_{jh}^{\alpha} \lambda_{i\alpha}^{m} + \lambda_{hi}^{\alpha} \lambda_{j\alpha}^{m} + \lambda_{ij}^{\alpha} \lambda_{h\alpha}^{m}. \tag{1.5.6}$$

Therefore

$$J(v_m) = (\sum_{p<i} \lambda_{ip}^{m} \lambda_{jh}^{p} + \sum_{i<q} \lambda_{iq}^{m} \lambda_{jh}^{q}) + (\sum_{p<j} \lambda_{jp}^{m} \lambda_{hi}^{p} + \sum_{j<q} \lambda_{jq}^{m} \lambda_{hi}^{q}) + (\sum_{p<h} \lambda_{hp}^{m} \lambda_{ij}^{p} + \sum_{h<q} \lambda_{hq}^{m} \lambda_{ij}^{q}).$$

We will denote:

$$\left.\begin{aligned} J(v_m)_{jh} &= -\sum_{p<i} \lambda^m_{pi}\lambda^p_{jh} + \sum_{i<q} \lambda^m_{iq}\lambda^q_{jh}, \\ J(v_m)_{hi} &= -\sum_{p<j} \lambda^m_{pj}\lambda^p_{hi} + \sum_{j<q} \lambda^m_{jq}\lambda^q_{hi}, \\ J(v_m)_{ij} &= -\sum_{p<h} \lambda^m_{ph}\lambda^p_{ij} + \sum_{h<q} \lambda^m_{hq}\lambda^q_{ij}. \end{aligned}\right\} \tag{1.5.7}$$

Thus,

$$J(v_m) = J(v_m)_{jh} + J(v_m)_{hi} + J(v_m)_{ij}. \tag{1.5.8}$$

On the other hand, we compute:

$$\begin{aligned} d \circ d(v_m) &= d\Big(\sum_{p<q} \lambda^m_{pq} v_p \wedge v_q\Big) \\ &= \sum_{p<q} \lambda^m_{pq}(d(v_p) \wedge v_q - v_p \wedge d(v_q)) \\ &= \sum_{p<q} \lambda^m_{pq}(\sum_{s<t} \lambda^p_{st} v_s \wedge v_t) \wedge v_q - \sum_{p<q} \lambda^m_{pq} v_p \wedge (\sum_{s<t} \lambda^q_{st} v_s \wedge v_t) \\ &= \sum_{s<t}\sum_{p<q} \lambda^m_{pq}\lambda^p_{st} v_s \wedge v_t \wedge v_q - \sum_{s<t}\sum_{p<q} \lambda^m_{pq}\lambda^q_{st} v_p \wedge v_s \wedge v_t. \end{aligned}$$

For $i < j < h$ given, we find the term $v_i \wedge v_j \wedge v_h$ taking into account possible choices of s, t and p, q:

$$\begin{aligned} d \circ d(v_m)_{ijh} = &\sum_{p<h} \lambda^m_{ph}\lambda^p_{ij} v_i \wedge v_j \wedge v_h - \sum_{h<q} \lambda^m_{hq}\lambda^q_{ij} v_h \wedge v_i \wedge v_j + \\ &\sum_{p<j} \lambda^m_{pj}\lambda^p_{ih} v_i \wedge v_h \wedge v_j - \sum_{j<q} \lambda^m_{jq}\lambda^q_{ih} v_j \wedge v_i \wedge v_h + \\ &\sum_{p<q} \lambda^m_{pi}\lambda^p_{jh} v_j \wedge v_h \wedge v_i - \sum_{p<q} \lambda^m_{iq}\lambda^q_{jh} v_i \wedge v_j \wedge v_h \\ = &(\sum_{p<h} \lambda^m_{ph}\lambda^p_{ij} - \sum_{h<q} \lambda^m_{hq}\lambda^q_{ij})\, v_i \wedge v_j \wedge v_h + \\ &(\sum_{p<j} \lambda^m_{pj}\lambda^p_{ih} - \sum_{j<q} \lambda^m_{jq}\lambda^q_{ih})\, v_i \wedge v_h \wedge v_j + \\ &(\sum_{p<i} \lambda^m_{pi}\lambda^p_{jh} - \sum_{i<q} \lambda^m_{iq}\lambda^q_{jh})\, v_j \wedge v_h \wedge v_i. \end{aligned}$$

Thus, equalities (1.5.4) and (1.5.7), imply that:

$$d \circ d(v_m)_{ijh} = (-J(m)_{ij} - J(m)_{hi} - J(m)_{jh})\, v_i \wedge v_j \wedge v_h = J(v_m)\, v_i \wedge v_j \wedge v_h. \tag{1.5.9}$$

Now, since d is a differential, then $d \circ d(v_m) = 0$, and therefore the component $d \circ d(v_m)_{ijh} = 0$. In a similar manner we obtain the following equalities:

$$0 = d(v_m)_{ihj} = (J(v_m)_{hi} + J(v_m)_{ij} + J(v_m)_{jh})\, v_i \wedge v_h \wedge v_j, \quad \text{for } i < h < j;$$

$$0 = d(v_m)_{jih} = (J(v_m)_{ij} + J(v_m)_{jh} + J(v_m)_{hi})\, v_i \wedge v_h \wedge v_j, \quad \text{for } j < i < h;$$

$$0 = d(v_m)_{jhi} = (-J(v_m)_{jh} - J(v_m)_{ij} - J(v_m)_{hi})\, v_j \wedge v_h \wedge v_i, \quad \text{for } j < h < i$$

$$0 = d(v_m)_{hij} = (-J(v_m)_{hi} - J(v_m)_{jh} - J(v_m)_{ij})\, v_h \wedge v_i \wedge v_j, \quad \text{for } h < i < j$$

$$0 = d(v_m)_{hij} = (J(v_m)_{jh} + J(v_m)_{hi} + J(v_m)_{ij})\, v_h \wedge v_i \wedge v_j, \quad \text{for } h < j < i.$$

Thus, replacing in to equation (1.5.8), we have that $J(v_m) = 0$, and therefore $J = 0$.

We now turn to the case $k \neq 1$, so $A_{\leq k} = \Lambda(V^1 \oplus \cdots \oplus V^k)$, and we let $\{x_\beta\}_{\beta \in B}$ be a basis of V^k. Since $d(x_j) \in \Lambda^{\geq 2} V$ then $d(x_j) \in A^1 A_{\leq k} \oplus A_{\leq k-1} A_{\leq k-1}$, for $x_j \in V^k$; because, by (1.5.2)

$$(A_{\leq k})^{k+1} = V^{k+1} \oplus (\oplus_{i+j=k+1} V^i \wedge V^j) \oplus (\oplus_{i+j+h=k+1} V^i \wedge V^j \wedge V^h) \oplus \cdots,$$

. Also, as $A_{\leq k-1}$ is a subalgebra, we conclude that: $d(x_j) \in A^1 V^k + A_{\leq k-1}$, that is, we can write

$$d(x_j) = \sum_{\beta \in B} \theta^k_{j\beta} \wedge x_\beta + a_j, \quad \text{where } a_j \in A^{k+1}_{\leq k-1} \text{ and } \theta^k_{j\beta} \in A^1. \tag{1.5.10}$$

We regard V^k as a graded vector space concentrated in degree zero, and we define a linear transformation

$$\begin{array}{rccc} \Theta^k : & V^k & \to & A^1 \otimes V^k \\ & x_j & \mapsto & \sum_{\beta \in B} \theta^k_{j\beta} \otimes x_\beta. \end{array}$$

It follows that can write (1.5.10) in the form:

$$(d|_{V^k} - m \circ \Theta^k)(x_j) = a_j, \tag{1.5.11}$$

where m is the multiplication in the cdga A, this is

$$\begin{array}{rccc} m : & A \otimes A & \to & A \\ & x \otimes y & \mapsto & x \wedge y. \end{array}$$

Furthermore, we have the following equalities:

$$\left.\begin{aligned} 0 = d \circ d(x_j) &= \sum_{\beta \in B} d(\theta^k_{j\beta} \wedge x_\beta) + d(a_j) \\ &= \sum_{\beta \in B} \Big(d(\theta^k_{j\beta}) \wedge x_\beta - \theta^k_{j\beta} \wedge d(x_\beta) \Big) + d(a_j) \\ &= \sum_{\beta \in B} \Big(d(\theta^k_{j\beta}) \wedge x_\beta - \theta^k_{j\beta} \wedge (\sum_{l \in B} \theta^k_{\beta l} \wedge x_l + a_\beta) \Big) + d(a_j) \\ &= \sum_{\beta \in B} \Big(d(\theta^k_{j\beta}) - \sum_{l \in B} \theta^k_{jl} \wedge \theta^k_{l\beta} \Big) \wedge x_\beta - \sum_{\beta \in B} \theta^k_{j\beta} \wedge a_\beta + d(a_j) \end{aligned}\right\} \tag{1.5.12}$$

Since A is free of relations, we have:

$$\sum_{\beta \in B} \Big(d(\theta^k_{j\beta}) - \sum_{l \in B} \theta^k_{jl} \wedge \theta^k_{l\beta} \Big) \wedge x_\beta = 0, \tag{1.5.13}$$

$$\sum_{\beta \in B} -\theta^k_{j\beta} \wedge a_\beta + d(a_j) = 0. \tag{1.5.14}$$

From (1.5.13) we conclude that Θ^k is a twisting matrix (with V replaced by V^k and A by $A_{\leq 1}$ in Definition 1.4.2) and we can consider $Hom(V^k; A_{\leq k-1})$ as a $A_{\leq 1}$-*dgm* with differential $d_{\Theta^k}(g) = d \circ g - \Phi_g \circ \Theta^k$ for all $g \in Hom(V^k; A_{\leq k-1})$ (see 1.4.6 and Theorem 1.4.3).

Finally, we construct a d_{Θ^k} cohomology class. If we consider the linear map $f_k = d|_{V^k} - m \circ \Theta^k$ given by the equality (1.5.11), this is:

$$\begin{array}{rccc} f_k: & V^k & \rightarrow & A_{\leq k-1} \\ & x_j & \mapsto & a_j, \end{array}$$

then, from the equality (1.5.14) we have that

$$d_{\Theta^k}(f_k)(x_j) = (d \circ f_k - \Phi_{f_k} \circ \Theta^k)(x_j) = d(a_j) - \sum_{\beta \in B} \theta^k_{j\beta} \wedge f_k(x_\beta) = 0.$$

Therefore $[f_k] \in H^{k+1}_{Hom,\Theta^k}(V^k; A_{\leq k-1})$.

□

Now we state the converse of the last Theorem. First we observe that the above proof allows us to establish a bijection between free cdga's on V^1 with $dim(V^1) < \infty$ and the Lie algebra structures on the dual vector space of V^1. In fact, as above we denote by L the vector space dual of V^1 with dual basis $\{v_i^*\}_{i \in I}$, and we denote the bracket of L by: $[v_i^*, v_j^*](v_\alpha) = \lambda^\alpha_{ij}$, and we define of differential d by $d(v_\alpha) := \sum_{i<j} \lambda^\alpha_{ij} v_i \wedge v_j$ and extended by linearity and so that Leibnitz rule is satisfied. Therefore, by the Jacobi identity we have $J(v_m) = 0$ in (1.5.6) then $d \circ d(v_m) = 0$ in (1.5.9).

By the above observation, the following theorem is the converse of Theorem (1.5.3), We construct a Sullivan decomposable algebra. We consider $V = \{V^k\}_{k \geq 0}$ a graded vector space of finite type such that $V^0 = \{0\}$, and we denote by $V^{\leq k}$ the graded vector subspace $V^{\leq k} = \{V^i\}_{i \leq k}$; we wish to find a differential $d_{\Lambda V}$ on ΛV such that it is compatible with $\check{d}_{\leq 1}$ a given differential on ΛV^1, this is $\check{d}_{\Lambda V}|_{\lambda V^{\leq 1}} = \check{d}_{\leq 1}$.

Theorem 1.5.2. *Let $V = \{V^k\}_{k \geq 0}$ be a graded vector space of finite type such that $V^0 = \{0\}$. Let us assume that $(\Lambda V^1, \check{d}_{\leq 1})$ is a cdga. Then for $k \geq 2$ the free commutative graded algebra $\Lambda V^{\leq k}$ has structure of cdga via the linear maps d_{V^k} defined recursively by*

$$\begin{array}{rccc} d_{V^k}: & V^k & \rightarrow & \Lambda V^{\leq k} \\ & x_j & \mapsto & m \circ \Theta^k(x_j) + f_k(x_j) \end{array}$$

where $\{x_\beta\}_{\beta \in B^k}$ is a basis of V^k, Θ^k is a twisting matrix given by $\begin{array}{rccc} \Theta^k: & V^k & \rightarrow & (\Lambda V^{\leq k})^1 \otimes V^k \\ & x_j & \mapsto & \sum_{\beta \in B^k} \theta^k_{j\beta} \otimes x_\beta \end{array}$ *,*

m is the exterior product on graded algebra $\Lambda V^{\leq k}$ given by $m\colon \begin{array}{ccc} \Lambda V^{\leq k}\otimes \Lambda V^{\leq k} & \to & \Lambda V^{\leq k} \\ x\otimes y & \mapsto & x\wedge y \end{array}$

and $f_k\colon V^k \to \Lambda V^{\leq k-1}$ is such that $[f_k]\in H^{k+1}_{Hom,\Theta^k}(V^k, \Lambda V^{\leq k-1})$.

Moreover $\Lambda V = \bigcup_k \Lambda(V^{k\geq 0})$ is a Sullivan decomposable algebra.

Proof. First we define the linear map $\check{d}_{\leq\, k}$ and by induction on k we show that $(\Lambda V^{\leq k}, \check{d}_{\leq\, k})$ is a *cdga*. The case $k=2$ is similar to the general case, for this reason we assume that this proposition is true for $k=i-1$ and we prove that it is true for $k=i$.

We define $d_{\leq\, i}\colon V^{\leq i}\to \Lambda V^{\leq i}$ by

$$d_{\leq\, i} = \begin{cases} d_{\leq\, j}(v), & \text{if } v\in V^j, \text{ for } j\leq i-1,\\ d_{V^i}(v), & \text{if } v\in V^i.\end{cases}$$

This is a linear map of degree 1, and therefore $d_{\leq\, i}$ can be extended to a unique derivation $\check{d}_{\leq\, i}$ of degree 1 on $\Lambda V^{\leq i}$ (see 1.2.2) and this derivation is compatible with differential $\widehat{d}_{\leq\, i\,-\,1}$ this is $\check{d}_{\leq\, i}|_{\Lambda V^{\leq\, i}} = \check{d}_{\leq\, i\,-\,1}$ and this property can be reduced to $\check{d}_{\leq\, i}|_{V^{\leq\, i}} = \check{d}_{\leq\, i\,-\,1}$, by the Leibnitz rule (1.2.3) and since

$$(\Lambda V^{\leq i-1})^n = \bigoplus_{1\leq r\leq n}\ \bigoplus_{\substack{0<j_1,\dots,j_r<i-1\\ j_1+\dots+j_r=n}} V^{j_1}\wedge\dots\wedge V^{j_r}.$$

For the last equality see (1.5.2).

Again by the Leibnitz rule the derivation $\check{d}_{\leq\, i}$ is completely characterized by elements in $V^{\leq i}$, since

$$(\Lambda V^{\leq i})^n = \bigoplus_{1\leq r\leq n}\ \bigoplus_{\substack{0<j_1,\dots,j_r<i\\ j_1+\dots+j_r=n}} V^{j_1}\wedge\dots\wedge V^{j_r}.$$

To check that $\check{d}_{\leq\, i}\circ \check{d}_{\leq\, i} = 0$, it is sufficient to show this property for a basis of V^i and since for $v\in V^{\leq i-1}$, $\check{d}_{\leq\, i}\circ\check{d}_{\leq\, i}(v) = \check{d}_{\leq\, i\,-\,1}\circ\check{d}_{\leq\, i\,-\,1}(v)=0$. Let $\{x_\beta\}_{\beta\in B^i}$ be a basis of V^i:

$$\begin{aligned}
\check{d}_{\leq\, i}\circ\check{d}_{\leq\, i}(x_j) &= \check{d}_{\leq\, i}(m\circ\Theta^i(x_j)+f_i(x_j))\\
&= \sum_{\beta\in B^i}\Big(\check{d}_{\leq\, 1}(\theta^i_{j\beta})\wedge x_\beta - \theta^i_{j\beta}\wedge \check{d}_{\leq\, i}(x_\beta)\Big) + \check{d}_{\leq\, i\,-\,1}(f_i(x_j))\\
&= \sum_{\beta\in B^i}\Big(\check{d}_{\leq\, 1}(\theta^i_{j\beta})\wedge x_\beta - \theta^i_{j\beta}\wedge\big(\sum_{\gamma\in B^i}\theta^i_{\beta\gamma}\wedge x_\gamma\big)+f_i(x_\beta)\Big)+\check{d}_{\leq\, i\,-\,1}(f_i(x_j))\\
&= \sum_{\beta\in B^i}\check{d}_{\leq\, 1}(\theta^i_{j\beta})\wedge x_\beta - \sum_{\beta\in B^i}\sum_{\gamma\in B^i}\theta^i_{j\beta}\wedge\theta^i_{\beta\gamma}\wedge x_\gamma\\
&\quad - \sum_{\beta\in B^i}\theta^i_{j\beta}\wedge(f_i(x_\beta)) + \check{d}_{\leq\, i\,-\,1}(f_i(x_j))\\
&= m(\check{d}_{\leq\, 1}\Theta^i - \Theta^i\circ\Theta^i(x_j)) - d_{\Theta^i}(f_i)(x_j)\\
&= 0
\end{aligned}$$

The last equality holds, because Θ^i is a twisting matrix (see 1.4.7) and $[f_i]$ is a class of cohomology in $H^{i+1}_{Hom,\Theta^i}(V^i;\Lambda(V^{\leq i-1})$; then $\check{d}_{\leq\ i}$ is a differential and therefore $(\Lambda V^{\leq i},\check{d}_{\leq\ i})$ is a *cdga* and it is a Sullivan decomposable algebra. In fact:

$$f_i(v)\in(\Lambda V^{\leq i-1})^{i+1}\subseteq\Lambda^{\geq 2}V^{\leq i}\quad\text{and}\quad\theta^i_{j\beta}\wedge x_\beta\in V^1\wedge V^i\subseteq\Lambda^2V^{\leq i},$$

then $\check{d}_{\leq\ i}(v)\in\Lambda^{\geq 2}V^{\leq i}$.

Now we have the linear map $d_{\Lambda V}\colon\Lambda V\to\Lambda V$ defined by $d_{\Lambda V}|_{\Lambda V^{\leq k}}=\check{d}_{\leq\ k}$, so for $v\in V^k$

$$d_{\Lambda V}(v)=m\circ\Theta^k(v)+f_k(v),$$

as $f_k(v)\in\Lambda V^{\leq k-1}$ and has degree $k+1$, then $f_k(v)\in\Lambda^{\geq 2}V$ and, $m\circ\Theta^k(v)\in V^1\wedge V^k\subseteq\Lambda^2V$, therefore ΛV is a Sullivan decomposable algebra.

□

Chapter 2

Minimal model

In this chapter we introduce one of the most important concepts in rational homotopy theory, the notion of *minimal model of a cdga*. This minimal model is a Sullivan minimal algebra (a special case of a Sullivan decomposable algebra, see Definition 2.1.2 below) which is quasi-isomorphic to the original *cdga* A

The existence of a minimal model is guaranteed by Sullivan's Theorem of which he presented a constructive proof. Here we give a detailed proof following [25], [8], [10], and [15]. In these papers there appear proofs of Sullivan's Theorem under certain technical assumptions which do not always hold. For example, in last reference we find the case in which the homology of the *cdga* is of finite type and in degree one not zero. We do not make these assumptions. The proof in [15] is made via "Hirsch extensions"that are a particular case of our definition of extensions of a *dga* (Definition 2.1.1). This definition permits us to prove Sullivan's theorem in full generality. We finish this chapter relating this demonstration with the twisting matrices and twisted cohomology of chapter 1.

2.1 Sullivan Minimal Algebra

If V and W are graded vector spaces, we have that

$$\Lambda(V \oplus W) = k \oplus (V \oplus W) \oplus \{(V \oplus W) \wedge (V \oplus W)\} \oplus \cdots$$

where we recall that $(V \oplus W)^n = V^n \oplus W^n$, this is, if $(v, w) \in V \oplus W$ then $|v| = |w|$, and we also note that $(v, w) \wedge (v', w') \in (\Lambda(V \oplus W))^{|v|+|v'|}$ with $v, v' \in V$ and $w, w' \in W$. Let $\{v_i\}$ and $\{w_j\}$ be homogeneous base of V and W respectively, so $|(v_i, 0)| = |v_i| = |0|$, $|(0, w_j)| = |w_j| = |0|$; this notation allows us to write:

$$(v, w) = (v, 0) + (0, w)$$

$$\begin{aligned} (v, w) \wedge (v', w') &= ((v, 0) + (0, w)) \wedge ((v', 0) + (0, w')) \\ &= (v, 0) \wedge (v', 0) + (v, 0) \wedge (0, w') + (0, w) \wedge (v', 0) + (0, w) \wedge (0, w') \end{aligned}$$

There exists a canonical isomorphism $\widehat{\varphi}$ of *cga's*

$$\Lambda(V \oplus W) \overset{\widehat{\varphi}}{\cong} \Lambda V \otimes \Lambda W, \tag{2.1.1}$$

defined by $(v, 0) \mapsto v \otimes 1$ for any $v \in V$ and $(0, w) \mapsto 1 \otimes w$ for any $w \in W$. We note that indeed, if $\varphi\colon V \oplus W \to \Lambda V \otimes \Lambda W$ is the linear map of degree zero determined by the extending linearly the assignment $\varphi(v_i, 0) = v_i \otimes 1$ and $\varphi(0, w_j) = 1 \otimes w_j$, via

$$\varphi(v, w) = \varphi((v, 0) + (0, w)) = \varphi(v, 0) + \varphi(0, w) = v \otimes 1 + 1 \otimes w,$$

then we can extend φ to a unique morphism $\widehat{\varphi}$ of commutative graded algebras (see section 1.2). By the multiplication rule in $\Lambda V \otimes \Lambda W$ (see equation 1.2.1) we have:

$$\widehat{\varphi}(1) = 1 \otimes 1, \;\; \widehat{\varphi}((v,0) \wedge (v',0)) = (v \otimes 1)(v' \otimes 1) = v \wedge v' \otimes 1, \;\; \text{for } \; v, v' \in V$$
$$\widehat{\varphi}((0,w) \wedge (0,w')) = (1 \otimes w)(1 \otimes w') = 1 \otimes w \wedge w', \;\; \text{for } \; w, w' \in W,$$
$$\widehat{\varphi}((v,0) \wedge (0,w) = (v \otimes 1)(1 \otimes w) = v \otimes w, \;\; \text{for } \; v \in V \text{ and } w \in W.$$

The inverse morphism of $\widehat{\varphi}$ is such that $v \otimes 1 \mapsto (v,0)$ for any $v \in V$ and $1 \otimes w \mapsto (0,w)$ for any $w \in W$.

Definition 2.1.1. *A **degree** n **extension** of a cdga (A,d), is a cdga of the form $A \otimes_\varepsilon (\Lambda V)$, where V is a vector space considered as a graded vector space concentrated in degree n, and*

$$\varepsilon \colon V \to A$$

is a linear map of degree 1 such that $d \circ \varepsilon = 0$. $A \otimes_\varepsilon (\Lambda V) = A \otimes \Lambda V$ as graded algebras, but the differential is "twisted"" by ε, this is, it is defined by linearity and the multiplication rules

$$d(a \otimes 1) = d(a) \otimes 1, \quad d(1 \otimes v) = \varepsilon(v) \otimes 1.$$

From last definition we can deduce that:

$$\begin{aligned} d(a \otimes v) &= d((a \otimes 1)(1 \otimes v)) \\ &= d(a \otimes 1)(1 \otimes v) + (-1)^{|a \otimes 1|}(a \otimes 1)(\varepsilon(v) \otimes 1) \\ &= da \otimes v + (-1)^{|a|}(a\, \varepsilon(v) \otimes 1). \end{aligned}$$

In a similar manner we have:

$d(a\, a' \otimes 1) = d((a \otimes 1)(a' \otimes 1)) = d(a\, a') \otimes 1,$

$d(1 \otimes (v \wedge v')) = d((1 \otimes v)(1 \otimes v')) = \varepsilon(v) \otimes v' + (-1)^{|v|} v \otimes \varepsilon(v')$ and

$d(a \otimes (v \wedge w)) = d((a \otimes v)(1 \otimes w)) = da \otimes (v \wedge w) + (-1)^{|a|} a\, \varepsilon(v) \otimes w + (-1)^{|a \otimes v| + |v||\varepsilon(w)|} a\, \varepsilon(w) \otimes v.$

When V is a finite dimensional vector space, $A \otimes_\varepsilon (\Lambda V)$ is called a *Hirsch extension* of A or an *elementary extension* of A (see [15] and [10]).

We define a particular extension which we will use in the proof of Sullivan's Theorem on the existence of minimal models: We assume that $V == \{V^t\}_{t \geq 0}$ is a graded vector, we take $A = \Lambda(V^{\leq t-1})$ and $V = V^t$ in Definition 2.1.1. The extension of degree t of a *cdga* $\Lambda(V^{\leq t-1})$ is

$$\Lambda(V^{\leq t-1}) \otimes_\varepsilon \Lambda V^t,$$

where V^t is consider as a graded vector space concentrated in degree t. This extension (as a graded algebra, see 2.1.1) satisfies that

$$\Lambda(V^{\leq t-1} \oplus_\varepsilon V^t) \overset{\widehat{\varphi}}{\cong} \Lambda(V^{\leq t-1}) \otimes_\varepsilon \Lambda V^t, \tag{2.1.2}$$

in the which $\oplus_\varepsilon$ means that we assume that the differential d in $\Lambda(V^{\leq t-1} \oplus V^t)$ is determined by:

$$d(v,0) = (d(v),0) \quad \text{for} \quad v \in V \quad \text{and} \quad d(0,w) = (\varepsilon(w),0) \quad \text{for} \quad w \in V^t,$$

and extended by linearity and so that Leibnitz rule is satisfied.

Remember that $\varepsilon(w) \in \Lambda(V^{\leq t-1})$, then $\widehat{\varphi}$ is an isomorphism of *cdga's* because for $v, v' \in \Lambda(V^{\leq t-1})$ with $|v| = n$, $|v'| = m$ and $w, w' \in V^t$ we have:

$$\widehat{\varphi}^{n+1} \circ d^n(v,0) = \widehat{\varphi}^{n+1}(d^n(v),0) = d^n(v) \otimes 1 = d^n \circ \widehat{\varphi}^n(v,0),$$

$$\widehat{\varphi}^{t+1} \circ d^t(0,w) = \widehat{\varphi}^{t+1}(\varepsilon(w),0) = \varepsilon(w) \otimes 1 = d^t \circ \widehat{\varphi}^t(0,w),$$

$$\begin{aligned}\widehat{\varphi}^{n+m+1} \circ d^{n+m}((v,0) \wedge (v',0)) &= \widehat{\varphi}^{n+m+1}(d^n(v,0) \wedge (v',0) + (-1)^{|v|}(v,0) \wedge d^m(v',0)) \\ &= d^{n+m} \circ \widehat{\varphi}^{n+m}((v,0) \wedge (v',0))\end{aligned}$$

$$\begin{aligned}\widehat{\varphi}^{2t+1} \circ d^{2t}((0,w) \wedge (0,w')) &= \widehat{\varphi}^{2t+1}(\varepsilon(w),0) \wedge (0,w') + (-1)^{|w|}(0,w) \wedge (\varepsilon(w'),0) \\ &= (\varepsilon(w) \otimes 1)(1 \otimes w') + (-1)^{|w|}(1 \otimes w)(\varepsilon(w') \otimes 1)) \\ &= d^{2t}((1 \otimes w)(1 \otimes w')) \\ &= d^{2t} \circ \widehat{\varphi}^{2t}(w \wedge w')\end{aligned}$$

$$\begin{aligned}\widehat{\varphi}^{n+t+1} \circ d^{n+t}((v,0) \wedge (0,w)) &= \widehat{\varphi}^{n+t+1}(d^n(v,0) \wedge (0,w) + (-1)^{|v|}(v,0) \wedge d^t(0,w)) \\ &= \widehat{\varphi}^{n+t+1}((d^n(v),0) \wedge (0,w)) + (-1)^{|v|}((v,0) \wedge (\varepsilon(w),0)) \\ &= (d^n(v) \otimes 1)(1 \otimes w) + (-1)^{|v|}(v \otimes 1)(\varepsilon(w) \otimes 1) \\ &= d^{n+t}((v \otimes 1)(1 \otimes w)) \\ &= d^{n+t} \circ \widehat{\varphi}^{n+t}((v,0) \wedge (0,w)).\end{aligned}$$

Definition 2.1.2. *A* ***Sullivan Minimal Algebra*** *is a cdga* $(\Lambda V, d)$ *for which* $V = \{V^n\}_{n\geq 1}$ *admits a basis* $\{v_\alpha\}_{\alpha\in I}$*, indexed by a well-ordered set* I*, of homogeneous elements, where the order of* I *is compatible with the degree (that is, if* $\beta < \alpha$*, then* $|v_\beta| \leq |v_\alpha|$*), and such that*

$$d(v_\alpha) \in \Lambda V_{<\alpha}, \quad \text{for each } \alpha \in I.$$

Here $V_{<\alpha}$ *denotes the subspace of* V *spanned by* $\{v_\beta\}_{\beta<\alpha}$*.*

Remark 2.1.1.

- *We observe that in the last Definition, If* $V = span\{v_1, v_2, v_3\}$ *with* $|v_1| = |v_2| = 5, |v_3| = 9$ *and* $I = \{1,2,3\}$ *with order* $1 < 2 < 3$*; the compatibility between of the well-order set* I *and the degree of the homogeneous elements is satisfied. In particular for* $1 < 2, |v_1| = |v_2|$ *and* $2 < 3, |v_2| < |v_3|$*. But if* $|v_1| = 3, |v_2| = 1$ *and* $|v_3| = 2$ *this compatibility is not satisfies because while* $1 < 2$*, we have that* $|v_1| > |v_2|$*.*
- *A Sullivan minimal algebra is a Sullivan decomposable algebra. Indeed let us assume that* V *admits a basis* $\{v_\alpha\}_{\alpha\in I}$ *as in Definition (2.1.2); then* $|d(v_\alpha)| = |v_\alpha| + 1$ *and* $d(v_\alpha) \in \Lambda V_{<\alpha}$*,*

such that $d(v_\alpha)$ is not a word of length 1, this is $d(v_\alpha) \notin \Lambda^1 V_{<\alpha}$ since $\Lambda^1 V_{<\alpha} = V_{<\alpha}$ and the elements of $V_{<\alpha}$ has elements degree less or equal that $|v_\alpha|$ (see compatibility). Therefore,

$$|d(v_\alpha)| = |v_\alpha| + 1 \quad \textit{and} \quad d(v_\alpha) \in \Lambda V_{<\alpha}, \textit{ then } |d(v_\alpha) \in \Lambda^{\geq 2} V. \tag{2.1.3}$$

This is, $d(V) \subset \Lambda^{\geq 2} V$.

- *A Sullivan decomposable algebra with $V^0 = V^1 = \{0\}$ is a Sullivan minimal algebra. In fact if $v_\alpha \in V$, then $|v_\alpha| = r \geq 2$ and by (1.5.2)*

$$(\Lambda V)^{r+1} = V^{r+1} \oplus (\oplus_{i+j=r+1} V^i \wedge V^j) \oplus (\oplus_{i+j+h=r+1} V^i \wedge V^j \wedge V^h) \oplus \cdots,$$

  then $d(v_\alpha) \in V^1 \wedge V^r \oplus \Lambda V^{\leq r-1}$. Since $V^1 = \{0\}$, we have that $d(v_\alpha) \in \Lambda V^{\leq r-1} \subset \Lambda V_{<\alpha}$ (Recall that $V_{<\alpha}$ has elements of degree less or equal to the degree of $|v_\alpha|$).

- *Let us assume that $V = span\{v_1, v_2, v_3\}$ with $|v_1| = |v_2| = |v_3| = 1$ and $I = \{1, 2, 3\}$ with order $1 < 2 < 3$. We have that $(\Lambda(v_1, v_2, v_3), d)$ with $d(v_1) = v_2 \wedge v_3$, $d(v_2) = v_1 \wedge v_3$, $d(v_3) = v_1 \wedge v_2$, is not a Sullivan minimal algebra, because $d(v_1) \notin V_{<1}$ (In this case $d(v_1)$ must be 0) and $d(v_1) \notin V_{<2}$ since $v_3 \notin V_{<2}$; but it is a Sullivan decomposable algebra. On the other hand, $(\Lambda(v_1, v_2, v_3), d)$ with $|v_1| = |v_2| = 2$ and $|v_3| = 3$, $d(v_1) = d(v_2) = 0$ and $d(v_3) = v_1 \wedge v_1 - v_2 \wedge v_2$ is Sullivan minimal algebra.*

- *If A is a Sullivan minimal algebra, then $d_0 = 0$ where d_0 is the linear part of d (see Example 1.2.2-6), because for all $v \in V$, $d(v)$ must have length greater than or equal to 2.*

- *If $(\Lambda V, d)$ is a Sullivan minimal algebra, then $(\Lambda V, d_1)$ (where d_1 is quadratic part of d) is also a Sullivan minimal algebra. In fact, we can prove that d_1 satisfies $d_1 \circ d_1 = 0$: since if $d_1 \circ d_1 \neq 0$, then $d_1 \circ d_1$ increases word length by exactly 2, and since $d = d_1 + \cdots + d_p$ for some $p \in \mathbb{N}$, then $d \circ d - d_1 \circ d_1$ increases word length at least 3, but, as $d \circ d = 0$ then $d_1 \circ d_1$ increases word length by at least 3, a contradiction; therefore $d_1 \circ d_1 = 0$. Also, since $d(v_\alpha) \in \Lambda V_{<\alpha}$, then $d_1(v_\alpha) \in \Lambda^{\geq 2} V_{<\alpha}$.*

In some works as [8], we find the definition of a "Sullivan *cdga*" as a *cdga* $(\Lambda V, d)$ whose underling algebra is free commutative, with $V = \{V^n\}_{n \geq 1}$, and such that V admits a basis $\{v_\alpha\}_{\alpha \in I}$ indexed by a well-ordered set I, satisfying that $d(v_\alpha) \in \Lambda(v_\beta)_{\beta < \alpha}$. And define a Sullivan minimal *cdga* as a Sullivan *cdga* $(\Lambda V, d)$ satisfying the additional property that $d(V) \subset \Lambda^{\geq 2} V$. Our definition of a Sullivan minimal algebra is different in what we require the basis $\{v_\alpha\}$ to be compatible with the graduation; this allows us to recognize in an easier way the algebra $\Lambda(v_\beta)_{\beta < \alpha}$, since we do not need exclude the terms v_β with $|v_\beta| > |v_\alpha| + 1$ and it allows us to prove that the minimality property is satisfied (see 2.1.3).

Definition 2.1.3. *Let A be a dga and let i be a non-negative integer. A morphism of dga's*

$$\rho_{\leq i} \colon M_{\leq i} \to A,$$

*where $M_{\leq i}$ is a Sullivan minimal algebra generated by elements of degrees smaller than or equal to i, is called an **i-minimal model** of A if $H^*(\rho_{\leq i})\colon H^*(M_{\leq i}) \to H^*(A)$ is an isomorphism for $* \leq i$, and it is injective for $* = i+1$.*

Definition 2.1.4. *Let A be a dga; a morphism $\rho\colon M \to A$ is a **minimal model** of A if M is a Sullivan minimal algebra and ρ is a quasi-isomorphism.*

The existence of an i-minimal model and a minimal model of a *dga* and also the relation between these definitions, will be proven later. Now we present a preliminary result:

Definition 2.1.5. *Let $f\colon A \to B$ be a morphism of dga's. We define the dgm called **dgm-cone** of f, denoted $C(f)$, as follows: $C^n(f) = A^n \oplus B^{n-1}$ and*

$$\begin{array}{rccc} d^n: & C^n(f) & \to & C^{n+1}(f) \\ & (x,y) & \mapsto & (-d_A^n(x), d_B^{n-1}(y) + f^n(x)). \end{array}$$

According to the above, we have $d^{n+1} \circ d^n(x,y) = (0,0)$ since $d^{n+1} f^n(x) = f^{n+1} d^n(x)$. Note that the *dgm-cone* of f is not a *dga* because multiplication is not defined in it. We denote the cohomology $H(C(f))$ of the *dgm-cone* of f by $H(A,B)$.

Theorem 2.1.1. *Let $f\colon A \to B$ a morphism of dga's and let $C(f)$ be its dgm-cone. Then we have the following exact sequence:*

$$\cdots \to H^*(A,B) \xrightarrow{H^*(-p_1)} H^*(A) \xrightarrow{H^*(f)} H^*(B) \xrightarrow{H^*(i_2)} H^{*+1}(A,B) \to \cdots$$

where p_1 and i_2 denote the projection to the first factor and the inclusion to the second factor respectively.

Proof. It is sufficient to note the following equalities:

$$H^*(A,B) = \{[(x,y)] : d_A^*(x) = 0,\ d_B^{*-1}(y) = -f^*(x)\}$$

$$Im\big(H^*(-p_1)\big) = \{[-x] :\ \exists y \mid d_B^{*-1}(y) = -f^*(x)\} = ker\big((H^*(f)\big)$$

$$\begin{aligned} ker\big(H^*(i_2)\big) &= \{[b] :\ \exists (x,y) \in C^*(f),\ d^*(x,y) = (0,b)\} \\ &= \{[b] : d_A^*(x) = 0 \text{ and } d_B^{*-1}(y) + f^*(x) = b\} \\ &= \{[b] : d_A^*(x) = 0 \text{ and } [f^*(x)] = [b]\} \\ &= Im\big(H^*(f)\big). \end{aligned}$$

□

Now we can state Sullivan's theorem on minimal models.

Theorem 2.1.2. *Let A be a dga such that $H^0(A) = k$, then there exists a i-minimal model, for any $i \geq 0$ and a minimal model of A.*

We note that $H^0(A) = k$ is the restriction which allows the existence of the 0-minimal of A. This proof is a little laborious, so, for the Readers' convenience, we will present below a sketch of proof; we show a full proof in the next section.

Sketch of proof: We show first the existence of i-minimal models of A of the form $\rho_{\leq\, i} \colon M_{\leq\, i} \to A$ where $M_{\leq\, i} = \Lambda(V^0 \oplus V^1 \oplus V^2 \oplus \cdots \oplus V^i)$. The minimal model of A will be $M = \bigcup_{i=0}^{\infty} M_{\leq\, i}$ and $\rho_{\leq\, i} = \rho|_{M_{\leq\, i}}$.

We need to identify appropriate vector spaces $V^0, V^1, V^2, \ldots, V^i$. For $i = 0$ we set $V^0 = \{0\}$, since $H^0(A) = k$ we obtain that $k \subset A$ and we define $\rho_{\leq\, 0}$ as the inclusion therefore $H^0(\rho_{\leq\, 0})$ is an isomorphism and $H^*(\rho_{\leq\, 0})$ is injective for $* \geq 1$. Then $k = \Lambda V^0 = M_{\leq\, 0}$. By induction on i, assuming that there is a $t-1$-minimal model of A and we apply the exact sequence of Theorem (2.1.1) to the $t-1$ minimal model $\rho_{\leq\, t\,-\,1} \colon M_{\leq\, t\,-\,1} \to A$. This exact sequence allows us to identify a vector space $V^{t-1,0}$ that we add to $M_{\leq\, t\,-\,1}$. We set

$$M_{\leq\, t\,-\,1,\,0} = M_{\leq\, t\,-\,1} \otimes_{\varepsilon_0} V^{t-1,0}$$

and $\rho_{\leq\, t\,-\,1,\,0}$ is a extension of $\rho_{\leq\, t\,-\,1}$, where:

$$V^{t-1,0} \cong H^{t\,+\,1}(M_{\leq\, t\,-\,1}, A) \cong Ker(H^{t+1}(\rho_{\leq\, t\,-\,1})) \oplus Coker(H^t(\rho_{\leq\, t\,-\,1})) \cong \langle \oplus[(\eta_k, w_k)] \rangle \oplus \langle \oplus[(0, u_j)] \rangle,$$

where $Ker(H^{t+1}(\rho_{\leq\, t\,-\,1}))$ has basis $\{[\eta_k]\}$, w_k is determined by $\rho^{t\,+\,1}_{\leq\, t\,-\,1}(\eta_k) = d^t_A(w_k)$ and

$Coker(H^t(\rho_{\leq\, t\,-\,1})) = H^t(A)/Im(H^t(\rho_{\leq\, t\,-\,1}))$ has basis $\{\overline{[u_j]}\}$.

If we assume that $H^1(A) = 0$, then it follows from that $H^{t+1}(\rho_{\leq\, t\,-\,1,\,0})$ is an injective mapping. Therefore $M_{\leq\, t} = M_{\leq\, t\,-\,1,\,0}$ and $\rho_{\leq\, t}$ is a extension of $\rho_{\leq\, t\,-\,1}$, so we denote $V^{t-1,0} = V^t$.

On the other hand, in the case that $H^1(A) \neq 0$ the mapping $H^{t+1}(\rho_{\leq\, t\,-\,1,\,0})$ need not be injective. We need add elements to $M_{\leq\, t\,-\,1,\,0}$, in order to kill the kernel of $H^{t+1}(\rho_{\leq\, t\,-\,1,\,0})$ so, we again use the exact sequence in the Theorem (2.1.1) with $f = \rho_{\leq\, t\,-\,1,\,0}$, and we obtain the vector space $V^{t-1,1}$ that we add to $M_{\leq\, t\,-\,1,\,0}$ for obtaining $M_{\leq\, t}$. In the proof we denote this extension by $M_{\leq\, t\,-\,1,\,1} = M_{\leq\, t\,-\,1,\,0} \otimes_{\varepsilon_1} V^{t-1,1}$ and $\rho_{\leq\, t\,-\,1,\,1}$ is a extension of $\rho_{\leq\, t\,-\,1,\,0}$.

If $H^{t+1}(\rho_{\leq\, t\,-\,1,\,1})$ is injective then we set $M_{\leq\, t} = M_{\leq\, t\,-\,1,\,1}$ but if this is not true, we must continue. This process as above, in order to kill the kernel of $H^{t+1}(\rho_{\leq\, t\,-\,1,\,1})$, this process can be finite and in this case the $M_{\leq\, t} = M_{\leq\, t\,-\,1,\,n}$ for some $n \in \mathbb{N}$, or it may nor stop; in this case we define $M_{\leq\, t} = \bigcup_{j=0}^{\infty} M_{\leq\, t\,-\,1,\,j}$ and $\rho_{\leq\, t}$ such that $\rho_{\leq\, t,\,j} = \rho_{\leq\, t}|_{M_{\leq\, t\,-\,1,\,j}}$.

2.2 Proof of Sullivan Theorem

In this section, we prove the fundamental theorem on existence of i-minimal and minimal models.

Proof. First we show the existence of i-minimal models. The proof is by induction on i. If $i = 0$, we consider k as a *dga* concentrated in degree zero; we obtain that $k \subset A$ and we define $\rho_{\leq\, 0}$ as the inclusion. We have the diagrams

$$\begin{array}{ccccccc} k & \xrightarrow{d_M} & 0 & \xrightarrow{d_M} & 0 & \longrightarrow & \dots \\ \downarrow{\scriptstyle \rho^0_{\leq 0}} & & \downarrow{\scriptstyle \rho^1_{\leq 0}} & & \downarrow{\scriptstyle \rho^2_{\leq 0}} & & \\ A^0 & \xrightarrow{d_A} & A^1 & \xrightarrow{d_A} & A^2 & \longrightarrow & \dots \end{array} \qquad \begin{array}{ccccccc} k & \xrightarrow{\overline{d}_M} & 0 & \xrightarrow{\overline{d}_M} & 0 & \longrightarrow & \dots \\ \downarrow{\scriptstyle H^0(\rho_{\leq 0})} & & \downarrow{\scriptstyle H^1(\rho_{\leq 0})} & & \downarrow{\scriptstyle H^2(\rho_{\leq 0})} & & \\ H^0(A) & \xrightarrow{\overline{d}_A} & H^1(A) & \xrightarrow{\overline{d}_A} & H^2(A) & \longrightarrow & \dots \end{array}$$

Figure 2.1.

Since $H^0(A) = k$ we conclude that, $H^0(\rho_{\leq\, 0})$ is an isomorphism and $H^*(\rho_{\leq\, 0})$ is injective for $* \geq 1$. Moreover $k = \Lambda V^0$, where $V^0 = \{0\}$.

Now, we assume that $i = t-1$, $t \geq 1$. Our induction hypothesis is that (as in Definition 2.1.3), $M_{\leq\, t\,-\,1}$ is generated by elements of degrees smaller than or equal to $t-1$ this is, (as in Definition 2.1.2) there is a graded vector space $V^0 \oplus V^1 \oplus \cdots \oplus V^{t-1}$ such that $M_{\leq\, t\,-\,1} = \Lambda(V^0 \oplus V^1 \oplus \cdots \oplus V^{t-1})$ and there exists a morphism $\rho_{\leq\, t\,-\,1} \colon M_{\leq\, t\,-\,1} \to A$ which is an $(t-1)$-minimal model of A.

We will construct $\rho_{\leq\, t\,-\,1} \colon M_{\leq\, t\,-\,1} \to A$, a t-minimal model of A. By the last theorem applied to the morphism $\rho_{\leq\, t} \colon M_{\leq\, t} \to A$, with domain $M_{\leq\, t}$ and codomain A, we have the exact sequence of Figure 2.2, where p_1 and i_2 denote the projection to the first factor and the inclusion into the second factor respectively.

We have $H^t(M_{\leq\, t\,-\,1}, A) = 0$. In fact, by hypothesis $ker(H^{t-1}(i_2)) = Im(H^{t-1}(\rho_{\leq\, t\,-\,1})) = H^{t-1}(A)$, so $ker(H^t(-p_1)) = 0$, moreover $ImH^t(-p_1) = Ker(H^t(\rho_{\leq\, t\,-\,1})) = 0$.

We need to add cohomology to $M_{\leq\, t\,-\,1}$ (if necessary), so that the extension of the map $H^t(\rho_{\leq\, t\,-\,1}) \colon H^{t-1}(M_{\leq\, t\,-\,1}) \to H^{t-1}(A)$ become surjective; but at the same time, these added variables must kill the kernel on cohomology in degree $t+1$, so that the extension of $H^{t+1}(\rho_{\leq\, t\,-\,1})$ becomes injective.

The question now is:

What are the elements that we have to add to $M_{\leq\, t\,-\,1}$, and how do we add them? From the long exact sequence (Figure 2.2) coming from Theorem (2.1.1) we get a short exact sequence:

$$0 \longrightarrow Coker(H^t(\rho_{\leq\, t\,-\,1})) \xrightarrow{H^t(i_2)} H^{t+1}(M_{\leq\, t\,-\,1}, A) \underset{s}{\overset{H^{t+1}(-p_1)}{\rightleftarrows}} Ker(H^{t+1}(\rho_{\leq\, t\,-\,1})) \longrightarrow 0 \tag{2.2.1}$$

$$\cdots \to H^{t-1}(M_{\leq\, t\,-\,1}) \xrightarrow{H^{t-1}(\rho_{\leq\, t\,-\,1})} H^{t-1}(A) \xrightarrow{H^{t-1}(i_2)} H^{t}(M_{\leq\, t\,-\,1}, A) \xrightarrow{H^{t}(-p_1)} H^{t}(M_{\leq\, t\,-\,1}) \xrightarrow{H^{t}(\rho_{\leq\, t\,-\,1})} H^{t}(A) \xrightarrow{H^{t}(i_2)} H^{t+1}(M_{\leq\, t\,-\,1}, A) \xrightarrow{H^{t+1}(-p_1)} H^{t+1}(M_{\leq\, t\,-\,1}) \xrightarrow{H^{t+1}(\rho_{\leq\, t\,-\,1})} H^{t+1}(A) \to \cdots$$

Figure 2.2.

As indicated in (2.2.1), this sequence splits. Indeed, let us consider:

$$\begin{array}{rccc} s: & Ker(H^{t+1}(\rho_{\leq\, t\,-\,1})) & \to & H^{t+1}(M_{\leq\, t\,-\,1}, A) \\ & [\eta] & \mapsto & [(-\eta, w)] \end{array}$$

where η and w are determined by $\rho^{t\,+\,1}_{\leq\, t\,-\,1}(\eta) = d^t_A(w)$. This function is well defined, since

$$\begin{aligned} [\eta] = [m] &\Rightarrow m = \eta + d^t_{M_{\leq\, t\,-\,1}}(x), \text{ for some } x \in M^t_{\leq\, t\,-\,1} \\ &\Rightarrow \rho^{t\,+\,1}_{\leq\, t\,-\,1}(m) = \rho^{t\,+\,1}_{\leq\, t\,-\,1}(\eta) + \rho^{t\,+\,1}_{\leq\, t\,-\,1}\big(d^t_{M_{\leq\, t\,-\,1}}(x)\big) \\ &\Rightarrow \rho^{t\,+\,1}_{\leq\, t\,-\,1}(m) = d^t_A(w + \rho^t_{\leq\, t\,-\,1}(x)) \\ &\Rightarrow s([m]) = [(-m, w + \rho^t_{\leq\, t\,-\,1}(x))], \end{aligned}$$

and so $[(-\eta, w)] = [(-m, w + \rho^t_{\leq\, t\,-\,1}(x))]$; in fact $(x, 0) \in C^t(\rho_{\leq\, t\,-\,1})$ such that

$$\begin{aligned} &(-m, w + \rho^t_{\leq\, t\,-\,1}(x)) + d^t(-x, 0) \\ &= (-m, w + \rho^t_{\leq\, t\,-\,1}(x)) + (-d^t_{M_{\leq\, t\,-\,1}}(-x), \rho^t_{\leq\, t\,-\,1}(-x)) = (-m + d^t_{M_{\leq\, t\,-\,1}}(x), w) = (-\eta, w). \end{aligned}$$

Besides, if we change w by $w + d^{t\,-\,1}_A(\tilde{w})$ we have that $[(-\eta, w)] = [(-m, w + d^{t\,-\,1}_A(\tilde{w}))]$. In fact:

$$(-\eta, w + d^{t\,-\,1}_A(\tilde{w})) + d^t(0, -\tilde{w}) = (-\eta, w + d^{t\,-\,1}_A(\tilde{w})) + (0, -d^{t\,-\,1}_A(\tilde{w}) + \rho^t_{\leq\, t\,-\,1}(0)) = (-\eta, w).$$

Moreover $H^{t+1}(-p_1) \circ s([\eta]) = H^{t+1}(-p_1)([(-\eta, w)]) = [\eta]$. Therefore

$$H^{t\,+\,1}(M_{\leq\, t\,-\,1}, A) \cong Ker(H^{t+1}(\rho_{\leq\, t\,-\,1})) \oplus Coker(H^t(\rho_{\leq\, t\,-\,1})) \cong \langle \oplus[(\eta_k, w_k)]\rangle \oplus \langle \oplus[(0, u_j)]\rangle, \quad (2.2.2)$$

where $Ker(H^{t+1}(\rho_{\leq\, t\,-\,1}))$ has basis $\{[\eta_k]\}$ and $Coker(H^t(\rho_{\leq\, t\,-\,1})) = H^t(A)/Im(H^t(\rho_{\leq\, t\,-\,1}))$ has basis $\{\overline{[u_j]}\}$.

Now we consider $V^{t-1,0}$ as a graded vector space concentrated in degree t with basis

$$\{[(\eta_k, w_k)], [(0, u_j)]\}_{k,j},$$

and we add elements to $M_{\leq\, t\,-\,1}$ by means of the following extension:

$M_{\leq\, t\,-\,1,\,0} = M_{\leq\, t\,-\,1} \otimes_{\varepsilon_0} V^{t-1,0}$ in which

$$\begin{array}{rccl} \varepsilon_0 : & V^{t-1,0} & \to & M_{\leq\, t\,-\,1} \\ & [(\eta_k, w_k)] & \mapsto & \eta_k \\ & [(0, u_j)] & \mapsto & 0 \end{array}$$

As we saw in Definition (2.1.1) we consider an extension of degree t of $M_{\leq\, t\,-\,1}$ of the form $M_{\leq\, t\,-\,1} \otimes_{\varepsilon_0} (\Lambda V^{t-1,0})$ and we have the identification (2.1.2) this is

$$M_{\leq\, t\,-\,1,\,0} \cong \Lambda(V^1 \oplus \cdots \oplus V^{t-1} \oplus V^{t-1,0}) \quad \text{and} \quad d_{M_{\leq\, t\,-\,1,\,0}}|_{V^{t-1,0}} = \varepsilon_0.$$

Note that $M_{\leq\, t\,-\,1,\,0}$ is a Sullivan minimal algebra, because by the inductive hypothesis so is $M_{\leq\, t\,-\,1}$ and $d(v) \in (\Lambda V^1 \oplus \cdots \oplus V^{t-1})$ for all $v \in V^{t-1,0}$. Now, we define a map $\rho_{\leq\, t\,-\,1,\,0} \colon M_{\leq\, t\,-\,1,\,0} \to A$, which is an extension of degree t of morphism $\rho_{\leq\, t\,-\,1}$, so for the degree t:

$$\rho^t_{\leq\, t\,-\,1,\,0}([(\eta_k, w_k)]) = w_k \quad \text{and} \quad \rho^t_{\leq\, t\,-\,1,\,0}([(0, u_j)]) = u_j.$$

Then $\rho_{\leq\, t\,-\,1,\,0}$ is a morphism of *dga's*, in fact:

$$d^t_A \circ \rho^t_{\leq\, t\,-\,1,\,0}([(\eta_k, w_k)]) = d^t_A(w_k) = \rho^{t\,+\,1}_{\leq\, t\,-\,1}(\eta_k) = \rho^{t\,+\,1}_{\leq\, t\,-\,1,\,0} \circ d^t_{M_{\leq\, t\,-\,1,\,0}}([(\eta_k, w_k)]),$$

$$d^t_A \circ \rho^t_{\leq\, t\,-\,1,\,0}([(0, u_j)]) = d^t_A(u_j) = 0 = \rho^{t\,+\,1}_{\leq\, t\,-\,1}(0) = \rho^{t\,+\,1}_{\leq\, t\,-\,1,\,0} \circ d^t_{M_{\leq\, t\,-\,1,\,0}}([(0, u_j)]).$$

We observe that:

$$\begin{aligned} \boldsymbol{M^i_{\leq\, t\,-\,1,\,0}} &= M^i_{\leq\, t\,-\,1} \otimes k = (\Lambda(V^1 \oplus \cdots \oplus V^{t-1}))^i \quad \text{for all} \quad i \leq t-1, \\ \boldsymbol{M^t_{\leq\, t\,-\,1,\,0}} &= (k \otimes V^{t-1,0}) \oplus (M^t_{\leq\, t\,-\,1} \otimes k) = (\Lambda(V^1 \oplus \cdots \oplus V^{t-1}) \otimes \Lambda V^{t-1,0})^t = (\Lambda(V^1 \oplus \cdots \oplus V^{t-1,0}))^t, \\ \boldsymbol{M^{t\,+\,1}_{\leq\, t\,-\,1,\,0}} &= (k \otimes (\Lambda V^{t-1,0})^{t+1}) \oplus (M^1_{\leq\, t\,-\,1} \otimes V^{t-1,0}) \oplus (M^{t\,+\,1}_{\leq\, t\,-\,1} \otimes k) = (\Lambda(V^1 \oplus \cdots \oplus V^{t-1,0}))^{t+1} \end{aligned} \tag{2.2.3}$$

Then $H^i(M_{\leq\, t\,-\,1,\,0}) = H^i(M_{\leq\, t\,-\,1})$ for all $i \leq t-1$ and

$$Ker(d^t_{M_{\leq\, t\,-\,1,\,0}}) = Ker(d^t_{M_{\leq\, t\,-\,1}}) \oplus \langle \oplus [(0, u_j)] \rangle, \quad Im(d^{t\,-\,1}_{M_{\leq\, t\,-\,1,\,0}}) = Im(d^{t\,-\,1}_{M_{\leq\, t\,-\,1}});$$

this implies that $H^t(M_{\leq\, t\,-\,1,\,0}) = H^t(M_{\leq t-1}) \oplus \langle \oplus [(0, u_j)] \rangle$, and we have the following Figure:

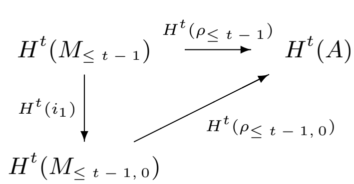


Figure 2.3.

Since $H^t(\rho_{\leq\, t\,-\,1})$ is injective and $H^t(\rho_{\leq\, t\,-\,1,\,0})([(0,u_j)]) = [u_j]$, where $\{\overline{[u_j]}\}$ is a basis of $Coker(H^t(\rho_{\leq\, t\,-\,1}))$ so, $[u_j] \notin H^t(A)$, therefore $H^t(\rho_{\leq\, t\,-\,1,\,0})$ is an isomorphism.

Now we need that $H^{t+1}(\rho_{\leq\, t\,-\,1,\,0})$ be injective:

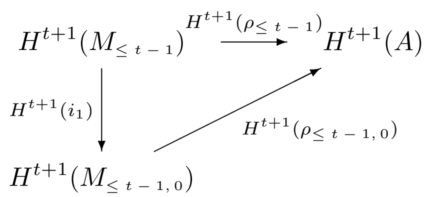


Figure 2.4.

First, note that $H^{t+1}(i_1)([\eta_k]) = 0$ for each $[\eta_k] \in Ker(H^{t+1}(\rho_{\leq\, t\,-\,1}))$, since

$$d^{t+1}_{\leq\, t\,-\,1}(\eta_k) = 0 \quad \text{and} \quad d^t_{M_{\leq\, t\,-\,1,\,0}}([(\eta_k, w_k)]) = \varepsilon_0([(\eta_k, w_k)]) = \eta_k. \tag{2.2.4}$$

Therefore the map $H^{t+1}(\rho_{\leq\, t\,-\,1,\,0})$ restricted to the image of $H^{t+1}(i_1)$ is injective: if $[m]$, $[n]$ belong to $Im(H^{t+1}(i_1))$, they are different and $[\rho_{\leq\, t\,-\,1,\,0}(m)] = [\rho_{\leq\, t\,-\,1,\,0}(n)]$, then $[\rho_{\leq\, t\,-\,1}(m-n)] = 0$, this is $[m-n] \in Ker(H^{t+1}(\rho_{\leq\, t\,-\,1}))$, and for the last affirmation $[m] = [n]$ which is a contradiction.

In the case of $t = 1$, we have

$$H^2(M_{\leq\, 0}, A) \cong Ker(H^2(\rho_{\leq\, 0})) \oplus Coker(H^1(\rho_{\leq\, 0})) \cong H^1(A),$$

then $V^{0,0} = H^1(A)$.

Now we assume that $H^1(A) = 0$. Then $M_{\leq\, 0,\,0} = M_{\leq\, 0} \otimes \Lambda(\{0\})$ as $\Lambda(\{0\}) = k$, and it follows that $M_{\leq\, 0,\,0} = M_{\leq\, 1} = k$ is a 1-minimal model of A, (this is in $M_{\leq\, 1}$ there is not elements of degree 1) and therefore $V^1 = \{0\}$.

Then by construction $M_{\leq\, t\,-\,1,\,0}$ does not have elements of degree 1, so by (2.2.3)

$$M^{t\,+\,1}_{\leq\, t\,-\,1,\,0} = M^{t\,+\,1}_{\leq\, t\,-\,1} \otimes k = (\Lambda(V^1 \oplus \cdots \oplus V^{t-1}))^{t+1}.$$

Therefore $H^{t+1}(M_{\leq\, t\,-\,1,\,0}) = H^{t+1}(M_{\leq\, t\,-\,1})$. For this equality $H^{t+1}(i_1)$ is surjective, and it is sufficient to show that $H^{t+1}(\rho_{\leq\, t\,-\,1,\,0})$ is injective (as we saw above).

This guarantees that $\rho_{\leq\, t\,-\,1,\,0}$ is a t-minimal model of A. We denote $V^t = V^{t-1,0}$, $M_{\leq\, t} = M_{\leq\, t\,-\,1,\,0}$, and $\rho_{\leq\, t} = \rho_{\leq\, t\,-\,1,\,0}$. By the induction hypothesis and the foregoing construction:

$$k = M_{\leq\, 0} = M_{\leq\, 1} \subset M_{\leq\, 2} \subset \cdots \subset M_{\leq\, t\,-\,1} \subset M_{\leq\, t} \subset \cdots$$

We conclude that for each i there exists a i-minimal model of A. Now we define $M = \bigcup_{i=0}^{\infty} M_{\leq\, i}$, so that $M^i = \bigcup_{i=0}^{\infty} M^i_{\leq\, i} = M^i_{\leq\, i}$.

What is the differential in M?

$$d^t_M \colon M^t \to M^{t+1} \;\text{ such that }\; d^t_M|_{M_{\leq\, t}} = d^t_{M_{\leq\, t}}.$$

What is the multiplication in M? if $x \in M^i$ and $y \in M^j$ then $x \in M^i_{\leq i}$ and $y \in M^j_{\leq j}$, it is $x \wedge y \in M^{i+j}_{\leq i+j}$ so $x \wedge y \in M^{i+j}$.

Now, we define $\rho \colon M \to A$ such that $\rho(m) = \rho_{\leq i}(m)$, if $m \in M^i$ as $M^i_{\leq i} \subset M^i_{\leq i+1}$ then ρ is a morphism since:

$$\begin{aligned} d^i_A \circ \rho^i(m) &= d^i_A \circ \rho^i_{\leq i}(m) \\ &= d^i_A \circ \rho^i_{\leq i+1}(m) \\ &= \rho^{i+1}_{\leq i+1} \circ d^i_{M_{\leq i}}(m) \\ &= \rho^{i+1} \circ d^i_M(m) \end{aligned}$$

Thus we get $H^i(M) = H^i(M_{\leq i})$ and $H^i(\rho) = H^i(\rho_{\leq i})$ then ρ is a quasi-isomorphism, therefore $\rho \colon M \to A$ is a minimal model of A.

In the case $H^1(A) \neq 0$. We cannot guarantee that $H^{t+1}(\rho_{\leq t-1,0})$ be injective in the diagram of the Figure 2.4.

We have seen that $H^{t+1}(\rho_{\leq t-1,0})|_{Im(H^{t+1}(i_1))}$ is injective, but in this case (namely, $H^1(A) \neq 0$) the map $H^{t+1}(i_1)$ is not necessarily surjective. We need to add elements to $M_{\leq t-1,0}$ of degree t to obtain a t-minimal model of A by means of extensions we proceed as follows:

If the map $H^{t+1}(\rho_{\leq t-1,0})$ is injective, we have that $\rho_{\leq t-1,0}$ is a t-minimal model of A, but if $H^{t+1}(\rho_{\leq t-1,0})$ is not injective, then we need add elements to $M_{\leq t-1,0}$ in order to kill the kernel of $H^{t+1}(\rho_{\leq t-1,0})$, this is, we need to construct an extension $\rho_{\leq t-1,1}$ of $\rho_{\leq t-1,0}$ such that $H^{t+1}(\rho_{\leq t-1,1})|_{Im(H^{t+1}(i_1))}$ is injective (see the following Figure). For this construction we apply the exact sequence in the Theorem (2.1.1) with $f = \rho_{\leq t-1,0}$ and we obtain the short exact splitting sequence

$$0 \longrightarrow Coker(H^t(\rho_{\leq t-1,0})) \xrightarrow{H^t(i_2)} H^{t+1}(M_{\leq t-1,0}, A) \underset{s}{\overset{H^{t+1}(-p_1)}{\rightleftarrows}} Ker(H^{t+1}(\rho_{\leq t-1,0})) \longrightarrow 0$$

reasoning as in (2.2.1).

As $H^t(\rho_{\leq t-1,0})$ is an isomorphism then $Coker(H^t(\rho_{\leq t-1,0})) = 0$ and we set

$$V^{t-1,1} = H^{t+1}(M_{\leq t-1,0}, A) \cong Ker(H^{t+1}(\rho_{\leq t-1,0})) \cong \langle \oplus [(\eta_k, w_k)] \rangle.$$

As above we add elements to $M_{\leq t-1,0}$ by means of the following extension:

$M_{\leq t-1,1} = M_{\leq t-1,0} \otimes_{\varepsilon_1} V^{t-1,1}$ and

$$\begin{aligned} \varepsilon_1 : \quad V^{t-1,1} &\to M_{\leq t-1,0} \\ [(\eta_k, w_k)] &\mapsto \eta_k \end{aligned}$$

As we saw in (2.1.1) this is:

$$M_{\leq t-1,1} = \Lambda(V^1 \oplus \cdots \oplus V^{t-1} \oplus V^{t-1,0} \oplus V^{t-1,1}) \quad \text{and} \quad d^t_{\leq t-1,1}([(\eta_k, w_k)]) = \eta_k,$$

and we define the map $\rho_{\leq\, t\,-\,1,\,1}\colon M_{\leq\, t\,-\,1,\,1} \to A$, which is an extension of degree t of $\rho_{\leq\, t\,-\,1}$, such that in degree t, we have $\rho^t_{\leq\, t\,-\,1,\,1}([(\eta_k, w_k)]) = w_k$ is a morphism of *dga's*.

If $H^{t+1}(\rho_{\leq\, t\,-\,1,\,1})$ is injective then $\rho_{\leq\, t\,-\,1,\,1}$ is a t-minimal model of A, but if this is not true, then we must continue this process: If for some n we have that $\rho_{\leq\, t\,-\,1,\,n}\colon M_{\leq\, t\,-\,1,\,n} \to A$ satisfies the property that $H^{t+1}(\rho_{\leq\, t\,-\,1,\,n})$ is injective, then $\rho_{\leq\, t\,-\,1,\,n}$ is a t-minimal model of A; if there is not such n, then, we define $M_{\leq\, t} = \bigcup_{j=0}^{\infty} M_{\leq\, t\,-\,1,\,j}$ and $\rho_{\leq\, t}$ such that $\rho_{\leq\, t,\,j} = \rho_{\leq\, t}|_{M_{\leq\, t\,-\,1,\,j}}$. Thus obtain the diagram of the Figure 2.5.

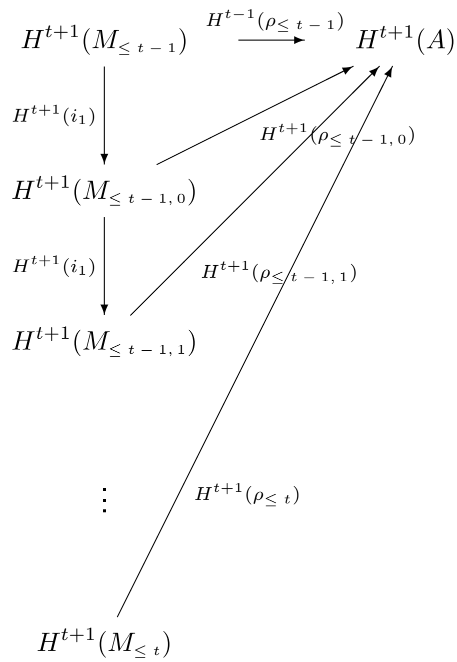


Figure 2.5.

We note that this problem occurs already from $M_{\leq\, 0}$ to obtain $M_{\leq\, 1}$. Now we see that the map $H^{t+1}(\rho_{\leq\, t})\colon H^{t+1}(M_{\leq\, t}) \to H^{t+1}(A)$ is injective:

$$\begin{aligned}
[\rho_{\leq\, t}(m)] = 0 \Rightarrow &\ \text{ There is } j \text{ such that } [\rho_{\leq\, t\,-\,1,\,j}(m)] = 0 \\
&\Rightarrow [m] \in Ker(H^{t+1}(\rho_{\leq\, t\,-\,1,\,j})) \\
&\Rightarrow [m] \in Ker(H^{t+1}(i_1)) \ \ \text{(see 2.2.4)} \\
&\Rightarrow [m] \in H^{t+1}(M_{t\,-\,1,\,j\,+\,1}) \ \text{ and } \ [m] = 0 \\
&\Rightarrow [m] \in H^{t+1}(M_t) \ \text{ and } \ [m] = 0.
\end{aligned}$$

Moreover, $H^t(\rho_{\leq\, t\,-\,1,\,j})$ is a isomorphism for all $j \geq 0$, so is $H^t(\rho_{\leq\, t})$. In fact if $H^t(\rho_{\leq\, t})([m]) = 0$ then $[\rho_{\leq\, t\,-\,1,\,j}(m)] = 0$ and therefore $[m] = 0$; on the other hand, if $[a] \in H^t(A)$, then there exists $[m] \in H^t(M_{\leq\, t\,-\,1,\,j})$ such that $\rho_{\leq\, t\,-\,1,\,j}([m]) = [a]$, so $\rho_{\leq\, t}([m]) = [a]$.

We conclude that, $\rho_{\leq\,t}\colon M_{\leq\,t} \to A$ is a t-minimal model:

$$M_{\leq\,0} = k \subset M_{\leq\,0,\,0} \subset M_{\leq\,0,\,1} \subset \cdots \subset M_{\leq\,1} = \bigcup_{j_0=0}^{\infty} M_{0,j_0}$$

$$M_{\leq\,1} \subset M_{\leq\,1,\,0} \subset M_{\leq\,1,\,1} \subset \cdots \subset M_{\leq\,2} = \bigcup_{j_1=0}^{\infty} M_{1,j_1}$$

$$\vdots$$

$$M_{\leq\,t\,-\,1} \subset M_{\leq\,t\,-\,1,\,0} \subset M_{\leq\,t\,-\,1,\,1} \subset \cdots \subset M_{\leq\,t\,-\,1,\,n} \subset \cdots \subset M_{\leq\,t} = \bigcup_{j}^{\infty} M_{\leq\,t\,-\,1,\,j}$$

Finally, the union $M = \bigcup_i^{\infty} M_{\leq\,i}$ of all these i-minimal models becomes a Sullivan minimal algebra, as we saw in the case $H^1(A) = 0$.

$\square$

2.3 Minimal model and Twisting cohomology

From the proof of Theorem (2.1.2) (section 2.2) we obtain a minimal model of A denoted by $\rho\colon M \to A$, so that M is a Sullivan minimal algebra. In Remark (2.1.1) we showed that a Sullivan minimal algebra is a Sullivan decomposable algebra. It follows that, *if M is of finite type from Theorem (1.5.1) we can determine M by a Lie algebra and a sequence of the twisted cohomology class.*

Now we remark that, from the definition of minimal model of A, if $H(A)$ is of finite type then $H(M)$ is of finite type. Moreover by the construction carried out in the proof of Sullivan's Theorem for the ***case*** $\boldsymbol{H^1(A) = 0}$ each extension as in Definition (2.1.1) is a Hirsch extension, we have the following Theorem.

Theorem 2.3.1. *Let A be a dga such that $H(A)$ is of finite type, with $H^0(A) = k$ and $H^1(A) = 0$; then there is $\rho\colon M \to A$, a minimal model of A, such that M is of finite type.*

Proof. From the proof of Sullivan Theorem, we have $M = \Lambda(V^1 \oplus \cdots \oplus V^t \oplus \cdots)$ with $M^1 = \{0\}$. Now, we must show that each V^i is finite dimensional.

We argue by induction on $i \geq 0$. For $i = 0$ or $i = 1$, we proved that $V^0 = V^1 = \{0\}$. Let us now suppose that the result holds for every $i \leq t-1$, with $t \geq 1$ and prove it when $i = t$. By construction $\rho_{\leq\,t\,-\,1}\colon M_{\leq\,t\,-\,1} \to A$ is a $t-1$-minimal model of A where $M_{\leq\,t\,-\,1} = \Lambda(V^1 \oplus \cdots \oplus V^{t-1})$. By the induction hypothesis we have $(M_{\leq\,t\,-\,1})^{t+1}$ is finite dimensional, then $Ker(H^{t+1}(\rho_{\leq\,t\,-\,1}))$ is finite dimensional; also $H^t(\rho_{\leq\,t\,-\,1})$ is injective then $Coker(\rho_{\leq\,t\,-\,1})$ is finite dimensional. Therefore, by isomorphism (2.2.2) $V^t \cong Ker(H^{t+1}(\rho_{\leq\,t\,-\,1})) \oplus Coker(H^t(\rho_{\leq\,t\,-\,1}))$ is finite dimensional.

$\square$

With the hypothesis of above Theorem, we conclude that M is a Sullivan decomposable algebra, for which the Lie algebra is the trivial algebra. Thus the t-twisting matrix is:

$$\begin{array}{rccc} \Theta^t: & V^t & \to & M^1 \otimes V^t \\ & v & \mapsto & 0 \end{array}$$

and the t-twisted cohomology class is $[f_t] \in H_{Hom,\Theta^t}(V^t; \Lambda(V^1 \oplus \cdots \oplus V^{t-1}))$ where $f_t = \varepsilon_0$ with the notation in the proof of Sullivan Theorem:

$$\begin{array}{rccc} f_t: & V^t & \to & (\Lambda(V^1 \oplus \cdots \oplus V^{t-1}))^{t+1} \\ & [(\eta_k, w_k)] & \mapsto & \eta_k \\ & [(0, u_j)] & \mapsto & 0 \end{array}$$

in this case $f_t = \varepsilon_0$.

Remark 2.3.1. *From the proof of Sullivan Theorem for the* **case $H^1(A) \neq 0$** *such that $H(A)$ is of finite type, we observe that $M = \Lambda(V^1 \oplus V^2 \oplus \cdots \oplus V^t \oplus \cdots)$, where $M^1 \neq 0$ and*

$$V^i = V^{i-1,0} \oplus V^{i-1,1} \oplus \cdots \oplus V^{i-1,n_i} \quad \text{for some } n \in \mathbb{N} \text{ or} \tag{2.3.1}$$

$$V^i = V^{i,0} \oplus V^{i,1} \oplus \cdots \tag{2.3.2}$$

Let us assume that $M_{\leq\, t\,-\,1} = \Lambda(V^1 \oplus V^2 \oplus \cdots \oplus V^{t-1})$ is a $t-1$-minimal model of finite type, this is, each V^i with $i \leq t-1$ is finite dimensional by the construction in the of proof of Theorem (2.1.2) we already know that:

- *$V^{t-1,0} \cong Ker(H^{t+1}(\rho_{\leq\, t\,-\,1})) \oplus Coker(H^t(\rho_{\leq\, t\,-\,1}))$ and so $V^{t-1,0}$ is finite dimensional (as in the last proof). Moreover If $v \in V^{t-1,0}$, then $d(v) \in M^{t+1}_{\leq\, t\,-\,1}$; this is $d(v) \in M_{\leq\, t\,-\,1} M_{\leq\, t\,-\,1}$ because $M_{\leq\, t\,-\,1}$ is generated by elements of degrees smaller than or equal to t.*
- *For $j \neq 0$ the vector space $V^{t-1,j} \cong Ker(H^{t+1}(\rho_{\leq\, t\,-\,1,\, j\,-\,1}))$ and so $V^{t-1,j}$ is finite dimensional, since by construction $M_{\leq\, t\,-\,1,\, j\,-\,1}$ is finite dimensional. Also, if $v \in V^{t-1,j}$ with $j \neq 0$, then*

  $$d(v) \in M^{t+1}_{\leq\, t\,-\,1,\, j\,-\,1} \cong \Lambda(V^1 \oplus \cdots \oplus V^{t-1} \oplus V^{t-1,0} \oplus \cdots \oplus V^{t-1,j-1});$$

  this means that $d(v) \in M^1_{\leq\, t\,-\,1} V^{t-1,\leq j-1} + M^{t+1}_{\leq\, t\,-\,1}$. Therefore

  $$d(v) = \sum_{\gamma \in B} \theta^t_\gamma \wedge v_\gamma + a_v$$

  where $\{v_\gamma\}_{\gamma \in B}$ is a basis of $V^{t-1,\leq j-1}$, $\theta^t_\gamma \in M^1_{\leq\, t\,-\,1}$, and $a_v \in M^{t+1}_{\leq\, t\,-\,1}$.

  We recall that $V^{t-1,0}, \ldots, V^{t-1,j-1}$ are disjunct vector spaces, because by construction

  $$H^{t+1}(\rho_{\leq\, t\,-\,1,\, k})|_{Im(H^{t+1}(i_1))}$$

  is injective, for $k = 1, \ldots, j-1$

For the case in which V^t becomes finite dimensional as (2.3.1) we have the twisting matrix Θ^t given by:

$$\begin{array}{rcl}\Theta^t: \quad V^t & \to & M^1_{\leq\, t\,-\,1}\otimes V^t\\ v\in V^{t-1,0} & \mapsto & 0\\ v\in V^{t-1,j} & \mapsto & \sum\limits_{\gamma\in B}\theta^t_{\gamma}\otimes v_\gamma\end{array}$$

and the t-twisted cohomology class is $[f_t]\in H_{Hom,\Theta^t}(V^t;M_{\leq\, t\,-\,1})$ where:

$$\begin{array}{rcl}f_t: \quad V^t & \to & (M_{\leq\, t\,-\,1})^{t+1}\\ v\in V^{t-1,0} & \mapsto & d(v)\\ v\in V^{t-1,j} & \mapsto & a_v.\end{array}$$

The above two statements, since: for $v\in V^{t-1,0}$, we have

$\Theta^t\circ\Theta^t(v)=d\Theta^t(v)=0$, *and* $d_{\Theta^t}(f_t)(v)=(d\circ f_t-\Phi_{f_t}\circ\Theta^t)(v)=d(d(v))=0.$

And, for $v_k\in V^{t-1,j}$ a basic element

$$\begin{aligned}0=d\circ d(v_k)&=\sum_{\gamma\in B}d(\theta^t_{k\gamma}\wedge v_\gamma)+d(a_{v_k})\\&=\sum_{\gamma\in B}\Big(d(\theta^t_{k\gamma})\wedge v_\gamma-\theta^t_{k\gamma}\wedge d(v_\gamma)\big)+d(a_{v_k})\\&=\sum_{\gamma\in B}\Big(d(\theta^t_{k\gamma})\wedge v_\gamma-\theta^t_{k\gamma}\wedge(\sum_{l\in B}\theta^t_{\gamma l}\wedge v_l+a_{v_\gamma})\Big)+d(a_{v_k})\\&=\sum_{\gamma\in B}\Big(d(\theta^t_{k\gamma})-\sum_{l\in B}\theta^t_{kl}\wedge\theta^t_{l\gamma}\Big)\wedge v_\gamma-\sum_{\gamma\in B}\theta^t_{k\gamma}\wedge a_{v_\gamma}+d(a_{v_k})\end{aligned}$$

Since $M_{\leq\, t\,-\,1}$ is free of relations, we have:

$$\sum_{\gamma\in B}\Big(d(\theta^t_{k\gamma})-\sum_{l\in B}\theta^t_{kl}\wedge\theta^t_{l\gamma}\Big)\wedge v_\gamma=0, \tag{2.3.3}$$

then, Θ^t is a twisting matrix and

$$d_{\Theta^t}(f_t)(v_k)=(d\circ f_t-\Phi_{f_t}\circ\Theta^t)(v_k)=\sum_{\gamma\in B}-\theta^k_{k\gamma}\wedge a_{v_\gamma}+d(a_{v_k})=0. \tag{2.3.4}$$

then, $[f_t]\in H_{Hom,\Theta^t}(V^t;M_{\leq\, t\,-\,1})$.

We have the following Theorem.

Theorem 2.3.2. *Let A be a dga such that $H(A)$ is of finite type, with $H^0(A)=k$ and $H^1(A)\neq 0$; then there is $\rho\colon M\to A$, a minimal model of A, such that $M=\Lambda(V^1\oplus V^2\oplus\cdots\oplus V^t\oplus\cdots)$. If each V^t is finite dimensional, finite type, then M is characterized by the Lie algebra over the dual space of V^1 and the sequence of twisted cohomology class $[f_t]\in H_{Hom,\Theta^t}(V^t;\Lambda(V^1\oplus\cdots\oplus V^{t-1}))$ as above.*

Example 2.3.1. *An example of an algebra satisfying the hypothesis of Theorem (2.3.1) is the algebra of smooth forms on a simply connected compact manifold* X *(see Example 1.2.2-4). Indeed, if* $A = \Omega^*(X)$ *and* X *is connected then* $Ker^0(A)$ *is the set of constant functions* $f\colon X \to \mathbb{R}$*. Furthermore, by the deRham Theorem* $H^*_{DR}(X) \cong H^*(X, \mathbb{R})$*, where* $H^*(X, \mathbb{R})$ *is the singular cohomology with real coefficients of* X *then, if* X *is simply connected* $H^1(A) = 0$*. The compactness of* X *guarantees that* $H^*_{DR}(X)$ *is of finite type; for more information on this topic see [4] and [23].*

The minimal model M_X *of* A *is called the* ***(real) minimal model for the manifold*** X*, the morphism* $\rho\colon M_X \to A$ *induces the isomorphism* $H(\rho)\colon H(M_X) \to H(X, \mathbb{R})$ *(by the deRham Theorem).*

We present other example of an algebra satisfying the hypothesis of Theorem (2.3.1) in the chapter 4, section 4.4 (The sine-Gordon Equation) and Theorem (4.4.1).

Chapter 3

Gauge cohomology and Twisted cohomology

The aim of this chapter is to find properties about twisting matrices determined by certain PDEs using the fact that they generate a submanifold of a infinite Jet bundle. For this reason we have compiled in the first section some basic facts about Geometry of Infinite Jets manifold and the Variational Bicomplex; for this part we refer the reader to [22], [21] and [1]. Besides we introduce the notion of $\mathfrak{g}$-valued differential forms in which $\mathfrak{g}$ is a Lie algebra. Then, we consider a submanifold of a infinite Jet bundle determined by PDEs, to finally prove that the horizontal gauge cohomology studied for Marvan in [12] and [13] is twisted cohomology with coefficients in $\mathfrak{g}$. Also we introduce the notion of a *double dgm* and we relate it with elements that satisfy a *Maurer-Cartan condition*. The last section we give a example of the $\mathfrak{sl}(2)$-valued zero curvature representation and generate the twisting matrix. In the next chapter, we conclude that gauge cohomology of PDEs generate Sullivan decomposable algebras.

3.1 Geometry of Infinite Jets and the Variational bicomplex

Definition 3.1.1. *Let $\pi\colon E \to M$ be a smooth onto map of manifolds, where $dimM = m$ and $dimE = m+n$. Then, π is a* ***bundle*** *if for every $p \in M$ there exists a neighborhood $W \subset M$ of p, a manifold F (called a* ***typical fibre*** *of π) and a diffeomorphism $t\colon \pi^{-1}(W) \to W \times F$ such that $p_1 \circ t = \pi|_{\pi^{-1}(W)}$. Where $p_1\colon W \times F \to W$ denote the projection onto the first component.*

We call E the total space, M the basis, $\pi^{-1}(p)$ the ***fiber*** over p and t a ***local trivialization***. The map π partitions the domain thus: $E = \bigcup E_p$ where $E_p = \pi^{-1}(p)$.

The last definition tell us that E has a local product structure. Let $y\colon U \to \mathbb{R}^{m+n}$ be a coordinate system on the open set $U \subset E$; then y is called an ***adapted coordinate system*** if, whenever $a, b \in U$ and $\pi(a) = \pi(b) = p$ then $p_1(y(a)) = p_1(y(b))$ (where $p_1\colon \mathbb{R}^{m+n} \to \mathbb{R}^m$ denote as always projection onto the first component). It is important to underline that an adapted coordinate system can be constructed from the local product structure, since if $x\colon N \to \mathbb{R}^m$ and $v\colon V \to \mathbb{R}^n$ are coordinates system in M and F respectively, where $N \subset W$ and $V \subset F$, and $t\colon \pi^{-1}(W) \to W \times F$ is a local trivialization then

$$\begin{array}{rccl} y: & t^{-1}(N \times V) \subset \pi^{-1}(W) & \to & \mathbb{R}^{m+n} \\ & z & \mapsto & (x \circ p_1 \circ t(z), v \circ p_2 \circ t(z)) \\ & & & (x \circ \pi(z), v \circ p_2 \circ t(z)) \end{array}$$

is an adapted coordinate system.

We can write $y = (x \circ \pi, u)$, where $u = v \circ p_2 \circ t$ and $p_2\colon W \times F \to F$ denote the projection onto the second component.

Conversely, any adapted coordinate system $y\colon U \to \mathbb{R}^{m+n}$ on E yields a coordinate system $x\colon \pi(U) \to \mathbb{R}^m$ by setting $x(p) = p_1 \circ y(a)$, where $a \in E_p \cap U$; this is independent of the choice of a, since y is adapted coordinate system; thus, in terms of coordinate function of x and y, we have that $(x^i \circ \pi)(a) = y^i(a)$, for $1 \leq i \leq m$. The following diagram describes the maps in the above argument:

$$\begin{array}{ccc} U \subset E^{m+n} & \xrightarrow{\pi} & M^m \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ \mathbb{R}^{m+n} & \xrightarrow[p_1]{} & \mathbb{R}^m \end{array}$$

We adopt the following notation: If x^i are the coordinates functions on M, then the coordinates functions on E will denoted:

$$\begin{array}{rccc} (x^i, u^\alpha): & U \subset E^{m+n} & \to & \mathbb{R}^{m+n} \\ & a & \mapsto & (x^i \circ \pi(a), u^\alpha(a)) \end{array} \qquad (3.1.1)$$

where $1 \leq i \leq m$ and $1 \leq \alpha \leq n$. So, the same symbol x^i will be used both for a function $\pi(U) \to \mathbb{R}$ and for the composition $U \to \pi(U) \to \mathbb{R}$. The functions x_i are called independent variables and the functions u^α dependent variables of the typical fiber.

The following definition is a particular case of a bundle:

Definition 3.1.2. *Let $\pi\colon E \to M$ be a smooth onto map of manifolds. Then π is a **trivial bundle** if there exists a manifold F and a diffeomorphism $t\colon E \to M \times F$ such that $p_1 \circ t = \pi$.*

The functions x_i are called independent variables and the functions u^α dependent variables of the typical fiber.

Definition 3.1.3. *A **local section** of π is a smooth map $\phi\colon W \to E$, where W is an open submanifold of M, satisfying the condition $\pi \circ \phi = Id_W$. The set of all local sections of π whose domains contain $p \in M$ will be denoted $\Gamma_p(\pi)$.*

A map $\phi\colon W \to E$ is called a ***section*** or a ***global section*** if $W = M$, and the set of all sections of π is denoted by $\Gamma(\pi)$. It may happen that a bundle does not have sections, for example the *slit tangent bundle* of S^2 (see following Remark, item 4).

The set $\Gamma(\pi)$ is a real vector space by defining addition and scalar multiplication as follows: for $\phi, \psi \in \Gamma(\pi)$ and $a \in \mathbb{R}$,

$$(\phi + \psi)(p) = \phi(p) + \psi(p), \quad a\phi(p) = a\phi(p).$$

For $\phi \in \Gamma(\pi)$ and for $f \in C^\infty(M)$, we can defined $(f\phi)(p) = f(p)\phi(p)$ making $\Gamma(\pi)$ into a module over the ring $C^\infty(M)$.

A section of π may be described in terms of coordinates as follows. If $\phi \in \Gamma(\pi)$ and (x, u) is a system of coordinates in E with (x^i, u^α) coordinate functions around $\phi(a) \in E$, then

$$x^i(\phi(a)) = (x^i \circ \pi)(\phi(a)) = x^i(a), \text{ for } 1 \leq i \leq m$$

so the first m coordinates of $\phi(a)$ are determined by the coordinates of a. Hence only the last n coordinates determine ϕ in this coordinate system; these are the real-valued functions

$$\phi^\alpha = u^\alpha \circ \phi, \text{ for } 1 \leq \alpha \leq n,$$

where in this equality ϕ actually represents the restriction of the section ϕ to the domain of the appropriate coordinate system in M.

As with sections, a local section may be represented in coordinates by the functions $u^\alpha \circ \phi$ for $1 \leq \alpha \leq n$, in which we restrict the section $\phi \in \Gamma_p(\pi)$ to the domain of the appropriate coordinate system in W, this is $(N \cap W, x|_{N \cap W})$, if $x \colon N \to \mathbb{R}^m$ is a coordinate system in M. We have the diagram

$$\begin{array}{ccc} E & \underset{\phi}{\overset{\pi}{\rightleftarrows}} & W \subseteq M \\ {\scriptstyle (x^i, u^\alpha)} \downarrow & & \downarrow {\scriptstyle x^i} \\ \mathbb{R}^{m+n} & \xrightarrow{p_1} & \mathbb{R}^m \end{array}$$

Remark 3.1.1.

1. *If (E, π, M) is a bundle. A bundle (E', π', M) is a **sub-bundle** of π if $E' \subset E$ is a submanifold such that the fibers satisfies that $E_p \subset E'_p$ for each $p \in M$.*

2. *If M and F are manifolds then $(M \times F, p_1, M)$ is a trivial bundle and we denote it by $E = M \times F$; the fiber over p is $E_p = \{p\} \times F$. Any smooth map $\phi \colon M \to E$ which is a section is the graph of the map $f = p_2 \circ \phi$. Thus sections of π are in a natural bijective correspondence with continuous functions $M \to F$.*

3. *Let $TM = \bigcup_{p \in M} T_pM$ be the **tangent manifold of the m-dimensional manifold** M, and let us define the natural projection map $\tau_M(v_p) = p$, with $v_p \in T_pM$. Then, (TM, τ_M, M) form a bundle such that $\tau_M^{-1}(p) = T_pM$ is the fiber; TM is called the **tangent bundle** and the typical fiber is $\mathbb{R}^m$. A section of the tangent bundle $TM \to M$ is a **vector field** on M.*

   Now let us consider a bundle (E, π, M) and a adapted coordinate system (x^i, u^α) on E; then a vector field X on E (a section in (TE, τ_E, E)) is expressed locally by

$$X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{n} Y^\alpha \frac{\partial}{\partial u^\alpha},$$

   where X^i and Y^α are the coordinates of X. As we saw above, these are defined on a coordinate domain U of E. Therefore if (x^i, u^α) are coordinates functions define on U then, for any point $a \in U$,

$$\left\{ \frac{\partial}{\partial x^i}(a), \ \frac{\partial}{\partial u^\alpha}(a) \right\}_{1 \leq i \leq m, \ 1 \leq \alpha \leq n}$$

   becomes a basis of the tangent space T_aE.

4. *Let S^2 be the 2-sphere and let T^0S^2 be the open subset of TS^2 containing all non-zero tangent vectors. The triple $(T^0S, \tau_{S^2}|_{T^0S^2}, S^2)$ is a bundle called the **slit tangent bundle of S^2** with typical fibre $\mathbb{R}^2 - \{0\}$. If ϕ were a section of this bundle then it would define a vector field on S^2 which was never zero, contradicting the well know theorem stating that all vector field on S^2 must be zero at some point in S^2.*

5. *Associated with the tangent bundle, there is a dual bundle (T^*M, τ_M^*, M) called the **cotangent bundle**. Its fibers are the vector spaces $T_p^*M = Hom(T_pM, \mathbb{R})$. The sections of τ_M^* are precisely the **differential 1-forms** on M*

$$\begin{array}{rccc} \sigma: & M & \to & T^*M \\ & p & \mapsto & \sigma_p : T_pM \to \mathbb{R}. \end{array}$$

Locally, a 1-form $\omega \in \Omega^1(E)$ (the set of sections in (T^*E, τ_E^*, E)) is expressed (in an adapted basis) by

$$\sigma = \sum_{i=1}^{m} \sigma_i dx^i + \sum_{\alpha=1}^{n} \varepsilon_\alpha du^\alpha.$$

where σ_i and ε_α are the coordinates of σ, this is smooth maps defined in the domain U of system coordinate (x^i, u^α) in E, so for any point $a \in U$,

$$\left\{ dx^i(a),\ du^\alpha(a) \right\}_{1\leq i\leq m,\ 1\leq\alpha\leq n}$$

becomes a basis of T_a^*E.

6. *Let (T^*M, τ_M^*, M) be the cotangent bundle with fibres T_p^*M. The bundle with fibres $\Lambda^r T_p^*M$ (see Example (1.2.1)-5) denoted by $(\Lambda^r(T^*M), \Lambda^r(\tau_M^*), M)$ where $\Lambda^r(\tau_M^*)(\mathrm{w}_p) = p$ for $\mathrm{w}_p \in \Lambda^r T_p^*M$, is called it the **$r$-exterior bundle**. A section of this bundle is a **differential r-form** in M,*

$$\begin{array}{rccc} \omega: & M & \to & \Lambda^r(T^*M) \\ & p & \mapsto & \omega_p. \end{array}$$

We denote the set of all differential r-forms in M by $\Omega^r(M)$. If $\omega \in \Omega^r(E)$ (the set of sections in $(\Lambda^r(T^*E), \Lambda^r(\tau^*E), E)$). ω is expressed locally by (in an adapted basis)

$$\omega = \sum_{\substack{i_1<\cdots<i_k \\ \alpha_1<\cdots<\alpha_l \\ k+l=r}} f_{i_1\ldots i_k\alpha_1\ldots\alpha_l} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge du^{\alpha_1} \wedge \cdots \wedge du^{\alpha_l},$$

where $f_{i_1\ldots i_k\alpha_1\ldots\alpha_l} : U \to \mathbb{R}$ is a smooth function in the domain U of a system coordinate (x^i, u^α) in E.

Definition 3.1.4. *Let $\pi : E \to M$ be a bundle. The **k-th Jet manifold of π** is the set:*

$$J^k\pi = \{j_p^k\phi : p \in M,\ \phi \in \Gamma_p(\pi)\}$$

where $j_p^k\phi$ is the equivalence class of the equivalence relation in $\Gamma_p(\pi)$ given by $\phi \sim \psi$ iff $\phi(p) = \psi(p)$ and if (x^i, u^α) is a adapted coordinate system around $\phi(p)$ then

$$\left.\frac{\partial^{|I|}(u^\alpha \circ \phi)}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}(u^\alpha \circ \psi)}{\partial x^I}\right|_p$$

for all $1 \le |I| \le k$*, where* I *is a multi-index.*

We recall that a ***multi-index*** is an m-tuple of natural numbers; the components of I are denoted by $I(i)$ where $1 \le i \le m$, so $(I \pm J)(i) = I(i) + J(i)$; $I! = \prod_{i=1}^{m}(I(i))!$, and $|I| = \sum_{i=1}^{m} I(i)$. The symbol $\dfrac{\partial^{|I|}}{\partial x^I}$ is defined by $\prod_{i=1}^{m}(\frac{\partial}{\partial x^i})^{I(i)}$, where the symbol $\prod$ is composition. We denote by $\mathbf{1_j}$ the multi-index such that $\mathbf{1_j}(j) = 1$ and $\mathbf{1_j}(i) = 0$ for all $i \ne j$.

It may be proven that Definition (3.1.4) does not depend on the coordinate system used [22].

Let (U, y) be an adapted coordinate system on E, where $y = (x^i, u^\alpha)$ for $1 \le i \le m$ and $1 \le \alpha \le n$. It induces a coordinate system (U^k, y^k) on $J^k\pi$ defined by:

$$U^k = \{j^k_p\phi : \phi(p) \in U\}, \quad y^k = (x^i, u^\alpha, u^\alpha_I) \quad \text{for } 1 \le i \le m,\ 1 \le \alpha \le n \ \text{ and } \ 1 \le |I| \le k \tag{3.1.2}$$

where:

$$\begin{array}{rccc} x^i : & U^k & \to & \mathbb{R} \\ & j^k_p\phi & \mapsto & x^i(p) \end{array} \qquad \begin{array}{rccc} u^\alpha : & U^k & \to & \mathbb{R} \\ & j^k_p\phi & \mapsto & u^\alpha(\phi(p)) \end{array} \qquad \begin{array}{rccc} u^\alpha_I : & U^k & \to & \mathbb{R} \\ & j^k_p\phi & \mapsto & \left.\frac{\partial^{|I|}(u^\alpha\circ\phi)}{\partial x^I}\right|_p . \end{array}$$

In multi-index notation $u^\alpha_I = u^\alpha_{(I(1),\dots,I(m))}$ where the component $I(i)$ represents the number of occasions that $u^\alpha \circ \phi$ is derived with respect to x^i. In the following example, we introduce the notation symmetric indices that will also we use for convenience in some cases.

Example 3.1.1. *Let* π *be the trivial bundle* $(R^3 \times R^2, p_1, R)$ *with coordinates* $(x^1, x^2, x^3; u^1, u^2)$*; the first derivative coordinates on* J^2p_1 *are:*

$$u^1_{(1,0,0)},\ u^1_{(0,1,0)},\ u^1_{(0,0,1)},\ u^2_{(1,0,0)},\ u^2_{(0,1,0)},\ u^2_{(0,0,1)},$$

and the second derivative are

$$u^1_{(2,0,0)},\ u^1_{(1,1,0)},\ u^1_{(1,0,1)},\ u^1_{(0,2,0)},\ u^1_{(0,1,1)},\ u^1_{(0,0,2)},\ u^2_{(2,0,0)},\ u^2_{(1,1,0)},\ u^2_{(1,0,1)},\ u^2_{(0,2,0)},\ u^2_{(0,1,1)},\ u^2_{(0,0,2)}.$$

In some cases, we will use the **notation of symmetric indices***, so the first derivative coordinates would be written as*

$$u^1_{x^1},\ u^1_{x^2},\ u^1_{x^3},\ u^2_{x^1},\ u^2_{x^2},\ u^2_{x^3},$$

an the second derivative coordinates as

$$u^1_{x^1x^1},\ u^1_{x^1x^2},\ u^1_{x^1x^3},\ u^1_{x^2x^2},\ u^1_{x^2x^3},\ u^1_{x^3x^3},\ u^2_{x^1x^1},\ u^2_{x^1x^2},\ u^2_{x^1x^3},\ u^2_{x^2x^2},\ u^2_{x^2x^3},\ u^2_{x^3x^3}.$$

In general, the coordinate u^α_I can be written as,

$$\begin{array}{rccc} u^\alpha_{x^{i_1}x^{i_2}\dots x^{i_l}} : & U^k & \to & \mathbb{R} \\ & j^k_p\phi & \mapsto & \left.\frac{\partial^l(u^\alpha\circ\phi)}{\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_l}}\right|_p \end{array}$$

for $0 \leq l \leq k$, where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq m$. Then, the coordinate functions in (3.1.2) can be written as:

$$y^k = (x^i, u^\alpha, u^\alpha_{x^i}, \ldots, u^\alpha_{x^{i_1}x^{i_2}\ldots x^{i_k}}) \quad \text{for } 1 \leq i \leq m,\ 1 \leq \alpha \leq n \ \text{ and } \ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq m. \tag{3.1.3}$$

$J^k\pi$ is a manifold of dimension $m + n + n(\binom{m+k}{k} - 1)$. The projection $\pi_k \colon J^k\pi \to M$ such that $j^k_p\phi \mapsto p$ induces a fiber bundle structure $(J^k\pi, \pi_k, M)$. We also have the projections (for $1 \leq l < k$):

$$\begin{array}{cccc} \pi_{k,0}: & J^k\pi & \to & E \\ & j^k_p\phi & \mapsto & \phi(p) \end{array} \qquad\qquad \begin{array}{cccc} \pi_{k,l}: & J^k\pi & \to & J^l\pi \\ & j^k_p\phi & \mapsto & j^l_p\phi \end{array}$$

which allow us to consider the bundle $(J^k\pi, \pi_{k,l}, J^l\pi)$ if $0 \leq l < k$.

Definition 3.1.5. *Let $\pi \colon E \to M$ be a bundle. The* ***infinite Jet manifold of π*** *is the set*

$$J^\infty\pi = \{j^\infty_p\phi : p \in M,\ \phi \in \Gamma_p(\pi)\},$$

where $j^\infty_p\phi$ is the equivalence class of the equivalence relation in $\Gamma_p(\pi)$ given by $\phi \sim \psi$ if $\phi(p) = \psi(p)$ and if (x^i, u^α) is a adapted coordinate system around $\phi(p)$ then

$$\left.\frac{\partial^{|I|}(u^\alpha \circ \phi)}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}(u^\alpha \circ \psi)}{\partial x^I}\right|_p$$

for all $1 \leq |I| < \infty$ and $1 \leq \alpha \leq n$.

It is possible to show that the particular choice of coordinate system in the last definition does not matter, we refer the reader to reference [22].

Let (U, y) be an adapted coordinate system on E, where $y = (x^i, u^\alpha)$. We induce a coordinate system (U^∞, y^∞) on $J^\infty\pi$ as follows:

$$U^\infty = \{j^\infty_p\phi : \phi(p) \in U, \phi \in \Gamma_p(\pi)\},\ y^\infty = (x^i, u^\alpha, u^\alpha_I) \quad \text{for } 1 \leq i \leq m,\ 1 \leq \alpha \leq n \ \text{ and } \ 1 \leq |I| < \infty \tag{3.1.4}$$

where:

$$\begin{array}{cccc} x^i: & U^\infty & \to & \mathbb{R} \\ & j^\infty_p\phi & \mapsto & x^i(p) \end{array} \qquad \begin{array}{cccc} u^\alpha: & U^\infty & \to & \mathbb{R} \\ & j^\infty_p\phi & \mapsto & u^\alpha(\phi(p)) \end{array} \qquad \begin{array}{cccc} u^\alpha_I: & U^\infty & \to & \mathbb{R} \\ & j^\infty_p\phi & \mapsto & \left.\frac{\partial^{|I|}(u^\alpha \circ \phi)}{\partial x^I}\right|_p. \end{array}$$

Then, the coordinate functions y^∞ in (3.1.4) can be represented by:

$$y^\infty = (x^i, u^\alpha, u^\alpha_{x^i}, \ldots, u^\alpha_{x^{i_1}x^{i_2}\ldots x^{i_k}}, \ldots) \quad \text{for } 1 \leq i \leq m,\ 1 \leq \alpha \leq n \ \text{ and } \ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq m. \tag{3.1.5}$$

$J^\infty\pi$ is an infinite-dimensional manifold, for this generalization see [3]; the projection $\pi_\infty \colon J^\infty\pi \to M$, $j^\infty_p\phi \mapsto p$ makes $(J^\infty\pi, \pi_\infty, M)$ a *bundle*. As above, we also have the projections:

$$\begin{array}{cccc} \pi_{\infty,0}: & J^\infty\pi & \to & E \\ & j^\infty_p\phi & \mapsto & \phi(p) \end{array} \qquad\qquad \begin{array}{cccc} \pi_{\infty,l}: & J^\infty\pi & \to & J^l\pi \\ & j^\infty_p\phi & \mapsto & j^l_p\phi \end{array}$$

Let $f\colon J^\infty\pi \to \mathbb{R}$ be a ***real-valued function*** on $J^\infty\pi$. We say that f is ***smooth*** if f is factored through some finite-order Jet bundle $J^k\pi$, that is, if there is a function $\widehat{f}\colon J^k\pi \to \mathbb{R}$ such that $f = \widehat{f}\circ\pi_{\infty,k}$, where $\pi_{\infty,k}\colon J^\infty\pi \to J^k\pi$ is such that $\pi_{\infty,k}(j_p^\infty\phi) = j_p^k\phi$. Therefore, if $f \in C^\infty(J^\infty\pi)$ and $f = \widehat{f}\circ\pi_{\infty,k}$ for $\widehat{f}$ on $J^k\pi$, then on each coordinate neighborhood U^∞ and each point $j^\infty\phi \in U^\infty$ with k-jet coordinates given by the expression in (3.1.2), we have

$$f(j_p^\infty) = \widehat{f}(x^i, u^\alpha, u_I^\alpha), \quad \text{where } \ |I| \le k.$$

We call k the order of f. if f ***is of order k***, then it also is of any order greater that k

Let ϕ be a local section of π with domain $W \subset M$; the ***infinite prolongation of ϕ*** is the map:

$$\begin{array}{rccc} j^\infty\phi: & W & \to & J^\infty\pi \\ & p & \mapsto & j_p^\infty\phi. \end{array}$$

Note that $j^\infty\phi$ is a local section of $(J^\infty\pi, \pi_\infty, M)$. We say that $j^\infty\phi$ is an ***holonomic section***, to distinguish it from arbitrary section which do not need to be prolongations of local sections of (E, π, M). To find the coordinate representation of $j^\infty\phi$, we must examine its composition with the coordinate functions u^α and u_I^α for $1 \le |I| < \infty$:

$$u^\alpha \circ j^\infty\phi(p) = u^\alpha(j_p^\infty\phi) = u^\alpha \circ \phi(p),$$

$$u_I^\alpha \circ j^\infty\phi(p) = u_I^\alpha(j_p^\infty\phi) = \left.\frac{\partial^{|I|}(u^\alpha\circ\phi)}{\partial x^I}\right|_p.$$

The coordinate representation of $j^\infty\phi$ is therefore $\left(u^\alpha\circ\phi, \frac{\partial^{|I|}(u^\alpha\circ\phi)}{\partial x^I}\right)$ for $1 \le |I| < \infty$.

Henceforth, if $|I| = 0$ then $\frac{\partial^{|I|}}{\partial x^I}$ is the identity operator. Thus, in (3.1.2) and (3.1.4) the coordinate functions would be written respectively as:

$$y^k = (x^i, u_I^\alpha) \ \text{ for } 0 \le |I| \le k, \quad y^\infty = (x^i, u_I^\alpha) \ \text{ for } 0 \le |I| < \infty.$$

A ***smooth vector field*** X on $J^\infty\pi$ is a formal series of the form:

$$X = \sum_{i=1}^m A_i \frac{\partial}{\partial x^i} + \sum_{|I|=0}^\infty \sum_{\alpha=1}^n B_I^\alpha \frac{\partial}{\partial u_I^\alpha},$$

in which A_i, B_I^α are smooth functions on U^∞, where (U^∞, y^∞) is a coordinate system on $J^\infty\pi$.

A differential r-form ω on $J^\infty\pi$ has not necessarily a finite number of terms [22] (Page 206, Example 7.2.13), but we will use a more restricted notion of differential forms (which is in agreement with our notion of smooth functions). We say that ω is called a r-***form of finite order k***, if it is the pull-back of a differential form on $J^k\pi$ this is $\omega \in \pi_{\infty,k}^*(\Omega^r(J^k\pi))$ for some $k \in \mathbb{N}$, in which

$$\begin{array}{rcccccl} \pi_{\infty,k}^*: & \Omega^r(J^k\pi) & \to & \Omega^r(J^\infty\pi) & & & \\ & \widehat{\omega} & \mapsto & \pi_{\infty,k}^*(\widehat{\omega}): & J^\infty\pi & \to & \Lambda^r(T^*J^\infty\pi) \\ & & & & p & \mapsto & \widehat{\omega}_p(D\pi_{\infty,k}(j_p^\infty\phi)(v_1), \ldots, D\pi_{\infty,k}(j_p^\infty\phi)(v_r)). \end{array}$$

where $D\pi_{\infty,k}$ is the differential of $\pi_{\infty,k}$, it is calculated formally, writing v_i as the derivative of a curve, $v_i \in T_{j_p^\infty\phi}J^\infty\pi$ for $1 \leq i \leq r$. .

In local coordinates a 1-form σ on $J^\infty\pi$ of order k, for instance, takes the form:

$$\sigma = \sum_{i=1}^{m} \sigma_i dx^i + \sum_{|I|=0}^{k} \sum_{\alpha=1}^{n} \varepsilon_\alpha^I du_I^\alpha, \tag{3.1.6}$$

where σ_i and ε_α^I are smooth functions on U^∞ of order $\leq k$ and $|I| \leq k$.

Also, a ***p-form ω on $J^\infty\pi$ of order*** k in local coordinates, is a finite sum of terms of the type:

$$Adx^{i_1} \wedge \cdots \wedge dx^{i_a} \wedge du_{I_1}^{\alpha_1} \wedge \cdots \wedge du_{I_b}^{\alpha_b},$$

where $a + b = p$ and the coefficient A is a smooth function on U^∞ which depends of x^i and u_I^α, with $|I| \leq k$ and $|I_j| \leq k$ for $1 \leq j \leq b$.

For example, if the coordinates of $J^\infty\pi$ are $y^\infty = (x, u, u_x, u_{xx}, \ldots u_{xxx\cdots x}, \ldots)$ then, $u_{xx}dx \wedge du_x$ is a 2-form and $u_x du_{xxx} \wedge dx$ is a 2-form of order 3.

The exterior differentiation $d\colon \Omega^p(J^\infty\pi) \to \Omega^{p+1}(J^\infty\pi)$ is defined via representatives: If ω is a p-form on $J^\infty\pi$ represented by $\widehat{\omega}$ on $J^k\pi$ for some k, then $d\omega$ is the $(p+1)$-form on $J^\infty\pi$ represented by $d\widehat{\omega}$. In local coordinates, the differential df of a function f of order k is given by:

$$df = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} dx^i + \sum_{|I|=0}^{k} \sum_{\alpha=1}^{n} \frac{\partial f}{\partial u_I^\alpha} du_I^\alpha. \tag{3.1.7}$$

If $\sigma \in \Omega(J^\infty\pi)$, we say that σ is a **contact form** if the pull-back of σ by holonomic sections $j^\infty\phi$ satisfies that

$$(j^\infty\phi)^*(\sigma) = 0 \tag{3.1.8}$$

for all local section ϕ of E. In local coordinates a contact one-form may be written as a finite sum of the following kind:

$$\sigma = \sum_{|I|=0}^{k} \sum_{\alpha=1}^{n} \varepsilon_\alpha^I (du_I^\alpha - \sum_{i=1}^{m} u_{I+\mathbf{1}_i}^\alpha dx^i)$$

for some smooth functions ε_α^I on U^∞. In fact,

$$\begin{aligned}
Dj^\infty\phi(p)\left(\frac{\partial}{\partial x^j}\bigg|_p\right) &= \sum_{i=1}^{\infty} \frac{\partial(y^i \circ j^\infty\phi \circ x^{-1})}{\partial x^j}(x(p))\ \frac{\partial}{\partial y^i}\bigg|_{j_p^\infty\phi} \\
&= \sum_{i=1}^{m} \frac{\partial(x^i \circ j^\infty\phi \circ x^{-1})}{\partial x^j}(x(p))\ \frac{\partial}{\partial x^i}\bigg|_{j_p^\infty\phi} + \sum_{|I|=0}^{\infty} \sum_{\alpha=1}^{m} + \frac{\partial(u_I^\alpha \circ j^\infty\phi \circ x^{-1})}{\partial x^j}(x(p))\ \frac{\partial}{\partial u_I^\alpha}\bigg|_{j_p^\infty\phi} \\
&= \frac{\partial}{\partial x^j}\bigg|_{j_p^\infty\phi} + \sum_{|I|=0}^{\infty} \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^j}\bigg|_p \left(\frac{\partial(u_I^\alpha \circ \phi)}{\partial x^I}\right) \frac{\partial}{\partial u_I^\alpha}\bigg|_{j_p^\infty\phi} \\
&= \frac{\partial}{\partial x^j}\bigg|_{j_p^\infty\phi} + \sum_{|I|=0}^{\infty} \sum_{\alpha=1}^{n} u_{I+\mathbf{1}_j}^\alpha (j_p^\infty\phi)\ \frac{\partial}{\partial u_I^\alpha}\bigg|_{j_p^\infty\phi}.
\end{aligned}$$

We have from equality (3.1.6) and property (3.1.8) that:

$$0=(j^{\infty}\phi)^*(\sigma)(\Big(\frac{\partial}{\partial x^j}\Big|_p\Big)=\sigma_{j^{\infty}_p\phi}\Big(Dj^{\infty}\phi(p)\Big(\frac{\partial}{\partial x^j}\Big|_p\Big)\Big)=\sigma_j+\sum_{|I|=0}^{k}\sum_{\alpha=1}^{n}\varepsilon^I_\alpha\,(u^\alpha_{I+\mathbf{1_j}}(j^\infty_p\phi));$$

therefore,

$$\begin{aligned}\sigma&=\sum_{i=1}^{m}\Big(-\sum_{|I|=0}^{k}\sum_{\alpha=1}^{n}\sigma^I_\alpha\,(u^\alpha_{I+\mathbf{1_i}}(j^\infty_p\phi))\Big)dx^i+\sum_{|I|=0}^{k}\sum_{\alpha=1}^{n}\sigma^I_\alpha du^\alpha_I\\ &=\sum_{|I|=0}^{k}\sum_{\alpha=1}^{n}\varepsilon^I_\alpha\,(du^\alpha_I-\sum_{i=1}^{m}u^\alpha_{I+\mathbf{1_i}}dx^i).\end{aligned}$$

We define the ***basic contact one-forms***: θ^α_I as $\theta^\alpha_I=du^\alpha_I-\sum\limits_{i=1}^{m}u^\alpha_{I+\mathbf{1_i}}dx^i$, where $|I|=0,1,2,\ldots$, and we call $|I|$ the order of the contact form θ^α_I, even though this form is defined on a coordinate neighborhood of $J^{|I|+1}\pi$. So, we have:

$$\begin{aligned}d(\theta^\alpha_I)&=d(du^\alpha_I-\sum_{i=1}^{m}u^\alpha_{I+\mathbf{1_i}}dx^i)\\ &=-(\sum_{i=1}^{m}d(u^\alpha_{I+\mathbf{1_i}})\wedge dx^i)\\ &=-(\sum_{i=1}^{m}(\theta^\alpha_{I+\mathbf{1_i}}+\sum_{j=1}^{m}u^\alpha_{I+\mathbf{1_i}+\mathbf{1_j}}dx^j)\wedge dx^i)\\ &=-\sum_{i=1}^{m}\theta^\alpha_{I+\mathbf{1_i}}\wedge dx^i-\sum_{i=1}^{m}\sum_{j=1}^{m}u^\alpha_{I+\mathbf{1_i}+\mathbf{1_j}}dx^j\wedge dx^i\\ &=-\sum_{i=1}^{m}\theta^\alpha_{I+\mathbf{1_i}}\wedge dx^i.\end{aligned}$$

The last equality is a consequence of the fact that $u_{I+\mathbf{1_i}+\mathbf{1_j}}=u_{I+\mathbf{1_j}+\mathbf{1_i}}$ and $dx^j\wedge dx^i=-dx^i\wedge dx^j$.

For example, with respect to the coordinates (x,y,u) on $p_1\colon\mathbb{R}^3\to\mathbb{R}^2$, the contact 1-forms of order zero and one are:

In multi-index notation: $\theta=du-u_{(1,0)}\,dx-u_{(0,1)}\,dy$ and

$\theta_{(1,0)}=du_{(1,0)}-u_{(2,0)}\,dx-u_{(1,1)}\,dy;\quad \theta_{(0,1)}=du_{(0,1)}-u_{(1,1)}\,dx-u_{(0,2)}\,dy.$

And in symmetric indices notation: $\theta=du-u_x\,dx-u_y\,dy;$ and

$\theta_x=du_x-u_{xx}\,dx-u_{xy}\,dy;\quad \theta_y=du_y-u_{xy}\,dx-u_{yy}\,dy.$

We define ***the variational bicomplex*** after [1]. We consider $\Omega^{r,s}(J^\infty\pi)$ as the subspace of $\Omega^p(J^\infty\pi)$ such that if $\omega\in\Omega^{r,s}(J^\infty\pi)$ then it can be written as a finite sum of term of the form

$$Adx^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge\theta^{\alpha_1}_{I_1}\wedge\cdots\wedge\theta^{\alpha_s}_{I_s}\quad\text{with } r+s=p, \tag{3.1.9}$$

where A is a smooth function over U^∞.

We have the direct sum decomposition $\Omega^p(J^\infty\pi) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty\pi)$, and for any non-negative integers r and s such that $r+s=p$, we let $\pi^{r,s}\colon \Omega^p(J^\infty\pi) \to \Omega^{r,s}(J^\infty\pi)$ be the projection map. Moreover, the substitution of $du_I^\alpha = \theta_I^\alpha + \sum_{i=1}^{m} u_{I+\mathbf{1}_i}^\alpha dx^i$ into the equality (3.1.7) leads to:

$$df = \underbrace{\sum_{i=1}^{m}\Big(\frac{\partial f}{\partial x^i} + \sum_{|I|=0}^{k}\sum_{\alpha=1}^{n}\frac{\partial f}{\partial u_I^\alpha}u_{I+\mathbf{1}_i}^\alpha\Big)dx^i}_{d_H(f)\in\Omega^{1,0}(J^\infty\pi)} + \underbrace{\sum_{|I|=0}^{k}\sum_{\alpha=1}^{n}\frac{\partial f}{\partial u_I^\alpha}\theta_I^\alpha}_{d_V(f)\in\Omega^{0,1}(J^\infty\pi)} := d_H f + d_V f \tag{3.1.10}$$

The component of $d_H(f)$ with respect to dx^i, denoted by $D_{x_i}(f)$ is called the total derivative of f with respect to x^i. Thus,

$$d_H(f) = \sum_{i=1}^{m} D_{x^i}(f)dx^i \text{ and } D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k}\sum_{\alpha=1}^{n} u_{I+\mathbf{1}_i}^\alpha \frac{\partial}{\partial u_I^\alpha}. \tag{3.1.11}$$

In symmetric indices notation:

$$\begin{aligned} D_{x^i} &= \tfrac{\partial}{\partial x^i} + \sum_{\alpha=1}^{n}\Big(u_{x^i}^\alpha \tfrac{\partial}{\partial u^\alpha} + \sum_{1\le i_1\le m} u_{x^i x^{i_1}}^\alpha \tfrac{\partial}{\partial u_{x^{i_1}}^\alpha} + \sum_{1\le i_1\le i_2\le m} u_{x^i x^{i_1} x^{i_2}}^\alpha \tfrac{\partial}{\partial u_{x^{i_1}x^{i_2}}^\alpha} + \cdots \\ &\quad + \sum_{1\le i_1\le i_2\le\cdots\le i_k\le m} u_{x^i x^{i_1} x^{i_2}\ldots x^{i_k}}^\alpha \tfrac{\partial}{\partial u_{x^{i_1}x^{i_2}\ldots x^{i_k}}^\alpha}\Big) \end{aligned} \tag{3.1.12}$$

and

$$\begin{aligned} d_V(f) &= \sum_{\alpha=1}^{n}\Big(\tfrac{\partial f}{\partial u^\alpha}\theta^\alpha + \sum_{1\le i_1\le m} \tfrac{\partial f}{\partial u_{x^{i_1}}^\alpha}\theta_{x^{i_1}}^\alpha + \sum_{1\le i_1\le i_2\le m} \tfrac{\partial f}{\partial u_{x^{i_1}x^{i_2}}^\alpha}\theta_{x^{i_1}x^{i_2}}^\alpha + \cdots \\ &\quad + \sum_{1\le i_1\le i_2\le\cdots\le i_k\le m} \tfrac{\partial f}{\partial u_{x^{i_1}x^{i_2}\ldots x^{i_k}}^\alpha}\theta_{x^i x^{i_1}x^{i_2}\ldots x^{i_k}}^\alpha\Big). \end{aligned} \tag{3.1.13}$$

We have $d(\Omega^{r,s}(J^\infty\pi)) \subset \Omega^{r+1,s}(J^\infty\pi) \oplus \Omega^{r,s+1}(J^\infty\pi)$ and therefore we can write $d = d_H + d_V$, in which:

$$\begin{aligned} d_H:\ \ \Omega^{r,s}(J^\infty\pi) &\to \Omega^{r+1,s}(J^\infty\pi) \\ \omega &\mapsto \pi^{r+1,s}(d(\omega)) \end{aligned}$$

$$\begin{aligned} d_V:\ \ \Omega^{r,s}(J^\infty\pi) &\to \Omega^{r,s+1}(J^\infty\pi) \\ \omega &\mapsto \pi^{r,s+1}(d(\omega)) \end{aligned}$$

Of course, $d\circ d = 0$ implies that $d_H\circ d_H = d_V\circ d_V = 0$ and $d_H\circ d_V + d_V\circ d_H = 0$. The operators d_H and d_V are differentials called ***horizontal*** and ***vertical differentials*** respectively. As $d(dx^i)=0$, we have $d_H(dx^i)=0$ and also $d_V(dx^i)=0$; on the other hand $d_H(\theta_I^\alpha) = \sum_{i=1}^{m} dx^i\wedge\theta_{I+\mathbf{1}_i}^\alpha$, $d_V(\theta_I^\alpha)=0$ and $d_H(u_I^\alpha) = \sum_{i=1}^{m} u_{I+\mathbf{1}_i}^\alpha dx^i$, $d_V(u_I^\alpha)=\theta_I^\alpha$.

We conclude that, if $\omega = f dx^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge\theta_{I_1}^{\alpha_1}\wedge\cdots\wedge\theta_{I_s}^{\alpha_s}$ then its differential satisfies:

$$
\begin{aligned}
d(\omega) &= (d_H(f)+d_V(f))\wedge(dx^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge\theta^{\alpha_1}_{I_1}\wedge\cdots\wedge\theta^{\alpha_s}_{I_s}) + f\wedge d(dx^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge\theta^{\alpha_1}_{I_1}\wedge\cdots\wedge\theta^{\alpha_s}_{I_s})\\
&= \underbrace{d_H(f)\wedge(x^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge\theta^{\alpha_1}_{I_1}\wedge\cdots\wedge\theta^{\alpha_s}_{I_s}) + f\wedge d_H(dx^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge\theta^{\alpha_1}_{I_1}\wedge\cdots\wedge\theta^{\alpha_s}_{I_s})}_{d_H(w)\in\Omega^{r+1,s}(J^\infty\pi)}\\
&+\underbrace{d_V(f)\wedge(dx^{i_1}\wedge\cdots\wedge dx^{i_r}\wedge\theta^{\alpha_1}_{I_1}\wedge\cdots\wedge\theta^{\alpha_s}_{I_s})}_{d_V(\omega)\in\Omega^{r,s+1}(J^\infty\pi)}
\end{aligned}
$$

Now we introduce two algebraic structures that we will use later:

Definition 3.1.6. *A* ***bigraded dgm over a ring*** $\boldsymbol{k}$ *is a dgm* (M,d) *over* k*, together with a direct sum decomposition* $M=\bigoplus M^{p,q}$ *such that* $d(M^{p,q})\subset M^{p+1,q}\oplus M^{p,q+1}$.

The ***bidegree*** of $x\in M^{p,q}$ is $|x|=(p,q)$, and its ***total degree*** is $|x|=p+q$. A ***bigraded dga over a ring*** $\boldsymbol{k}$ is a *dga* (A,d) over k, which is a *bigraded dgm* over k and satisfies that $A^{p,q}A^{p',q'}\subset A^{p+p',q+q'}$. The base ring k is consider as a *bigraded dga* of bidegree(0,0).

Definition 3.1.7. *A* **double dgm** *is a bigraded dgm* (M,d) *over* k *equipped with two differentials* $d'\colon M^{p,q}\to M^{p+1,q}$ *and* $d''\colon M^{p,q}\to M^{p,q+1}$*, such that* $d=d'+d''$ *and* $d'd''+d''d'=0$

In this way, a **double (c)dga** is a *bigraded (c)dga* which is a *double dgm*.

Let $\pi\colon E\to M$ be a bundle and let $J^\infty\pi$ be the infinite Jet manifold of π. The space of the differential forms on $J^\infty\pi$ with exterior product and exterior differential is a *bigraded cdga* $(\Omega(J^\infty\pi),d)$ over $\mathbb{R}$ with $\Omega^p(J^\infty\pi)=\bigoplus_{r+s=p}\Omega^{r,s}(J^\infty\pi)$; this *bigraded cdga* equipped with the horizontal differential d_H and the vertical differential d_V is a *double dgm*, which is denoted by $(\Omega^{*,*}(J^\infty\pi),d_H,d_V)$ and this is called ***The variational bicomplex*** for the bundle $\pi\colon E\to M$. We have the diagram

$$
\begin{array}{ccccccccccccc}
 & & & & \vdots & & \vdots & & \vdots & \cdots & \vdots & & \vdots\\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V}\\
 & & 0 & \longrightarrow & \Omega^{0,2}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{1,2}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{2,2}(J^\infty\pi) & \xrightarrow{d_H}\cdots\xrightarrow{d_H} & \Omega^{n-1,2}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{n,2}(J^\infty\pi)\\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V}\\
 & & 0 & \longrightarrow & \Omega^{0,1}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{1,1}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{2,1}(J^\infty\pi) & \xrightarrow{d_H}\cdots\xrightarrow{d_H} & \Omega^{n-1,1}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{n,1}(J^\infty\pi)\\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V}\\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^{0,0}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{1,0}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{2,0}(J^\infty\pi) & \xrightarrow{d_H}\cdots\xrightarrow{d_H} & \Omega^{n-1,0}(J^\infty\pi) & \xrightarrow{d_H} & \Omega^{n,0}(J^\infty\pi)
\end{array}
$$

Figure 3.1.

3.2 V-Valued Differential Form

Now we are interested in relating the so-called Linear Gauge Complex [12, 13] to the twisting matrices of the chapter 2. This complex uses $\mathfrak{g}$-valued differential forms as elements, where $\mathfrak{g}$ is a Lie algebra. Precise definitions follow.

Definition 3.2.1. *Let E, M be smooth manifolds, and $\pi\colon E \to M$ a surjective smooth function. We say (E, π, M) is a **vector bundle of rank n** if for every $p \in M$*

1. *$E_p = \pi^{-1}(p)$ has the structure of an n-dimensional real vector space.*
2. *For each $p \in M$ there exists an open neighborhood U of p and a diffeomorphism*
$$t_U\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$$
such that for each point $q \in U$ its restriction to $\pi^{-1}(p)$ gives a linear isomorphism:
$$t_U(\pi^{-1}(p)) = \{p\} \times \mathbb{R}^n.$$

Note that a vector bundle is a bundle, and t_U is a *local trivialization*. Then we have the definition of sections as in Definition (3.1.3).

Now suppose there are given two open sets U_α, U_β with $p \in U_\alpha \cap U_\beta$, and local trivializations $t_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n,\ t_\beta\colon \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^n$; then the composition map:
$$t_\alpha \circ t_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$
can be written in the form:
$$t_\alpha \circ t_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)(v)).$$

Here $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$, this is $t_\alpha \circ t_\beta^{-1}\colon \{p\} \times \mathbb{R}^n \to \{p\} \times \mathbb{R}^n$ is a linear isomorphism (where $GL(n, \mathbb{R})$ is the Lie group of invertible real matrices $n \times n$). The map $g_{\alpha\beta}$ is smooth and expresses the "shift" of the two trivializations on $U_\alpha \cap U_\beta$. It is called a **transition function**.

Definition 3.2.2. *Let M be a smooth manifold and V a fixed real vector space of dimension n. $(M \times V, p_1, M)$ is a **trivial vector bundle of rank n** over M.*

As we saw in the Remark (3.1.1-2), we can identify the set of all sections of p_1 with $C^\infty(M, V)$, the set of continuous functions $M \to V$. Besides, if $g \in C^\infty(M, V)$ then $g(p) = \sum\limits_{i=1}^{n} \lambda_i(p) v_i$, where $\lambda_i(p) \in \mathbb{R}$. Thus we define smooth functions $\boldsymbol{\lambda}_i\colon M \to \mathbb{R}$ such that $\boldsymbol{\lambda}_i(p) = \lambda_i(p)$ for each $i = 1, \ldots, n$; therefore
$$g = \sum_{i=1}^{n} \boldsymbol{\lambda}_i v_i. \tag{3.2.1}$$
By identification of v_i with the constant map $g_i\colon M \to V$ such that $g_i(p) = v_i$ for $i = 1, \ldots, n$, we can write
$$g = \sum_{i=1}^{n} \boldsymbol{\lambda}_i g_i. \tag{3.2.2}$$

Theorem 3.2.1. *Let V be a real vector space of dimension n and M be a smooth manifold of dimension m. Then $C^\infty(M,V) \cong C^\infty(M) \otimes_{\mathbb{R}} V$.*

Proof. This isomorphism can be obtained as follows: for every $p \in M$, let us define the map

$$\begin{array}{rccccc} \gamma: & C^\infty(M) \times V & \to & C^\infty(M) \otimes_{\mathbb{R}} V & & \\ & (f,v) & \mapsto & \gamma(f,v): \ M & \to & V \\ & & & p & \mapsto & f(p)v \end{array}$$

since $C^\infty(M,V)$ is a real vector space and γ is a bilinear map, then by applying the universal property of the tensor product for vector space, there exists a unique linear map $\widetilde{\gamma}$ such that the following diagram is commutative:

$$\begin{array}{ccc} C^\infty(M) \times V & \xrightarrow{\Pi} & C^\infty(M) \otimes_{\mathbb{R}} V \\ & \searrow^{\gamma} & \downarrow \widetilde{\gamma} \\ & & C^\infty(M,V) \end{array}$$

let $g \in C^\infty(M,V)$; then, from the equation (3.2.1) $g = \sum_{i=1}^{n} \boldsymbol{\lambda}_i v_i$ and $\widetilde{\gamma}$ is an isomorphism with inverse

$$\begin{array}{rccc} \widetilde{\rho}: & C^\infty(M,V) & \to & C^\infty(M) \otimes_{\mathbb{R}} V \\ & g & \mapsto & \sum_{i=1}^{n} \boldsymbol{\lambda}_i \otimes v_i. \end{array}$$

Indeed, $\widetilde{\rho}$ is linear map and satisfies that: $\widetilde{\gamma} \circ \widetilde{\rho}(g) = \widetilde{\gamma}(\sum_{i=1}^{n} \boldsymbol{\lambda}_i \otimes v_i) = \sum_{i=1}^{n} \gamma(\boldsymbol{\lambda}_i, v_i) = g$, the last equality since $\gamma(\boldsymbol{\lambda}_i, v_i)(p) = \boldsymbol{\lambda}_i(p) v_i$. And, $\widetilde{\rho} \circ \widetilde{\gamma}(f \otimes v_i) = \widetilde{\rho}\,(f v_i) = f \otimes v_i$.

□

This Theorem allows us to conclude, that for the trivial bundle $(M \times V, p_1, M)$ the set of all sections of p_1 satisfy that $\Gamma(p_1) \cong C^\infty(M) \otimes_{\mathbb{R}} V$.

Remark 3.2.1.

1. *The tangent bundle $\tau_M : TM \to M$ and the cotangent bundle $\tau_M^* : T^*M \to M$ (see Remark 3.1.1) are vector bundles, with local trivializations t_U and $\mathbf{t}_U$ given respectively as follows: If we have a coordinate system $\varphi : U \to \mathbb{R}^m$ and coordinate functions $\varphi_i : U \to \mathbb{R}$ then*

$$\begin{array}{rccc} t_U: & \tau_M^{-1}(U) & \to & U \times \mathbb{R}^m \\ & v_p & \mapsto & (\tau_M(v_p), D_p\varphi(v_p)) \end{array}$$

and

$$\begin{array}{rccc} \mathbf{t}_U: & (\tau_M^*)^{-1}(U) & \to & U \times \mathbb{R}^m \\ & \eta_p & \mapsto & (\tau_M^*(\eta_p), \partial_\varphi(\eta_p)) \end{array}$$

where

$$\begin{array}{rccl} \partial_\varphi : & T_p^*U & \to & \mathbb{R}^m \\ & \eta_p & \mapsto & \left(\frac{\partial f}{\partial \varphi_1}\Big|_p, \ldots, \frac{\partial f}{\partial \varphi_m}\Big|_p \right) \end{array}$$

for $f\colon U \to R$ a smooth function such that $D_p f = \eta$ and $\frac{\partial f}{\partial \varphi_i}\Big|_p = D_{\varphi(p)}(f \circ \varphi^{-1})(e_i)$. For example if $\eta_p = \delta_{ij}\colon T_pU \to \mathbb{R}^n$, this is $\delta_{ij}(\frac{\partial}{\partial \varphi^i}(p)) = 1$ and $\delta_{ij}(\frac{\partial}{\partial \varphi^j}(p)) = 0$ for $i \neq j$, then $f = \varphi_i$.

2. *Let (E, π, M) and (F, ρ, M) be vector bundles with fibres E_p, F_p respectively. The* ***tensor bundle of*** *π and ρ is the vector bundle with fibres $E_p \otimes F_p$ and it is denoted $(E \otimes F, \pi \otimes \rho, M)$. Moreover, the* ***r-fold alternating product*** *of π is the vector bundle with fibres $\Lambda^r(E_p)$ (see 1.2.1-5) and it is denoted by $(\Lambda^r(E), \Lambda^r(\pi), M)$. So, as in the Remark (3.1.1), the r-exterior bundle $(\Lambda^r(T^*M), \Lambda^r(\tau_M^*), M)$ is a vector bundle.*

Definition 3.2.3. *Let V be a n-dimensional real vector space, and $(M \times V, p_1, M)$ be a trivial vector bundle of rank n over a m-dimensional manifold M. A r-form on M with values in V, or a* ***V-valued differential form*** *of degree r, is a section of the vector bundle*

$$(\Lambda^r(T^*M) \otimes (M \times V), \Lambda^r(\tau_M^*) \otimes p_1, M).$$

The set of all r-forms with values in V is denoted by $\Omega^r(M, V)$. This is a generalization of the set of the sections of the vector bundle $(\Lambda^r(T^*M), \Lambda^r(\tau_M^*), M)$ which we denoted by $\Omega^r(M)$.

We now introduce a new notation: for W, V vector spaces, $A_r(W, V)$ is the vector space of all alternating r-multilinear maps, that is, the space of r-multilinear maps $f\colon \underbrace{W \times W \times \cdots \times W}_{r-\text{times}} \to V$ such that for all σ in the permutation group S_r,

$$f(w_{\sigma(1)}, \ldots, w_{\sigma(r)}) = (sgn\sigma) f(w_1, \ldots, w_r),$$

in which $sgn\sigma$ is the sign of σ.

In the case $V = \mathbb{R}$, we denoted this set in the Example (1.2.1) item 5, by $A_r(W)$.

With respect to the notations introduced so far, we prove the following theorem, which describes the r-forms on M with values in V in terms of the r-forms on M and the sections of the trivial bundle $(M \times V, p_1, M)$.

Theorem 3.2.2. *Let V be a real vector space of dimension n and M be a smooth manifold of dimension m. Then, for every $p \in M$, we have:*

1. *$A_r(T_pM) \otimes V \cong A_r(T_pM, V)$ of real vector spaces,*
2. *$\Omega^r(M) \otimes_{C^\infty(M)} C^\infty(M, V) \cong \Omega^r(M, V)$ of $C^\infty(M)$-modules,*
3. *$\Omega^r(M) \otimes_{\mathbb{R}} V \cong \Omega^r(M, V)$ of real vector spaces.*

Proof. The first isomorphism can be obtained as follows: for every $p \in M$, let us define the map

$$\begin{array}{rccccc} \varphi: & A_r(T_pM) \times V & \to & A_r(T_pM, V) & & \\ & (h, v) & \mapsto & \varphi(h,v): \ T_pM \times \cdots \times T_pM & \to & V \\ & & & (\xi_1, \dots, \xi_r) & \mapsto & h(\xi_1, \dots, \xi_r)v. \end{array}$$

Since $A_r(T_pM, V)$ is a vector space and φ is a bilinear map, then by applying the universal property of the tensor product for vector spaces, there exists a unique linear map $\widetilde{\varphi}$ such that the following diagram is commutative:

$$\begin{array}{ccc} A_r(T_pM) \times V & \xrightarrow{\ \Pi\ } & A_r(T_pM) \otimes V \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \widetilde{\varphi}} \\ & & A_r(T_pM, V) \end{array}$$

where $\Pi(h, v) = h \otimes v$.

Now, let $\{v_i\}_{i=1}^n$ be a fixed basis of V. If $f \in A_r(T_pM, V)$, then

$$f(\xi_1, \dots, \xi_r) = \sum_{i=1}^{n} \alpha_i(\xi_1, \dots, \xi_r) v_i, \quad \text{where} \quad \alpha_i(\xi_1, \dots, \xi_r) \in \mathbb{R}.$$

Thus we define for each $i = 1, \dots, n$, the maps

$$f_i\colon \underbrace{T_pM \times \cdots \times T_pM}_{r-times} \to \mathbb{R}, \quad \text{such that} \quad f_i(\xi_1, \dots, \xi_r) = \alpha_i(\xi_1, \dots, \xi_r);$$

therefore

$$f = \sum_{i=1}^{n} f_i v_i, \tag{3.2.3}$$

where each $f_i \in A_r(T_pM)$. In fact, as f is an alternating map

$$f(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}) = (sgn\sigma) f(\xi_1, \dots, \xi_r), \ \text{for each } \xi_{\sigma(1)}, \dots, \xi_{\sigma(r)} \in \underbrace{T_pM \times \cdots \times T_pM}_{r-times};$$

then, $\sum\limits_{i=1}^{n} f_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}) v_i = \sum\limits_{i=1}^{n} (sgn\sigma) f_i(\xi_1, \dots, \xi_r) v_i$.

Hence, $f_i(\xi_{\sigma(1)}, \dots, \xi_{\sigma(r)}) = (sgn\sigma) f_i(\xi_1, \dots, \xi_r)$.

We are able to show that $\widetilde{\varphi}$ is an isomorphism. From the last fact we define the map $\widetilde{\psi}$ by property (3.2.3):

$$\begin{array}{rccc} \widetilde{\psi}: & A_r(T_pM, V) & \to & A_r(T_pM) \otimes V \\ & f & \mapsto & \sum\limits_{i=1}^{n} f_i \otimes v_i \end{array}$$

this map is linear and it satisfies: $\widetilde{\varphi} \circ \widetilde{\psi}(f) = \widetilde{\varphi}(\sum\limits_{i=1}^{n} f_i \otimes v_i) = \sum\limits_{i=1}^{n} \varphi(f_i, v_i) = f$; the last equality holds $\sum\limits_{i=1}^{n} \varphi(f_i, v_i)(\xi_1, \dots, \xi_r) = \sum\limits_{i=1}^{n} f_i(\xi_1, \dots, \xi_r) v_i$. And, $\widetilde{\psi} \circ \widetilde{\varphi}(h \otimes v_i) = \widetilde{\psi}(\varphi(h, v_i)) = h \otimes v_i$, since $\varphi(h, v_i)(\xi_1, \dots, \xi_r) = h(\xi_1, \dots, \xi_r) v_i$. Therefore, $\widetilde{\psi} = (\widetilde{\varphi})^{-1}$, and $A_r(T_pM, V) \cong A_r(T_pM) \otimes V$.

On the other hand, by definition of vector tensor bundle

$$\Lambda^r(T^*M) \otimes (M \times V) = \bigcup_{p \in M} \Lambda^r(T_p^*M) \otimes \big(\{p\} \times V\big)$$

and as $\Lambda^r(T_p^*M) \cong A_r(T_pM)$, we have $\Lambda^r(T^*M) \otimes (M \times V) \cong \bigcup_{p\in M} A_r(T_pM) \otimes V$; moreover by the isomorphism φ we can identify

$$\Lambda^r(T^*M) \otimes (M \times V) \cong \bigcup_{p \in M} A_r(T_pM, V). \tag{3.2.4}$$

For the second part of the theorem, we define the following map:

$$\begin{array}{llllllll}
\eta : & \Omega^r(M) \times C^\infty(M,V) & \to & \Omega^r(M,V) & & & & \\
 & (\omega, g) & \mapsto & \eta(\omega,g) : M & \to & \bigcup_{p\in M} A_r(T_pM,V) & & \\
 & & & p & \mapsto & \eta(\omega,g)_p : T_pM \times \cdots \times T_pM & \to & V \\
 & & & & & (\xi_1,\ldots,\xi_r) & \mapsto & \omega_p(\xi_1,\ldots,\xi_r)g(p).
\end{array}$$

This map is $C^\infty(M)$-bilinear. Hence, by the universal property of the tensor product for modules there exists a unique linear map $\widetilde{\eta}$ such that the following is a commutative diagram:

$$\begin{array}{ccc}
\Omega^r(M) \times C^\infty(M,V) & \xrightarrow{\ \Pi\ } & \Omega^r(M) \otimes_{C^\infty(M)} C^\infty(M,V) \\
 & \searrow^{\eta} & \downarrow \widetilde{\eta} \\
 & & \Omega^r(M,V)
\end{array}$$

Now, let $\Theta \in \Omega^r(M,V)$; then, from equality (3.2.3) $\Theta_p = \sum_{i=1}^{n} (\Theta_p)_i v_i$ where $(\Theta_p)_i \in A_r(T_pM)$. Thus, we can define the r-form on M

$$\begin{array}{llll}
\Theta_i : & M & \to & \bigcup_{p\in M} A_r(T_pM) \\
 & p & \mapsto & (\Theta_p)_i .
\end{array}$$

By identification of v_i with the constant map $g_i \colon M \to V$ such that $g_i(p) = v_i$ for $i = 1, \ldots, n$, we can write

$$\Theta = \sum_{i=1}^{n} \Theta_i g_i .$$

Finally, we can deduce that $\widetilde{\eta}$ is an isomorphism with inverse

$$\begin{array}{llll}
\widetilde{\mu} : & \Omega^r(M,V) & \to & \Omega^r(M) \otimes_{C^\infty(M)} C^\infty(M,V) \\
 & \Theta & \mapsto & \sum_{i=1}^{n} \Theta_i \otimes g_i ,
\end{array}$$

since, $\widetilde{\mu}$ is $C^\infty(M)$-linear map and satisfies that $\widetilde{\eta} \circ \widetilde{\mu}(\Theta) = \widetilde{\eta}(\sum_{i=1}^{n} \Theta_i \otimes g_i) = \sum_{i=1}^{n} \eta(\Theta_i, g_i) = \Theta$, the last equality following from the fact that $\sum_{i=1}^{n} \eta(\Theta_i, g_i)_p(\xi_1, \ldots, \xi_r) = \sum_{i=1}^{n} (\Theta_i)_p(\xi_1, \ldots, \xi_r)(g_i)(p)$. We note that $(\Theta_i)_p = (\Theta_p)_i$ and $g_i(p) = v_i$. From the property (3.2.1) we have $\widetilde{\mu} \circ \widetilde{\eta}(\omega \otimes g) = \widetilde{\mu} \circ \widetilde{\eta}(\sum_{i=1}^{n} \omega \otimes \boldsymbol{\lambda}_i g_i)$, and so then $\widetilde{\mu} \circ \widetilde{\eta}(\omega \otimes g) = \sum_{i=1}^{n} \boldsymbol{\lambda}_i \widetilde{\mu}(\eta(\omega, g_i)) = \omega \otimes g$, the last equality, holding since $\eta(\omega, g_i) = \omega g_i$.

The thirst part of theorem follows from the second part and the Theorem (3.2.1):

$$\Omega^r(M,V) \cong \Omega^r(M) \otimes_{C^\infty(M)} C^\infty(M,V) \cong \Omega^r(M) \otimes_{C^\infty(M)} (C^\infty(M) \otimes_{\mathbb{R}} V) \cong \Omega^r(M) \otimes_{\mathbb{R}} V.$$

The last isomorphism is a consequence of the fact that $\mathbb{R}$ is a sub-ring of $C^\infty(M)$ via the constant functions and thus this isomorphism is obtained by mappings:

$\omega \otimes (f \otimes v) \mapsto \omega f \otimes v,$ for $\omega \otimes (f \otimes v) \in \Omega^r(M) \otimes_{C^\infty(M)} (C^\infty(M) \otimes_{\mathbb{R}} V)$, and

$\omega \otimes v \mapsto \omega \otimes (1 \otimes v),$ for $\omega \otimes v \in \Omega^r(M) \otimes_{\mathbb{R}} V$ and 1 is the constant map.

This ends the proof. □

The isomorphism given in (3.2.4) implies that a V-valued differential form of degree r say ω can be described as a map

$$\begin{array}{rccl} \omega: & M & \to & \bigcup_{p\in M} A_r(T_pM, V) \\ & p & \mapsto & \omega_p \colon T_pM \times \cdots \times T_pM \to V \end{array} \tag{3.2.5}$$

Moreover, the last theorem permits us to state that an element in $\Omega^r(M,V)$ is a finite linear combination of tensors of the form $\omega \otimes v$ where $\omega \in \Omega^r(M)$ and $v \in V$. We define the degree of $\omega \otimes v$ as the degree of ω.

It is possible to define the ***wedge product of two*** V**-valued forms**

$$\begin{array}{rcccl} \wedge: & \Omega^p(M,V) \times \Omega^q(M,V) & \to & \Omega^{p+q}(M, V \otimes V) & \\ & (\omega, \eta) & \mapsto & \omega \wedge \eta : M & \to \bigcup_{\mathfrak{z}\in M} A_r(T_\mathfrak{z}M, V\otimes V) \\ & & & \mathfrak{z} & \mapsto (\omega \wedge \eta)_\mathfrak{z} \end{array} \tag{3.2.6}$$

where

$$\begin{array}{rcl} (\omega\wedge\eta)_\mathfrak{z} : T_\mathfrak{z}M \times \cdots \times T_\mathfrak{z}M & \to & V \otimes V \\ (\xi_1, \dots, \xi_{p+q}) & \mapsto & \frac{1}{(p+q)!} \sum\limits_{\sigma\in S_{p+q}} sgn(\sigma)\omega_\mathfrak{z}(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}) \otimes \eta_\mathfrak{z}(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}). \end{array}$$

For any vector space V there is a natural ***exterior differential on the space of*** V***-valued forms***, Thus, if $\rho \otimes v \in \Omega^p(M,V)$

$$\mathbf{d}(\rho \otimes v) = d(\rho) \otimes v \;\in \Omega^{p+1}(M,V). \tag{3.2.7}$$

The exterior differential on V-valued forms is completely characterized by the usual relations for $\omega, \eta \in \Omega(M,V)$:

$$\mathbf{d}(\mathbf{d}(\omega)) = 0, \qquad \mathbf{d}(\omega + \eta) = \mathbf{d}(\omega) + \mathbf{d}(\eta).$$

$\mathfrak{g}$-valued differential forms

If $V = \mathfrak{g}$ is a Lie algebra one can define a bilinear operation on $\Omega^*(M,\mathfrak{g})$ by the composite $\Omega^p(M,\mathfrak{g}) \times \Omega^q(M,\mathfrak{g}) \to \Omega^{p+q}(M,\mathfrak{g}\otimes\mathfrak{g}) \to \Omega^{p+q}(M,\mathfrak{g})$. Here the first map is the above defined exterior product, and the second map is induced by the bracket $\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ of the Lie algebra. Thus, using (3.2.6), we set

$$\begin{array}{rcll} [\ ,\ ]: \quad \Omega^p(M,\mathfrak{g}) \times \Omega^q(M,\mathfrak{g}) & \to & \Omega^{p+q}(M,\mathfrak{g}) \\ (\rho\otimes a, \beta\otimes b) & \mapsto & (\rho\wedge\beta)\otimes[a,b]. \end{array} \tag{3.2.8}$$

Since, for $\mathfrak{z}\in M$

$$[\omega,\eta]_{\mathfrak{z}}(\xi_1,\dots,\xi_{p+q}) = \frac{1}{(p+q)!}\sum_{\sigma\in S_{p+q}} sgn(\sigma)[\omega_{\mathfrak{z}}(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)}),\eta_{\mathfrak{z}}(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)})].$$

Thus,

$$[\rho\otimes a,\beta\otimes b]_{\mathfrak{z}}(\xi_1,\dots,\xi_{p+q}) = \frac{1}{(p+q)!}\sum_{\sigma\in S_{p+q}} sgn(\sigma)[\rho_{\mathfrak{z}}(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)})a,\eta_{\mathfrak{z}}(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)})b] = (\rho\wedge\beta)_{\mathfrak{z}}[a,b].$$

Furthermore, for $\omega\in\Omega^1(M,\mathfrak{g})$, $\mathfrak{z}\in M$ and $\xi_1,\xi_2\in T_{\mathfrak{z}}M$ we have

$$[\omega,\omega]_{\mathfrak{z}}(\xi_1,\xi_2) = \frac{1}{2}\big([\omega_{\mathfrak{z}}(\xi_1),\omega_{\mathfrak{z}}(\xi_2)] - [\omega_{\mathfrak{z}}(\xi_2),\omega_{\mathfrak{z}}(\xi_1)]\big) = [\omega_{\mathfrak{z}}(\xi_1),\omega_{\mathfrak{z}}(\xi_2)]. \tag{3.2.9}$$

For $\omega,\eta,\gamma\in\Omega(M,\mathfrak{g})$, this bilinear operation on $\Omega(M,\mathfrak{g})$ satisfies:

$$[\omega,\eta] = (-1)^{|\omega||\eta|+1}[\eta,\omega], \tag{3.2.10}$$

$$(-1)^{|\omega||\gamma|}[\omega,[\eta,\gamma]] + (-1)^{|\eta||\omega|}[\eta,[\gamma,\omega]] + (-1)^{|\gamma||\eta|}[\gamma,[\omega,\eta]] = 0. \tag{3.2.11}$$

The first equality, since:

$$\begin{array}{rcl} [\rho\otimes a,\beta\otimes b] & = & (\rho\wedge\beta)[a,b] \\ & = & (-1)^{|\rho||\beta|+1}(\beta\wedge\rho)[b,a] \\ & = & (-1)^{|\omega||\eta|+1}[\beta\otimes b,\rho\otimes a]. \end{array}$$

And, the second equality because:

$$\begin{array}{rcl} A = (-1)^{|\omega||\gamma|}[\rho\otimes a,[\beta\otimes b,\varphi\otimes c]] & = & (-1)^{|\omega||\gamma|}[\rho\otimes a,\beta\wedge\varphi\otimes[b,c]] \\ & = & (-1)^{|\rho||\varphi|}\rho\wedge(\beta\wedge\varphi)\otimes[a,[b,c]] \end{array}$$

$$\begin{array}{rcl} B = (-1)^{|\eta||\omega|}[\beta\otimes b,[\varphi\otimes c,\rho\otimes a]] & = & (-1)^{|\eta||\omega|}[\beta\otimes b,\varphi\wedge\rho\otimes[c,a]] \\ & = & (-1)^{|\beta||\rho|}\beta\wedge(\varphi\wedge\rho)\otimes[b,[c,a]] \\ & = & (-1)^{|\beta||\rho|}(-1)^{|\varphi||\rho|}\beta\wedge(\rho\wedge\varphi)\otimes[b,[c,a]] \\ & = & (-1)^{|\varphi||\rho|}\rho\wedge(\beta\wedge\varphi)\otimes[b,[c,a]] \end{array}$$

$$\begin{array}{rcl} C = (-1)^{|\gamma||\eta|}[\varphi\otimes c,[\rho\otimes a,\beta\otimes b]] & = & (-1)^{|\gamma||\eta|}[\varphi\otimes c,\rho\wedge\beta\otimes[a,b]] \\ & = & (-1)^{|\varphi||\beta|}\varphi\wedge(\rho\wedge\beta)\otimes[c,[a,b]] \\ & = & (-1)^{|\varphi||\beta|}(-1)^{(|\rho|+|\beta|)|\varphi|}(\rho\wedge\beta)\wedge\varphi\otimes[c,[a,b]] \\ & = & (-1)^{|\rho||\varphi|}\rho\wedge(\beta\wedge\varphi)\otimes[c,[a,b]] \end{array}$$

$$A+B+C=(-1)^{|\rho||\varphi|}\left(\rho\wedge(\beta\wedge\varphi)\otimes([a,[b,c]]+[b,[c,a]]+[c,[a,b]])\right)=0.$$

As in the equality (3.2.7) we define the ***exterior differential*** on the space of $\mathfrak{g}$-valued forms. It satisfies that:

$$\mathbf{d}([\omega,\eta])=[\mathbf{d}(\omega),\eta]+(-1)^{|\omega|}[\omega,\mathbf{d}(\eta)],\ \text{where } \omega,\eta\in\Omega(M,\mathfrak{g}). \tag{3.2.12}$$

Indeed,

$$\begin{aligned}\mathbf{d}([\rho\otimes a,\beta\otimes b]) &= d(\rho\wedge\beta)\otimes[a,b]\\ &= (d(\rho)\wedge\beta+(-1)^{|\rho|}\rho\wedge d(\beta))\otimes[a,b]\\ &= d(\rho)\wedge\beta\otimes[a,b]+(-1)^{|\rho|}\rho\wedge d(\beta)\otimes[a,b]\\ &= [d(\rho)\otimes a,\beta\otimes b]+(-1)^{|\rho|}[\rho\otimes a,d(\beta)\otimes b].\end{aligned}$$

On the other hand, we note that $\Omega(M,\mathfrak{g})$ is not a differential graded algebra because the property (3.2.11) tells us that $[\ ,\ ]$ is not associative. But it is a differential graded Lie algebra, because it satisfies the following definition.

Definition 3.2.4. *Let $L=\{L^i\}_{i\geq 0}$ be a graded vector space over k. A **differential graded Lie algebra** is a graded module M together with a bilinear map of degree zero $[\ ,\ ]\colon L^i\otimes L^j\to L^{i+j}$, $x\otimes y\mapsto[x,y]$, and a differential $d\colon L^i\to L^{i+1}$ satisfying:*

(i) $[x,y]=(-1)^{|x||y|+1}[y,x]$,

(ii) $(-1)^{|x||z|}[x,[y,z]]+(-1)^{|y||x|}[y,[z,x]]+(-1)^{|z||y|}[z,[x,y]]=0$,

(iii) *and the graded Leibnitz rule* $d([x,y])=[d(x),y]+(-1)^{|x|}[x,d(y)]$.

Matrix-valued differential form

If V is a subalgebra of the space of all $n\times n$ real matrices $M_{n\times n}$, (the Lie algebra of $GL(n,\mathbb{R})$, the Lie group of invertible $n\times n$ matrices over $\mathbb{R}$), then $\Omega(M,V)$ has a multiplication induced by ordinary matrix multiplication:

$$\begin{aligned}\wedge:\quad \Omega^p(M,V)\times\Omega^q(M,V) &\to \Omega^{p+q}(M,V)\\ (\rho\otimes A,\beta\otimes B) &\mapsto (\rho\wedge\beta)\otimes AB.\end{aligned}$$

Here, we interpret the function $\wedge$ as the composite of exterior product given by in (3.2.6) and the map induced by the matrix multiplication, this is $\Omega^p(M,V)\times\Omega^q(M,V)\to\Omega^{p+q}(M,V\otimes V)\to\Omega^{p+q}(M,V)$. In fact,

$$(\omega\wedge\eta)_{\mathfrak{z}}(\xi_1,\ldots,\xi_{p+q})=\frac{1}{(p+q)!}\sum_{\sigma\in S_{p+q}} sgn(\sigma)\ \omega_{\mathfrak{z}}\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})\ \eta_{\mathfrak{z}}(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)}). \tag{3.2.13}$$

Thus,

$$((\rho\otimes A)\wedge(\beta\otimes B))_{\mathfrak{z}}(\xi_1,\ldots,\xi_{p+q})=\frac{1}{(p+q)!}\sum_{\sigma\in S_{p+q}} sgn(\sigma)\rho_{\mathfrak{z}}(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})A\,\eta_{\mathfrak{z}}(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)})B=(\rho\wedge\beta)_{\mathfrak{z}}AB.$$

We note that $\wedge$ is associative and it has identity element $1\otimes I$, where $1\in\Omega^0(M)$ is a function constant, but this multiplication is not necessarily commutative

$$(\rho\otimes A)\wedge(\beta\otimes B)=\rho\wedge\beta\otimes AB=(-1)^{|\rho||\beta|}\beta\wedge\rho\otimes AB.$$

Thus with the exterior differential defined in the equation (3.2.8) we have $(\Omega(M,V),\wedge,d)$ is a *dga*, since:

$$\mathbf{d}(\omega\wedge\eta)=\mathbf{d}(\omega)\wedge\eta+(-1)^{|\omega|}\omega\wedge\mathbf{d}(\eta). \tag{3.2.14}$$

Indeed,

$$\begin{aligned}\mathbf{d}(\rho\otimes A\wedge\eta\otimes B) &= d(\rho\wedge\beta)\otimes AB\\ &= (d(\rho)\wedge\beta+(-1)^{|\rho|}\rho\wedge d(\beta))\otimes AB\\ &= \mathbf{d}(\rho\otimes A)\wedge(\beta\otimes B)+(-1)^{|\omega|}(\rho\otimes A)\wedge\mathbf{d}(\beta\otimes B).\end{aligned}$$

In this case, we can also define the Lie bracket on V by $[A,B]:=AB-BA$ where AB and BA denote the usual matrix multiplication. Moreover $(\Omega(M,V),\wedge,d)$ is a differential graded Lie algebra. And the bracket on $\Omega(M,V)$ defined by the function (3.2.8) satisfies

$$[\omega,\eta]=\omega\wedge\eta-(-1)^{|\omega||\eta|}\eta\wedge\omega, \tag{3.2.15}$$

$$[\gamma,\omega\wedge\eta]=[\gamma,\omega]\wedge\eta+(-1)^{|\gamma||\omega|}\omega\wedge[\gamma,\eta]. \tag{3.2.16}$$

Therefore, If $\omega\in\Omega^1(M,V)$ then $[\omega,\omega]=2(\omega\wedge\omega)$.

The first equality holds because,

$$\begin{aligned}[\rho\otimes A,\beta\otimes B]=\rho\wedge\beta\otimes[A,B] &= \rho\wedge\beta\otimes(AB-BA)\\ &= \rho\wedge\beta\otimes AB-\rho\wedge\beta\otimes BA\\ &= (\rho\otimes A)\wedge(\beta\otimes B)-(-1)^{|\rho||\beta|}(\beta\wedge\rho)\otimes BA\\ &= (\rho\otimes A)\wedge(\beta\otimes B)-(-1)^{|\rho||\beta|}(\beta\otimes B)\wedge(\rho\otimes A).\end{aligned}$$

And, using (3.2.15), we have the second equality:

$$\begin{aligned}[\gamma,\omega\wedge\eta] &= \gamma\wedge(\omega\wedge\eta)-(-1)^{|\gamma||\omega\wedge\eta|}(\omega\wedge\eta)\wedge\gamma\\ &= (\gamma\wedge\omega)\wedge\eta-(-1)^{|\gamma||\omega\wedge\eta|}\omega\wedge(\eta\wedge\gamma)\\ &= ([\gamma,\omega]+(-1)^{|\gamma||\omega|}\omega\wedge\gamma)\wedge\eta-(-1)^{|\gamma|(|\omega|+|\eta|)}\omega\wedge([\eta,\gamma]+(-1)^{|\eta||\gamma|}\gamma\wedge\eta)\\ &= [\gamma,\omega]\wedge\eta-(-1)^{|\gamma|(|\omega|+|\eta|)}\omega\wedge[\eta,\gamma]\\ &= [\gamma,\omega]\wedge\eta-(-1)^{|\gamma|(|\omega|+|\eta|)}(-1)^{|\eta||\gamma|+1}\omega\wedge[\gamma,\eta]\\ &= [\gamma,\omega]\wedge\eta+(-1)^{|\gamma||\omega|}\omega\wedge[\gamma,\eta]\end{aligned}$$

Remark 3.2.2. *If V is a subalgebra of $M_{n\times n}$A matrix-valued differential form can be expressed as a matrix whose entries are differential forms on some manifold M, in the following precise sense: Let $(a_{ij})\in V$ and $\rho\in\Omega(M)$; if $\omega=\rho\otimes(a_{ij})\in\Omega(M,V)$, then the component $\omega_{ij}=\rho a_{ij}$. Therefore the components ij of the matrix exterior product and the matrix differential are respectively*

$$(\omega\wedge\eta)_{ik}=\sum_j\omega_{ij}\wedge\eta_{jk}\text{ and }(d\omega)_{ij}=(\mathbf{d}\omega_{ij}).$$

3.3 Submanifolds of $J^{\infty}\pi$ determined by a finite system of PDEs of kth order

In this section we observe that a kth order differential equation (hereafter understood as either a scalar equation or finite system) $\Xi = 0$, in which Ξ depends on independent variable $x_1, \ldots, x_m$, dependent variables $u^1, \ldots, u^n$ and a finite number of partial derivatives $\frac{\partial u}{\partial x_j}$, determinate a manifold.

Now, we present some basic constructions following the works of Anderson and Kamran in [2] and Reyes in [21], for determining this manifold. We start with a second order partial differential equation, and we end up presenting a generalization of this construction to a system of PDEs of order kth.

Le us consider a given second-order partial differential equation

$$\Xi\left(x,\, y,\, u,\, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial x}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial y}\right) = 0, \tag{3.3.1}$$

in two independent variables x, y and one depend variable u.

Now, we consider the trivial bundle $\pi\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2,\ \pi(x, y, u) = (x, y)$, then the coordinates on $J^2(\pi)$ are denoted by $y^2 = (x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ (see expression 3.1.3).

The equation (3.3.1) define a locus in $J^2\pi$

$$\mathcal{L}_E = \left\{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \in J^2(\pi) \mid E\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = 0.\right\}$$

We restrict this locus to a submanifold $S^{(2)}$ of $J^2\pi$, and we assume that the function Ξ is smooth on a neighbourhood of $S^{(2)}$. We further ask that $\pi_2\colon S^{(2)} \to \mathbb{R}^2$ be a sub-bundle of the bundle

$$\begin{array}{rccc} \pi_2 : & J^2\pi & \to & \mathbb{R}^2 \\ & j^2_{(p,q)}\phi & \mapsto & (p, q) \end{array}$$

and that the following diagram commutes:

$$\begin{array}{ccc} S^{(2)} & \xrightarrow{i^{(2)}} & J^2\pi \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{R}^2 & \xrightarrow{id} & \mathbb{R}^2 \end{array}$$

We can assume that $\Xi(x, y, \ldots, u_{yy}) = 0$ can be solved for one of the variables u_{xx}, u_{xy}, u_{yy}, so $S^{(2)}$ is defined by a subset of the locus of the equation, for example

$$u_{xx} + f(x, y, u, u_x, u_y, u_{xy}, u_{yy}) = 0. \tag{3.3.2}$$

Then $S^{(2)}$ is a 7-dimensional submanifold of $J^2(\pi)$. This sub-bundle $\pi_2\colon S^{(2)} \to \mathbb{R}^2$ is called ***the equation manifold*** of $\Xi = 0$.

The successive ***prolongations of*** S^2 are defined by the total derivatives of $\Xi(x, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ that is

$$S^{(3)} = \left\{ j^3_{(p,q)}\phi \mid j^2_{(p,q)}\phi \in S^{(2)} \text{ and } (D_x\Xi)(j^3_{(p,q)}\phi) = (D_y\Xi)(j^3_{(p,q)}) = 0 \right\}$$

and

$$S^{(4)} = \left\{ j^4_{(p,q)}\phi \mid j^3_{(p,q)}\phi \in S^{(3)} \text{ and } (D_{xx}\Xi)(j^4_{(p,q)}\phi) = (D_{xy}\Xi)(j^4_{(p,q)}\phi) = (D_{yy}\Xi)(j^4_{(p,q)}\phi) = 0 \right\}$$

and so on. We assume that each $S^{(l+1)}$ $(l \geq 2)$ is a submanifold of $J^{l+1}\pi$ which fiber over $S^{(l)}$, that is, the following diagram is commutative:

$$\begin{array}{ccc} S^{(l+1)} & \xrightarrow{i^{(l+1)}} & J^{l+1}\pi \\ \Big\downarrow \pi_{l+1,l} & & \Big\downarrow \pi_{l+1,l} \\ S^{(l)} & \xrightarrow{i^{(l)}} & J^{l}\pi \end{array}$$

Let $i\colon S^{(\infty)} \to J^{\infty}\pi$ be the infinite prolongation of $S^{(2)}$:

$$S^{(\infty)} = \left\{ j^\infty_{(p,q)}\phi \mid j^\infty_{(p,q)}\phi \in S^{(k)} \text{ and } (D_I\Xi)(j^\infty_{(p,q)}\phi) = 0 \text{ for all } |I| \geq 0 \right\}$$

Locally, $S^{(\infty)}$ is the set of infinite jets in $J^\infty\pi$ satisfying $\Xi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ and all its total derivatives. And $\pi_\infty\colon S^{(\infty)} \to \mathbb{R}^2$ is a sub-bundle of $\pi_\infty\colon J^\infty\pi \to \mathbb{R}^2$, called the ***full equation manifold*** of $\Xi = 0$.

Let u be a local solution to (3.3.1) then we obtain a section of $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by the graph of u,

$$\begin{array}{rccc} \phi: & W & \to & W \times \mathbb{R} \\ & (p,q) & \mapsto & (p, q, u(p,q)). \end{array}$$

Thus, ***a local solution*** of Ξ is a local section of $\pi_2 : S^2 \to \mathbb{R}^2$ which is the 2-prolongation of a local section ϕ.

$$\begin{array}{rccc} j^2\phi: & W & \to & S^2 \\ & (p,q) & \mapsto & j^2_{(p,q)}\phi \end{array}$$

since $j^2_{(p,q)}\phi \in S^{(2)}$, in fact:

$$\begin{array}{rccc} x: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \mapsto & x(p,q) = p \end{array} \qquad \begin{array}{rccc} y: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \mapsto & y(p,q) = q \end{array}$$

$$\begin{array}{rccc} u: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \mapsto & u \circ \phi(p,q) = u(p,q) \end{array} \qquad \begin{array}{rccc} u_x: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \mapsto & \frac{\partial(u\circ\phi)}{\partial x} = \frac{\partial u}{\partial x}(p,q) \end{array}$$

$$\begin{array}{rccc} u_y: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \longmapsto & \frac{\partial(u\circ\phi)}{\partial y} = \frac{\partial}{\partial y}u(p,q) \end{array} \qquad \begin{array}{rccc} u_{xx}: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \longmapsto & \frac{\partial(u\circ\phi)}{\partial xx} = \frac{\partial u}{\partial xx}(p,q) \end{array}$$

$$\begin{array}{rccc} u_{xy}: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \longmapsto & \frac{\partial(u\circ\phi)}{\partial xy} = \frac{\partial u}{\partial xy}(p,q) \end{array} \qquad \begin{array}{rccc} u_{yy}: & J^2\pi & \to & \mathbb{R} \\ & j^2_{(p,q)}\phi & \mapsto & \frac{\partial(u\circ\phi)}{\partial yy} = \frac{\partial u}{\partial yy}(p,q). \end{array}$$

A ***local smooth solution*** of Ξ is a local section of the full equation manifold of Ξ, $\pi_\infty\colon S^{(\infty)} \to \mathbb{R}^2$, which is the infinite prolongation of a local section of $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$

$$\begin{array}{rccc} \phi: & W & \to & W \times \mathbb{R} \\ & (p,q) & \mapsto & (p,q,u(p,q)) \end{array} \qquad \begin{array}{rccc} j^\infty\pi: & W & \to & J^\infty\pi \\ & (p,q) & \mapsto & j^\infty_{(p,q)}\phi \end{array}$$

Finally, the vector fields and differential forms on $S^{(k)}$ and $S^{(\infty)}$ are defined via pull-back by the canonical inclusions $i^{(k)}\colon S^k \to J^k\pi$, $i : S^{(\infty)} \to J^\infty\pi$. Thus, for instance, if $\sigma \in J^\infty\pi$ is a contact form then $i^*(\sigma)$ is a contact form in $S^{(\infty)}$. Then differential forms on $S^{(\infty)}$ can now be bi-graded as in (3.1.9). And we have the following definition.

Definition 3.3.1. *The **variational bicomplex** for the bundle $\pi_\infty\colon S^{(\infty)} \to \mathbb{R}^2$ is the pull-back by the inclusion $i\colon S^{(\infty)\to J^\infty\pi}$ of the variational bicomplex $(\Omega^{*,*}(J^\infty\pi), d_H, d_V)$ to $S^{(\infty)}$. We have the diagram:*

$$\begin{array}{ccccccccccccccc}
 & & & & \vdots & & \vdots & & \vdots & & \cdots & & \vdots & & \vdots \\
 & & & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V \\
 & & 0 & \to & \Omega^{0,2}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{1,2}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{2,2}(S^{(\infty)}) & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{n-1,2}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{n,2}(S^{(\infty)}) \\
 & & & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V \\
 & & 0 & \to & \Omega^{0,1}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{1,1}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{2,1}(S^{(\infty)}) & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{n-1,1}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{n,1}(S^{(\infty)}) \\
 & & & & \uparrow d_V & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V \\
0 & \to & \mathrm{R} & \to & \Omega^{0,0}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{1,0}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{2,0}(S^{(\infty)}) & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{n-1,0}(S^{(\infty)}) & \xrightarrow{d_H} & \Omega^{n,0}(S^{(\infty)})
\end{array}$$

Figure 3.2.

If we write the differential equation as in (3.1.5), the natural coordinates for $S^{(\infty)}$ are

$$(x, y, u, u_x, u_y, u_{xy}, u_{yy}, \ldots, u_{xy^{k-1}}, u_{y^k}, \ldots)$$

The total derivatives with respect x and y on $S^{(\infty)}$ (see 3.1.12) are

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} - f\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y} - D_y f\frac{\partial}{\partial u_{xy}} + u_{xyy}\frac{\partial}{\partial u_{yy}} + \cdots \\ D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y} + u_{xyy}\frac{\partial}{\partial u_{xy}} + u_{yyy}\frac{\partial}{\partial u_{yyy}} \cdots \end{aligned}$$

And basic contact one-forms are $\theta_I = du_I - \sum_{i=1}^{2} u_{I+\mathbf{1}_i} dx^i$ where $|I| = 0, 1, 2, \ldots$

In symmetric indices notation we have:

$$\begin{aligned}
\theta &= du - u_x dx - u_y dy \\
\theta_x &= du_x - u_{xx} dx - u_{xy} dy = du_x + f dx - u_{xy} dy \\
\theta_y &= du_y - u_{xy} dx - u_{yy} dy \\
\theta_{xy^{k-1}} &= du_{xy^{k-1}} - u_{xxy^{k-1}} dx - u_{xy^k} dy \\
&= du_{xy^{k-1}} - u_{y^{k-1}xx} dx - u_{xy^k} dy \\
&= du_{xy^{k-1}} - u_{y^{k-1}}(-f) dx - u_{xy^k} dy \\
&= du_{xy^{k-1}} + D_y^{k-1}(f) dx - u_{xy^k} dy.
\end{aligned}$$

If $g = g(x, u, u_x, u_y, u_{xy}, u_{yy}, \ldots, u_{xy^{k-1}}, u_{y^k})$ is a smooth function on $S^{(\infty)}$, the horizontal derivative of g (see 3.1.11) is

$$d_H(g) = (D_x g) dx + (D_y g) dy$$

and the vertical differential (see 3.1.13) is

$$d_V(g) = \frac{\partial g}{\partial u}\theta + \frac{\partial g}{\partial u_x}\theta_x + \frac{\partial g}{\partial u_y}\theta_y + \frac{\partial g}{\partial u_{xy}}\theta_{xy} + \frac{\partial g}{\partial u_{yy}}\theta_{yy} + \cdots$$

The differentials d_H and d_V satisfy: $d_H(dx^i) = 0$, $d_V(dx^i) = 0$, $d_V(\theta_I^\alpha) = 0$, and

$$\begin{aligned}
d_H(\theta_{xy^{k-1}}) &= d_H(du_{xy^{k-1}} + D_y^{k-1}(f) dx - u_{xy^k}\, dy) \\
&= d_H(d_H u_{xy^{k-1}} + d_V u_{xy^{k-1}} + (D_y^{k-1})(f) dx - u_{xy^k} dy) \\
&= d_H d_V(u_{xy^{k-1}}) + d_H(D_y^{k-1}(f) dx) - d_H(u_{xy^k} dy) \\
&= d_H d_V(u_{xy^{k-1}}) + D_y(D_y^{k-1}(f)) dy \wedge dx - u_{xy^k x} dx \wedge dy \\
&= d_H d_V(u_{xy^{k-1}}) + (D_y^k(f)) dy \wedge dx - u_{y^k xx} dx \wedge dy \\
&= d_H d_V(u_{xy^{k-1}}) + (D_y^k(f)) dy \wedge dx + (D_y^k(f)) dx \wedge dy \\
&= d_H d_V(u_{xy^{k-1}}) \\
&= -d_V d_H(u_{xy^{k-1}}) \\
&= -d_V(u_{xxy^{k-1}} dx + u_{xy^k} dy) \\
&= d_V(D_y^{k-1}(f)) dx - \theta_{xy^k} dy.
\end{aligned}$$

Now we consider the generalization of last construction to a finite system of kth order system of PDEs

$$\Xi^l\left(x^i,\, u^\alpha,\, \frac{\partial u^\alpha}{\partial x^i}, \ldots \frac{\partial^k u^\alpha}{\partial x^{i_1} \ldots \partial x^{i_k}}\right) = 0, \tag{3.3.3}$$

where $l = 1, \ldots, r$, x^i, $(1 \leq i \leq m)$ are the independent variables, and u^α, $(1 \leq \alpha \leq n)$ are unknown functions.

This system determines a submanifold $S^{(k)}$ of $J^k\pi$, in which $\pi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and the functions $\Xi^l(x^i, u^\alpha, u_I^\alpha) = 0$ are smooth on a neighbourhood of $S^{(k)}$, where I is a multi-index with $|I| \leq k$ (see discussion after the Definition 3.1.4).

We also define $S^{(\infty)}$ as the submanifold of $J^{\infty}\pi$ constructed thus:

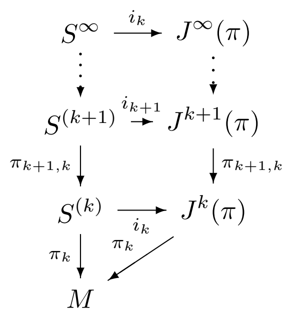
$$S^{(k+1)} = \{j_p^{k+1}\phi \in J^{k+1}\pi : j_p^k\phi \in S^{(k)} \text{ and } (D_i\Xi^l)(j_p^{k+1}\phi = 0, \text{ for all } i = 1, 2, ..., m\}$$

$$S^{(k+2)} = \{j_p^{k+2}\phi \in J^{k+2}\pi : j_p^{k+1}\phi \in S^{(k+1)} \text{ and } (D_I\Xi^l)(j_p^{k+2}\phi = 0, \text{ for all } |I| \leq 2\}$$

$$\vdots$$

$$S^{(\infty)} = \{j_p^{\infty}(\phi) \in J^{\infty}\pi : j_p^k\phi \in S^{(k)} \text{ and } (D_I\Xi^l)(j_p^{\infty}\phi) = 0, \text{ for all } |I| \geq 0\}$$

We assume that the tower is well defined, that is, $S^{(l+1)}$, $(l \geq k)$ is a submanifold of $J^{l+1}(\pi)$ which fibers over $S^{(l)}$.



Below, we present two examples:

Example 3.3.1. *Let* $\frac{\partial u}{\partial t} = F(x, t, u, \ldots, \frac{\partial^k u}{\partial x^k})$ *be an evolution equation of order* k*, in two independent variables. This equation determines a submanifold* $S^{(k)}$ *of* $J^k\pi$*, with* π *the trivial bundle* $\pi\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ *The coordinates on* S^k *are* $(x, t, u, u_x, , u_{x^k})$*, and the coordinates on* $S^{(\infty)}$ *are*

$$(x, t, u, u_x, , u_{xx\cdots x}, \ldots)$$

The total derivatives restricted to $S^{(\infty)}$ *are:*

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_{xx}} + u_{xxx}\frac{\partial}{\partial u_{xx}} + \cdots \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{xxt}\frac{\partial}{\partial u_{xx}} + \cdots \\ &= \frac{\partial}{\partial t} + F\frac{\partial}{\partial u} + D_x(F)\frac{\partial}{\partial u_x} + D_x^2(F)\frac{\partial}{\partial u_{xx}} + \cdots \end{aligned}$$

and the basic contact forms on $S^{(\infty)}$ *become:*

$$\begin{aligned} \theta &= du - u_x dx - u_t dt = du - u_x dx - F dt \\ \theta_{x^k} &= du_{x^k} - u_{x^{k+1}}dx - u_{x^{k+1}t}dt = du_{x^k} - u_{x^{k+1}}dx - D_x^{k+1}(F)dt \end{aligned}$$

Example 3.3.2. *We consider the system:*

$$\begin{cases} \frac{\partial^2 u}{\partial zz^2} + \frac{\partial^2 u}{\partial x\partial y} + u &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= 0 \\ \frac{\partial u}{\partial y} - u^2 &= 0 \end{cases}$$

This system defines a locus in $J^2\pi$, where $\pi\colon \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is de trivial bundle, such that:

$$\begin{cases} u_{zz} + u_{xy} + u & = & 0 \\ u_x + u_t & = & 0 \\ u_y - u^2 & = & 0 \end{cases}$$

Thus, $S^{(2)}$ and S^∞ have coordinates respectively given by

$$(x, y, t, z, u, u_x, u_z, u_{xx}, u_{xz})$$

$$(x, y, t, z, u, u_x, u_z, u_{xx}, u_{xz}, u_{xxx}, u_{xxz}, u_{xxxx}, u_{xxxz}, \ldots, u_{x^k}, u_{x^{k-1}z}, \ldots)$$

And, the basic contact one-forms on S^∞ are:

$$\begin{aligned} \theta \quad & = du - u_x dx - u_y dy - u_t dt - u_z dz = du - u_x dx - u^2 dy + u_x dt - u_z dz \\ \theta_{x^k} \quad & = du_{x^k} - u_{x^{k+1}} dx - u_{x^k y} dy - u_{x^k t} dt - u_{x^k z} dz \\ & = du_{x^k} - u_{x^{k+1}} dx - D_x^k(u^2) dy - D_x^k(-u_x) dt - u_{x^k z} dz \\ \theta_{x^{k-1}z} \quad & = du_{x^{k-1}z} - u_{x^k z} dx - u_{x^k yz} dy - u_{x^k zz} dz - u_{x^k zt} dt \\ & = du_{x^{k-1}} - u_{x^k z} dx - D_z D_x^k(u^2) dy + (D_x^{k+1} y + u_{x^k}) dz - D_x^k(2uu_z) \end{aligned}$$

3.4 Linear Gauge Complex and Twisting matrix

Now we let $\mathfrak{g}$ be a Lie algebra; we can consider the $\mathfrak{g}$-valued forms $\Omega(J^\infty\pi, \mathfrak{g}) = \Omega(J^\infty\pi) \otimes_{\mathbb{R}} \mathfrak{g}$, and we have a new *double dgm* given by $\Omega(J^\infty\pi, \mathfrak{g}) = \bigoplus \Omega^{r,s}(J^\infty\pi, \mathfrak{g})$ with differentials:

$$\begin{aligned} \mathbf{d_H} : \quad \Omega^{r,s}(J^\infty\pi, \mathfrak{g}) & \rightarrow \quad \Omega^{r+1,s}(J^\infty\pi, \mathfrak{g}) \\ \rho \otimes a \quad & \mapsto \quad d_H(\rho) \otimes a, \end{aligned}$$

$$\begin{aligned} \mathbf{d_V} : \quad \Omega^{r,s}(J^\infty\pi, \mathfrak{g}) & \rightarrow \quad \Omega^{r,s+1}(J^\infty\pi, \mathfrak{g}) \\ \rho \otimes a \quad & \mapsto \quad d_V(\rho) \otimes a. \end{aligned}$$

By the property (3.2.12) $\mathbf{d_H}([\omega, \eta]) = [\mathbf{d_H}(\omega), \eta] + (-1)^{|\omega|}[\omega, \mathbf{d_H}(\eta)]$, and therefore $(\Omega(J^\infty\pi, \mathfrak{g}), [\ ,\ ], \mathbf{d_H})$ is a differential graded Lie algebra.

Moreover, if $\mathfrak{g}$ is a subalgebra of $M_{n\times n}$, from (3.2.14) we have

$$\mathbf{d_H}(\omega \wedge \eta) = \mathbf{d_H}(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge \mathbf{d_H}(\eta),$$

therefore $(\Omega(M, \mathfrak{g}), \wedge, \mathbf{d_H})$ is a *dga* which is not necessary commutative.

The same is true for the differential $\mathbf{d_V}$.

Remark 3.4.1. *By the Remark (3.2.2-1) if $\mathfrak{g}$ is a subalgebra of $M_{n\times n}$*

$$\mathbf{d_H}(\omega)_{jk} = (\mathbf{d_H}(\omega_{jk})).$$

In particular if $f \in \Omega^0(J^\infty\pi, \mathfrak{g})$, we have

$$d_H(f)_{jk} = \sum_{i=1}^{m}(D_{x^i}(f_{jk}))dx^i.$$

In [12] and [13] Michael Marvan define the pth (linear) Gauge complex, considering a submanifold $S^{(\infty)} \subseteq J^\infty\pi$ which is determined by a finite system of PDEs kth order. For this, Michael Marvan uses the *double dgm* $\Omega(S^{(\infty)}, \mathfrak{g}) = \bigoplus \Omega^{r,s}(S^{(\infty)}, \mathfrak{g})$. In this section we use the notation ε instead of $S^{(\infty)}$.

Definition 3.4.1. *A form $\alpha \in \Omega^{1,0}(\varepsilon, \mathfrak{g})$ is called a $\mathfrak{g}$-**valued zero curvature representation for** ε (for short zcr for ε) if α is a Maurer-Cartan element, this is:*

$$\mathbf{d_H}(\alpha) = \frac{1}{2}[\alpha, \alpha].$$

Give a fixed *zcr* α, we can consider the linear maps:

$$\begin{array}{rccc} ad_\alpha: & \Omega(\varepsilon,\mathfrak{g}) & \to & \Omega(\varepsilon,\mathfrak{g}) \\ & \omega & \mapsto & [\alpha,\omega] \end{array} \qquad \begin{array}{rccc} \partial_\alpha: & \Omega(\varepsilon,\mathfrak{g}) & \to & \Omega(\varepsilon,\mathfrak{g}) \\ & \omega & \mapsto & \mathbf{d_H}(\omega)-[\alpha,\omega] \end{array}$$

Then $ad_\alpha(\Omega^{p,q}(\varepsilon, \mathfrak{g})) \subset \Omega^{p+1,q}(\varepsilon, \mathfrak{g})$ for fixed $q \geq 0$, ∂_α is a differential on the graded module $\Omega^{\square,q}(\varepsilon, \mathfrak{g}) = \{\Omega^{i,q}(\varepsilon, \mathfrak{g})\}_{i\geq 0}$ because:

$$\begin{aligned}
\partial_\alpha \circ \partial_\alpha(\omega) &= \partial_\alpha(\mathbf{d_H}(\omega) - [\alpha,\omega]) \\
&= \mathbf{d_H}(\mathbf{d_H}(\omega) - [\alpha,\omega]) - [\alpha, \mathbf{d_H}(\omega) - [\alpha,\omega]] \\
&= -\mathbf{d_H}([\alpha,\omega]) - [\alpha, \mathbf{d_H}(\omega)] + [\alpha,[\alpha,\omega]] \\
&= -[\mathbf{d_H}(\alpha),\omega] + [\alpha, \mathbf{d_H}(\omega)] - [\alpha, \mathbf{d_H}(\omega)] + [\alpha,[\alpha,\omega]] \\
&= [-\mathbf{d_H}(\alpha),\omega] + \frac{1}{2}[[\alpha,\alpha],\omega] \\
&= [-\mathbf{d_H}(\alpha) + \frac{1}{2}[\alpha,\alpha],\omega] \\
&= 0.
\end{aligned}$$

The fifth equality is obtained by means of the properties (3.2.10) and (3.2.11). In fact:

$$\begin{aligned}
0 &= (-1)^{|\alpha||\omega|}[\alpha,[\alpha,\omega]] + (-1)^{|\alpha||\alpha|}[\alpha,[\omega,\alpha]] + (-1)^{|\omega||\alpha|}[\omega,[\alpha,\alpha]] \\
&= (-1)^{|\omega|}[\alpha,[\alpha,\omega]] - [\alpha,(-1)^{|\omega||\alpha|+1}[\alpha,\omega]] + (-1)^{|\omega|}[\omega,[\alpha,\alpha]] \\
&= (-1)^{|\omega|}\big([\alpha,[\alpha,\omega]] + [\alpha,[\alpha,\omega]] + [\omega,[\alpha,\alpha]]\big).
\end{aligned}$$

Therefore, $2[\alpha,[\alpha,\omega]] = -[\omega,[\alpha,\alpha]] = -(-1)^{|\omega|(|\alpha|+|\alpha|)+1}[[\alpha,\alpha],\omega] = [[\alpha,\alpha],\omega]$.

For a fixed $q \geq 0$, the complex $(\Omega^{\square,q}(\varepsilon, \mathfrak{g}), \partial_\alpha)$ is called qth-***linear complex*** of ε, or the qth-***differential graded module*** of ε. The the homology groups $(p \geq 0)$ $H^{p,q}_\alpha(\varepsilon, \mathfrak{g}) = Ker\partial^p_\alpha / Im\partial^{p-1}_\alpha$

are called the q-***horizontal Gauge cohomology groups with respect to*** α .

$$\Omega^{0,q}(\varepsilon,\mathfrak{g}) \xrightarrow{\partial_\alpha} \Omega^{1,q}(\varepsilon,\mathfrak{g}) \xrightarrow{\partial_\alpha} \Omega^{2,q}(\varepsilon,\mathfrak{g}) \longrightarrow \cdots$$

Remark 3.4.2.

- $(\Omega^{\Box,q}(\varepsilon,\mathfrak{g}),\partial_\alpha)$ *is a graded Lie algebra, in fact:*

$$\begin{aligned}\partial_\alpha([\omega,\eta]) &= \mathbf{d_H}([\omega,\eta]) - [\alpha,[\omega,\eta]]\\ &= [\mathbf{d_H}(\omega),\eta] + (-1)^{|\omega|}[\omega,\mathbf{d_H}(\eta)] - (-1)^{|\omega|}[\omega,[\alpha,\eta]] - [[\alpha,\omega],\eta]\\ &= [\mathbf{d_H}(\omega) - [\alpha,\omega],\eta] + (-1)^{|\omega|}([\omega,\mathbf{d_H}(\eta) - [\alpha,\eta]])\\ &= [\partial_\alpha(\omega),\eta] + (-1)^{|\omega|}[\omega,\partial_\alpha(\eta)].\end{aligned}$$

  The second equality is obtained from properties (3.2.10) and (3.2.11); since $(-1)^{|\eta|}[\alpha,[\omega,\eta]] + (-1)^{|\omega|}[\omega,[\eta,\alpha]] + (-1)^{|\eta||\omega|}[\eta,[\alpha,\omega]] = 0$*, then*

$$\begin{aligned}&-[\alpha,[\omega,\eta]]\\ &= (-1)^{|\omega|+|\eta|}[\omega,[\eta,\alpha]] + (-1)^{|\eta|(|\omega|+1)}[\eta,[\alpha,\omega]] - [\alpha,[\omega,\eta]]\\ &= (-1)^{|\omega|+|\eta|}[\omega,(-1)^{(|\eta||\alpha|+1)}[\alpha,\eta]] + (-1)^{|\eta|(|\omega|+1)}(-1)^{|\eta|(|\alpha|+|\omega|+1)}[[\alpha,\omega],\eta] - [\alpha,[\omega,\eta]]\\ &= -(-1)^{|\omega|}[\omega,[\alpha,\eta]] - [[\alpha,\omega],\eta].\end{aligned}$$

- *If V is a subalgebra of $M_{n\times n}$, we saw that $(\Omega(\varepsilon,\mathfrak{g}),\wedge,d)$ is a dga. Moreover we have that $(\Omega(\varepsilon,\mathfrak{g}),\wedge,\partial_\alpha)$ is a dga. In fact:*

$$\begin{aligned}\partial_\alpha(\omega\wedge\eta) &= \mathbf{d_H}(\omega\wedge\eta) - [\alpha,\omega\wedge\eta]\\ &= \mathbf{d_H}(\omega)\wedge\eta + (-1)^{|\omega|}\omega\wedge\mathbf{d_H}(\eta) - ([\alpha,\omega]\wedge\eta + (-1)^{|\omega|}\omega\wedge[\alpha,\eta]) \quad (\textit{by } 3.2.15)\\ &= ([\mathbf{d_H}(\omega) - [\alpha,\omega])\wedge\eta + (-1)^{|\omega|}\omega\wedge(\mathbf{d_H}(\eta) - [\alpha,\eta])\\ &= \partial_\alpha(\omega)\wedge\eta + (-1)^{|\omega|}\omega\wedge\partial_\alpha(\eta).\end{aligned}$$

Our goal is to represent the horizontal Gauge cohomology groups $H^{\Box,q}_\alpha(\varepsilon,\mathfrak{g})$ using twisted cohomology of $\Omega^{\Box,q}(\varepsilon)$ with coefficients in $\mathfrak{g}$. That is, we wish to write $\partial_\alpha = d_{H_{\Theta_\alpha}}$ for some twisting matrix Θ_α on $\mathfrak{g}$ into $\Omega^{1,0}(\varepsilon,\mathfrak{g})$.

Theorem 3.4.1. *Let us assume that α in $\Omega^{1,0}(\varepsilon,\mathfrak{g})$ is a zcr for ε. For a fixed $q\geq 0$, the horizontal cohomology $H^{\Box,q}_\alpha(\varepsilon,\mathfrak{g})$ is the twisted cohomology of $\Omega^{\Box,q}(\varepsilon)$ with coefficients in $\mathfrak{g}$ and twisting matrix*

$$\begin{aligned}\Theta_\alpha:\ \mathfrak{g} &\to \Omega^{1,0}(\varepsilon,\mathfrak{g})\\ a &\mapsto \sum_i \alpha_i\otimes[a,a_i],\end{aligned} \tag{3.4.1}$$

where $\alpha = \sum_i \alpha_i\otimes a_i$ is the zcr for ε, with $\alpha_i\in\Omega^{1,0}(\varepsilon)$ and $a_i\in\mathfrak{g}$.

Proof. First we recall that $\Omega^{\star,0}(\varepsilon) = \{\Omega^{i,0}(\varepsilon,\mathfrak{g})\}_{i\geq 0}$, $\Omega^{\Box,q}(\varepsilon) = \{\Omega^{i,q}(\varepsilon,\mathfrak{g})\}_{i\geq 0}$ and $(\Omega^{\star,0}(\varepsilon),\wedge,d_H)$ is a dga. Now, $(\Omega^{\Box,q}(\varepsilon),d_H)$ is a $\Omega^{\star,0}(\varepsilon)$-*dgm*. In fact the following linear map of degree zero satisfies the Definition (1.3.1)

$$\begin{array}{rcl} \Omega^{\star,0}(\varepsilon)\otimes\Omega^{\Box,q}(\varepsilon) & \to & \Omega^{\star+\Box,q}(\varepsilon) \\ \gamma\otimes\omega & \mapsto & \gamma\wedge\omega. \end{array}$$

Thus $(\Omega^{\Box,q}(\varepsilon)\otimes\mathfrak{g}, d_{H_{\Theta_\alpha}})$ is a $\Omega^{\star,0}(\varepsilon)$-*dgm* (see equation(1.4.3)) with:

$$\begin{aligned} d_{H_{\Theta_\alpha}}(\rho\otimes a) &= d_H(\rho)\otimes a + (-1)^{|\rho|}\Psi_\rho\circ\Theta_\alpha(a) \\ &= \mathbf{d_H}(\rho\otimes a) + (-1)^{|\rho|}\Psi_\rho\big(\sum_i \alpha_i\otimes[a,a_i]\big) \\ &= \mathbf{d_H}(\rho\otimes a) + (-1)^{|\rho|}\sum_i(\rho\wedge\alpha_i)\otimes[a,a_i] \\ &= \mathbf{d_H}(\rho\otimes a) - \sum_i(\alpha_i\wedge\rho)\otimes[a_i,a] \\ &= \mathbf{d_H}(\rho\otimes a) - [\sum_i \alpha_i\otimes\alpha_i, \rho\otimes a] \\ &= \mathbf{d_H}(\rho\otimes a) - [\alpha,\rho\otimes a] \\ &= \partial_\alpha(\rho\otimes a). \end{aligned}$$

Therefore Θ_α is a twisting matrix, for the Theorem (1.4.2), and we conclude that the horizontal gauge cohomology $H_\alpha^{\Box,q}(\varepsilon,\mathfrak{g})$ is the twisted cohomology of $\Omega^{\Box,q}(\varepsilon)$ with coefficients in $\mathfrak{g}$, this is $H_\alpha^{\Box,q}(\varepsilon,\mathfrak{g}) = H_{\otimes,\Theta_\alpha}(\Omega^{-,q}(\varepsilon);\mathfrak{g})$. □

Similarly, we can define the pth-differential graded module $(\Omega^{p,-}(\varepsilon,\mathfrak{g}),\delta_\beta)$ with $\beta\in\Omega^{0,1}(\varepsilon,\mathfrak{g})$ satisfying $\mathbf{d_V}(\beta) = \frac{1}{2}[\beta,\beta]$ and $\delta_\beta = \mathbf{d_V}(\omega) - ad_\beta$. As $(\Omega^{0,-}(\varepsilon), d_V)$ is a *dga* and $\Omega^{p,-}(\varepsilon)$ is a $\Omega^{0,-}(\varepsilon)$-*dgm*, we have that $(\Omega^{p,-}(\varepsilon,\mathfrak{g}),\delta_\beta) = (\Omega^{p,-}(\varepsilon,\mathfrak{g}), d_{V_{\Theta_\beta}})$ where:

$$\begin{array}{rrcl} \Theta_\beta: & \mathfrak{g} & \to & \Omega^{0,1}(\varepsilon)\otimes\mathfrak{g} \\ & a & \mapsto & \sum_i \rho_i\otimes[a,b_i] \end{array}$$

with $\beta = \sum_i \rho_i\otimes b_i$ for $\rho_i\in\Omega^{0,1}(\varepsilon)$ and $b_i\in\mathfrak{g}$. So, we have two twisted differentials $d_{H_{\Theta_\alpha}}$ and $d_{V_{\Theta_\beta}}$ in the following diagram:

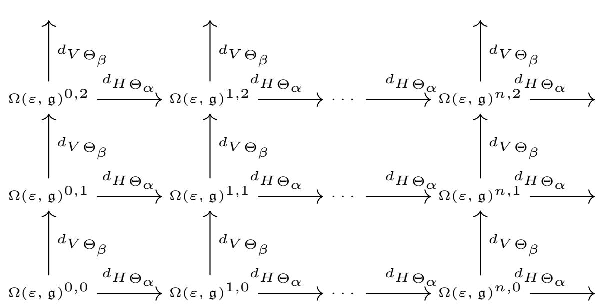


Figure 3.3.

We want to find conditions assuming that $d_{H_{\Theta_\alpha}} d_{V_{\Theta_\beta}} + d_{V_{\Theta_\beta}} d_{H_{\Theta_\alpha}} = 0$. Since $\Omega(\varepsilon, \mathfrak{g}) = \bigoplus \Omega^{p,q}(\varepsilon, \mathfrak{g})$ we would then conclude that $(\Omega(\varepsilon, \mathfrak{g}), d_{H_{\Theta_\alpha}} + d_{V_{\Theta_\beta}})$ is a *double dgm*.

Theorem 3.4.2. *If* $\mathbf{d_H}(\beta) + \mathbf{d_V}(\alpha) = [\alpha, \beta]$ *then* $(\Omega(\varepsilon, \mathfrak{g}), d_{H_{\Theta_\alpha}} + d_{V_{\Theta_\beta}})$ *is a double dgm.*

Proof. Note that:

$$
\begin{aligned}
& d_{H_{\Theta_\alpha}} d_{V_{\Theta_\beta}} + d_{V_{\Theta_\beta}} d_{H_{\Theta_\alpha}} (\rho \otimes a) \\
&= d_{H_{\Theta_\alpha}} (d_V(\rho) \otimes a - [\beta, \rho \otimes a]) + d_{V_{\Theta_\beta}} (d_H(\rho) \otimes a - [\alpha, \rho \otimes a]) \\
&= \mathbf{d_H}(d_V(\rho) \otimes a - [\beta, \rho \otimes a]) - [\alpha, d_V(\rho) \otimes a - [\beta, \rho \otimes a]] + \mathbf{d_V}(d_H(\rho) \otimes a - [\alpha, \rho \otimes a]) - \\
&\quad [\beta, d_H(\rho) \otimes a - [\alpha, \rho \otimes a]] \\
&= d_H d_V(\rho) \otimes a - \mathbf{d_H}([\beta, \rho \otimes a]) - [\alpha, d_V(\rho) \otimes a] + [\alpha, [\beta, \rho \otimes a]] + d_V d_H(\rho) \otimes a - \mathbf{d_V}([\alpha, \rho \otimes a]) - \\
&\quad [\beta, d_H(\rho) \otimes a] + [\beta, [\alpha, \rho \otimes a]] \\
&= -[\mathbf{d_H}(\beta), \rho \otimes a] + [\beta, d_H(\rho) \otimes a] - [\alpha, d_V(\rho) \otimes a] + [\alpha, [\beta, \rho \otimes a]] - [\mathbf{d_V}(\alpha), \rho \otimes a] + [\alpha, d_V(\rho) \otimes a] - \\
&\quad [\beta, d_H(\rho) \otimes a] + [\beta, [\alpha, \rho \otimes a]] \\
&= -[\mathbf{d_H}(\beta), \rho \otimes a] - [\mathbf{d_V}(\alpha), \rho \otimes a] + [\alpha, [\beta, \rho \otimes a]] + [\beta, [\alpha, \rho \otimes a]] \\
&= [-\mathbf{d_H}(\beta) - \mathbf{d_V}(\alpha), \rho \otimes a] + [[\alpha, \beta], \rho \otimes a] \\
&= [[\alpha, \beta] - (\mathbf{d_H}(\beta) + \mathbf{d_V}(\alpha)), \rho \otimes a] \\
&= 0,
\end{aligned}
$$

Where we have used that $[\alpha, [\beta, \rho \otimes a]] + [\beta, [\alpha, \rho \otimes a]] = [[\alpha, \beta], \rho \otimes a]$, for the properties (3.2.10) and (3.2.11).

□

3.4.1 Special Case

Let $SL(2)$ be the Lie group of all 2×2 real matrices with determinant 1. We consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$ of $SL(2)$, that is $\mathfrak{sl}(2)$ consists of traceless 2×2 matrices with entries in $\mathbb{R}$; the Lie bracket is defined by $[X, Y] := XY - YX$, where XY and YX denote the usual matrix multiplication. The elements:

$$
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

constitute the standard basis and the Lie bracket with respect to the standard basis is given by:

$$
[H, X] = 2X, \qquad [X, Y] = H, \qquad [H, Y] = -2Y.
$$

In the article [12] Marvan gives an example about non linear Klein-Gordon equation

$$
\frac{\partial u}{\partial x \partial y} = g(u). \tag{3.4.2}
$$

The manifold ε has coordinates $(x, y, u, u_x, u_{xx}, ..., u_y, u_{yy}, ...)$, and the total derivatives become

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + ... + u_{yx} \frac{\partial}{\partial u_y} + u_{yyx} \frac{\partial}{\partial u_{yy}} + ...$$

$$D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + ... + u_{yy} \frac{\partial}{\partial u_y} + u_{yyy} \frac{\partial}{\partial u_{yyy}} + ...$$

where $u_{xyy} = \frac{\partial u_{xy}}{\partial y} = g'(u) u_y$ and $u_{xyx} = \frac{\partial u_{xy}}{\partial x} = g'(u) u_x$.

In order to admit zero curvature representation Marvan found that for M, N, K, c constants, the function g must be of the form $\frac{2}{c}(Ne^{cu} - Me^{-cu})$. then α, the $\mathfrak{sl}(2)$-valued one-form *zcr* for ε, is given by the following formula:

$$\alpha = Ne^{cu} dx \otimes X + Ke^{-cu} dx \otimes Y + \frac{cu_y}{2} dy \otimes H + \frac{M}{K} dy \otimes X + dy \otimes Y. \tag{3.4.3}$$

Recall that for the property (3.1.11) we have $d_H(f) = D_x(f)dx + D_y(f)dy$, then:

$$\begin{aligned} \mathbf{d_H}(\alpha) &= D_y(Ne^{cu}) dy \wedge dx \otimes X + D_y(Ke^{-cu}) dy \wedge dx \otimes Y + D_x\left(\frac{cu_y}{2}\right) dx \wedge dy \otimes H \\ &= Ne^{cu} cu_y dy \wedge dx \otimes X + (-Ke^{-cu} cu_y) dy \wedge dx \otimes Y + \frac{cu_{xy}}{2} dx \wedge dy \otimes H \\ &= -Ne^{cu} cu_y dx \wedge dy \otimes X + Ke^{-cu} cu_y dx \wedge dy \otimes Y + \frac{cu_{xy}}{2} dx \wedge dy \otimes H \end{aligned}$$

and on the other hand

$$\begin{aligned} [\alpha, \alpha] &= Ne^{cu} cu_y dx \wedge dy \otimes [X, H] + 2Ne^{cu} \frac{M}{K} dx \wedge dy \otimes [X, X] + 2Ne^{cu} dx \wedge dy \otimes [X, Y] + \\ &\quad Ke^{-cu} cu_y dx \wedge dy \otimes [Y, H] + 2Me^{-cu} dx \wedge dy \otimes [Y, X] + 2Ke^{-cu} dx \wedge dy \otimes [Y, Y] \\ &= -2Ne^{cu} cu_y dx \wedge dy \otimes X + 2Ke^{-cu} cu_y dx \wedge dy \otimes Y + (2Ne^{cu} - 2Me^{-cu}) dx \wedge dy \otimes H \\ &= -2Ne^{cu} cu_y dx \wedge dy \otimes X + 2Ke^{-cu} cu_y dx \wedge dy \otimes Y + cu_{xy} dx \wedge dy \otimes H. \end{aligned}$$

Therefore, $\mathbf{d_H}(\alpha) = \frac{1}{2}[\alpha, \alpha]$ on the full equation manifold of (3.4.2). Therefore from Theorem (3.4.1) we have the twisting matrix:

$$\begin{aligned} \Theta_\alpha : \quad & \mathfrak{sl}(2) \quad \to \quad \Omega^{1,0}(\varepsilon) \otimes \mathfrak{sl}(2) \\ & A \quad \mapsto \quad Ne^{cu} dx \otimes [A, X] + Ke^{-cu} dx \otimes [A, Y] + \frac{cu_y}{2} dy \otimes [A, H] + \frac{M}{K} dy \otimes [A, X] + dy \otimes [A, Y]. \end{aligned}$$

Substituting the matrices X, H, Y in last map, we have:

$$\left.\begin{aligned} \Theta_\alpha(X) &= Ke^{-cu} dx \otimes H - cu_y dy \otimes X + dy \otimes H, \\ \Theta_\alpha(H) &= 2Ne^{cu} dx \otimes X - 2Ke^{-cu} dx \otimes Y + \frac{2M}{K} dy \otimes X - 2dy \otimes Y, \\ \Theta_\alpha(Y) &= -Ne^{cu} dx \otimes H + cu_y dy \otimes Y - \frac{M}{K} dy \otimes H. \end{aligned}\right\} \tag{3.4.4}$$

In terms of $\mathfrak{sl}(2)$-valued forms (see Remark 3.2.2) the element α in equality (3.4.3) can be represented by the following matrices:

$$\alpha = \begin{pmatrix} 0 & Ne^{-cu}dx \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ Ke^{-cu}dx & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}cu_y dy & 0 \\ 0 & -\frac{1}{2}cu_y dy \end{pmatrix} + \begin{pmatrix} 0 & \frac{M}{K}dy \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ dy & 0 \end{pmatrix}.$$

Thus α can be written as:

$$\alpha = \underbrace{\begin{pmatrix} 0 & Ne^{-cu} \\ Ke^{-cu} & 0 \end{pmatrix}}_{A} dx + \underbrace{\begin{pmatrix} \frac{1}{2}cu_y & \frac{M}{K} \\ 1 & -\frac{1}{2}cu_y \end{pmatrix}}_{B} dy \tag{3.4.5}$$

where, $A, B \in \Omega^0(\varepsilon, \mathfrak{sl}(2))$, Moreover, with the last equality and Remarks (3.2.2, 3.4.1) and equality (3.2.15):

$$\mathbf{d_H}(\alpha) = \mathbf{d_H}(Adx + Bdy) = (D_y A)dy \wedge dx + (D_x B)dx \wedge dy = (D_y A - D_x B)dy \wedge dx,$$

$$\frac{1}{2}[\alpha, \alpha] = \alpha \wedge \alpha = ABdx \wedge dy + BAdx \wedge dy = (AB - BA)\, dx \wedge dy = [A, B]dx \wedge dy.$$

Therefore $D_y A - D_x B + [A, B] = 0$, the usual “zero curvature representation” of (3.4.2).

We find that our twisting matrix satisfies (see 3.4.4):

$$\begin{array}{cccc} \Theta_\alpha : & \mathfrak{sl}(2) & \to & \Omega^{1,0}(\varepsilon) \otimes \mathfrak{sl}(2) \\ & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \mapsto & \begin{pmatrix} Ke^{-cu} & 0 \\ 0 & -Ke^{-cu} \end{pmatrix} dx + \begin{pmatrix} 1 & -cu_y \\ 0 & -1 \end{pmatrix} dy \\ & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \mapsto & \begin{pmatrix} 0 & 2Ne^{cu} \\ -2Ke^{-cu} & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & \frac{2M}{K} \\ -2 & 0 \end{pmatrix} dy \\ & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \mapsto & \begin{pmatrix} -Ne^{cu} & 0 \\ 0 & Ne^{cu} \end{pmatrix} dx + \begin{pmatrix} -\frac{M}{K} & 0 \\ cu_y & \frac{M}{K} \end{pmatrix} dy. \end{array}$$

We find an element $\beta \in \Omega^{0,1}(\varepsilon, \mathfrak{g})$ such that $\mathbf{d_V}(\beta) = \frac{1}{2}[\beta, \beta]$, as saw after of the Theorem (3.4.1). Let us write the general form of a term β, with $p, q \in \mathbb{N}$

$$\begin{aligned} \beta \;=\; & (a_{1,0}\theta + a_{1,1}\theta_x + a_{1,2}\theta_{xx} + a_{1,3}\theta_{xxx} + \cdots + a_{1,p}\theta_{x^p} + b_{1,1}\theta_y + \cdots + b_{1,q}\theta_{y^q}) \otimes X + \\ & (a_{2,0}\theta + a_{2,1}\theta_x + a_{2,2}\theta_{xx} + a_{2,3}\theta_{xxx} + \cdots + a_{2,p}\theta_{x^n} + b_{2,1}\theta_y + \cdots + b_{2,q}\theta_{y^q}) \otimes Y + \\ & (a_{3,0}\theta + a_{3,1}\theta_x + a_{3,2}\theta_{xx} + a_{3,3}\theta_{xxx} + \cdots + a_{3,p}\theta_{x^n} + b_{3,1}\theta_y + \cdots + b_{3,q}\theta_{y^q}) \otimes H \\ =\; & a \otimes X + b \otimes Y + e \otimes H, \end{aligned}$$

thus

$$\begin{aligned}
\tfrac{1}{2}[\beta,\beta] &= \tfrac{1}{2}\left\{ab\otimes[X,Y]+ae\otimes[X,H]+ba\otimes[Y,X]+be\otimes[Y,H]+ea\otimes[H,X]+eb\otimes[H,Y]\right\}\\
&= \tfrac{1}{2}\left\{ab\otimes H+ae\otimes(-2X)+ba\otimes(-H)+be\otimes(2Y)+ea\otimes(2X)+eb\otimes(-2Y)\right\}\\
&= \tfrac{1}{2}\left\{(ab-ba)\otimes H+(-ae+ea)\otimes X+(be-eb)\otimes Y\right\}\\
&= \tfrac{1}{2}\left\{2ab\otimes H+4ea\otimes X+4be\otimes Y\right\}\\
&= ab\otimes H+2ea\otimes X+2be\otimes Y.
\end{aligned}$$

We consider the element:

$$\beta=\left(\frac{cN\theta}{2}+\frac{Ne^{cu}}{4}\theta x\right)\otimes X+\left(\frac{c}{2N}\theta-\frac{e^{cu}}{4N}\theta x\right)\otimes Y+\left(-\frac{e^{cu}}{4}\theta x\right)\otimes H,$$

then

$$\mathbf{d}_V(\beta)=\frac{cNe^{cu}}{4}\theta\wedge\theta_x\otimes X-\frac{cNe^{cu}}{4N}\theta\wedge\theta_x\otimes Y-\frac{ce^{cu}}{4}\theta\wedge\theta_x\otimes H$$

On the other hand:

$$\begin{aligned}
2ea &= 2\left(-\frac{e^{cu}}{4}\theta_x\right)\wedge\left(\frac{cN\theta}{2}+\frac{Ne^{cu}}{4}\theta_x\right) &&= \frac{cNe^{cu}}{4}\theta\wedge\theta_x\\
2be &= 2\left(\frac{c}{2N}\theta-\frac{e^{cu}}{4N}\theta_x\right)\wedge\left(-\frac{e^{cu}}{4}\theta_x\right) &&= -\frac{ce^{cu}}{4N}\theta\wedge\theta_x\\
ab &= \left(\frac{cN}{2}\theta+\frac{Ne^{cu}}{4}\theta_x\right)\wedge\left(\frac{c}{2N}\theta-\frac{e^{cu}}{4N}\theta_x\right) &&= \left(-\frac{ce^{cu}}{8}-\frac{ce^{cu}}{8}\right)\theta\wedge\theta_x = -\frac{ce^{cu}}{4}\theta\wedge\theta_x
\end{aligned}$$

Therefore, $\mathbf{d}_\mathbf{V}(\beta)=\frac{1}{2}[\beta,\beta]$.

Chapter 4

Manifold of Pseudo-spherical type and Generation of Sullivan decomposable Algebras

In this chapter we wish to find twisting matrices and to generate Sullivan decomposable algebras starting from certain forms ω^i, $i = 1, 2, 3$ determined by a manifold of pseudo-spherical type, this is, a special kind of submanifold of a infinite Jet bundle.

4.1 Manifold of Pseudo-spherical type and twisting matrix

Definition 4.1.1. *Let* $\Xi = 0$ *be a scalar differential equation*

$$\Xi(x,\ t,\ u,\ \frac{\partial u}{\partial x},\ \dots,\ \frac{\partial u}{\partial x^n t^m}) = 0 \tag{4.1.1}$$

in two independent variables x, t. *The full equation manifold* $S^{(\infty)}$ *of (4.1.1) is called of* ***pseudo-spherical type*** *if there exist one-forms* ω^i, $i = 1, 2, 3$,

$$\omega^i = f_{i,1}(x, t, u, \dots, u_{x^n t^m})dx + f_{i,2}(x, t, u, \dots, u_{x^s t^q})dt \tag{4.1.2}$$

whose coefficients f_{ij} *are smooth functions on (a neighborhood of)* $S^{(\infty)}$, *such that they satisfy the independence condition* $\omega^1 \wedge \omega^2 \neq 0$ *and the equations*

$$d_H(\omega^1) = \omega^3 \wedge \omega^2, \quad d_H(\omega^2) = \omega^1 \wedge \omega^3, \quad d_H(\omega^3) = \omega^1 \wedge \omega^2. \tag{4.1.3}$$

The class of differential equations of pseudo-spherical type was introduced by S.S.Chern and K. Teneblat [6] in terms of solution of differential equation $\Xi = 0$. If the equalities in $S^{(\infty)}$ in (4.1.3) are replaced by one-forms $\overline{\omega} = \omega(u(x,t))$ we obtain:

$$d(\overline{\omega}^1) = \overline{\omega}^3 \wedge \overline{\omega}^2, \quad d(\overline{\omega}^2) = \overline{\omega}^1 \wedge \overline{\omega}^3, \quad d(\overline{\omega}^3) = \overline{\omega}^1 \wedge \overline{\omega}^2.$$

whenever $u(x,t)$ is a solution of $\Xi = 0$. If $u\colon M \subset \mathbb{R}^2 \to \mathbb{R}$ satisfies $(\omega^1 \wedge \omega^2)(u(x,t)) \neq 0$ then M is called pseudo-spherical surface.

The purpose of this section is to relate the $S^{(\infty)}$ manifold determined by a differential equation and Sullivan descomposable algebras, so we privilege the structure defined in (4.1.1) instead of the definition introduced by Chern and Teneblat, since they give structure of a pseudo-spherical surface of open subset of $\mathbb{R}^2$.

The one-forms ω^i allow us to construct a form $\alpha \in \Omega^{1,0}(\varepsilon, \mathfrak{sl}(2))$;

$$\alpha = \frac{1}{2}\{\omega^2 \otimes H + (\omega^1 - \omega^3) \otimes X + (\omega^1 + \omega^3) \otimes Y\}, \tag{4.1.4}$$

where

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By Remark (3.2.2-1) α can be written as:

$$\alpha = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix} \tag{4.1.5}$$

or even

$$\alpha = \underbrace{\frac{1}{2}\begin{pmatrix} f_{21} & f_{11} - f_{31} \\ f_{11} + f_{31} & -f_{21} \end{pmatrix}}_{X} dx + \underbrace{\frac{1}{2}\begin{pmatrix} f_{22} & f_{12} - f_{32} \\ f_{12} + f_{32} & -f_{22} \end{pmatrix}}_{T} dt = Xdx + Tdt, \tag{4.1.6}$$

where $X, T \in \Omega^0(\varepsilon, \mathfrak{sl}(2))$. Remarks (3.2.2-1) and (3.4.1) imply:

$\alpha \wedge \alpha = XTdx \wedge dt + TXdt \wedge dx = -[X, T]dt \wedge dx,$

$d_H(\alpha) = (D_t X)dt \wedge dx + (D_x T)dx \wedge dt = (D_t X - D_x T)dt \wedge dx,$

therefore $D_t X - D_x T + [X, T] = 0$.

The following theorem shows that α is a $\mathfrak{sl}(2)$-valued zero curvature representation for ε.

Theorem 4.1.1. *Let ε be a manifold of pseudo-spherical type determined by a scalar differential equation $\Xi(x, t, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial u}{\partial x^n t^m}) = 0$ with one-forms ω^i, $i = 1, 2, 3$; then*

$$\alpha = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}$$

is a $\mathfrak{sl}(2)$-valued zero curvature representation for ε.

Proof. Here we use the property (3.2.15), thus $[\alpha, \alpha] = 2(\alpha \wedge \alpha)$ and by the Remark (3.2.2-1), we have the following computations:

$$\frac{1}{2}[\alpha, \alpha] = \alpha \wedge \alpha = \frac{1}{4}\begin{pmatrix} \omega^2 \wedge \omega^2 + (\omega^1 - \omega^3) \wedge (\omega^1 + \omega^3) & \omega^2 \wedge (\omega^1 - \omega^3) - (\omega^1 - \omega^3) \wedge \omega^2 \\ (\omega^1 + \omega^3) \wedge \omega^2 - \omega^2 \wedge (\omega^1 + \omega^3) & (\omega^1 + \omega^3) \wedge (\omega^1 - \omega^3) + \omega^2 \wedge \omega^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\omega^1 \wedge \omega^3}{2} & \frac{\omega^3 \wedge \omega^2 - \omega^1 \wedge \omega^2}{2} \\ \frac{\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^2}{2} & \frac{-\omega^1 \wedge \omega^3}{2} \end{pmatrix}$$

$$\mathbf{d_H}(\alpha) = \frac{1}{2}\begin{pmatrix} d_H(\omega^2) & d_H(\omega^1 - \omega^3) \\ d_H(\omega^1 + \omega^3) & -d_H(\omega^2) \end{pmatrix} = \begin{pmatrix} \frac{\omega^1 \wedge \omega^3}{2} & \frac{\omega^3 \wedge \omega^2 - \omega^1 \wedge \omega^2}{2} \\ \frac{\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^2}{2} & \frac{-\omega^1 \wedge \omega^3}{2} \end{pmatrix}$$

Therefore $\mathbf{d_H}(\alpha) = \frac{1}{2}[\alpha, \alpha]$. This proves that α is a $\mathfrak{sl}(2)$-valued zero curvature representation for ε by Definition (3.4.1). □

Since α is a $\mathfrak{sl}(2)$-valued *zcr* for ε, by Theorem (3.4.1), for a fixed $q \geq 0$ the qth-linear complex of ε $(\Omega^{\square,q}(\varepsilon,\mathfrak{g}), \partial_\alpha) = (\Omega^{\square,q}(\varepsilon,\mathfrak{g}), d_{H_{\Theta_\alpha}})$ is a $\Omega^{\star,0}(\varepsilon)$-*dgm* and by Theorem (1.4.2), Θ_α constructed in (4.1.7) below is a twisting matrix:

$$
\left.\begin{array}{rccl}
\Theta_\alpha : & \mathfrak{sl}(2) & \to & \Omega^{1,0}(\varepsilon)\otimes \mathfrak{sl}(2) \\
\text{in which} & & & \\
& A & \mapsto & \frac{1}{2}\Big(\omega^2\otimes[A,H] + (\omega^1-\omega^3)\otimes[A,X] + (\omega^1+\omega^3)\otimes[A,Y]\Big) \\
& X & \mapsto & -\omega^2\otimes X + \frac{1}{2}(\omega^1+\omega^3)\otimes H = \begin{pmatrix} \frac{\omega^1+\omega^3}{2} & -\omega^2 \\ 0 & -\frac{(\omega^1+\omega^3)}{2}\end{pmatrix} \\
& Y & \mapsto & \omega^2\otimes Y - \frac{1}{2}(\omega^1-\omega^3)\otimes H = \begin{pmatrix} -\frac{(\omega^1-\omega^3)}{2} & 0 \\ \omega^2 & -\frac{(\omega^1-\omega^3)}{2}\end{pmatrix} \\
& H & \mapsto & (\omega^1-\omega^3)\otimes X - (\omega^1+\omega^3)\otimes Y = \begin{pmatrix} 0 & \omega^1-\omega^3 \\ -\omega^1-\omega^3 & 0\end{pmatrix}
\end{array}\right\} \quad (4.1.7)
$$

Let us describe the miltiplication operation and differential of $(\Omega^{\square,q}(\varepsilon)\otimes\mathfrak{sl}(2), d_{H_{\Theta_\alpha}})$ as a $\Omega^{\star,0}(\varepsilon)$-*dgm*, (see equalities 1.4.3). They are given by the linear maps:

$$\rho(\eta\otimes A) = \rho\wedge\eta\otimes A,$$

$$d_{H_{\Theta_\alpha}}(\eta\otimes A) = d_H(\eta)\otimes A + \frac{(-1)^{|\eta|}}{2}\{\eta\wedge\omega^2\otimes[A,H] + \eta\wedge(\omega^1-\omega^3)\otimes[A,X] + \eta\wedge(\omega^1+\omega^3)\otimes[A,Y]\}.$$

Thus,

$$
\begin{aligned}
d_{H_{\Theta_\alpha}}(\eta\otimes X) &= d_H(\eta)\otimes X + \frac{(-1)^{|\eta|}}{2}\Big(\eta\wedge\omega^2\otimes(-2X) + \eta\wedge(\omega^1+\omega^3)\otimes H\Big) \\
&= \begin{pmatrix} \frac{(-1)^{|\eta|}}{2}\,\eta\wedge(\omega^1+\omega^3) & d_H(\eta) - (-1)^{|\eta|}\eta\wedge\omega^2 \\ 0 & -\frac{(-1)^{|\eta|}}{2}\,\eta\wedge(\omega^1+\omega^3)\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
d_{H_{\Theta_\alpha}}(\eta\otimes Y) &= d_H(\eta)\otimes Y + \frac{(-1)^{|\eta|}}{2}\Big(\eta\wedge\omega^2\otimes(2Y) + \eta\wedge(\omega^1-\omega^3)\otimes(-H)\Big) \\
&= \begin{pmatrix} -\frac{(-1)^{|\eta|}}{2}\,\eta\wedge(\omega^1-\omega^3) & 0 \\ d_H(\eta) + (-1)^{|\eta|}\eta\wedge\omega^2 & \frac{(-1)^{|\eta|}}{2}\,\eta\wedge(\omega^1-\omega^3)\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
d_{H_{\Theta_\alpha}}(\eta\otimes H) &= d_H(\eta)\otimes H + \frac{(-1)^{|\eta|}}{2}\Big(\eta\wedge(\omega^1-\omega^3)\otimes(2X) + \eta\wedge(\omega^1+\omega^3)\otimes(-2Y)\Big) \\
&= \begin{pmatrix} d_H(\eta) & (-1)^{|\eta|}\,\eta\wedge(\omega^1-\omega^3) \\ -(-1)^{|\eta|}\eta\wedge(\omega^1+\omega^3) & -d_H(\eta)\end{pmatrix}
\end{aligned}
$$

On the other hand, by Theorem (1.4.3) the module $Hom(\mathfrak{sl}(2), \Omega^{-,q}(\varepsilon))$ is a $\Omega^{-,0}(\varepsilon)$-*dgm* (see equalities 1.4.6) with linear maps:

$$(\rho g)(v) = \rho \wedge g(v),$$

$$d_{H_{\Theta_\alpha}}(g) = d_{H_{\Theta_\alpha}} \circ g - \frac{1}{2}\Big\{\omega^2 \wedge g([A,H]) + (\omega^1 - \omega^3) \wedge g([A,X]) + (\omega^1 + \omega^3) \wedge g([A,Y])\Big\}$$

Then,

$$d_{H_{\Theta_\alpha}}(g)(X) = d_{H_{\Theta_\alpha}} \circ g(X) + \omega^2 \wedge g(X) - \tfrac{1}{2}(\omega^1 + \omega^3) \wedge g(H),$$

$$d_{H_{\Theta_\alpha}}(g)(Y) = d_{H_{\Theta_\alpha}} \circ g(Y) - \omega^2 \wedge g(Y) + (\omega^1 - \omega^3) \wedge g(H),$$

$$d_{H_{\Theta_\alpha}}(g)(H) = d_{H_{\Theta_\alpha}} \circ g(H) - (\omega^1 - \omega^3) \wedge g(X) + (\omega^1 + \omega^3) \wedge g(Y).$$

4.2 Generation of Sullivan decomposable algebras

Using the twisting matrix Θ_α defined in (4.1.7), now we shall construct a Sullivan decomposable algebra by means of Theorem (1.5.2).

As in the last section, let ε be the manifold of pseudo-spherical type determined by a scalar differential equation $\Xi(x,\ t,\ u,\ \frac{\partial u}{\partial x},\ \ldots,\ \frac{\partial u}{\partial x^n t^m}) = 0$, with one-forms ω^α, $\alpha = 1,2,3$, and let

$$\alpha = \frac{1}{2}\{\omega^2 \otimes H + (\omega^1 - \omega^3) \otimes X + (\omega^1 + \omega^3) \otimes Y\}$$

in $\Omega^{1,0}(\varepsilon, \mathfrak{sl}(2))$ be a *zcr for* ε.

Now, we consider henceforth

$$W = span_{\mathbb{R}}\{\omega^1, \omega^2, \omega^3\} \tag{4.2.1}$$

the real subspace of $\Omega^{1,0}(\varepsilon)$.

Therefore, we can define the *cdga* $(\Lambda W, d)$ with differential given by:

$$d(\omega^1) = \omega^3 \wedge \omega^2, \quad d(\omega^2) = \omega^1 \wedge \omega^3, \quad d(\omega^3) = \omega^1 \wedge \omega^2. \tag{4.2.2}$$

It is important to note that the sign $\wedge$ in (4.2.2) is the operation in the algebra ΛW and not the operation in the algebra $\Omega^{1,0}(\varepsilon)$.

Then, we restrict the codomain of the twisting matrix Θ_α in (4.1.7) induced by the one form α to

$$\begin{array}{rccl} \Theta_\alpha: & \mathfrak{sl}(2) & \to & W \otimes \mathfrak{sl}(2) \\ & A & \mapsto & \frac{1}{2}(\omega^2 \otimes [A,H] + (\omega^1 - \omega^3) \otimes [A,X] + (\omega^1 + \omega^3) \otimes [A,Y]). \end{array} \tag{4.2.3}$$

This restriction allows us to obtain for the *cdga* $(\Lambda W, d)$ and the vector space $\mathfrak{sl}(2)$ a new twisting matrix by Definition (1.4.2). Since $d\Theta_\alpha + \Theta_\alpha \circ \Theta_\alpha = 0$. Indeed, by the maps in (1.4.7):

$$\begin{array}{llll} d\Theta_\alpha : & \mathfrak{sl}(2) & \to & (\Lambda W)^2 \otimes \mathfrak{sl}(2) \\ & A & \mapsto & \frac{1}{2}(\omega^1 \wedge \omega^3 \otimes [A,H] + (\omega^3 \wedge \omega^2 - \omega^1 \wedge \omega^2) \otimes [A,X] + (\omega^3 \wedge \omega^2 + \omega^1 \wedge \omega^2) \otimes [A,Y]) \\ & X & \mapsto & -\omega^1 \wedge \omega^3 \otimes X + \frac{1}{2}(\omega^3 + \omega^1) \wedge \omega^2 \otimes H \\ & Y & \mapsto & \omega^1 \wedge \omega^3 \otimes Y - \frac{1}{2}(\omega^3 - \omega^1) \wedge \omega^2 \otimes H \\ & H & \mapsto & (\omega^3 - \omega^1) \wedge \omega^2 \otimes X - (\omega^3 + \omega^1) \wedge \omega^2 \otimes Y \end{array}$$

$$\begin{array}{llll} \Theta_\alpha \circ \Theta_\alpha : & \mathfrak{sl}(2) & \to & (\Lambda W)^2 \otimes \mathfrak{sl}(2)) \\ & A & \mapsto & \frac{1}{2}(\omega^2 \otimes \Theta_\alpha([A,H]) + (\omega^1 - \omega^3) \otimes \Theta_\alpha([A,X]) + (\omega^1 + \omega^3) \otimes \Theta_\alpha([A,Y]) \end{array}$$

Then,

$$\begin{aligned} \Theta_\alpha \circ \Theta_\alpha(X) &= -\omega^2 \Theta_\alpha(X) + \frac{1}{2}(\omega^1 + \omega^3)\Theta_\alpha(H) \\ &= -\omega^2(-\omega^2 \otimes X + \frac{1}{2}(\omega^1 + \omega^3) \otimes H) + \frac{1}{2}(\omega^1 + \omega^3)((\omega^1 - \omega^3) \otimes X - (\omega^1 + \omega^3) \otimes Y) \\ &= -\omega^1 \wedge \omega^3 \otimes X + \frac{1}{2}(\omega^3 + \omega^1) \wedge \omega^2 \otimes H \end{aligned}$$

$$\begin{aligned} \Theta_\alpha \circ \Theta_\alpha(Y) &= \omega^2 \Theta_\alpha(Y) - \frac{1}{2}(\omega^1 - \omega^3)\Theta_\alpha(H) \\ &= \omega^2(\omega^2 \otimes Y - \frac{1}{2}(\omega^1 - \omega^3) \otimes H) - \frac{1}{2}(\omega^1 - \omega^3)((\omega^1 - \omega^3) \otimes X - (\omega^1 + \omega^3) \otimes Y) \\ &= \omega^1 \wedge \omega^3 \otimes Y - \frac{1}{2}(\omega^3 - \omega^1) \wedge \omega^2 \otimes H \end{aligned}$$

$$\begin{aligned} \Theta_\alpha \circ \Theta_\alpha(H) &= (\omega^1 - \omega^3)\Theta_\alpha(X) - (\omega^1 + \omega^3)\Theta_\alpha(Y) \\ &= (\omega^1 - \omega^3)(-\omega^2 \otimes X + \frac{1}{2}(\omega^1 + \omega^3) \otimes H) - (\omega^1 + \omega^3)(\omega^2 \otimes Y - \frac{1}{2}(\omega^1 - \omega^3) \otimes H) \\ &= (\omega^3 - \omega^1) \wedge \omega^2 \otimes X - (\omega^3 + \omega^1) \wedge \omega^2 \otimes Y). \end{aligned}$$

Theorem 4.2.1. *Let $V = \{V^k\}_{k \geq 0}$ be a graded vector space of finite type such that $V^0 = \{0\}$, $V^1 = W$, $V^2 = \mathfrak{sl}(2)$ and $V^k = \{0\}$ for all $k \geq 3$. Then, algebra $\Lambda(W \oplus \mathfrak{sl}(2))$ has structure of cdga via the linear map $d_{\mathfrak{sl}(2)}$ defined by*

$$\begin{array}{llll} d_{\mathfrak{sl}(2)} : & \mathfrak{sl}(2) & \to & \Lambda(W \oplus \mathfrak{sl}(2)) \\ & A & \mapsto & m \circ \Theta_\alpha(A) + f_2(A) \end{array} \tag{4.2.4}$$

where m is the exterior product on graded algebra $\Lambda(W \oplus \mathfrak{sl}(2))$ and $f_2 \colon \mathfrak{sl}(2) \to (\Lambda W)^3$ is a linear map. Moreover $\Lambda V = \Lambda(W \oplus \mathfrak{sl}(2))$ is a Sullivan decomposable algebra.

Proof. Since $(\Lambda W, d)$ is a *cdga* the Theorem (1.5.2) states that for $k = 2$, the free commutative graded algebra $\Lambda(W \oplus \mathfrak{sl}(2))$ has structure of *cdga* via the linear map $d_{\mathfrak{sl}(2)}$, where the function $f_2 \colon \mathfrak{sl}(2) \to (\Lambda W)^3$ is a linear map such that $[f_2] \in H^3_{Hom,\Theta_\alpha}(\mathfrak{sl}(2), \Lambda W)$.

In this case, any linear map f_2 satisfies the last property. To prove the previous assertion, first we recall some the notation introduced in the section (1.4.2): $(Hom(\mathfrak{sl}(2), \Lambda W)$ is a ΛW-***dgm*** with differential for $g \in Hom(\mathfrak{sl}(2), \Lambda W)$ given by $d_{\Theta_\alpha}(g) = d_H \circ f_2 - \Phi_g \circ \Theta_\alpha$, where

$$\begin{array}{rccl} \Phi_g : & \Lambda W \otimes \mathfrak{sl}(2) & \to & \Lambda W \\ & \omega \otimes A & \mapsto & \omega \wedge g(A). \end{array}$$

Thus, $[f_2] \in H^3_{Hom,\Theta_\alpha}(\mathfrak{sl}(2), \Lambda W)$ if for each $A \in \mathfrak{sl}(2)$ is satisfied that:

$$d_{\Theta_\alpha}(f_2)(A) = d(f_2(A)) - \frac{1}{2}(\omega^2 \wedge f_2([A,H]) + (\omega^1 - \omega^3) \wedge f_2([A,X]) + (\omega^1 + \omega^3) \wedge f_2([A,Y])) = 0.$$

Now, we observe that $(\Lambda W)^3 = span_{\mathbb{R}}\{\omega^1 \wedge \omega^2 \wedge \omega^3\}$ and $(\Lambda W)^4 = \{0\}$, as $d_{\Theta_\alpha}(f_2)(A) \in (\Lambda W)^4$ then $d_{\Theta_\alpha}(f_2) = 0$, for any f_2 linear map.

Therefore, $\Lambda V = \Lambda(W \oplus \mathfrak{sl}(2))$ is a Sullivan decomposable algebra.

□

The following examples appear in [20]; we apply the above results to the Burgers' equation and to the Sine-Gordon equation.

4.3 Burgers' Equation

Let ε be the manifold determined by the scalar differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x \partial x} + u\frac{\partial u}{\partial x}.$$

This equation is a ***manifold of pseudo-spherical type*** because there exist one-forms ω^i, $i = 1, 2, 3$,

$$\omega^1 = \big(\frac{1}{2}u - \frac{\beta}{\lambda}\big)\, dx + \frac{1}{2}\big(u_x + \frac{1}{2}u^2\big)\, dt \qquad \omega^2 = \lambda\, dx + \big(\frac{\lambda}{2}u + \beta\big)\, dt \qquad \omega^3 = -\omega^2, \tag{4.3.1}$$

in which λ is a nonzero parameter and $\beta\colon \varepsilon \to \mathbb{R}$ is defined by $j^\infty_{(p,q)}\phi \mapsto \beta(p)$, where $\beta(x)$ is a solution to the equation $\beta^2 - \lambda\beta_x = 0$, therefore the coefficients of ω^i are smooth functions, such that satisfy the independence condition $\omega^1 \wedge \omega^2 \neq 0$ and the structure equations (4.1.3). In fact:

$$\begin{aligned} \omega^1 \wedge \omega^2 &= (\tfrac{1}{2}u - \tfrac{\beta}{\lambda})(\tfrac{\lambda}{2}u + \beta)\, dx \wedge dt + \tfrac{\lambda}{2}(u_x + \tfrac{1}{2}u^2)\, dt \wedge dx \\ &= (\tfrac{1}{4}u^2\lambda - \tfrac{\beta^2}{\lambda})\, dx \wedge dt - (\tfrac{1}{2}\lambda u_x + \tfrac{1}{4}u^2\lambda)\, dx \wedge dt \\ &= (-\tfrac{\beta^2}{\lambda} - \tfrac{1}{2}\lambda u_x)\, dx \wedge dt, \\ \omega^1 \wedge \omega^3 &= -\omega^1 \wedge \omega^2, \\ \omega^3 \wedge \omega^2 &= -\omega^2 \wedge \omega^2 = 0. \end{aligned}$$

$$
\begin{aligned}
d(\omega^1) &= D_t(\tfrac{1}{2}u - \tfrac{\beta}{\lambda})\, dt\wedge dx + \tfrac{1}{2}D_x(u_x + \tfrac{1}{2}u^2)\, dx\wedge dt\\
&= (\tfrac{1}{2}u_t)\, dt\wedge dx + \tfrac{1}{2}(u_{xx} + u_x u)\, dt\wedge dx\\
&= 0,\\
d(\omega^2) &= D_x(\tfrac{\lambda}{2}u + \beta)\, dx\wedge dt\\
&= (\tfrac{\lambda}{2}u_x + \tfrac{\beta^2}{\lambda})\, dx\wedge dt,
\end{aligned}
$$

$$
d(\omega^3) = -d(\omega^2).
$$

Then, by the Theorem (4.1.1) we have the $\mathfrak{sl}(2)$-valued zero curvature representation for ε given by:

$$
\begin{aligned}
\alpha &= \tfrac{1}{2}\begin{pmatrix} \omega^2 & \omega^1+\omega^2\\ \omega^1-\omega^2 & -\omega^2\end{pmatrix}\\
&= \tfrac{1}{2}\{\omega^2\otimes H + (\omega^1+\omega^2)\otimes X + (\omega^1-\omega^2)\otimes Y\}\\
&= \tfrac{1}{2}\Big\{\big(\lambda dx + (\tfrac{\lambda}{2}u+\beta)dt\big)\otimes H + \Big(\big(\tfrac{1}{2}u - \tfrac{\beta}{\lambda}+\lambda\big)dx + \big(\tfrac{1}{2}u_x + \tfrac{1}{4}u^2 + \tfrac{\lambda}{2}u + \beta\big)dt\Big)\otimes X\\
&+\Big(\big(\tfrac{1}{2}u - \tfrac{\beta}{\lambda} - \lambda\big)dx + \big(\tfrac{1}{2}u_x + \tfrac{1}{4}u^2 - \tfrac{\lambda}{2}u - \beta\big)dt\Big)\otimes Y\Big\}.
\end{aligned}
$$

In this case $W = span_{\mathbb{R}}\{\omega^1, \omega^2\}$ and from the Theorem (4.2.1), we have that $\Lambda(W\oplus\mathfrak{sl}(2))$ has structure of *cdga* via the linear map $d_{\mathfrak{sl}(2)}$, defined by equation (4.2.4), which is:

$$
\begin{array}{rccl}
d_{\mathfrak{sl}(2)}: & \mathfrak{sl}(2) & \to & \Lambda(W\oplus\mathfrak{sl}(2))\\
& X & \mapsto & -\omega^2\wedge X + (\omega^1-\omega^2)\wedge\frac{H}{2} + f_2(X)\\
& Y & \mapsto & \omega^2\wedge Y - (\omega^1+\omega^2)\wedge\frac{H}{2} + f_2(X)\\
& H & \mapsto & (\omega^1+\omega^2)\wedge X - (\omega^1-\omega^2)\wedge Y + f_2(X).
\end{array}
$$

On the other hand, since $(\Lambda W)^3 = 0$, we have that $f_2\colon\mathfrak{sl}(2)\to(\Lambda W)^3$ is the null map and:

$$(\Lambda(W\oplus\mathfrak{sl}(2)))^0 = \mathbb{R}$$

$$(\Lambda(W\oplus\mathfrak{sl}(2)))^1 = W$$

$$(\Lambda(W\oplus\mathfrak{sl}(2)))^2 = W\wedge W\oplus\mathfrak{sl}(2)$$

$$(\Lambda(W\oplus\mathfrak{sl}(2)))^{2n+1} = W\wedge\underbrace{\mathfrak{sl}(2))\wedge\cdots\wedge\mathfrak{sl}(2))}_{n-times},\ \text{ for }\ n\geq 1$$

$$(\Lambda(W\oplus\mathfrak{sl}(2)))^{2n} = W\wedge W\wedge\underbrace{\mathfrak{sl}(2))\wedge\cdots\wedge\mathfrak{sl}(2))}_{(n-1)-times}\oplus\underbrace{\mathfrak{sl}(2))\wedge\cdots\wedge\mathfrak{sl}(2))}_{n-times},\ \text{ for }\ n\geq 2.$$

Let us denote δ the differential in the algebra $\Lambda(W\oplus\mathfrak{sl}(2))$, that is $\delta(\omega^i) = d(\omega^i)$, for $i=1,2$ and $\delta(A) = d_{\mathfrak{sl}(2)}(A)$, for $A\in\mathfrak{sl}(2)$. We observe that:

- $Ker(\delta^0) = \mathbb{R}$

- $Ker(\delta^1) = \{a\omega^1 : a \in \mathbb{R}\} \cong \mathbb{R}$. Indeed, for $a, b \in \mathbb{R}$:

$$\delta^1(a\omega_1 + b\omega^2) = -b\omega^1 \wedge \omega^2 = 0 \Leftrightarrow b = 0$$

- $Ker(\delta^2) = W \wedge W$. In fact, $\delta^2(\omega^1 \wedge \omega^2) = d(\omega^1) \wedge \omega^2 - \omega^1 \wedge d(\omega^2) = \omega^1 \wedge \omega^1 \wedge \omega^2 = 0$, and for $a, b, c \in \mathbb{R}$ we have:

  $\delta^2(aX + bY + cH) =$
  $c\omega^1 \wedge X + (c-a)\omega^2 \wedge X - c\omega^1 \wedge Y + (b+c)\omega^2 \wedge Y + (a-b)\omega^1 \wedge \frac{H}{2} - (a+b)\omega^2 \wedge \frac{H}{2}$,

  since $\{\omega^1 \wedge X,\ \omega^1 \wedge H,\ \omega^1 \wedge Y,\ \omega^2 \wedge X,\ \omega^2 \wedge H,\ \omega^2 \wedge Y\}$ is a basis of $W \wedge \mathfrak{sl}(2)$, then

$$\delta^2(aX + bY + cH) = 0 \Leftrightarrow a = b = c = 0.$$

  Moreover, $Im(\delta^0) = \{0\}$ and $Im(\delta^1) = W \wedge W$,

  then $H^0(\Lambda(W \oplus \mathfrak{sl}(2)), \delta) = \mathbb{R}$, $H^1(\Lambda(W \oplus \mathfrak{sl}(2)), \delta) = \mathbb{R}$ and $H^2(\Lambda(W \oplus \mathfrak{sl}(2)), \delta) = \{0\}$.

Remark 4.3.1.

1. *If $A \in \mathfrak{sl}(2))$, then $\delta^{2n}(\underbrace{A \wedge \cdots \wedge A \wedge A}_{n-times}) = n\delta^2(A) \wedge \underbrace{A \wedge \cdots \wedge A \wedge A}_{(n-1)-times}$, for $n \geq 2$.*

   In fact, since $|A| = 2$, then $\delta^4(A \wedge A) = \delta^2(A) \wedge A + A \wedge \delta^2(A) = 2\delta^2(A) \wedge A$, and

$$\begin{aligned}
&\delta^{2n}(\underbrace{A \wedge \cdots \wedge A \wedge A}_{n-times}) \\
&= \delta^{2(n-1)}(\underbrace{A \wedge \cdots \wedge A \wedge A}_{(n-1)-times}) \wedge A + \underbrace{A \wedge \cdots \wedge A \wedge A}_{(n-1)-times} \wedge \delta^2(A) \\
&= (n-1)\delta^2(A) \wedge (\underbrace{A \wedge \cdots \wedge A \wedge A}_{(n-2)-times}) \wedge A + \underbrace{A \wedge \cdots \wedge A \wedge A}_{(n-1)-times} \wedge \delta^2(A) \quad \textit{by inductive hypothesis} \\
&= n\delta^2(A) \wedge \underbrace{A \wedge \cdots \wedge A \wedge A}_{(n-1)-times}
\end{aligned}$$

   Thus, for $A \in \mathfrak{sl}(2)$, such that $A \neq 0$ and $\delta^2(A) \neq 0$, then $(\underbrace{A \wedge \cdots \wedge A \wedge A}_{n-times}) \notin Ker(\delta^{2n})$.

2. *Let $A_i \in \mathfrak{sl}(2)$, for $i = 1, 2 \ldots, n$, then $\delta^{2n+2}(A_1 \wedge A_2 \wedge \cdots \wedge A_n \wedge \omega^1 \wedge \omega^2) = 0$, since*

$$\delta^{2n}(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \in W \wedge \mathfrak{sl}(2)$$

   this means that $W \wedge W \wedge \underbrace{\mathfrak{sl}(2)) \wedge \cdots \wedge \mathfrak{sl}(2))}_{n-times} \subseteq Ker(\delta^{2n+2})$.

4.4 The sine-Gordon Equation

Let ε be the manifold determined by the scalar differential equation

$$\frac{\partial^2 u}{\partial x \partial t} = \sin u.$$

This equation is of ***pseudo-spherical type***, because there exist one-forms ω^i, $i = 1, 2, 3$,

$$\omega^1 = \frac{1}{\lambda} \sin u \, dt, \qquad \omega^2 = \lambda \, dx + \frac{1}{\lambda} \cos u \, dt, \qquad \omega^3 = u_x \, dx. \tag{4.4.1}$$

These differential forms satisfy the independence condition $\omega^1 \wedge \omega^2 \neq 0$ and the structure equations (4.1.3). In fact:

$$\omega^1 \wedge \omega^2 = -\sin u \; dx \wedge dt,$$

$$\omega^1 \wedge \omega^3 = -\tfrac{1}{\lambda} u_x \sin u \; dx \wedge dt,$$

$$\omega^3 \wedge \omega^2 = \tfrac{1}{\lambda} u_x \cos u \; dx \wedge dt,$$

$$d(\omega^1) = D_x(\tfrac{1}{\lambda}\sin u)\; dx \wedge dt = \tfrac{1}{\lambda} u_x \cos u \; dx \wedge dt$$

$$d(\omega^2) = D_x(\tfrac{1}{\lambda}\cos u)\; dx \wedge dt = -\tfrac{1}{\lambda} u_x \sin u \; dx \wedge dt$$

$$d(\omega^3) = D_t(u_x)\; dt \wedge dx = u_{xt}\; dt \wedge dx = -\sin u \; dx \wedge dt$$

Then, by the Theorem (4.1.1) we have the $\mathfrak{sl}(2)$-valued zero curvature representation for ε given by:

$$\alpha = \frac{1}{2}\{(\lambda dx + \frac{1}{\lambda}\cos u dt) \otimes H + (\frac{1}{\lambda}\sin u dt - u_x dx) \otimes X + (\frac{1}{\lambda}\sin u dt + u_x dx) \otimes Y\}$$

In this case $W = span_{\mathbb{R}}\{\omega^1, \omega^2, \omega^3\}$ and from the Theorem (4.2.1) we have that $\Lambda(W \oplus \mathfrak{sl}(2))$ has structure of *cdga* via the linear map $d_{\mathfrak{sl}(2)}$, defined by equation (4.2.4):

$$\begin{array}{rcll} d_{\mathfrak{sl}(2)}: & \mathfrak{sl}(2) & \rightarrow & \Lambda(W \oplus \mathfrak{sl}(2)) \\ & X & \mapsto & -\omega^2 \wedge X + (\omega^1 + \omega^3) \wedge \frac{H}{2} + f_2(X) \\ & Y & \mapsto & \omega^2 \wedge Y - (\omega^1 - \omega^3) \wedge \frac{H}{2} + f_2(Y) \\ & H & \mapsto & (\omega^1 - \omega^3) \wedge X - (\omega^1 + \omega^3) \wedge Y + f_2(H) \end{array}$$

On the other hand, since $(\Lambda W)^3 = span_{\mathbb{R}}\{\omega^1 \wedge \omega^2 \wedge \omega^3\}$, the linear map $f_2 \colon \mathfrak{sl}(2) \rightarrow (\Lambda W)^3$ satisfies that $f_2(A) = \gamma \omega^1 \wedge \omega^2 \wedge \omega^3$, for some $\gamma \in \mathbb{R}$. Moreover, as $(\Lambda W)^4 = 0$, we have:

$$(\Lambda(W \oplus \mathfrak{sl}(2)))^0 = \mathbb{R}$$

$$(\Lambda(W \oplus \mathfrak{sl}(2)))^1 = W$$

$$(\Lambda(W \oplus \mathfrak{sl}(2)))^2 = W \wedge W \oplus \mathfrak{sl}(2)$$

$$(\Lambda(W \oplus \mathfrak{sl}(2)))^3 = W \wedge \mathfrak{sl}(2)) \oplus W \wedge W \wedge W,$$

$$(\Lambda(W \oplus \mathfrak{sl}(2)))^{2n+1} = W \wedge \underbrace{\mathfrak{sl}(2)) \wedge \cdots \wedge \mathfrak{sl}(2))}_{n-times} \oplus W \wedge W \wedge W \wedge \underbrace{\mathfrak{sl}(2)) \wedge \cdots \wedge \mathfrak{sl}(2))}_{(n-1)-times}, \quad \text{for } n \geq 2$$

$$(\Lambda(W \oplus \mathfrak{sl}(2)))^{2n} = W \wedge W \wedge \underbrace{\mathfrak{sl}(2)) \wedge \cdots \wedge \mathfrak{sl}(2))}_{(n-1)-times} \oplus \underbrace{\mathfrak{sl}(2)) \wedge \cdots \wedge \mathfrak{sl}(2))}_{n-times}, \quad \text{for } n \geq 2$$

Let us denote δ the differential in the algebra $\Lambda(W \oplus \mathfrak{sl}(2))$, that is $\delta(\omega^i) = d(\omega^i)$, for $i = 1, 2, 3$ and $\delta(A) = d_{\mathfrak{sl}(2)}(A)$, for $A \in \mathfrak{sl}(2)$. We observe that:

- $Ker(\delta^0) = \mathbb{R}$

- $Ker(\delta^1)=\{0\}$, since $\{\omega^3\wedge\omega^2,\ \omega^1\wedge\omega^3,\ \omega^1\wedge\omega^2\}$ is a basis of $W\wedge W$. Thus, $a,b,c\in\mathbb{R}$:

$$\delta^1(a\omega^1+b\omega^2+c\omega^3)=a\omega^3\wedge\omega^2+b\omega^1\wedge\omega^3+c\omega^1\wedge\omega^2=0\Leftrightarrow a,b,c=0$$

- $Ker(\delta^2)=W^2$. In fact,

$$\delta^2(\omega^1\wedge\omega^2)=d(\omega^1)\wedge\omega^2-\omega^1\wedge d(\omega^2)=\omega^3\wedge\omega^2\wedge\omega^2-\omega^1\wedge\omega^1\wedge\omega^3=0,$$
$$\delta^2(\omega^1\wedge\omega^3)=d(\omega^1)\wedge\omega^3-\omega^1\wedge d(\omega^3)=\omega^3\wedge\omega^2\wedge\omega^3-\omega^1\wedge\omega^1\wedge\omega^2=0,$$
$$\delta^2(\omega^3\wedge\omega^2)=d(\omega^3)\wedge\omega^2-\omega^3\wedge d(\omega^2)=\omega^1\wedge\omega^2\wedge\omega^2-\omega^3\wedge\omega^1\wedge\omega^3=0.$$

And for $a,b,c\in\mathbb{R}$, we have:

$$\begin{aligned}\delta^2(aX+bY+cH)\ &=\omega^2\wedge(-aX+bY)+\omega^1\wedge(a\tfrac{H}{2}-b\tfrac{H}{2}+cX-cY)\\ &+\omega^3\wedge(a\tfrac{H}{2}+b\tfrac{H}{2}-cX-cY)+af_2(X)+bf_2(Y)+cf_2(H).\end{aligned}$$

Since $\{\omega^1\wedge X,\ \omega^1\wedge H,\ \omega^1\wedge Y,\omega^2\wedge X,\ \omega^2\wedge H,\ \omega^2\wedge Y,\ \omega^3\wedge X,\omega^3\wedge H,\ \omega^3\wedge Y\}$ is a basis of $W\wedge\mathfrak{sl}(2)$ and $af_2(X)+bf_2(Y)+cf_2(H)=\gamma\omega^1\wedge\omega^2\wedge\omega^3$, for some $\gamma\in\mathbb{R}$, then

$$\delta^2(aX+bY+cH)=0\Leftrightarrow a=b=c=0.$$

Moreover, $Im(\delta^0)=\{0\}$ and $Im(\delta^1)=W\wedge W$,

then $H^0(\Lambda(W\oplus\mathfrak{sl}(2)),\delta)=\mathbb{R}$, $H^1(\Lambda(W\oplus\mathfrak{sl}(2)),\delta)=\{0\}$ and $H^2(\Lambda(W\oplus\mathfrak{sl}(2)),\delta)=\{0\}$.

In this case the item 1 of Remark (4.3.1) is true, but the item 2 is not satisfied, here are some calculations for δ^4:

$$\delta^4(\omega^1\wedge\omega^2\wedge X)=\omega^1\wedge\omega^2\wedge\omega^3\wedge\frac{H}{2}$$
$$\delta^4(\omega^1\wedge\omega^2\wedge Y)=\omega^1\wedge\omega^2\wedge\omega^3\wedge\frac{H}{2}$$
$$\delta^4(\omega^1\wedge\omega^2\wedge H)=-\omega^1\wedge\omega^2\wedge\omega^3\wedge(X+Y)$$
$$\delta^4(\omega^1\wedge\omega^3\wedge X)=-\omega^1\wedge\omega^3\wedge\omega^2\wedge X$$
$$\delta^4(\omega^1\wedge\omega^3\wedge Y)=\omega^1\wedge\omega^3\wedge\omega^2\wedge Y$$
$$\delta^4(\omega^1\wedge\omega^3\wedge H)=0$$
$$\delta^4(\omega^3\wedge\omega^2\wedge X)=\omega^3\wedge\omega^2\wedge\omega^1\wedge\frac{H}{2}$$
$$\delta^4(\omega^3\wedge\omega^2\wedge Y)=-\omega^3\wedge\omega^2\wedge\omega^1\wedge\frac{H}{2}$$
$$\delta^4(\omega^3\wedge\omega^2\wedge H)=\omega^3\wedge\omega^2\wedge\omega^1\wedge(X-Y)$$

The following theorem allows us to identify some homologies for the algebra $\Lambda(W\oplus\mathfrak{sl}(2))$. We recall that $W=span_{\mathbb{R}}\{\omega^1,\omega^2,\omega^3\}$ as in (4.2.1).

Theorem 4.4.1. *If $\{\omega^1,\omega^2,\omega^3\}$ are linearly independent, then*

$$H^0(\Lambda(W\oplus\mathfrak{sl}(2)),\delta)\cong\mathbb{R},\ H^1(\Lambda(W\oplus\mathfrak{sl}(2)),\delta)\cong 0\ \text{ and }\ H^2(\Lambda(W\oplus\mathfrak{sl}(2)),\delta)\cong 0.$$

And if $\{\omega^1, \omega^2, \omega^3\}$ *are linearly dependent, this is* $\omega^3 = \lambda\omega^1 + \beta\omega^2$*, then*

$H^0(\Lambda(W \oplus \mathfrak{sl}(2)), \delta) \cong \mathbb{R},\ H^1(\Lambda(W \oplus \mathfrak{sl}(2)), \delta) \cong \mathbb{R}$ *when* $\lambda = 0$ *or* $\beta = 0$ *and* $H^2(\Lambda(W \oplus \mathfrak{sl}(2)), \delta) \cong 0.$

Proof. The first part of theorem is obtained by observing that $Ker(\delta^0) = \mathbb{R},\ \ Ker(\delta^1) = \{0\}$, $Ker(\delta^2) = W \wedge W,\ Im(\delta^0) = \{0\}$ and $Im(\delta^1) = W \wedge W$. as in the sine-Gordon equation.

The second part is a generalization of the Burgers' equation case. For this we note that:

- $Ker(\delta^0) = \mathbb{R}$
- $Ker(\delta^1) = \{a\omega^1 : a \in \mathbb{R}\} \cong \mathbb{R}$. Indeed,

  $$\delta^1(\omega^1) = (\lambda\omega^1 + \beta\omega^2) \wedge \omega^2 = \lambda\omega^1 \wedge \omega^2$$

  $$\delta^1(\omega^1) = \omega^1 \wedge (\lambda\omega^1 + \beta\omega^2) = \beta\omega^1 \wedge \omega^2$$

  and for $a, b, c \in \mathbb{R}$:

  $$\delta^1(a\omega^1 + b\omega^2) = (a\lambda + b\beta)\omega^1 \wedge \omega^2 = 0 \Leftrightarrow a\lambda + b\beta = 0$$

  Observe that $\lambda = \beta = 0$ is impossible, since $d(\omega^3) = \omega^1 \wedge \omega^2 = 0$.

  If $\lambda = 0$ and $\beta \neq 0$, then $Ker(\delta^1) = \{a\omega^1 : a \in \mathbb{R}\} \cong \mathbb{R}$.

  If $\beta = 0$ and $\lambda \neq 0$, then $Ker(\delta^1) = \{b\omega^2 : b \in \mathbb{R}\} \cong \mathbb{R}$.

  If $\lambda \neq 0$ and $\beta \neq 0$, then $Ker(\delta^1) = \{\frac{-b\beta}{\lambda}\omega^1 + b\omega^2 : b \in \mathbb{R}\}$.
- $Ker(\delta^2) = W \wedge W$. In fact,

  $$\delta^2(\omega^1 \wedge \omega^2) = d(\omega^1) \wedge \omega^2 - \omega^1 \wedge d(\omega^2) = (\lambda\omega^1 \wedge \omega^2) \wedge \omega^2 + \omega^1 \wedge (\beta\omega^1 \wedge \omega^2) = 0,$$

  and for $a, b, c \in \mathbb{R}$, we have:

  $\delta^2(aX + bY + cH) =$
  $(c - c\lambda)\omega^1 \wedge X + (-a - c\beta)\omega^2 \wedge X + (-c - c\lambda)\omega^1 \wedge Y$
  $+(b - c\beta)\omega^2 \wedge Y + (a + a\lambda - b + b\lambda)\omega^1 \wedge \frac{H}{2} + (a\beta + b\beta)\omega^2 \wedge \frac{H}{2}.$

  Since $\{\omega^1 \wedge X,\ \omega^1 \wedge H,\ \omega^1 \wedge Y,\ \omega^2 \wedge X,\ \omega^2 \wedge H,\ \omega^2 \wedge Y\}$ is a basis of $W \wedge W$; then, for any case of λ, β:

  $$\delta^2(aX + bY + cH) = 0 \Leftrightarrow a = b = c = 0.$$

  Therefore $Ker(\delta^2) = W \wedge W$.

On the other hand $Im(\delta^0) = \{0\}$ and $Im(\delta^1) = W \wedge W$. This proves the theorem.

□

4.5 Gauge transformation

The main goal of this work is to generate Sullivan decomposable algebras and we saw that this can be achieved by using a twisting matrix. We introduce the notion of gauge transformation in such a way that the structure equation (4.1.3) are invariant under the gauge transformation. In this way, we have new twisting matrices, although the algebra the Sullivan decomposable generated are not isomorphic to the original.

Below we review some aspect of Lie groups with the goal of arriving to expression (4.5.8).

Definition 4.5.1. *Let G be a Lie group and R_g the right multiplication for each $g \in G$, given by*

$$\begin{array}{rccc} R_g: & G & \to & G \\ & h & \mapsto & hg. \end{array}$$

The $\mathfrak{g}$-valued 1-form over G

$$\begin{array}{rccc} \omega_G(g): & T_gG & \to & \mathfrak{g} \cong T_eG \\ & v_g & \mapsto & T_gR_g^{-1}(v_g), \end{array}$$

is called the ***right Maurer-Cartan form****.*

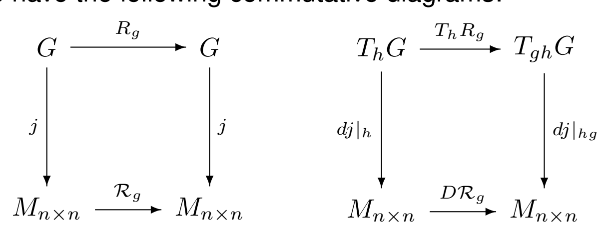
Let us see the right Maurer-Cartan form for the particular case in that G is a subgroup of $GL(n, \mathbb{R})$, the Lie group of invertible real matrices $n \times n$). We consider the inclusion map

$$\begin{array}{rccc} j: & G & \to & M_{n\times n} \\ & A & \mapsto & [a^i_j] \end{array} \tag{4.5.1}$$

where $M_{n\times n}$ is the set of $n \times n$ matrices. Then we have the differential $dj: TG \to M_{n\times n}$ and two right multiplications $R_g: G \to G$ and

$$\begin{array}{rccc} \mathcal{R}_g: & M_{n\times n} & \to & M_{n\times n} \\ & h & \mapsto & hg \end{array}$$

for $g \in G$. Moreover, we have the following commutative diagrams:



for any $h \in G$. Since $\mathcal{R}_g$ is linear, $D\mathcal{R}_g(A) = \mathcal{R}_g$ for all $A \in M_{n\times n}$, then for $v_g \in T_gG$

$$\begin{aligned} dj|_e\big(\omega_G(v_g)\big) &= dj|_e(T_gR_{g^{-1}}v_g) \\ &= \mathcal{R}_{g^{-1}}\big(dj|_g(v_g)\big) \\ &= dj|_g(v_g)g^{-1} \\ &= dj(v_g)\,(j \circ \pi(v_g))^{-1} \end{aligned} \tag{4.5.2}$$

where $\pi\colon TG \to G$ is the tangent bundle projection, this is:

$$\begin{array}{ccccc} TG & \xrightarrow{\pi} & G & \xrightarrow{j} & M_{n\times n} \\ v_g & \mapsto & g & \mapsto & g \end{array},$$

We note that the effect of $dj|_e$ is simply to interpret elements of T_eG as matrices, and so can suppressed it from equality (4.5.2). Taking this into account, and applying a reasonable abbreviation for

$$\omega_G(v_g) = dj(v_g)\,(j\circ\pi(v_g))^{-1}\,,$$

we can write that

$$\omega_G = dj j^{-1}. \tag{4.5.3}$$

But notice that if x^i_j are the coordinate functions on $M_{n\times n}$ defined by $x^i_j(A) = a^i_j$, where $A = [a^i_j]$, then j is none other that the map $[x^i_j]\colon A \mapsto [x^i_j(A)] = [a^i_j]$. So j^{-1} is the map $[x^i_j]^{-1}\colon A \to [a^i_j]^{-1}$ and $dj(A) = [dx^i_j(A)]$, since:

$$\begin{array}{rccl} dx^i_j(A): & T_AG & \to & \mathbb{R} \\ & \left.\dfrac{\partial}{\partial\varphi^k}\right|_A & \mapsto & \dfrac{\partial(id\circ x^i_j\circ\varphi^{-1})}{\partial\varphi^k}(\varphi(A)), \end{array}$$

where φ^k are coordinates functions on G, then $dx^i_j(A) = \sum\limits_{k=1}^{m} \dfrac{\partial(id\circ x^i_j\circ\varphi^{-1})}{\partial\varphi^k}(\varphi(A))d\varphi^k(A)$.

On the other hand

$$\begin{array}{rccl} dj(A): & T_AG & \to & T_{[a^i_j]}M_{n\times n}\cong\mathbb{R}^{n\times n} \\ & \left.\dfrac{\partial}{\partial\varphi^k}\right|_A & \mapsto & \left[\left(\dfrac{\partial(x^i_j\circ j\circ\varphi^{-1})}{\partial\varphi^k}(\varphi(A))\right)^i_j\right], \end{array}$$

therefore $dj(A)(v_A) = [dx^i_j(A)(v_A)]$. So that of (4.5.3) we arrive to the expression:

$$\omega_G(A) = [dx^i_j(A)][x^i_j]^{-1}(A) = dAA^{-1} \tag{4.5.4}$$

Example 4.5.1. *The Right Maurer-Cartan form of $G = SO(2)$, the Lie group of real orthogonal matrices with determinant 1, is given by:*

$$\begin{array}{ccc} j: SO(2) & \xrightarrow{[x^i_j]} & M_{2\times 2} \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\rho=(x^i_j)} \\ \mathbb{R} & \longrightarrow & \mathbb{R}^4 \end{array}$$

$$\begin{array}{rccc} \varphi^{-1}: & \mathbb{R} & \to & SO(2) \\ & \theta & \mapsto & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{array} \qquad \begin{array}{rccc} \rho: & M_{2\times 2} & \to & \mathbb{R}^4 \\ & \begin{pmatrix} x & y \\ z & w \end{pmatrix} & \mapsto & (x,y,z,w) \end{array}$$

Let $A \in G$*, so* $A = \begin{pmatrix} \cos(\varphi(A)) & -\sin(\varphi(A)) \\ \sin(\varphi(A)) & \cos(\varphi(A)) \end{pmatrix}$*, and*

$$\omega_G(A) = \begin{pmatrix} -\sin(\varphi(A))d\varphi(A) & -\cos(\varphi(A))d\varphi(A) \\ \cos(\varphi(A))d\varphi(A) & -\sin(\varphi(A))d\varphi(A) \end{pmatrix} \begin{pmatrix} \cos(\varphi(A)) & \sin(\varphi(A)) \\ -\sin(\varphi(A)) & \cos(\varphi(A)) \end{pmatrix} = \begin{pmatrix} 0 & -d\varphi(A) \\ d\varphi(A) & 0 \end{pmatrix}.$$

Now we consider the smooth map $f\colon M \to G$, where M is a manifold of dimension n, so for the $\mathfrak{g}$-valued one-form ω_G on M, the pullback $f^*(\omega_G)$ is given by

$$\begin{array}{rccc} f^*: & \Omega^1(G, \mathfrak{g}) & \to & \Omega^1(M, \mathfrak{g}) \\ & \omega_G & \mapsto & f^*(\omega_G), \end{array}$$

where

$$\begin{array}{rccc} f^*(\omega_G)(p): & T_pM & \to & \mathfrak{g} \\ & \xi_p & \mapsto & (\omega_G)_{(f(p))}(d(f)(p)(\xi_p)). \end{array}$$

Again let us consider the case where G is a subgroup of $GL(n, \mathbb{R})$ with the goal of arriving at the expression $f^*(\omega_G) = d(f)f^{-1}$. We use the map j defined in (4.5.1),

$$\begin{array}{rccc} j \circ f: & M & \to & M_{n\times n} \\ & p & \mapsto & [f(p)^i_j] \end{array}$$

As we saw before, and $j = [x^i_j]$, therefore:

$$\begin{array}{rccc} d(x^i_j \circ f)(p): & T_pM & \to & \mathbb{R} \\ & \left.\dfrac{\partial}{\partial \psi^k}\right|_A & \mapsto & \dfrac{\partial(id \circ x^i_j \circ f \circ \psi^{-1})}{\partial \psi^k}(\psi(A)), \end{array}$$

where ψ^k are coordinates functions on M.

Then $d(x^i_j \circ f)(p) = \sum\limits_{k=1}^{m} \dfrac{\partial(id \circ x^i_j \circ f \circ \psi^{-1})}{\partial \psi^k}(\psi(A))d\psi^k(p)$.

Therefore, $d(j \circ f)(p)(\xi_p) = [d(x^i_j \circ f)(p)(\xi_p)] \in G$, and by the rule chain,

$$d(j \circ f)(p)(\xi_p) = [d(x^i_j)(f(p))(d(f)(p)(\xi_p))] \tag{4.5.5}$$

Using the expression in (4.5.4), we have

$$(\omega_G)_{(f(p))}(d(f)(p)(\xi_p)) = [dx^i_j(f(p))(d(f)(p)(\xi_p))][x^i_j]^{-1}(f(p)),$$

and for the equality (4.5.5), we have the following identification:

$$(\omega_G)_{(f(p))}(d(f)(p)(\xi_p)) = (d(j \circ f)(p)(\xi_p))(j \circ f)^{-1}(p).$$

Thus, $f^*(\omega_G)_{(f(p))} = d(f)(p)(f(p))^{-1}$ and

$$f^*(\omega_G) = d(f)f^{-1}. \tag{4.5.6}$$

On the other hand, for some fixed $g \in G$ the map

$$\begin{array}{rccc} C_g: & G & \to & G \\ & x & \mapsto & gxg^{-1} \end{array}$$

is a Lie group automorphism called the ***conjugation map***, and the tangent map

$$\begin{array}{rccc} T_eC_g: & T_eG & \to & T_eG \\ & v_e & \mapsto & T_eC_g(v_e) \end{array}$$

$T_eC_g\colon \mathfrak{g} \to \mathfrak{g}$ by the isomorphism $T_eG \cong \mathfrak{g}$, is called the ***adjoint map*** and it is denoted Ad_g. In our case in which $G \subset GL(n, \mathbb{R})$, we have $Ad_A(B) = ABA^{-1}$, for $A \in G$ and $B \in \mathfrak{g}$ (see [11]).

Let $\alpha \in \Omega^1(M, \mathfrak{g})$. Then $(j \circ f)\alpha(j \circ f)^{-1} \in \Omega^1(M, M_{n\times n})$, in which we consider α and $j \circ f$ as $M_{n\times n}$-valued one-forms over M, so

$$\begin{array}{rccc} (j \circ f)\alpha(j \circ f)^{-1}(p): & T_pM & \to & M_{n\times n} \\ & \xi_p & \mapsto & [x^i_j \circ f(p)]\alpha_p(\xi_p)[x^i_j \circ f(p)]^{-1}. \end{array}$$

By the adjoint map of a Lie group G, we can ensure that for each $p \in M$ the term

$$[x^i_j \circ f(p)]\alpha_p[x^i_j \circ f(p)]^{-1} \in \mathfrak{g}$$

and therefore $(j \circ f)\alpha(j \circ f)^{-1} \in \Omega^1(M, \mathfrak{g})$, which can be identified with $f\alpha f^{-1}$.

In general, we obtain the last form, in the following manner: For a fixed element g in G, Ad_g induces the map

$$\begin{array}{rccc} *Ad_g: & \Omega^1(M, \mathfrak{g}) & \to & \Omega^1(M, \mathfrak{g}) \\ & \alpha & \mapsto & *Ad_g(\alpha) \end{array}$$

where, for $p \in M$

$$\begin{array}{rccc} *Ad_g(\alpha)(p): & T_pM & \to & \mathfrak{g} \\ & \xi_p & \mapsto & Ad_g(\alpha_p(\xi_p)) \end{array}$$

Finally, us define the $\mathfrak{g}$-valued 1-form over M:

$$\begin{array}{rccc} Ad_f(\alpha)(p): & \Omega^1(M, \mathfrak{g}) & \to & \Omega^1(M, \mathfrak{g}) \\ & \alpha & \mapsto & Ad_f(\alpha) \end{array}$$

$$\begin{array}{rccc} Ad_f(\alpha)(p): & T_pM & \to & \mathfrak{g} \\ & \xi_p & \mapsto & Ad_{f(p)}\big(\alpha_p(\xi_p)\big) \end{array}$$

In the case of linear Lie Groups $G \subset GL(n, \mathbb{R})$, we can write

$$\begin{array}{rccc} Ad_f(\alpha)(p): & T_pM & \to & \mathfrak{g} \\ & \xi_p & \mapsto & [x^i_j \circ f(p)]\alpha_p(\xi_p)[x^i_j \circ f(p)]^{-1}. \end{array}$$

So, we arrive at the identification

$$Ad_f(\alpha) = f\alpha f^{-1}. \tag{4.5.7}$$

Therefore, for a Lie group G with Lie algebra $\mathfrak{g}$, from (4.5.6) and (4.5.7), we have the $\mathfrak{g}$-valued one-form over M

$$f\alpha f^{-1} + d(f)f^{-1}, \text{ for } G \subset GL(n,\mathbb{R})$$

$$Ad_f(\alpha) + f^*(\omega_G), \text{ for any } G \text{ group of Lie } .$$

We can write the following definition:

Definition 4.5.2. *Let G be a group of lie such that $G \subset GL(n,\mathbb{R})$, with lie algebra $\mathfrak{g}$ and $f\colon M \to G$ in $C^\infty(M,G)$. Define the $\mathfrak{g}$-valued 1-form over M:*

$$f\alpha f^{-1} + d(f)f^{-1} \tag{4.5.8}$$

*This form is known as a **gauge transformation** on α by the map f.*

Now we apply the above remarks to equations of pseudo-spherical type. We assume that $f \in C^\infty(\varepsilon, SL(2))$, then $f = \begin{pmatrix} a & b \\ c & e \end{pmatrix}$ where $a, b, c, e \in C^\infty(\varepsilon,\mathbb{R})$. For the last identification of f as $j \circ f \in C^\infty(\varepsilon, M_{2\times 2})$ then

$$d(f) = \begin{pmatrix} d(a) & d(b) \\ d(c) & d(e) \end{pmatrix}$$

and we denoted

$$d_H(f) = \begin{pmatrix} d_H(a) & d_H(b) \\ d_H(c) & d_H(e) \end{pmatrix}$$

And if we consider $M_{2\times 2}$ as Lie algebra and exterior derivative on the $M_{2\times 2}$-valued form, then by Remark (3.4.1), we have that $d(f) = \mathbf{d}f$ and $d_H(f) = \mathbf{d_H}f$. Under these identifications we enunciate the following theorem:

Theorem 4.5.1. *Let be ε a pseudo-spherical manifold with associated one forms ω^i $(i = 1,2,3)$ and $\mathfrak{sl}(2)$-valued zero curvature representation*

$$\alpha = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix} \in \Omega^{1,0}(\varepsilon, \mathfrak{sl}(2)).$$

If $f \in C^\infty(\varepsilon, SL(2))$, then $\widehat{\alpha} = f\alpha f^{-1} + d_H(f)f^{-1} \in \Omega^{1,0}(\varepsilon,\mathfrak{sl}(2))$ and $\widehat{\alpha}$ is also a $\mathfrak{sl}(2)$-valued zero curvature representation for ε.

Proof. For the last definition $\widehat{\alpha} = f\alpha f^{-1} + d(f)f^{-1} \in \Omega^1(\varepsilon,\mathfrak{sl}(2))$. So, we can calculate the exterior derivative on the $\mathfrak{sl}(2)$-valued form $\widehat{\alpha}$, this is $\mathbf{d}(\widehat{\alpha})$.

Now, we observe that by applying the Leibnitz rule and using the constant map $h\colon \varepsilon \to G$ such that $h(\epsilon) = I$ where I is the identity matrix we have, this map can be obtained by the map f by the expression $d(ff^{-1}) = d(h) = 0$, and so $\mathbf{d}(f)f^{-1} + f\mathbf{d}(f^{-1}) = 0$, so

$$\mathbf{d}(f^{-1}) = -f^{-1}\mathbf{d}(f)f^{-1}, \tag{4.5.9}$$

and we have:

$$
\begin{aligned}
\mathbf{d}(\widehat{\alpha}) &= \mathbf{d}(f)\wedge\alpha f^{-1} + f\wedge\mathbf{d}(\alpha f^{-1}) + \mathbf{d}(\mathbf{d}(f))\wedge f^{-1} - \mathbf{d}(f)\wedge\mathbf{d}(f^{-1})\\
&= \mathbf{d}(f)\wedge\alpha f^{-1} + f\wedge\mathbf{d}(\alpha)\wedge f^{-1} - f\wedge\alpha\wedge\mathbf{d}(f^{-1}) - \mathbf{d}(f)\wedge\mathbf{d}(f^{-1})\\
&= \mathbf{d}(f)\wedge\alpha f^{-1} + f\mathbf{d}(\alpha)f^{-1} - f\alpha\wedge\mathbf{d}(f^{-1}) - \mathbf{d}(f)\wedge\mathbf{d}(f^{-1}).
\end{aligned}
$$

On the other hand, using the properties of exterior product and (4.5.9)

$$
\begin{aligned}
\widehat{\alpha}\wedge\widehat{\alpha} &= (f\alpha f^{-1})\wedge(f\alpha f^{-1}) + (f\alpha f^{-1})\wedge(\mathbf{d}(f)f^{-1}) + (\mathbf{d}(f)f^{-1})\wedge(f\alpha f^{-1})\\
&\quad + (\mathbf{d}(f)f^{-1})\wedge(\mathbf{d}(f)f^{-1})\\
&= f\alpha\wedge\alpha f^{-1} + f\alpha f^{-1}\wedge\mathbf{d}(f)f^{-1} + \mathbf{d}(f)\wedge\alpha f^{-1} + \mathbf{d}(f)f^{-1}\wedge\mathbf{d}(f)f^{-1}\\
&= f\alpha\wedge\alpha f^{-1} - f\alpha\wedge\mathbf{d}(f^{-1}) + \mathbf{d}(f)\wedge\alpha f^{-1} + \mathbf{d}(f)\wedge\mathbf{d}(f^{-1}).
\end{aligned}
$$

Moreover, $\alpha\in\Omega^{1,0}(\varepsilon, M_{2\times 2})$ then $f\alpha f^{-1} + \mathbf{d_H}(f)f^{-1}\in\Omega^{1,0}(\varepsilon, M_{2\times 2})$ and

$$
\mathbf{d}(\widehat{\alpha}) = \mathbf{d_H}(f)\wedge\alpha f^{-1} + f\mathbf{d_H}(\alpha)f^{-1} + f\alpha\wedge\mathbf{d_H}(f^{-1}) + \mathbf{d_H}(f)\wedge\mathbf{d_H}(f^{-1}).
$$

By hypothesis $\mathbf{d_H}(\alpha) = \alpha\wedge\alpha$, therefore $\mathbf{d_H}(\widehat{\alpha}) = \widehat{\alpha}\wedge\widehat{\alpha}$, this is, $\widehat{\alpha}$ is a $\mathfrak{sl}(2)$-valued zero curvature representation for ε (see Definition 3.4.1).

□

Let us observe explicitly $\widehat{\alpha}$ for $f\colon\varepsilon\to SL(2)$ such that $f = \begin{pmatrix} a & b\\ c & e\end{pmatrix}$ where $a, b, c, e\in C^\infty(\varepsilon,\mathbb{R})$

$$
\begin{aligned}
\widehat{\alpha} &= f\alpha f^{-1} + d_H(f)f^{-1}\\
&= \begin{pmatrix} a & b\\ c & e\end{pmatrix}\begin{pmatrix}\omega^2 & \omega^1-\omega^3\\ \omega^1+\omega^3 & -\omega^2\end{pmatrix}\begin{pmatrix} e & -b\\ -c & a\end{pmatrix} + \frac{1}{2}\begin{pmatrix} d_H(a) & d_H(b)\\ d_H(c) & d_H(e)\end{pmatrix}\begin{pmatrix} e & -b\\ -c & a\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}\widehat{\alpha}_{11} & \widehat{\alpha}_{12}\\ \widehat{\alpha}_{21} & \widehat{\alpha}_{22}\end{pmatrix}
\end{aligned}
$$

where

$\widehat{\alpha}_{11} = (ae+bc)\omega^2 + (be-ac)\omega^1 + (be+ac)\omega^3 + 2(d_H(a)e - d_H(b)c)$

$\widehat{\alpha}_{12} = -2ab\omega^2 + (a^2-b^2)\omega^1 - (b^2+a^2)\omega^3 - 2(d_H(a)b - d_H(b)a)$

$\widehat{\alpha}_{21} = 2ce\omega^2 + (e^2-c^2)\omega^1 - (e^2+c^2)\omega^3 + 2(d_H(c)e - d_H(e)c)$

$\widehat{\alpha}_{22} = -(ae+bc)\omega^2 - (be-ac)\omega^1 - (be+ac)\omega^3 - 2(d_H(c)b - d_H(e)a)$

Since, $f\in C^\infty(\varepsilon, SL(2))$ then $ae-cb=1$, therefore $d_H(ae-cb)=0$ and

$$
d_H(a)e - d_H(b)c = d_H(c)b - d_H(e)a,
$$

therefore as above $\widehat{\alpha} \in \Omega^{1,0}(\varepsilon, \mathfrak{sl}(2))$. Moreover $\widehat{\alpha}$ induces one-forms $\widehat{\omega}^i$, $i = 1, 2, 3$ such that

$$\widehat{\alpha} = \frac{1}{2}\begin{pmatrix} \widehat{\omega}^2 & \widehat{\omega}^1 - \widehat{\omega}^3 \\ \widehat{\omega}^1 + \widehat{\omega}^3 & -\widehat{\omega}^2 \end{pmatrix}.$$

Where,

$$\widehat{\omega}^1 = -2(ab - ce)\omega^2 + (a^2 - b^2 + e^2 - c^2)\omega^1 - (b^2 + a^2 - e^2 - c^2)\omega^3 - 2(d_H(a)b - d_H(b)a - d_H(c)e + d_H(e)c)$$

$$\widehat{\omega}^2 = (ae + bc)\omega^2 + (be - ac)\omega^1 + (be + ac)\omega^3 + 2(d_H(a)e - d_H(b)c)$$

$$\widehat{\omega}^3 = 2(ab + ce)\omega^2 - (a^2 - b^2 - e^2 + c^2)\omega^1 - (b^2 + a^2 + e^2 + c^2)\omega^3 - 2(d_H(a)b - d_H(b)a + d_H(c)e - d_H(e)c)$$

From the last theorem, since $\mathbf{d_H}(\widehat{\alpha}) = \widehat{\alpha} \wedge \widehat{\alpha}$, then:

$$d_H(\widehat{\omega}^1) = \widehat{\omega}^3 \wedge \widehat{\omega}^2$$
$$d_H(\widehat{\omega}^2) = \widehat{\omega}^1 \wedge \widehat{\omega}^3$$
$$d_H(\widehat{\omega}^3) = \widehat{\omega}^1 \wedge \widehat{\omega}^2.$$

Examples 4.5.1. *We present some case for the function $f\colon \varepsilon \to G$ where $\rho\colon \varepsilon \to \mathbb{R}$ is a smooth function:*

$$\begin{pmatrix} \cos(\frac{\rho}{2}) & -\sin(\frac{\rho}{2}) \\ \sin(\frac{\rho}{2}) & \cos(\frac{\rho}{2}) \end{pmatrix} \qquad \begin{cases} \widehat{\omega}^1 = \sin(\rho)\,\omega^2 + \cos(\rho)\,\omega^1 \\ \widehat{\omega}^2 = \cos(\rho)\,\omega^2 - \sin(\rho)\,\omega^1 \\ \widehat{\omega}^3 = \omega^3 + d_H(\rho) \end{cases}$$

$$\begin{pmatrix} \cosh(\frac{\rho}{2}) & \sinh(\frac{\rho}{2}) \\ \sinh(\frac{\rho}{2}) & \cosh(\frac{\rho}{2}) \end{pmatrix} \qquad \begin{cases} \widehat{\omega}^1 = \omega^1 + d_H\rho \\ \widehat{\omega}^2 = \cosh(\rho)\,\omega^2 + \sinh(\rho)\,\omega^3 \\ \widehat{\omega}^3 = \sinh(\rho)\,\omega^2 + \cosh(\rho)\,\omega^3 \end{cases}$$

$$\begin{pmatrix} \cosh(\frac{\rho}{2}) + \sinh(\frac{\rho}{2}) & 0 \\ 0 & \cosh(\frac{\rho}{2}) - \sinh(\frac{\rho}{2}) \end{pmatrix} \qquad \begin{cases} \widehat{\omega}^1 = \cosh(\rho)\,\omega^1 - \sinh(\rho)\,\omega^3 \\ \widehat{\omega}^2 = \omega^2 + d_H(\rho) \\ \widehat{\omega}^3 = -\sinh(\rho)\,\omega^1 + \cosh(\rho)\,\omega^3 \end{cases}$$

Thus, in our terminology of twisting matrices, $\Theta_{\widehat{\alpha}}$ determines a Sullivan decomposable algebra $\Lambda(span_{\mathbb{R}}\{\widehat{\omega}^1, \widehat{\omega}^2, \widehat{\omega}^3\} \oplus \mathfrak{sl}(2))$.

Given a $\mathfrak{sl}(2)$-valued zero curvature representation for ε say α, we introduce the following notations:

$$W_\alpha := span_{\mathbb{R}}\{\omega^1, \omega^2, \omega^3\},$$

$$(\Lambda W_\alpha, d) := (W_\alpha, d_\alpha), \quad \text{(see 4.2.2)},$$

$$\Lambda_\alpha := \Lambda(W_\alpha \oplus \mathfrak{sl}(2), \delta_\alpha)$$

and the map $d_{\mathfrak{sl}(2)}$ defined in (4.2.4) by $d_{\mathfrak{sl}(2)_\alpha}$.

Theorem 4.5.2. *Assume α is a zcr for ε and $\overline{\alpha}$ is a zcr for $\overline{\varepsilon}$ and consider the subvector spaces W_α and $W_{\overline{\alpha}}$ of $\Omega^{1,0}(\varepsilon)$ and $\Omega^{1,0}(\overline{\varepsilon})$ respectively. There is a isomorphism of vector spaces $W_\alpha \overset{\phi^1}{\cong} W_{\overline{\alpha}}$ if and only if ϕ^1 induces a isomorphism of cdga's $\Lambda_\alpha \overset{\widehat{\phi}}{\cong} \Lambda_{\overline{\alpha}}$, where*

$$d_{\mathfrak{sl}(2)_\alpha} = m \circ \Theta_\alpha + f_2 \text{ and } d_{\mathfrak{sl}(2)_{\overline{\alpha}}} = m \circ \Theta_\alpha + \eta \circ f_2,$$

with $f_2\colon \mathfrak{sl}(2) \to (\Lambda W)^3$ a linear map and $\eta\colon (\Lambda W_\alpha)^3 \to (\Lambda W_{\overline{\alpha}})^3$ such that

$$\eta(\omega^1 \wedge \omega^2 \wedge \omega^3) = \overline{\omega}^1 \wedge \overline{\omega}^2 \wedge \overline{\omega}^3.$$

Proof. Suppose that there is a isomorphism ϕ^1. We define the isomorphism $\phi = \{\phi^1, \phi^2\}$ of graded vector spaces $W_\alpha \oplus \mathfrak{sl}(2)$ and $W_{\overline{\alpha}} \oplus \mathfrak{sl}(2)$, such that $\phi^1(\omega^i) = \overline{\omega}^i$, $i = 1, 2, 3$ and $\phi^2 = Id$.

Since, ϕ extend to $\widehat{\phi}$ a morphism of commutative graded algebras, $\widehat{\phi}\colon \Lambda_\alpha \to \Lambda_{\overline{\alpha}}$ such that $\widehat{\phi}$ satisfies:

$\widehat{\phi} \circ d_\alpha(\omega^i) = d_{\overline{\alpha}} \circ \widehat{\phi}(\omega^i)$, for $i = 1, 2, 3$; in fact, if $i = 1$,

$$\widehat{\phi}(d_\alpha(\omega^1)) = \widehat{\phi}(\omega^3 \wedge \omega^2) = \overline{\omega}^3 \wedge \overline{\omega}^2 = d_{\overline{\alpha}}(\widehat{\phi}(\omega^1)).$$

$$\begin{aligned}
\widehat{\phi} \circ d_{\mathfrak{sl}(2)_\alpha}(A) &= \widehat{\phi}\Big(\frac{1}{2}\big(\omega^2 \otimes [A, H] + (\omega^1 - \omega^3) \otimes [A, X] + (\omega^1 + \omega^3) \otimes [A, Y]\big) + f_2(A)\Big) \\
&= \frac{1}{2}\big(\overline{\omega}^2 \otimes [A, H] + (\overline{\omega}^1 - \overline{\omega}^3) \otimes [A, X] + (\overline{\omega}^1 + \overline{\omega}^3) \otimes [A, Y]\big) + \widehat{\phi} \circ f_2(A) \\
&= d_{\mathfrak{sl}(2)_{\overline{\alpha}}} \circ \widehat{\phi}(A)
\end{aligned}$$

therefore, $\widehat{\phi}$ is a isomorphism of *cdga's*.

Now suppose that there is a isomorphismo of cdga's $\Lambda_\alpha \overset{\widehat{\phi}}{\cong} \Lambda_{\overline{\alpha}}$, then we have $\Lambda^1_\alpha \cong \Lambda^1_{\overline{\alpha}}$ and $\Lambda^1_\alpha = W_\alpha$, this is $W_\alpha \cong W_{\overline{\alpha}}$

□

Examples 4.5.2.

1. *The sine-Gordon equation determines the manifold ε of pseudo-spherical type then $\{\omega^1, \omega^2, \omega^3\}$ are linearly independent (see 4.4), and $\widehat{\alpha} = f\alpha f^{-1} + d_H(f)f^{-1})$ where $f = \begin{pmatrix} \cos(\frac{\varrho}{2}) & -\sin(\frac{\varrho}{2}) \\ \sin(\frac{\varrho}{2}) & \cos(\frac{\varrho}{2}) \end{pmatrix}$ induces the forms $\{\overline{\omega}^1, \widehat{\omega}^2, \overline{\omega}^3\}$ which also are linearly independent (see Example 4.5.1). By last theorem $\Lambda_\alpha \overset{\widehat{\phi}}{\cong} \Lambda_{\widehat{\alpha}}$.*
2. *The Burgers' equation $u_t = u_{xx} + uu_x$ determines the manifold ε of pseudo-spherical type then $\{\omega^1, \omega^2\}$ are linearly independent and $\omega^3 = -\omega^2$ (see 4.3), and $\widehat{\alpha} = f\alpha f^{-1} + d_H(f)f^{-1}$ where $f = \begin{pmatrix} \cosh(\frac{\varrho}{2}) & \sinh(\frac{\varrho}{2}) \\ \sinh(\frac{\varrho}{2}) & \cosh(\frac{\varrho}{2}) \end{pmatrix}$ induces the forms $\{\widehat{\omega}^1, \widehat{\omega}^2\}$ which also are linearly independent and $\omega^3 = -\omega^2$ (see Example 4.5.1). Indeed,*

$$\widehat{\omega}^1 = \omega^1 + d_H(\rho)$$
$$\widehat{\omega}^2 = \cosh(\rho)\,\omega^2 - \sinh(\rho)\,\omega^3$$
$$\widehat{\omega}^3 = \sinh(\rho)\,\omega^2 - \cosh(\rho)\,\omega^3.$$

By last theorem $\Lambda_\alpha \overset{\widehat{\phi}}{\cong} \Lambda_{\widehat{\alpha}}$.

3. *But, the last example if* $f = \begin{pmatrix} \cos(\frac{\rho}{2}) & -\sin(\frac{\rho}{2}) \\ \sin(\frac{\rho}{2}) & \cos(\frac{\rho}{2}) \end{pmatrix}$ *where* ρ *is not constant, then* $\{\widehat{\omega}^1, \widehat{\omega}^2, \widehat{\omega}^3\}$ *are linearly independent, since*

$$\widehat{\omega}^1 = \sin(\rho)\,\omega^2 + \cos(\rho)\,\omega^1$$
$$\widehat{\omega}^2 = \cos(\rho)\,\omega^2 - \sin(\rho)\,\omega^1$$
$$\widehat{\omega}^3 = -\omega^2 + d_H(\rho).$$

This allows us to conclude that the gauge transformations $\alpha \mapsto \widehat{\alpha}$ *does not always induce an isomorphism of algebras* $\Lambda_\alpha \cong \Lambda_{\widehat{\alpha}}$

4.6 Hierarchies of evolution equations of pseudo-spherical type

Hierarchies of evolution equations of pseudo-spherical type are introduced in [19] and [20]; this definition generalize the notion of a manifold of pseudo-spherical type determined by a single equation.

We present a Sullivan decomposable algebra generated by a hierarchy of pseudo-spherical type using the twisting matrix and Theorem (1.5.2), generalizing Theorem (4.2.1).

Definition 4.6.1. *A countable system of evolution equations of finite order*

$$\frac{\partial u}{\partial \tau_i} = F_i(x, \tau_i, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial u}{\partial x^{r_i}}) \tag{4.6.1}$$

in two independent variables x, τ_i *is a hierarchy pseudo-spherical type if there exist one-forms* $w_\alpha^{[n]}$, $\alpha = 1, 2, 3$, $n \geq 0$

$$\left.\begin{aligned} \omega_\alpha^{[0]} &= f_{\alpha 1} dx + f_{\alpha 2} dt \in \varepsilon^{[0]} \\ \omega_\alpha^{[n]} &= f_{\alpha 1} dx + f_{\alpha 2} dt + \sum_{k=1}^{n} h_{\alpha k} d\tau_k \in \varepsilon^{[n]} \end{aligned}\right\} \tag{4.6.2}$$

where $\varepsilon^{[n]}$ *is a manifold determines by the system* $u_{\tau_i} = F_i$, $i = 0, 1, \ldots, n$, *whose coefficients* $f_{\alpha 1}, f_{\alpha 2}, h_{\alpha k}$ *are differential functions, such that for each* $n \geq 0$,

$$d_H \omega_1^{[n]} = w_3^{[n]} \wedge \omega_2^{[n]}, \quad d_H \omega_2^{[n]} = w_1^{[n]} \wedge \omega_3^{[n]}, \quad d_H \omega_3^{[n]} = w_1^{[n]} \wedge \omega_2^{[n]}.$$

By this definition, we have for $n \geq 0$ the following forms in $\Omega^{1,0}(\varepsilon^{[n]}, \mathfrak{sl}(2))$:

$$\alpha^{[n]} = \frac{1}{2}\begin{pmatrix} \omega_2^{[n]} & \omega_1^{[n]} + \omega_2^{[n]} \\ \omega_1^{[n]} - \omega_2^{[n]} & -\omega_2^{[n]} \end{pmatrix} = \frac{1}{2}\{\omega_2^{[n]} \otimes H + (\omega_1^{[n]} + \omega_2^{[n]}) \otimes X + (\omega_1^{[n]} - \omega_2^{[n]}) \otimes Y\}$$

Moreover $\mathbf{d_H}(\alpha^{[n]}) = \alpha^{[n]} \wedge \alpha^{[n]}$ as in the proof of Theorem (4.1.1).

We can define the twisting matrix for $n \geq 0$

$$\begin{array}{rccl} \Theta_\alpha^{[n]}: & \mathfrak{sl}(2) & \rightarrow & \Omega^{1,0}(\varepsilon^{[n]}) \otimes \mathfrak{sl}(2) \\ & A & \mapsto & \frac{1}{2}\big(\omega_2^{[n]} \otimes [A,H] + (\omega_1^{[n]} - \omega_3^{[n]}) \otimes [A,X] + (\omega_1^{[n]} + \omega_3^{[n]}) \otimes [A,Y]\big) \end{array}$$

Thus, there is a *cdga* $(\Lambda W^{[n]}, d^{[n]})$, for each $n \geq 0$ as in (4.2.2), where

$$\begin{aligned} W^{[n]} &= span_{\mathbb{R}}\{\omega_1^{[n]}, \omega_2^{[n]}, \omega_3^{[n]}\}, \\ d^{[n]}(\omega_1^{[n]}) &= \omega_3^{[n]} \wedge \omega_2^{[n]}, \\ d^{[n]}(\omega_2^{[n]}) &= \omega_1^{[n]} \wedge \omega_3^{[n]}, \\ d^{[n]}(\omega_3^{[n]}) &= \omega_1^{[n]} \wedge \omega_2^{[n]}, \end{aligned}$$

and, we obtain the twisting matrix on $W^{[n]}$ (not on $\Omega^{1,0}(\varepsilon^{[n]})$) for each $n \geq 1$ given by:

$$\begin{array}{rccl} \Theta_\alpha^{[n]}: & \mathfrak{sl}(2) & \rightarrow & W^{[n]} \otimes \mathfrak{sl}(2) \\ & A & \mapsto & \frac{1}{2}\big(\omega_2^{[n]} \otimes [A,H] + (\omega_1^{[n]} - \omega_3^{[n]}) \otimes [A,X] + (\omega_1^{[n]} + \omega_3^{[n]}) \otimes [A,Y]\big) \end{array} \tag{4.6.3}$$

Now, we define the *cdga* $(\Lambda W^{\leq[n]}, d^{\leq[n]})$, for each $n \geq 0$ fixed, where

$$\begin{aligned} W^{\leq[n]} &= span_{\mathbb{R}}\{\omega_1^{[0]}, \omega_2^{[0]}, \omega_3^{[0]}, \ldots, \omega_1^{[n]}, \omega_2^{[n]}, \omega_3^{[n]}\}, \\ d^{\leq[n]}(\omega_\alpha^{[j]}) &= d^{[j]}(\omega_\alpha^{[j]}), \quad \text{for all } \ j \leq n \ \text{ and } \ \alpha = 1,2,3. \end{aligned}$$

By means of Theorem (1.5.2) for $n \geq 0$ fixed, we have the following Sullivan decomposable algebra: $\Lambda V = \Lambda(W^{\leq[n]} \oplus \underbrace{\mathfrak{sl}(2) \oplus \cdots \oplus \mathfrak{sl}(2)}_{n-times})$, where:

$$\begin{aligned} &V^0 = \{0\} \\ &V^1 = W^{\leq[n]} \\ &V^2 = \mathfrak{sl}(2) \\ &\quad\vdots \\ &V^{n+1} = \mathfrak{sl}(2) \\ &V^{n+2} = \mathfrak{sl}(2) \\ &V^k = \{0\}, \quad \text{for } k \geq n+3; \end{aligned}$$

via the linear maps d_{V^n} defined by

$$\begin{array}{rccl} d_{V^n}: & \mathfrak{sl}(2) & \rightarrow & \Lambda(W^{[n]} \oplus \mathfrak{sl}(2)) \\ & A & \mapsto & m \circ \Theta_\alpha^{[n]}(A) + f_{n+2}(A) \end{array} \tag{4.6.4}$$

where m is the exterior product on graded algebra ΛV, and we consider $f_{n+2}\colon \mathfrak{sl}(2) \rightarrow (\Lambda V^{\leq n+1})^{n+3}$ the null map for $n \geq 0$. Thus, we have that given a a hierarchy $\dfrac{\partial u}{\partial \tau_i} = F_i$ of pseudo-spherical type with associated one forms

$$\omega_\alpha^{[0]} = f_{\alpha 1}dx + f_{\alpha 2}dt \in \varepsilon^{[0]}, \text{ and } \omega_\alpha^{[n]} = f_{\alpha 1}dx + f_{\alpha 2}dt + \sum_{k=1}^{n} h_{\alpha k}d\tau_k \in \varepsilon^{[n]},$$

we obtain a Sullivan decomposable algebra.

Now, let us express the matrix $\alpha^{[n]}$ as in (4.1.5) and calculate $\mathbf{d_H}\left(\alpha^{[n]}\right)$, $\frac{1}{2}[\alpha^{[n]}, \alpha^{[n]}]$:

$$\begin{aligned}
\alpha^{[n]} &= \frac{1}{2}\begin{pmatrix} f_{21}dx + f_{22}dt + \sum\limits_{k=1}^{n} h_{2k}d\tau_k & \begin{matrix}(f_{11}-f_{31})dx + (f_{12}-f_{32})dt + \\ \sum\limits_{k=1}^{n}(h_{1k}-h_{3k})d\tau_k\end{matrix} \\ \begin{matrix}(f_{11}+f_{31})dx + (f_{12}+f_{32})dt + \\ \sum\limits_{k=1}^{n}(h_{1k}-h_{3k})d\tau_k\end{matrix} & -\left(f_{21}dx + f_{22}dt + \sum\limits_{k=1}^{n} h_{2k}d\tau_k\right) \end{pmatrix} \\
&= \underbrace{\frac{1}{2}\begin{pmatrix} f_{21} & f_{11}-f_{31} \\ f_{11}+f_{31} & -f_{21} \end{pmatrix}}_{X} dx + \underbrace{\frac{1}{2}\begin{pmatrix} f_{22} & f_{12}-f_{32} \\ f_{12}+f_{32} & -f_{22} \end{pmatrix}}_{T} dt \\
&\quad + \underbrace{\frac{1}{2}\begin{pmatrix} h_{21} & h_{11}-h_{31} \\ h_{11}+h_{31} & -h_{21} \end{pmatrix}}_{H_1^{[n]}} d\tau_1 + \underbrace{\frac{1}{2}\begin{pmatrix} h_{22} & h_{12}-h_{32} \\ h_{12}+h_{32} & -h_{22} \end{pmatrix}}_{H_2^{[n]}} d\tau_2 \\
&\quad + \underbrace{\frac{1}{2}\begin{pmatrix} h_{23} & h_{13}-h_{33} \\ h_{13}+h_{33} & -h_{23} \end{pmatrix}}_{H_3^{[n]}} d\tau_3 + \cdots + \underbrace{\frac{1}{2}\begin{pmatrix} h_{2n} & h_{1n}-h_{3n} \\ h_{1n}+h_{3n} & -h_{2n} \end{pmatrix}}_{H_k^{[n]}} d\tau_n \\
&= X dx + T dt + \sum_{k=1}^{n} H_k^{[n]} d\tau_n
\end{aligned}$$

We have:

$$\begin{aligned}
\mathbf{d_H}\left(\alpha^{[n]}\right) &= \mathbf{d_H}\left(X dx + T dt + \sum_{k=1}^{n} H_k^{[n]} d\tau_n\right) \\
&= D_t X dt \wedge dx + \sum_{k=1}^{n} D_{t_k} X d\tau_k \wedge dx + D_x T dx \wedge dt + \sum_{k=1}^{n} D_{t_k} T d\tau_k \wedge dt + \\
&\quad D_x H_1^{[n]} dx \wedge d\tau_1 + D_t H_1^{[n]} dt \wedge d\tau_1 + \sum_{k=1}^{n} D_{\tau_k} H_1^{[n]} d\tau_k \wedge d\tau_1 + \\
&\quad D_x H_2^{[n]} dx \wedge d\tau_2 + D_t H_2^{[n]} dt \wedge d\tau_2 + \sum_{k=1}^{n} D_{\tau_k} H_2^{[n]} d\tau_k \wedge d\tau_2 + \\
&\quad \vdots \\
&\quad D_x H_n^{[n]} dx \wedge d\tau_n + D_t H_n^{[n]} dt \wedge d\tau_n + \sum_{k=1}^{n} D_{\tau_k} H_n^{[n]} d\tau_k \wedge d\tau_n \\
&= (D_t X - D_x T)\, dt \wedge dx + \sum_{k=1}^{n}\left(D_{\tau_k} X - D_x H_k^{[n]}\right) d\tau_k \wedge dx + \\
&\quad \sum_{k=1}^{n}\left(D_{\tau_k} T - D_t H_k^{[n]}\right) d\tau_k \wedge dt + \sum_{k<j}\left(D_{\tau_j} H_k - D_{\tau_k} H_j\right) d\tau_j \wedge d\tau_k,
\end{aligned}$$

$$
\begin{aligned}
\tfrac{1}{2}[\alpha^{[n]},\alpha^{[n]}] &= \tfrac{1}{2}\left[Xdx+Tdt+\sum_{k=1}^{n}H_k^{[n]}d\tau_k \quad , \quad Xdx+Tdt+\sum_{k=1}^{n}H_k^{[n]}d\tau_k\right]\\
&= \tfrac{1}{2}\Bigg\{[X,T]\,dx\wedge dt+[T,X]\,dt\wedge dx+\sum_{k=1}^{n}\left[X,H_k^{[n]}\right]dx\wedge d\tau_k+\\
&\quad \sum_{k=1}^{n}\left[H_k^{[n]},X\right]d\tau_k\wedge dx+\sum_{k=1}^{n}\left[T,H_k^{[n]}\right]dt\wedge d\tau_k+\\
&\quad \sum_{k=1}^{n}\left[H_k^{[n]},T\right]d\tau_k\wedge dt+\sum_{k,j=1}^{n}\left[H_k^{[n]},H_j^{[n]}\right]d\tau_k\wedge d\tau_j\Bigg\}\\
&= \tfrac{1}{2}\Bigg\{-2\,[X,T]\,dt\wedge dx-2\sum_{k=1}^{n}\left[X,H_k^{[n]}\right]d\tau_k\wedge dx\\
&\quad -2\sum_{k=1}^{n}\left[T,H_k^{[n]}\right]d\tau_k\wedge dt-2\sum_{k<j}\left[H_k^{[n]},H_j^{[n]}\right]d\tau_j\wedge d\tau_k\Bigg\}\\
&= -\Bigg\{[X,T]\,dt\wedge dx+\sum_{k=1}^{n}\left[X,H_k^{[n]}\right]d\tau_k\wedge dx+\\
&\quad \sum_{k=1}^{n}\left[T,H_k^{[n]}\right]d\tau_k\wedge dt+\sum_{k<j}\left[H_k^{[n]},H_j^{[n]}\right]d\tau_j\wedge d\tau_k\Bigg\}.
\end{aligned}
$$

Then,

$$D_tX-D_xT+[X,T]=0$$

$$D_{\tau_k}X-D_xH_k^{[n]}+\left[X,H_k^{[n]}\right]=0,\quad \text{for } k=\{1,...,n\}$$

$$D_{\tau_k}T-D_tH_k^{[n]}+\left[T,H_k^{[n]}\right]=0,\quad \text{for } k=\{1,...,n\}$$

$$D_{\tau_j}H_k-D_{\tau_k}H_j+\left[H_k^{[n]},H_j^{[n]}\right]=0,\quad \text{for } k=\{1,...,n\},\ \ j=\{1,...,n\},k<j$$

thus we is conclude that:

1. The equation $\frac{\partial u}{\partial t}=F_0$ describes a manifold of pseudo-spherical type with associated one-forms $\omega_\alpha^{[0]}=f_{\alpha 1}dx+f_{\alpha 2}dt\in\varepsilon^{[0]}$, with *zcr for* $\varepsilon^{[0]}$ given by $\alpha^{[0]}=Xdx+Tdt\in\Omega^{1,0}(\varepsilon^{[0]})$ as we saw earlier.
2. The equation $\frac{\partial u}{\partial \tau_i}=F_i$ describes a manifold of pseudo-spherical type with associated one-forms

$$\omega_i^\alpha=f_{\alpha 1}dx+h_{\alpha i}d\tau_i\in\varepsilon^i$$

with *zcr for* ε^i given by $\alpha^i=Xdx+H_id\tau_i$. Therefore, for each equation $\dfrac{\partial u}{\partial \tau_i}=F_i$ of (4.6.1), we have by Theorem (4.2.1) the Sullivan decomposable algebras: $\Lambda V=\Lambda(W^i\oplus\mathfrak{sl}(2))$ where

$V_i^0 \ = \{0\}$
$V_i^1 \ = span_{\mathbb{R}}\{\omega_1^i,\ \omega_2^i,\ \omega_3^i\}$
$V_i^2 \ = \mathfrak{sl}(2)$
$V_i^k \ = \{0\}$ for $k\geq 3$

via the linear map $d_{\mathfrak{sl}(2)}$ defined by

$$\begin{array}{rcl} d_{\mathfrak{sl}(2)}: \quad \mathfrak{sl}(2) & \to & \Lambda(W^i \oplus \mathfrak{sl}(2)) \\ A & \mapsto & m \circ \Theta_{\alpha_i}(A) + f_2(A), \end{array} \tag{4.6.5}$$

where m is the exterior product on graded algebra $\Lambda(W \oplus \mathfrak{sl}(2))$ and $f_2 \colon \mathfrak{sl}(2) \to (\Lambda W^i)^3$ is a linear map.

3. The ones-forms $\sigma_j^\alpha = f_{\alpha 2}dt + h_{\alpha j}d\tau_j$ (j fixed) and $\sigma_{ij}^\alpha = h_{\alpha i}d\tau_i + h_{\alpha j}d\tau_j$, (i,j) fixed $i \neq j$ satisfy the equations (4.1.3).

Example 4.6.1. *In the paper [20] by Reyes, he presents the Korteweg-de Vries hierarchy of pseudo-spherical type; this example follows from the seminal paper by Chern and Peng [5]. The associated functions are:*

$$f_{11} = 1 - u, \qquad f_{12} = \lambda u_x - u_{xx} - 2u^2 + 2u - \lambda^2 u + \lambda^2;$$

$$f_{21} = \lambda, \qquad f_{22} = \lambda^3 + 2\lambda u - 2u_x;$$

$$f_{31} = -1 - u, \quad f_{32} = \lambda u_x - u_{xx} - \lambda^2 u - 2u^2 - \lambda^2 - 2u;$$

and

$$\begin{array}{rcl} h_{1i} & = & \frac{1}{2}\lambda B_x^{(i+1)} - \frac{1}{2}B_{xx}^{(i+1)} - uB^{(i+1)} + B^{(i+1)}; \\ h_{2i} & = & \lambda B^{(i+1)} - B_x^{(i+1)}; \\ h_{3i} & = & \frac{1}{2}\lambda B_x^{(i+1)} - \frac{1}{2}B_{xx}^{(i+1)} - uB^{(i+1)} - B^{(i+1)}, \end{array}$$

in which

$$B^{(i)} = \sum_{j=0}^{i} B_j \lambda^{2(i-j)} = B_0\lambda^{2i} + B_1\lambda^{2(i-1)} + \cdots + B_{i-1}\lambda + B_i,$$

and the functions B_j are defined recursively by means of the equations

$$\begin{array}{rcl} B_{0,x} & = & 0, \\ B_{j+1,x} & = & B_{j,xxx} + 4uB_{j,x} + 2u_x B_j, \ j \geq 0. \end{array}$$

The functions $F_i, \ i \geq 0$, are given by

$$F_i = \frac{1}{2}B_{i+1,xxx} + u_x B_{i+1} + 2uB_{i+1,x} = \frac{1}{2}B_{i+2,x}$$

or, equivalently in terms of the functions $B^{(i)}$, by

$$F_i = \frac{1}{2}B_{xxx}^{(i+1)} + u_x B^{(i+1)} + 2uB_x^{(i+1)} - \frac{1}{2}\lambda^2 B_x^{(i+1)}$$

For instance, if $B_0 = 1$ and all integration constants are set to zero, then

$$B_{1,x} = 2u_x \Rightarrow B_1 = 2u$$

$$B_{2,x} = 2u_{xxx} + 12u_x u \Rightarrow B_2 = 2u_{xx} + 6u^2$$

$$B_{3,x} = 2uxxxxx + 40u_x u_{xx} + 20uu_{xxx} + 60u^2 u_x \Rightarrow B_3 = 2u_{xxxx} + 20u^3 + 10u_x^2 + 20uu_{xx}$$

$$B^{(0)} = 1, \quad B^{(1)} = \lambda^2 + 2u, \quad B^{(2)} = \lambda^4 + 2\lambda^2 u + 2u_{xx} + 6u^2$$

and the equation $u_{\tau_0} = F_0$ *is the standard K-dV equation* $u_t = u_{xxx} + 6uu_x$*, the equations* $u_{\tau_1} = F_1$ *and* $u_{\tau_2} = F_2$ *can be also found easily. They are:*

$$u_{\tau_1} = u_{xxxxx} + 20u_x u_{xx} + 10uu_{xxx} + 30u^2 u_x$$

$$u_{\tau_2} = u_{xxxxxxx} + 70u_{xx}u_{xxx} + 42u_x u_{xxxx} + 14uu_{xxxxx} + 70u_x^3 + 280uu_x u_{xx} + 70u^2 u_{xxx} + 140u^3 u_x.$$

4.7 Algebra of polynomial differential forms

In this section section, we briefly explain some aspects for the construction of the Algebra of polynomial differential forms; this algebra allows us to define a functor A_{PL}, which construction is originally due to Sullivan [24], [25] we will apply this construction to our differential equations in the next section.

Definition 4.7.1. *A* ***simplicial object*** S *with values in a category* $\mathcal{C}$ *is a sequence* $\{S_n\}_{n\geq 0}$ *of objects in* $\mathcal{C}$*, together with* $\mathcal{C}$*-morphisms* $\partial_i : S_{n+1} \to S_n$*, for* $0 \leq i \leq n+1$ *and* $s_j : S_n \to S_{n+1}$*, for* $0 \leq j \leq n$*; satisfying the following relations:*

- $\partial_i \partial_j = \partial_{j-1}\partial_i, \quad i < j$
- $s_i s_j = s_{j+1} s_i, \quad i \leq j$
- $\partial_i s_j = \begin{cases} s_{j-1}\partial_i, & j < i \\ id, & i = j, \ j+1 \\ s_j \partial_{i-1}, & i > j+1 \end{cases}$

A **simplicial morphism** $\varphi \colon L \to K$ between two such simplicial objects is a sequence of $\mathcal{C}$-morphisms $\varphi \colon L_n \to K_n$ commuting with the ∂_i and s_j, this is, for each $n \geq 0$:

$$\varphi_n \partial_i^L = \partial_i^K \varphi_{n+1}, \ \text{ for } \ 0 \leq i \leq n+1, \ \text{ and } \ s_j^K \varphi_n = \varphi_{n+1} s_j^L, \ \text{ for } \ 0 \leq j \leq n.$$

In other words, we have the diagrams

$$\begin{array}{ccc} L_n & \xleftarrow{\partial_i^L} & L_{n+1} \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} \\ K_n & \xleftarrow{\partial_i^K} & K_{n+1} \end{array} \qquad \begin{array}{ccc} L_n & \xrightarrow{s_j^L} & L_{n+1} \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} \\ K_n & \xrightarrow{s_j^K} & K_{n+1} \end{array}$$

A **simplicial set** $K = \{K_n\}_{n\geq 0}$ is a simplicial object in the category of sets. In this case each K_n is a set, and ∂_i and s_j are functions between sets. We denote by ***sSET*** *the category of simplicial sets*.

Example 4.7.1. *Let* Δ^n *be a standard* n*-simplex, this is* Δ^n *is the convex hull of the canonical basis* $e_0, \ldots, e_n$ *of* $\mathbb{R}^{n+1}$*:*

$$\Delta^n = \{\sum_{i=0}^{n} t_i e_i; \ \ 0 \leq t_i \leq 1, \ \ \sum_{i=0}^{n} t_i = 1\},$$

and let X be a topological space; a singular n-simplex in X is a smooth function $\sigma\colon \Delta^n \to X$. The set of singular n-simpleces in X will be denoted by $S_n(X)$. The sequence $S(X) = \{S_n(X)\}_{n\geq 0}$ constitutes a simplicial set with operators ∂_i and s_j defined by the functions λ_i and q_j respectively:

$$\begin{array}{rccc} \partial_i : & S_{n+1}(X) & \to & S_n(X) \\ & \sigma & \mapsto & \sigma\circ\lambda_i, \end{array} \qquad and \qquad \begin{array}{rccc} s_j : & S_n & \to & S_{n+1} \\ & \sigma & \mapsto & \sigma\circ q_j, \end{array}$$

where

$$\begin{array}{rccc} \lambda_i : & \Delta^n & \to & \Delta^{n+1} \\ & (t_0,\ldots,t_n) & \mapsto & (t_0,\ldots,t_{i-1},0,t_i,...,t_n). \end{array}$$

$$\begin{array}{rccc} q_j : & \Delta^{n+1} & \to & \Delta^n \\ & (t_0,\ldots,t_{n+1}) & \mapsto & (t_0,\ldots,t_j+t_{j+1},t_{j+2}\ldots,t_{n+1}) \end{array}$$

In terms of category this assignation of X to $S(X)$ is the singular simplexes functor.

A **simplicial *dga*** $A = \{A_n\}_{n\geq 0}$ is a simplicial object in the category of *dga's*. In this case each A_n is a *dga* and ∂_i and s_j are morphisms of *dga's*. We will denote by ***sDGA*** the category of simplicial *dga's* .

Let us consider the free commutative graded algebra on the graded vector space over a field k, $V = \{V^n\}_{n\geq 0}$, where $V^0 = \langle t_0,\ldots,t_n\rangle$, $V^1 = \langle y_0,\ldots,y_n\rangle$ and $V^n = \{0\}$, for $n\geq 2$. Hence, from the isomorphism given in (2.1.1):

$$\Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n) \cong \Lambda(t_0,\ldots,t_n)\otimes\Lambda(y_0,\ldots,y_n).$$

Moreover, we note that:

$$\begin{aligned} \Lambda(t_0,\ldots,t_n) &= \Lambda\left(\langle t_0\rangle\oplus\cdots\oplus\langle t_n\rangle\right) \\ &= \Lambda(t_0)\otimes\cdots\otimes\Lambda(t_n) \\ &= \mathbb{K}[t_0]\otimes\cdots\otimes\mathbb{K}[t_n] \\ &= \mathbb{K}[t_0,\ldots,t_n] \end{aligned}$$

Thus, $\Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n)\cong\mathbb{K}[t_0,\ldots,t_n]\otimes\Lambda(y_0,\ldots,y_n)$.

There is a unique differential in this algebra specified by $dt_i \to y_i$ and $dy_j\to 0$. Now, we consider the ideal I of $\Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n)$ generated by the two elements $\sum\limits_{i=0}^{n} t_i - 1$ and $\sum\limits_{j=0}^{n} y_j$, this is $I = \left(\sum\limits_{i=0}^{n} t_i - 1, \sum\limits_{j=0}^{n} y_j\right)$. Observe that

$$I_n^k = \Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n)^k(\sum_{i=0}^{n} t_i - 1) + \Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n)^{k-1}\Big(\sum_{j=0}^{n} y_j\Big).$$

Moreover I_n is stable under the differential d. In fact, if $\alpha\in I_n$, then α has the form:

$$\alpha = x\left(\sum_{i=0}^{n} t_i - 1\right) + y\left(\sum_{j=0}^{n} y_j\right)$$

when $x \in \Lambda(t_0, \dots, t_n, y_0, \dots, y_n)^p$ and $y \in \Lambda(t_0, \dots, t_n, y_0, \dots, y_n)^{p-1}$.

Thus,

$$\begin{aligned} d(\alpha) &= d(x)\Big(\sum_{i=0}^{n} t_i - 1\Big) + (-1)^p x\, d\Big(\sum_{i=0}^{n} t_i - 1\Big) + d(y)\Big(\sum_{j=0}^{n} y_j\Big) + (-1)^{p-1} y\, d\Big(\sum_{j=0}^{n} y_j\Big) \\ &= d(x)\Big(\sum_{i=0}^{n} t_i - 1\Big) + (-1)^p x\Big(\sum_{j=0}^{n} y_j\Big) + d(y)\Big(\sum_{j=0}^{n} y_j\Big) \\ &= d(x)\Big(\sum_{i=0}^{n} t_i - 1\Big) + \big((-1)^p x + d(y)\big)\Big(\sum_{j=0}^{n} y_j\Big) \end{aligned}$$

Therefore, $d(\alpha) \in I_n^{p+1}$.

So, we can define the quotient differential algebra for each $n \geq 0$:

$$\Lambda(t_0, \dots, t_n, y_0, \dots, y_n) \Big/ \Big(\sum_{i=0}^{n} t_i - 1, \sum_{j=0}^{n} y_j\Big).$$

This algebras allow us to define de following simplicial *cdga*.

Definition 4.7.2. *The **algebra of polynomial differential forms with coefficient in** k , denoted $A_{PL} = \{(A_{PL})_n\}_{n\geq 0}$, is the simplicial cdga given by*

$$(A_{PL})_n = \Lambda(t_0, \dots, t_n, y_0, \dots, y_n) \Big/ (\sum_{i=0}^{n} t_i - 1, \sum_{j=0}^{n} y_j),$$

in which the basis elements t_i have degree zero and the basis elements y_j have degree 1 and the derivation is determined by $dt_i = y_i, dy_j = 0$ for all i, j, and the operators ∂_i and s_j are specified by:

$$\begin{array}{rcl} \partial_i : (A_{PL})_{n+1} & \to & (A_{PL})_n \\ t_k & \mapsto & \begin{cases} t_k, & k < i; \\ 0, & k = i; \\ t_{k-1}, & k > i. \end{cases} \end{array} \quad and \quad \begin{array}{rcl} s_j : (A_{PL})_n & \to & (A_{PL})_{n+1} \\ t_k & \mapsto & \begin{cases} t_k, & k < j; \\ t_k + t_{k+1}, & k = j; \\ t_{k+1}, & k > j. \end{cases} \end{array}$$

We note that the definition is complete as written, since the differential must commute with ∂_i, s_j, then is sufficient to define these functions in terms of t_k, in fact

$$\partial_i(y_k) = \partial_i(d(t_k)) = (\partial_i \circ d)(t_k) = (d \circ \partial_i)(t_k) = d(\partial_i(t_k))$$

and

$$s_j(y_k) = s_j(d(t_k)) = (s_j \circ d)(t_k) = (d \circ s_j)(t_k) = d(s_j(t_k)).$$

Remark 4.7.1. *The elements of $(A_{PL})_n$ are called **Polynomial differential forms with coefficients in** k, for the following reason. When $k = \mathbb{Q}$, the algebra $(A_{PL})_n$ is isomorphic to the algebra of $\mathbb{Q}$-polynomial forms on Δ^n, the last algebra is defined below:*

Definition 4.7.3.

1. *The restriction of any C^∞ differential form on $\mathbb{R}^{n+1}$ to Δ^n is called a C^∞ form on it. They are expressed as*
$$\omega = \sum_{I=(i_1<\cdots<i_r)} f_{i_1\ldots i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r} \tag{4.7.1}$$
where $f_{i_1\ldots i_r}$ is a smooth function on an open subset of $\mathbb{R}^{n+1}$ containing Δ^n.

2. *If $f_{i_1\ldots i_r}(t_0,\ldots,t_n) \in \mathbb{Q}[t_0,\ldots,t_n]$ in (4.7.1) this is $f_{i_1\ldots i_r}$ are polynomial with rational coefficients on Δ^n, then ω is called a $\mathbb{Q}$ polynomial r-form.*

This definition is used to define differential forms on a simplicial complex, see [15] and [16].

We denote by $A^*(\Delta^n)$ and $A^*_{\mathbb{Q}}(\Delta^n)$ the algebras of all C^∞ forms and $\mathbb{Q}$ polynomial forms on Δ^n. It is important, for polynomial forms over $\mathbb{Q}$, that the form $d(\sum_{i_1<\cdots<i_r} f_{i_1\ldots i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r})$ can be expressed on terms of the partial derivatives of the polynomial $f_{i_1\cdots i_r}$, which are also polynomials with rational coefficients on Δ^n. First, we observe that:

$$\begin{aligned}
(A_{PL})^0_n &= \Lambda(t_0,\ldots,t_n,y_0,\ldots,y_n)^0 \Big/ \left(\sum_{i=1}^n t_i - 1, \sum_{j=1}^n y_j\right)\\
&= \Lambda(t_0,\ldots,t_n) \Big/ \left(\sum_{i=1}^n t_i - 1\right)\\
&\cong \mathbb{Q}[t_0,\ldots,t_n] \Big/ \left(\sum_{i=1}^n t_i - 1\right)\\
&\cong A^0_{\mathbb{Q}}(\Delta^n).
\end{aligned}$$

Moreover, since $A/a \otimes B/b \cong A \otimes B/(a \otimes b)$, we have:

$$\begin{aligned}
(A_{PL})_n &\cong \mathbb{Q}[t_0,\ldots,t_n] \otimes \Lambda(y_0,\ldots,y_n) \Big/ \left(\left(\sum_{i=1}^n t_i - 1\right) \otimes 1,\ 1 \otimes \sum_{j=1}^n y_j\right)\\
&\cong \mathbb{Q}[t_0,\ldots,t_n] \Big/ \left(\sum_{i=1}^n t_i - 1\right) \otimes \Lambda(y_0,\ldots,y_n) \Big/ \left(\sum_{j=1}^n y_j\right)\\
&\cong (A_{PL})^0_n \otimes \Lambda(y_0,\ldots,y_n) \Big/ \left(\sum_{j=1}^n y_j\right)\\
&\cong C^\infty_{\mathbb{Q}}(\Delta^n) \otimes \Lambda(dt_0,\ldots,dt_n)\\
&\cong A^*_{\mathbb{Q}}(\Delta^n)
\end{aligned}$$

Thus, $A^*(\Delta^n) \cong C^\infty(\Delta^n) \otimes_{(A_{PL})^0_n} (A_{PL})_n$.

4.8 From topology to algebra and from algebra to topology

First, let us fix some notacion from category theory, since we will be using functors. Denote by ***TOP*** the category of spaces and continuous maps, and denote by ***CDGA*** the category of cdga's over a field k and morphisms of *cdga's*. Now we will present see the transition from the category of ***TOP*** to ***CDGA*** by the functor A_{PL}. These results allow us to relate our study of Sullivan descomposable algebras to the manifold of pseudo-spherical type.

Let $A_{PL} = \{(A_{PL})_n\}_{n\geq 0}$ be a simplicial *cdga*. Then for each fixed p, we have that $A^p_{PL} = \{(A_{PL})^p_n\}_{n\geq 0}$ is a simplicial set. We set

$$\begin{array}{rccc} F: & \textbf{\textit{sSET}} & \to & \textbf{\textit{CDGA}} \\ & K & \mapsto & \{A^p_{PL}(K)\}_{p\geq 0} \end{array}$$

where $A^p_{PL}(K) = \{\phi \colon K \to A^p_{PL} : \phi$ is a morphism of simplicial sets $\}$. Then, ϕ is a mapping that assigns to each n-simplex $\sigma \in K_n$ an element $\phi_\sigma \in (A_{PL})^p_n$, and $A^p_{PL}(K)$ is a *cdga* with operations given by as follows:

the addition is $(\phi + \psi)_\sigma = \phi_\sigma + \psi_\sigma$, scalar multiplication $(\lambda\phi)_\sigma = \lambda\phi_\sigma$, differential $(d\phi)_\sigma = d(\phi_\sigma)$ and multiplication $(\phi\psi)_\sigma = \phi_\sigma\psi_\sigma$.

By composition of the last functor and the singular simplexes functor (see Example 4.7.1) we obtain the following functor:

$$\begin{array}{rccccc} A_{PL}: & \textbf{\textit{TOP}} & \to & \textbf{\textit{sSET}} & \to & \textbf{\textit{CDGA}} \\ & X & \mapsto & S(X) & \mapsto & F(S(X)) \end{array}$$

For any topological space X, the algebra $A_{PL}(X) = F(S(X))$, is called the ***cdga of piecewise-linear de Rham forms on*** X. Therefore, an element of $(A_{PL}(X))^q$, this is a q-form ω, is a correspondence that assigns to each singular n-simplex σ of X, an element $\omega_\sigma \in (A_{PL})^q_n$, such that the following compatibility criteria is satisfied: $\omega_{\partial_i\sigma} = \partial_i\omega_\sigma$ and $\omega_{s_j\sigma} = s_j\omega_\sigma$. Moreover, if $f : X \to Y$ is any continuous map, then there is a morphism of *cdga's* $A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X)$. All the details of the construction of A_{PL} can be found in chapter 10 of [7].

The functor A_{PL} has the following important property: For any space X, the cohomology of $A_{PL}(X)$ is isomorphic to the rational singular cohomology of X, i.e.

$$H^*(X;k) \cong H^*(A_{PL}(X)). \tag{4.8.1}$$

For the proof to consult ([7]-corollary 10.10)

Conversely, there is a **spatial realization functor *CDGA*** $\to$ ***TOP***. It is obtained by composition of Sullivan's simplicial realization functor from ***CDGA*** $\to$ ***sSET*** (introduced in [25]) and Milnor's realization functor from simplicial sets, to ***TOP***, ***sSET*** $\to$ ***TOP*** (introduced in [14]).

The Sullivan's simplicial realization is the functor:

$$\begin{array}{rccc} \langle\,\rangle: & \textbf{\textit{CDGA}} & \to & \textbf{\textit{sSET}} \\ & (A,d) & \mapsto & \langle A,d\rangle \end{array}$$

where, $\langle A,d\rangle_n = \{f \colon A \to (A_{PL})_n : f$ is a *cdga* morphism$\}$ and the operators $\widetilde{\partial_i}$ and $\widetilde{s_j}$ are given by:

$$\begin{array}{rccc} \widetilde{\partial_i}: & \langle A,d\rangle_{n+1} & \to & \langle A,d\rangle_n \\ & \sigma : A \to (A_{PL})_{n+1} & \mapsto & \partial_i \circ \sigma \end{array} \quad and \quad \begin{array}{rccc} \widetilde{s_j}: & \langle A,d\rangle_n & \to & \langle A,d\rangle_{n+1} \\ & \sigma : A \to (A_{PL})_n & \mapsto & s_j \circ \sigma \end{array}$$

The Milnor Realization of a simplicial set is the functor:

$$\begin{array}{rccc} |\,|: & \textbf{\textit{sSET}} & \to & \textbf{\textit{TOP}} \\ & K & \mapsto & (\coprod_n K_n \times \Delta^n)/\sim \end{array}$$

where,

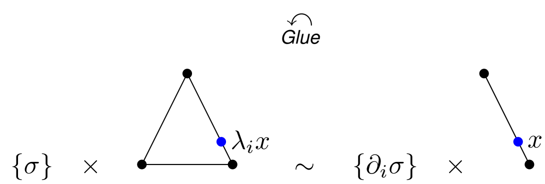
- $K = \{K_n\}_{n\geq 0}$,
- Each K_n is given the discrete topology,
- $\coprod_n K_n \times \Delta^n$ is the disjoint union of spaces $K_n \times \Delta^n$,
- $\sim$ is the equivalence relation generated by the relations:

$$\sigma \in K_{n+1}, \quad x \in \Delta^n, \qquad (\partial_i \sigma, x) \sim (\sigma, \lambda_i x)$$

$$\sigma \in K_n, \quad x \in \Delta^{n+1}, \qquad (s_j \sigma, x) \sim (\sigma, q_j x).$$

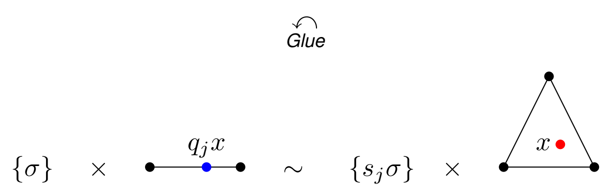
Remark 4.8.1.

- *The relation $(\sigma, \lambda_i x) \sim (\partial_i \sigma, x)$ identifies common faces:*



  So the identification just described takes the n-simplex corresponding to $\partial_i(\sigma)$ in $K_n \times \Delta^n$ and glues it as the $i-th$ face of the $(n+1)$-simplex assigned to σ in $K_{n+1} \times \Delta^{n+1}$.

- *The relation $(\sigma, q_j x) \sim (s_j \sigma, x)$ suppresses the degenerate simplex (a simplex $\tau \in K_{n+1}$ is degenerate, if $\tau = S_j \sigma$ for some $\sigma \in K_n$).*



  This relation tell us that given a degenerate $(n+1)$-simplex corresponding to $s_j(\sigma)$ and a point x in the "pre-collapse" n-simplex Δ^n, we should glue x to the $(n-1)$-simplex represented by σ at the point $q_j(x)$ in the image of the "collapse map" σ.

The spatial realization of a *cdga* is the composition:

$$\begin{array}{ccccc} \textbf{\textit{CDGA}} & \to & \textbf{\textit{sSET}} & \to & \textbf{\textit{TOP}} \\ A & \mapsto & \langle A, d \rangle & \mapsto & |\langle A, d \rangle| \end{array}$$

This functor has the following important property:

Theorem 4.8.1. *If $(\Lambda V, d)$ is a 1-connected and of finite type Sullivan algebra on a field k, then $|\langle \Lambda V, d\rangle|$ is simply connected and for each $n \geq 2$, there is a isomorphism of abelian groups:*

$$\pi_n(|\langle \Lambda V, d\rangle|) \cong Hom_k(V^n, k)$$

Moreover, If $k = \mathbb{Q}$, then for each $n \geq 1$ there is a vector space isomorphism

$$\pi_n(|\langle \Lambda V, d\rangle|) \cong Hom_{\mathbb{Q}}(V^n, \mathbb{Q}]$$

and there always exists a quasi-isomorphism $m : (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(|\langle \Lambda V, d\rangle|)$.

We refer the reader to theorem 17.10 of [7] for the details of the proof.

We can now make the crucial definition that relates Sullivan algebras to topological spaces.

Definition 4.8.1. *A Sullivan model of a path-connected space X is a Sullivan model for the cdga $A_{PL}(X)$; in other words, it is a quasi-isomorphism $m\colon (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$ for some Sullivan algebra $(\Lambda V, d)$.*

Often we, just refer to $(\Lambda V, d)$ as the algebraic model of X. And by the isomorphism given in (4.8.1), if $(\Lambda V, d)$ is a Sullivan model of X then $H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$.

It follows from Theorem (2.3.1) that if X is simply connected, and if every $H_i(X; \mathbb{Q})$ is finite dimensional, then the minimal model $(\Lambda V, d)$ of X has the property that $V = V^{\geq 2}$ and every V^i is finite dimensional.

We finish this work relating the Sullivan decomposable algebras generated by the Theorem (4.2.1) when $\{\omega_1, \omega_2, \omega_3\}$ are linearly independent with the previous theorem.

By Sullivan's Theorem (2.3.1) there exists $\rho\colon M \to \Lambda(W \oplus \mathfrak{sl}(2))$ a minimal model of $\Lambda(W \oplus \mathfrak{sl}(2))$, such that $M = \Lambda(V)$ is a Sullivan minimal algebra of finite type and simply connected (this is $H^1(M) = 0$. Then, the space $|\langle M, d\rangle|$ is simply connected, and there is an isomorphism of abelian groups for $n \geq 2$:

$$\pi_n(|\langle (M, d)\rangle|) \cong Hom_{\mathbb{R}}(V^n, \mathbb{R}).$$

Then $\pi_2(|\langle (M, d)\rangle|) \cong \{0\}$ since $V^2 = H^2(\Lambda(W \oplus \mathfrak{sl}(2))) \cong \{0\}$.

If we assume that W and $\mathfrak{sl}(2)$ are $\mathbb{Q}$-vector spaces then there is a quasi-isomorphism

$$(M, d) \xrightarrow{\simeq} A_{PL}(|\langle M, d\rangle|)$$

This means that our simply connected Sullivan minimal algebra of finite type constructed via the study of equations of pseudo-spherical type is the minimal model of a topological space such that:

$$H^*(A_{PL}(|\langle M, d\rangle|); \mathbb{Q}) \cong H^*(M, d) \text{ and } \pi_n(|\langle (M, d)\rangle|) \cong Hom_{\mathbb{Q}}(V^n, \mathbb{Q}).$$

Conclusions

The study of the concepts of twisting matrices and twisted cohomology (Hom version) in the Theorem (1.5.1), and the explicit proof of the converse (Theorem 1.5.2), has allowed us to establish a constructive way of generating Sullivan decomposable algebras. Moreover, in our general proof of Sullivan's Theorem (we stress the fact that we did not find such a detailed proof in the literature), we have related the concepts of minimal model and twisting cohomology.

On the other hand, we have applied the above mentioned homological tools in the context of submanifolds of infinite jet bundles generated by certain PDEs. The concept of twisted cohomology (tensor version) has allowed us to prove that the "horizontal gauge cohomology" studied for M. Marvan in [12] and [13] as an elaboration of the more standard theory of the variational bicomplex, is a twisted cohomology with coefficients in a Lie algebra $\mathfrak{g}$ (see Theorem)3.4.1)). This means that we should be able to investigate the Marvan cohomology via Sullivan theory. We have indeed investigated some relations between differential equations and Sullivan theory in the special case of equations of pseudo-spherical type (which generate what we call a manifold of pseudo-spherical type embedded in a manifold of infinite jets). We have shown that Theorem (4.1.1) and the twisting matrix (4.1.7) allow us to generate Sullivan decomposable algebras. More generally, we have been able to present a Sullivan decomposable algebra generated by a hierarchy of pseudo-spherical type using a sequence of twisting matrices.

We have left as an open problem to generate Sullivan decomposable algebras via gauge cohomology, this is, to generalize Theorem (4.2.1). Besides, we leave pending the development of minimal models based on modules over the ring of smooth functions of submanifolds of an infinite jet bundle. We would hope that such a development may capture analytical properties of equations related to overdetermined linear problems such as the equations studied by Marvan in [12] and [13], or the equations of pseudo-spherical type. We also leave as an open problem to investigate in detail the minimal model of a Sullivan decomposable algebra generated by a manifold of pseudo-spherical type, and to further exploit the properties of the functor A_{PL} in order to investigate topological aspects of this (in general infinite-dimensional) manifold.

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