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The Markus–Yamabe Conjecture in $\mathbb{R}^n$ and Couples of Transversal Nets with Singularities

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Introduction

The thesis focuses on the following two topics: global attractors of vector fields in $\mathbb{R}^n$ and quartic differential forms on surfaces associated to couples of transversal nets with common singularities.

The first topic is related to the **Markus–Yamabe Conjecture**. We say that a square matrix is **Hurwitz** if all its eigenvalues have negative real part. Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1- vector field with $X(0) = 0$. Consider the differential system

$$\dot{x} = X(x)\,. \tag{0.0.1}$$

We say that 0 is a **local attractor** of the dynamical system (0.0.1) or the vector field X if $\phi(t, x)$ is defined for all $t > 0$ and tends to 0 as t tends to infinity, for each x in a neighborhood U of 0 in $\mathbb{R}^n$. When $U = \mathbb{R}^n$, we say that 0 is a **global attractor**. Here $\phi(t, x)$ is the solution of (0.0.1) with initial condition $\phi(0, x) = x$.

If $X(0) = 0$ and $DX(0)$ is Hurwitz, then the origin is a local attractor by the Hartman–Grobman Theorem [C1–22]. The problem is what hypotheses do we have to add to X to ensure that the origin is a global attractor. In [C1–17], L. Markus and H. Yamabe state their well–known global stability conjecture.

The Markus–Yamabe Conjecture (MYC). Let X be a Hurwitz C^1- vector field in $\mathbb{R}^n$, that is, $DX(p)$ is Hurwitz for all $p \in \mathbb{R}^n$. If $X(0) = 0$, then 0 is a global attractor of the system $\dot{x} = X(x)$.

In [C1–17], this conjecture is shown in two special cases. One case is for $n = 2$, $X(x, y) = (f(x, y), g(x, y))$, $X(0) = 0$, and when one of the four partial derivatives

f_x, f_y, g_x, g_y vanishes in all $\mathbb{R}^2$. The other is the triangular case, or in other words when $X = (f_1, f_2, \cdots, f_n)$, $X(0) = 0$, and for every $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, we have $\frac{\partial f_i}{\partial x_j}(x) = 0$ for $j < i$ and $\frac{\partial f_i}{\partial x_i}(x) < 0$, each $i = 1, 2, \cdots, n$.

In [C1–21], C. Olech shows that when $n = 2$, the injectivity of X is equivalent to proving that 0 is a global attractor of the system $\dot{x} = X(x)$ under the assumption that X is Hurwitz. Following this idea, the conjecture is proved for planar polynomial maps by G.H. Meisters and C. Olech [C1–19]. For planar C^1-maps, C. Gutiérrez [C1–14], R. Feßler [C1–12], and A. A. Glutsyuk [C1–13] prove the conjecture independently. N. E. Barabanov [C1–2] and, subsequently, J. Bernat and J. Llibre [C1–4] give examples of smooth Hurwitz–vector fields defined on $\mathbb{R}^n$, for $n \geq 4$, having a periodic orbit. A. Cima et al. [C1–8] show an example of a Hurwitz–polynomial vector field defined on $\mathbb{R}^n$, for each $n \geq 3$, having orbits which scape to infinity. Further, a more general family of polynomial counterexamples is given in [C1–9]. Finally, F. Mañosas and D. Peralta–Salas [C1–16] show that if a Hurwitz–vector field X is gradient, or in other words $X = \nabla f$ (f of class C^2), then the basin of attraction of the singular point 0 is $\mathbb{R}^n$, and therefore implying that the Markus–Yamabe Conjecture is true for this class of vector fields.

Since the counterexamples that we know have orbits which tend to infinity, we may consider the additional assumption that infinity is a repellor. However, this assumption is not good enough, at least for the differentiable cases. In fact, we prove that the Bernat–Llibre Counterexample [C1–4] may be modified so as to obtain smooth counterexamples having a periodic orbit and infinity as a repellor.

In Chapter 1 of the Thesis we also consider a special class of Hurwitz–polynomial vector fields. For a negative real number λ and a positive integer n, we denote by $\mathcal{N}(\lambda, n)$ the set consisting of the polynomial vector fields in $\mathbb{R}^n$ of the form $X = \lambda I + H$, where I is the identity map and H is a vector field with nilpotent Jacobian matrix at every point. Observe that the counterexamples of [C1–9] are vector fields $X = \lambda I + H$ in $\mathcal{N}(\lambda, n)$, with $n \geq 3$, where H is a quasi–homogeneous vector field of degree one. Here we find a more general family of counterexamples for the MYC, consisting of the vector fields of the form $X = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$ where H is not necessarily a quasi–homogeneous vector field of degree one, which contains the family constructed in [C1–9]. In addition, we show that the vector fields $X = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$, with $H = H_k + H_m$, where H_k and H_m are homogeneous of degree k and m, respectively, with $1 \leq k < m$, are linearly triangularizable and, therefore, the origin is a global attractor.

Chapter 1 is organized as follows:

Section 1.1 contains an overview of the relevant results about the Markus–Yamabe Conjecture in the literature. We discuss the proofs of the conjecture in the case of C^1-planar vector fields due to C. Gutiérrez [C1–14], R. Feßler [C1–12], and A. A. Glutsyuk [C1–13].

Section 1.2 studies the triangular case. We reproduce the main result of [C1–17] where the conjecture for this class of vector fields is proved.

In Section 1.3, we outline the results of [C1–4] where J. Bernat and J. Llibre find a counterexample to the conjecture in $\mathbb{R}^4$ which has a periodic orbit.

In Section 1.4, we recall the concept of *bounded vector field*, that is, a vector field with infinity as a repellor. We establish conditions under which a smooth Hurwitz–vector field can be modified outside of a given compact neighborhood in order to obtain a bounded Hurwitz–vector field. We apply this result to the Bernat–Llibre Counterexample [C1–4] in order to obtain a smooth Hurwitz–vector field in $\mathbb{R}^4$ which is bounded and has a periodic orbit.

Section 1.5 contains our second main result for the first topic. We find a more general family of counterexamples for the MYC which contains the family constructed in [C1–9].

Section 1.6 contains our third main result for the first topic. Vector fields of the form $X = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$, with $H = H_k + H_m$ where H_k is homogeneous of degree $k > 1$ and H_m is homogeneous of degree $m > k$, are linearly triangularizable, and therefore the origin is a global attractor.

Chapter 2 is devoted to our second topic of study. Given a smooth, connected, oriented two–manifold M, we consider a class $\mathcal{Q}(M)$ consisting of all smooth quartic differential forms ω defined on M which have the following property. At each point p in M, there exist a local chart $(u,v) : U \subset M \longrightarrow \mathbb{R}^2$ and smooth maps $E, F, G : (u,v)(U) \to \mathbb{R}$, with $EG - F^2$ positive everywhere, such that if

$$(u,v)^*(\omega) \;=\; a_4 dv^4 \;+\; 4a_3 dv^3 du \;+\; 6a_2 dv^2 du^2 \;+\; 4a_1 dv du^3 \;+\; a_0 du^4\,, \qquad (0.0.2)$$

then

$$G\,(a_0,a_1,a_2) \;-\; 2F\,(a_1,a_2,a_3) \;+\; E\,(a_2,a_3,a_4) \equiv 0\,. \qquad (0.0.3)$$

We associate to each ω a pair of transversal nets, say $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$, with common singularities. These quartic forms are related to geometric objects such as curvature lines, asymptotic lines of surfaces immersed in $\mathbb{R}^4$. (See [C2–3], [C2–6], [C2–7], [C2–8], [C2–9], [C2–19] and [C2–20].)

Local problems around rank–2 singular points of the elements of $\mathcal{Q}(M)$, such as stability, normal forms, finite determinacy, versal unfoldings, are studied in [C2–4]. Our principal contribution related to this topic is the study of a rank–1 singular point, namely that of type H_{45}, which is the analogue of the saddle–node singularity of vector fields. For this singular point, we find the local phase portrait of the corresponding nets around the point , a normal form for the family $\omega(\mu)$ in $\mathcal{Q}(M)$, with parameter $\mu \in \mathbb{R}^k$, for which the origin is an $H_{45}-$singular point of $\omega(\bar{0})$ and a versal unfolding nd its corresponding bifurcation diagram.

Chapter 2 is organized as follows:

Section 2.1 defines the set $\mathcal{Q}(M)$ and the nets associated to each $\omega \in \mathcal{Q}(M)$.

In Section 2.2, we prove that the set $\mathcal{Q}(M)$ is well defined, or in other words that its definition is independent of charts chosen. We show that for any $\omega \in \mathcal{Q}(M)$ and any point $p \in M$, there exists a local chart (u,v), namely a **main chart**, where the quartic has the simple form

$$(u,v)^*(\omega) \;=\; 4a\,(du^2 - dv^2)du dv \;+\; b\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

In Section 2.3, we introduce the *simple singular points*. We show that they are both generic and persistent under perturbations of the quartic differential form in $\mathcal{Q}(M)$. We give the local configuration of the nets $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$ around this type of points, and we characterize those singular points which are locally stable.

In Section 2.4, we introduce the $H_{45}-$*singular point*. This is a rank–1 singular point which is the analogue of both the saddle–node singularity of vector fields and the D_{23} singular point for positive quadratic differential forms (see [C2–5]). We determine the local phase portrait of the corresponding nets around this point.

Section 2.5 considers smooth $k-$parameter families of quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$, establishing the notion of equivalence for families. We find versal unfoldings for the two different types of non–locally stable simple singular points, showing that one type is of codimension one and the other is of codimension two. Further, we show that the singular points of type H_{34} are of codimension one.

Section 2.6 is devoted to proving a crucial result used in Section 2.5: the existence of main charts for smooth $k-$parameter families of quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$. Our proof was inspired by the one given by M. Spivak in [C2–21, Addendum 1] for the existence of smooth isothermal coordinates (in the case where there are no parameters).

Contents

Chapter 1

The Markus-Yamabe Conjecture

1.1 The Markus-Yamabe Conjecture

The purpose of this section is to present the Markus–Yamabe Conjecture. We give an overview of known results on the subject.

Recall that a C^1-vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ is *Hurwitz* if for each $p \in \mathbb{R}^n$, all of the eigenvalues of the Jacobian matrix of X at p, denoted $JX(p)$, have negative real part. We record the conjecture which was first introduced in [C1–17].

The Markus–Yamabe Conjecture (MYC). Let X be a Hurwitz C^1- vector field in $\mathbb{R}^n$. If $X(0) = 0$, then 0 is a global attractor of the system $\dot{x} = X(x)$.

There L. Markus and H. Yamabe establish the conjecture in the case of triangular vector fields in $\mathbb{R}^n$.

An important result for this conjecture in the two–dimensional case is obtained by C. Olech in [C1–21]. He first shows that the origin is a global attractor of any Hurwitz C^1-vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$, with $X(0) = 0$, such that

$$\|X(x)\| \geq \rho \quad \text{if} \quad \|x\| \geq r \tag{1.1.1}$$

for positive constants ρ and r. Next he shows that any injective Hurwitz C^1-vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies condition (1.1.1). Therefore, in order to prove the Markus–

Yamabe Conjecture when $n = 2$, it suffices to show that any Hurwitz C^1-vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ is injective.

In 1987, G. H. Meisters and C. Olech [C1–19] used Olech's results to show the two–dimensional Markus–Yamabe Conjecture in the case of polynomial vector fields. In 1993, after several years without significant progress on the subject, R. Feßler [C1–12], C. Gutiérrez [C1–14], and A. A. Glutsyuk [C1–13] proved independently the conjecture in the case of C^1-vector fields in $\mathbb{R}^2$. The three authors showed the injectivity of any Hurwitz C^1-vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$.

The assumption that a C^1-vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ is Hurwitz consists of two inequalities: for all $q \in \mathbb{R}^2$, we have

(i) $\det JX(q) > 0$,

(ii) $\mathrm{trace} JX(q) < 0$.

Only A. A. Glutsyuk uses inequality (ii) explicitly. Assuming that Y is not injective, he cleverly constructs a bounded regular region so that the flow of the vector field through its boundary is positive. He thus contradicts inequality (ii).

However, the injectivity results are more general in the other two works. Feßler's main result is the following.

Theorem 1.1.1. *[C1–12, Theorem 1] Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-vector field such that:*

1. $\det JX(q) > 0$ *for all* $q \in \mathbb{R}^2$.

2. *There is a compact set $K \subset \mathbb{R}^2$ such that $DX(q)v \neq \lambda v$ for all $q \in \mathbb{R}^2 \setminus K$ and $\lambda \in [0, \infty[$.*

Then X is injective.

The proof of Theorem 1.1.1 follows Theorem 1.1.3, and is consequence of the next two results.

Theorem 1.1.2. *[C1–12, Theorem 2] Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-vector field which is not injective, and is such that $\det JX(q) > 0$ for all $q \in \mathbb{R}^2$. Then, for every compact set $K \subset \mathbb{R}^2$, there is a curve $\gamma : \mathbb{R} \to \mathbb{R}^2 \setminus K$ satisfying the following properties:*

1. *γ is injective, proper, and regular.*
2. *There is an $\varepsilon > 0$ such that, for every $s_1 \leq 0$ and $s_2 \geq 1$, the rotation of $(X \circ \gamma)'$ from s_1 to s_2 is at least $3\pi + \varepsilon$.*

Theorem 1.1.3. *[C1–12, Theorem 3] Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be injective, proper, and regular. Then for every $\varepsilon > 0$, there are $s_1 \leq 0$ and $s_2 \geq 1$ such that the rotation of $\gamma'(s)$ from s_1 to s_2 is less than $\pi + \varepsilon$.*

Suppose that there were a C^1-vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ which is not injective satisfying assertions 1 and 2 of Theorem 1.1.1. Then we could take the curve γ and $\varepsilon > 0$ of Theorem 1.1.2, and the s_1, s_2 of Theorem 1.1.3 to obtain the estimate

$$\begin{aligned}
(\angle\gamma'(s_2) \;-\; \angle(X\circ\gamma)'(s_2)) \;&-\; (\angle\gamma'(s_1) \;-\; \angle(X\circ\gamma)'(s_1)) \\
= \angle\gamma'(s_2) \;-\; \angle\gamma'(s_1) \;&-\; (\angle(X\circ\gamma)'(s_2) - \angle(X\circ\gamma)'(s_1)) \\
&< \pi + \varepsilon \;-\; (3\,\pi + \varepsilon) \;=\; -2\,\pi\,.
\end{aligned}$$

Then, according to assumption 2 of Theorem 1.1.1, we would have

$$(X\circ\gamma)'(s) \;=\; DX(\gamma(s)\,\gamma'(s) \neq \lambda\,\gamma'(s) \quad \text{for all} \quad \lambda > 0\,.$$

This would mean that $X \circ \gamma)'(s)$ and $\gamma'(s)$ never point in the same direction. Then there must be an open interval of length $2\,\pi$, say $]\alpha_0, \alpha_0 + 2\,\pi[$, such that $\angle\gamma'(s) \;-\; \angle(X\circ\gamma)'(s) \;\in]\alpha_0, \alpha_0 + 2\,\pi[$ for all s, which contradicts the estimate above. Therefore, X must be injective.

Two slightly more general results are proved by C. Gutiérrez in [C1–14], called Theorem C and Theorem D. Here we reproduce only Theorem C because it is more related with the Markus–Yamabe Conjecture. Let H_θ denote the rotation of $\mathbb{R}^2$ by an angle θ, and let $Y_\theta = H_\theta \circ Y \circ H_\theta^{-1} = (f_\theta, g_\theta)$. The result is the following.

Theorem 1.1.4. *[C1–14, Theorem C] Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-map such that, for all $p \in \mathbb{R}^2$, $DY(p)$ is invertible. Suppose that there exists $v \in \mathbb{R}^2$, with $\|v\| = 1$, such that the following (directional) obstruction property is satisfied: For all $\theta \in \mathbb{R}$ and for all $p \in \mathbb{R}^2$, with $\|p\| \geq \rho$,*

$$\nabla f_\theta(q) \cdot v \neq \|\nabla f_\theta(q)\| \ .$$

Then, Y is injective.

A consequence of this Theorem is the following result. Given $\rho \in [0, \infty)$ and a C^1-map $Y : \mathbb{R}^2 \to \mathbb{R}^2$, we say that Y satisfies the $\rho-$*eigenvalue condition* if, for all $q \in \mathbb{R}^2$, the determinant of $DY(q)$ is positive and, for all $p \in \mathbb{R}^2$, with $\|p\| \geq \rho$, the spectrum of $DY(p)$ is disjoint of the non–negative real half axis.

Theorem 1.1.5. *[C1–14, Theorem A] If $Y : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1-map that satisfies the $\rho-$eigenvalue condition, for some $\rho \in [0, \infty)$, then Y is injective.*

Since, any Hurwitz C^1-vector field $Y : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the condition of Theorem 1.1.5, the bi–dimesional Markus–Yamabe Conjecture follows of this result.

It was more or less known for some time that the conjecture is not true for $n > 3$. The counterexamples usually claimed the existence of a non–trivial periodic trajectory of the vector field in question, which made them rather difficult to construct and sometimes more difficult to check or believe in (see [C1–2], [C1–4], as well as Section 1.3). Another way of constructing counterexamples is to find Hurwitz–vector fields with unbounded orbits. In this context, an explicit polynomial counterexample to the MYC in dimension 3 was announced by A. Cima, A. van den Essen, A. Gasull, E. Hubbers and F. Mañosas in 1995. They proved that for the vector field

$$Y(x, y, z) \ = \ (-x + z(x + yz)^2, -y - (x + yz)^2, -z),$$

all of the eigenvalues of DY are constant and equal to $-1,$ while the vector field admits the unbounded trajectory $y(t) \ = \ (18e^t, -12e^{2t}, e^{-t})$.

1.2 The Triangular Case

A fundamental class of vector fields for which the Markus–Yamabe Conjecture is true is that consisting of the triangular vector fields. Since these vector fields play a key role in one of our main results (see Theorem 1.6.3), we reproduce in Theorem 1.2.1 the original result of Markus–Yamabe together with its proof (see [C1–17][Theorem 3]).

Let us first recall two definitions.

Definition 1.2.1. *A C^1-vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ is said to be* **triangular** *if it has the form*

$$X(x_1, x_2, \ldots, x_n) \;=\; (f_1(x_1, x_2, \ldots, x_n), f_2(x_2, \ldots, x_n), \ldots, f_n(x_n))\,. \tag{1.2.2}$$

In addition, X is said to be **linearly triangularizable** *if there exists a linear change of coordinates which makes X triangular.*

Remark 1.2.1. *Any triangular and any linearly triangularizable vector field is a Hurwitz–vector field.*

Theorem 1.2.1. *Consider a C^1-triangular vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ of the form (1.2.2) and the system*

$$\dot{x}_i = f_i(x_1, \ldots, x_n)\,, \quad \textit{with } i = 1, 2, \ldots, n\,. \tag{1.2.3}$$

Suppose that $X(x) = 0$ if and only if $x = 0$. Then each solution of system (1.2.3) is defined for all large t and tends to the origin as $t \to \infty$.

Proof. The theorem is trivial if $n = 1$. Now we proceed by induction to prove the theorem in the general case.

Suppose the theorem holds for differential systems in $\mathbb{R}^{n-1}$ satisfying the hypotheses.

Consider the system

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, \ldots, x_n) \\ \dot{x_2} &= f_2(x_2, x_3, \ldots, x_n) \\ &\;\;\vdots \\ \dot{x_n} &= f_n(x_n) \end{aligned} \tag{1.2.4}$$

in $\mathbb{R}^n$ which satisfies the hypotheses of the theorem.

If $(x_1^0, x_2^0, \dots, x_n^0)$ is a point in $\mathbb{R}^n$ at which

$$\begin{aligned} f_2(x_2^0, x_3^0, \dots, x_n^0) &= 0 \\ &\vdots \\ f_n(x_n^0) &= 0, \end{aligned}$$

since $f_n(0) = 0$ and $f_n'(x_n) < 0$ for all $x_n \in \mathbb{R}$, then we have that $x_n^0 = 0$. Hence

$$f_{n-1}(x_{n-1}^0, 0) = 0$$

and since $f_{n-1}(0,0) = 0$ and $\dfrac{\partial f_{n-1}}{\partial x_{n-1}} < 0$ we have $x_{n-1}^0 = 0$. Similarly

$$x_{n-2}^0 = x_{n-3}^0 = \cdots = x_2^0 = x_1^0 = 0.$$

Thus the last $(n-1)$ equations of 1.2.4 form a system

$$\begin{aligned} \dot{x_2} &= f_2(x_2, x_3, \dots, x_n) \\ &\vdots \\ \dot{x_n} &= f_n(x_n) \end{aligned} \tag{1.2.5}$$

which satisfies the hypothesis of the theorem in the $\mathbb{R}^{n-1}$ space $x_1 = 0$.

Let $S(t)$ with coordinates $x_1(t), x_2(t), \dots, x_n(t)$ on $0 \le t < \tau < \infty$ a solution of 1.2.4 in $\mathbb{R}^n$. Then $x_2(t), \dots x_n(t)$ form a solution of 1.2.5 and so can be extend over $0 \le t < \infty$.

Moreover

$$|x_2(t)|^2 + |x_3(t)|^2 + \cdots + |x_n(t)|^2 = \rho(t)^2$$

is bounded on $0 \le t < \infty$ and

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} x_3(t) = \cdots = \lim_{t\to\infty} x_n(t) = 0$$

by the induction hypothesis.

Set K be a compact subset of the $\mathbb{R}^{n-1}$ space $x_1 = 0$ which contains the curve

$$x_2(t), x_3(t), \dots, x_n(t) \;\text{ for }\; 0 \le t < \infty.$$

Since $|f_1(0, x_2, \dots, x_n)|$ is bounded in K and since $\dfrac{\partial f_1}{\partial x_1} < 0$ in $\mathbb{R}^n$, we find that $x_1(t)$ can be extended over $0 \le t < \infty$, so solution $S(t)$ of 1.2.4 exists on $0 \le t < \infty$.

Now there is a ball $\mathcal{B}$, centered at the origin of $\mathbb{R}^n$, such that $S(t)$ approaches the origin if $S(t)$ intersects $\mathcal{B}$. Moreover there is a tube in $\mathbb{R}^n$

$$T : \rho(t) < \rho_0$$

such that $S(t)$ intersects $\mathcal{B}$ if $S(t)$ intersects T.

But

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} x_3(t) = \cdots = \lim_{t\to\infty} x_n(t) = 0.$$

Hence $S(t)$ must intersects T. Therefore

$$\lim_{t\to\infty} x_1(t) = 0$$

and $S(t)$ approaches the origin of $\mathbb{R}^n$ as $t \to \infty$. $\square$

An immediate consequence is

Corollary 1.2.1. *Any* linearly triangularizable *vector field $X : \mathbb{R}^n \to \mathbb{R}^n$, with $X(0) = 0$, has the origin as a global attractor.*

1.3 The Bernat–Llibre Counterexample

In 1994, J. Bernat and J. Llibre [C1–4] spurred by an article of N. Barabanov [C1–2], were able to construct a C^1–Hurwitz vector field that has a periodic orbit. This is the unique counterexample that we know which has a periodic orbit. Actually, all other counterexamples does not verify the conjecture because they have orbits that scape to infinity. Since we can not exclude that Bernat–Llibre vector field have orbits that escape to infinity, in Theorem 1.4.3 we modify this counterexample to obtain a bounded Hurwitz–vector field with a periodic orbit. In this section we will outline the procedure they used to find its counterexample.

The Bernat–Llibre vector field belong to a special kind called *linear control vector fields* which depend of a *characteristic function* $\varphi : \mathbb{R} \to \mathbb{R}$. More precisely they consider vector fields in $\mathbb{R}^4$ of the form

$$X_\varphi(x_1, x_2, x_3, x_4) = (x_2, -x_4, x_1 - 2x_4 - k_1\,\varphi(x_4), x_1 + x_3 - x_4 - k_2\,\varphi(x_4)), \quad (1.3.6)$$

where $k_1 = \frac{9131}{900}$, $k_2 = \frac{1837}{180}$. They began with the piecewise characteristic function

$$\varphi(x) = \begin{cases} -u & \text{si} \quad x < -u\,, \\ x & \text{si} \quad |x| \leq u\,, \\ u & \text{si} \quad x > u\,, \end{cases} \qquad \text{with} \quad u = \frac{900}{9185}\,.$$

Observe that this vector field is Lipschitz and so the existence and uniqueness of solutions works. They prove that the vector field X_φ has a stable periodic orbit. We will outline the procedure they used to find the stable periodic orbit. Since $\varphi(-t) = -\varphi(t)$, the system is symmetric with respect to the origin of $\mathbb{R}^4$, that is, if $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ is a solution, then $-x(t)$ is also a solution. Therefore if a solution $x(t)$ pass through the points x_0 and $-x_0$, then $x(t)$ is a symmetric periodic solution. In this form it is sufficient to study two linear systems, the linear system in the region $|x_4| \leq u$ and the linear system in the region $x_4 < -u$. They prove the existence of a symmetric periodic orbit $\gamma(t)$ through a point $(a_1, a_2, a_3, -u)$ in the following way:

(1) Computing explicitly the solutions $y(t)$ in the region $|x_4| < u$ and the solutions $m(t)$ in the region $x_4 < -u$.

(2) Computing the time $s > 0$ which needs the solution $m(t)$ of the system in the region $x_4 < -u$ for going in forward time from the point $m(0) = (a_1, a_2, a_3, -u)$ to the hyperplane $x_4 = -u$.

(3) Computing the time $-\tau < 0$ which needs the solution $y(t)$ of the system in the region $|x_4| < u$ for going in forward time from the point $y(0) = (-a_1, -a_2, -a_3, u)$ to the hyperplane $x_4 = -u$.

(4) Finding $a_1, a_2, a_3, s > 0$ and $\tau > 0$ such that $m(s) = y(-\tau)$.

Let Γ_- (resp. Γ_+) denotes the hyperplane $x_4 = -u$ (resp. $x_4 = u$). Let $(z_0, -u)$ be the initial point $(a_1, a_2, a_3, -u) \in \Gamma_-$ of the periodic orbit $\gamma(t)$ constructed above. The Poincaré map defined on a neighborhood of $\gamma(t)$ is the composition of the following four maps. Let $(z_1, -u)$ be the first point of the periodic orbit on Γ_- after the orbit passes through the point $(z_0, -u)$. The maps are

$$\begin{aligned} T_1 &: B_r(z_0) \cap \Gamma_- \to \Gamma_- \ , \\ T_2 &: B_r(z_1) \cap \Gamma_- \to \Gamma_+ \ , \\ T_3 &: B_r(-z_0) \cap \Gamma_+ \to \Gamma_+ \quad \text{and} \\ T_4 &: B_r(-z_1) \cap \Gamma_+ \to \Gamma_- \ , \end{aligned}$$

where $r > 0$ is sufficiently small and $B_r(z)$ is the open ball in $\mathbb{R}^4$ with center the point $(z, -u)$ and radius r. Due to the symmetry of the solutions with respect to the origin we get that $T_3(z) = -T_1(-z)$ and $T_4(z) = -T_2(-z)$. Then the Poincaré map T in a neighborhood of $\gamma(t)$ can be defined as

$$T \ = \ T_4 \circ T_3 \circ T_2 \circ T_1 : B_r(z_0) \cap \Gamma_- \to \Gamma_- \,.$$

Since the ordinary differential systems which define the Poincaré map T_i for $i = 1, \ldots, 4$ are linear systems, the maps T_i and T are analytic. Then to prove that $\gamma(t)$ is locally stable, the authors show that all the eigenvalues of $DT(z_0)$ are real and have modulus smaller than 1. Since the point z_0 and the smallest time $t_0 > 0$ that the solution $\gamma(t)$ from $(z_0, -u)$ needs for to arrive Γ_- are not know exactly, they find an specific point

$$\tilde{z}_0 \ = \ (0.22275019594, -2.13366751029745, -1.395139155570) \in \Gamma_-$$

with the following properties:

1) $\|DT(z_0) - DT(\tilde{z}_0)\| \ \leq 6.75 \cdot 10^{-3}$.

2) The eigenvalues of $DT(\tilde{z}_0)$ are 0.30521, 0.00557788 and $9.11685 \cdot 10^{-6}$.

3) The matrix $D = P^{-1}\, DT(\tilde{z}_0)\, P$ is diagonal, with

$$P = \begin{pmatrix} 0.643732 & 0.646488 & -0.689503 \\ -0.331955 & -0.216012 & -0.731705 \\ -0.689503 & -0.00999182 & 0.483755 \end{pmatrix} .$$

Then to estimate the eigenvalues of $DT(z_0)$ the authors use the following result (see [C1–23][Theorem 6.9.6]).

Proposition 1.3.1. *If B is a diagonalizable $n \times n$ matrix, $B = P\,D\,P^{-1}$, and A is an arbitrary $n \times n$ matrix, then for each eigenvalue $\lambda(A)$ there is an eigenvalue $\lambda(B)$ such that*

$$|\lambda(A) \; - \; \lambda(B)| \leq \|P\| \left\|P^{-1}\right\| \|A - B\| \; .$$

Here $A = DT(z_0)$, $B = DT(\tilde{z}_0)$, $\|P\| < 2$ and $\|P^{-1}\| < 16.6$. Hence $|\lambda(A) \; - \; \lambda(B)| < 0.2241$, it follows that the eigenvalues of $DT(z_0)$ are real and with modulus smaller than 1.

Although, for $|x_4| > u$ the Jacobian matrix of X_φ has eigenvalues with zero real part, they prove that if the map φ is changed by any C^1-map ϕ with $0 < \phi'(x_4) < \frac{91310}{5511}$, then the vector field X_ϕ is Hurwitz. In this form they prove that there exists a C^1-map ψ such that the vector field X_ψ is Hurwitz and has a periodic orbit. Also they remark that the function ψ can be chosen C^r for all $r \geq 1$, C^∞ or analytic.

To be more specific, they have the following results.

Proposition 1.3.2. *[C1–4, Proposition 8.1] The vector field X_ϕ is Hurwitz if the characteristic function ϕ is C^1 and satisfies $0 < \phi'(x_4) < \frac{91310}{5511}$.*

Theorem 1.3.1. *[C1–4, Section 9] Given $\varepsilon > 0$ and $R > 0$ and $r \in \mathbb{Z}^+ \cup \{\infty\}$ there exists a characteristic function ψ with $\psi(0) = 0$ such that*

a) *ψ is of class C^r,*

b) *$0 < \psi'(x) < 10$ for all $x \in \mathbb{R}$,* *c)* $|\psi(x) - \varphi(x)| < \varepsilon$ *for all* $x \in [-R, R]$.

d) *The Poincaré map* T_ψ*, defined by the flow of* X_ψ *in a neighborhood of* $z_0 \in \Gamma$ *with* Γ *a transversal section, has a stable fixed point near* z_0.

Remark 1.3.1. *(1)* X_ψ *is a Hurwitz* C^r- *vector field such that* $X_\psi(0) = 0$ *and having an stable periodic orbit.*

(2) The vector field X_ψ *can have orbits that scape to infinity.*

1.4 Hurwitz–vector fields with periodic orbits

In this section we study the conditions under which a Hurwitz vector field X of class C^∞ in $\mathbb{R}^n$ can be arbitrarily approximate by bounded Hurwitz vector field in a compact neighborhood. Hence, we apply the result to the Bernat–Llibre vector field to obtain a bounded Hurwitz vector field which has a periodic orbit. We begin by definition of bounded vector field.

Definition 1.4.1. *Let* $X : \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^1-*vector field. We say that* X *is* **bounded** *if there exists a compact set* $K \subseteq \mathbb{R}^n$ *such that for all* $x \in \mathbb{R}^n$ *there exist* $t_x \in \mathbb{N}$ *such that* $\phi(t, x) \in K$ *for all* $t \geq t_x$.

The Euclidean inner product of two vector $p, q \in \mathbb{R}^n$ will denoted by $< p, q >$.

Remark 1.4.1. *Any* C^1-*vector field* $X : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $< X(p), p >< 0$ *for all* $\|p\| > K$*, is bounded.*

To simplify we introduce the following notation. Given a C^1-vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ we denote

$$S(X) \; = \; \sup_{p \in \mathbb{R}^n} \{\max\{\Re(\lambda) \, : \, \lambda \;\; \text{is an eigenvalue of } DX(p)\}\} \, .$$

Next we state the mail result of this section. Let $r \in \mathbb{Z}^+ \cup \{\infty\}$.

Theorem 1.4.1. *Let* $X : \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^r-*vector field such that* $S(X) \leq -c_0$*, for some* $c_0 > 0$*. Suppose further that exist positive numbers* C, R *and* $\alpha > 1$ *such that* $\|DX(p)\| \leq C$ *for all* $p \in \mathbb{R}^n$ *and* $< X(p), p > \ \leq \ \alpha < p, p >$ *for all* $\|p\| \geq R$*. Then, given* $0 < b < 1$ *and a compact neighborhood* U *of the origin , there exists a bounded* C^r-*vector field,* $Y : \mathbb{R}^n \to \mathbb{R}^n$ *with* $S(Y) \leq \dfrac{-c_0}{2}$ *such that* $Y(p) = X(p) - b\,p$ *for all* $p \in U$*.*

The following two Lemmas 1.4.1 and 1.4.2 are fundamental in the proof of Theorem 1.4.1.

Lemma 1.4.1. *Given* $R > 0$*,* $0 \leq b < a$ *and* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *and a smooth function*

$$\phi : [0, \infty) \to [b, a]$$

such that

(1) $\phi(r) = b$ *for all* $r \in [0, R]$

(2) $\phi^{'}(r) \geq 0$ *and* $\mid \phi^{'}(r) \cdot r \mid < \epsilon$ *for all* $r \geq 0$*.*

(3) $\phi(r) = a$ *for all* $r \geq R + N$*.*

Proof. First we define $\phi(r) = b$, for all $r \in [0, R]$. Hence, we consider the sequence $(S(n))$ defined for all positive integer n by

$$S(n) = b + \frac{\varepsilon}{8}\left(\frac{1}{R+1} + \frac{1}{R+2} + \cdots \frac{1}{R+n}\right).$$

Observe that $S(n) \to \infty$ as $n \to \infty$. Let N be the integer such that verifies $S(n) < a$ for $n = 1, \ldots, N-1$, and $S(N) \geq a$. For all positive integer $n \geq 1$ we define

$$\begin{cases} \phi(R+n) = S(n), & \text{if} \quad 1 \leq n \leq N-1 \\ \phi(R+n) = a, & \text{if} \quad n \geq N \end{cases}$$

For each integer n such that $1 \leq n \leq N-1$ we consider a smooth map C^∞ $\phi : [R+n, R+n+1] \to [S(n+1), S(n)]$ which is flat at $R+n$ y $R+n+1$ and defined as above and such that

$$0 \leq \phi'(r) =\mid \phi'(r) \mid = \frac{2(S(n+1) - S(n))}{(R+n+1) - (R+n)} = \frac{\varepsilon}{4(R+n+1)}.$$

Then the map ϕ satisfies the conditions because, for $n = 1, \ldots N$ y $R+n-1 \leq r \leq R+n$ we have

$$\mid \phi'(r) \cdot r \mid \leq \frac{\varepsilon}{4(R+n)} \cdot (R+n) = \frac{\varepsilon}{4} < \epsilon.$$

□

Lemma 1.4.2. *Set $M(n)$ be the space of $n \times n$ real matrices. Given $c_0 > 0$ and $C > 0$, set $\mathcal{A}$ be the compact set*

$$\mathcal{A} = \{A \in M(n) : \|A\| \leq C \text{ and } S(A) \leq -c_0\}.$$

Then, given $\varepsilon > 0$ exists $\delta > 0$ such that

$$S(A - B) \leq -c_0 + \varepsilon$$

for all $A \in \mathcal{A}$ and for all $B = cI + E$, with $c \geq 0$ y $\|E\| < \delta$.

Proof. Writting $B = cI + E$, we have $A - B = A - cI - E$ and

$$\det(A - B - \lambda I) = \det\left(A - E - (\lambda + c)I\right),$$

i.e.,

$$\lambda - c \in \operatorname{Spec}(A - B) \iff \lambda \in \operatorname{Spec}(A - E),$$

which implies

$$S(A - B) \leq S(A - E).$$

Then, it is sufficient to choose $\delta > 0$ such that

$$S(A - E) \leq -c_0 + \varepsilon,$$

for all $A \in \mathcal{A}$ and $\|E\| < \delta$. □

Theorem 1.4.2. *Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a C^r-vector field such that $S(X) \leq -c_0 < 0$. Suppose that exists $C > 0$ such that $\|DX(p)\| \leq C$ for all $p \in \mathbb{R}^n$. Given $R > 0$ and $0 \leq b < a$, there exist $N \in \mathbb{N}$ and a C^r- vector field $Y : \mathbb{R}^n \to \mathbb{R}^n$ such that:*

a) $S(Y) \leq \dfrac{-c_0}{2}$,

b) $Y(x) = X(x) - b\,x$ *for all* $\|x\| \leq R$,

c) $Y(x) = X(x) - a\,x$ *for all* $\|x\| \geq R + N$.

Proof. First choose $\varepsilon > 0$ such that $S(A - B) \leq -\frac{c_0}{2}$ for all $A \in \mathcal{A} = \{A \in M(n) : \|A\| \leq C \text{ and } S(A) \leq -c_0\}$ and all $B = cI + E$, with $c \geq 0$ and $\|E\| < \varepsilon$ (Lemma 1.4.2). Associated at this $\varepsilon > 0$, $R > 0$ and $0 \leq b < a$, we consider the function of class $\mathcal{C}^\infty$,

$$\phi : [0, \infty) \to [b, a]$$

of the Lemma 1.4.1 that verifies

(1) $\phi(r) = b$ for all $r \in [0, R]$

(2) $\phi'(r) \geq 0$ and $| \phi'(r) \cdot r | < \epsilon$ for all $r \geq 0$.

(3) $\phi(r) = a$ for all $r \geq R + N$.

Then the vector filed $Y(x) = X(x) - \phi(\|x\|)\,x$ is $\mathcal{C}^r$ and verifies the conditions b) and c). Finally if we define $f(x) = \phi(\|x\|)\,x$, we have $Df(x) = \phi(\|x\|)\,I + E(x)$, with $\|E(x)\| \leq |\phi'(\|x\|)\ \|x\|| < \varepsilon$. Then the condition a) is consequence of Lemma 1.4.2. $\square$

Proof of Theorem 1.4.1. Set $R > 0$ such that $\|X(p)\| \leq R$ for all $p \in U$ and $a > \alpha$. Since X verifies the conditions of Theorem 1.4.2, associated to the numbers R, b and a, there exists $N \in \mathbb{N}$ and a C^r-vector field $Y : \mathbb{R}^n \to \mathbb{R}^n$ that verify

a) $S(Y) \leq \dfrac{-c_0}{2}$,

b) $Y(x) = X(x) - b\,x$ for all $\|x\| \leq R$,

c) $Y(x) = X(x) - a\,x$ for all $\|x\| \geq R + N$.

This vector field Y is also bounded. In fact, for $\|p\| \geq R + N$ we have

$$\begin{aligned} < Y(p), p > &= < X(p) - a\,p, p > \\ &= < X(p), p > -a < p, p > \\ &< < X(p), p > -\alpha < p, p > \\ &\leq 0\,, \end{aligned}$$

and the proof is completed. $\square$

Consider now the Bernat–Llibre vector field X_ψ of Theorem 1.3.1.

Proposition 1.4.1. *There exists a Hurwitz C^r-vector field $X : \mathbb{R}^4 \to \mathbb{R}^4$ which has a stable periodic orbit that satisfies the following conditions:*

(1) $S(X) < -c_0$ for some $c_0 > 0$.

(2) $\|DX(p)\| < C$ for all $p \in \mathbb{R}^4$ for some $C > 0$.

(3) $< X(p), p >< \alpha < p, p >$ for all $\|p\| > R$, for some positive numbers α and R .

Proof. The vector field X_ψ satisfies (2) and (3). In fact

$$\|DX_\psi(p)\| = 2 + \frac{9131}{900}\,\psi'(x_4) < 2 + \frac{9131}{90}\,,$$

and if $p = (x_1, x_2, x_3, x_4)$, then

$$\begin{aligned} < X_{\psi)}(p), p > &= x_1x_2 - x_2x_4 + x_3x_1 - x_4x_3 + x_4x_1 - x_4^2 + \\ &\quad -(\frac{9131}{900}\,x_3 + \frac{1837}{180}\,x_4)\,\psi(x_4) \\ &< 208 < p, p >, \end{aligned}$$

because $|\psi(x_4)| \leq 10\,|x_4|$. Since it is not clear that X_ψ satisfies condition (1), we consider $X = X_\psi - \varepsilon I$, where I is the identity map and $\varepsilon > 0$ is sufficiently small. This vector field is a Hurwitz C^r-vector field which has a stable periodic orbit that verifies the three conditions. $\square$

Theorem 1.4.3. *There exists a bounded Hurwitz C^r−vector field $Y : \mathbb{R}^4 \to \mathbb{R}^4$ which has a stable periodic orbit.*

Proof. The vector field X of Proposition 1.4.1 satisfy all conditions of Theorem 1.4.1. Then associated to a compact ball $U \subset \mathbb{R}^4$ that contain the stable periodic orbit of X and a small positive number b, there exists a bounded Hurwitz C^r−vector field Y such that $Y(p) = X(p) - b\,p$ for all $p \in U$. Such vector field satisfies our Theorem. □

1.5 A New Family of Counterexamples

In [C1–8], a polynomial counterexample for the Markus–Yamabe Conjecture for dimension $n \geq 3$ is given. Subsequently, in [C1–9], the authors explain a way for obtaining a family of polynomial counterexamples containing the ones above. The construction is based on results about quasi–homogeneous vector fields of degree one (see [C1–9, Section 2]). We next record the resultI [C1–9, Theorem 3.2] for $n = 3$. We do not give its proof.

Theorem 1.5.1. *For each $a, b, \lambda \in \mathbb{R}$ and each $k, l, m \in \mathbb{N}$, the vector field*

$$X(x, y, z) \;=\; \lambda(x, y, z) \;+\; (axz^l + byz^m)^k\,(-bz^m, az^l, 0) \tag{1.5.7}$$

satisfies the following three properties:

(1) X is linear quasi–homogeneous with weights

$$(\alpha_1, \alpha_2, \alpha_3) = (m + kl, l + km, 1 - k)\,.$$

(2) For all $\lambda \in \mathbb{R}$, with $\lambda < 0$, the vector field $X \in \mathcal{N}(\lambda, 3)$, and

(3) for all $\lambda \in \mathbb{R}$, with $\lambda < 0$, k an even number, $l, k, l - m \in \mathbb{N}$ other than zero, and all $a, b \in \mathbb{R} - \{0\}$, the differential system $\dot{x} = X(x)$ has unbounded orbits.

Remark 1.5.1. *1) If $m = l$ and/or $ab = 0$, then the origin is a global attractor of the vector field (1.5.7).*

2) The unbounded orbits of assertion (3) may be found as follows. Consider the curve

$$\alpha(t) = (x(t), y(t), z(t)) = (x_0 e^{At}, y_0 e^{Bt}, z_0 e^{\lambda t})$$

with $A = \lambda\, \frac{m+kl}{1-k}$, $B = \lambda\, \frac{l+km}{1-k}$. *If* x_0, y_0, z_0 *satisfy*

$$\begin{aligned} \lambda\, x_0 \frac{m+kl}{1-k} &= \lambda\, x_0 \;-\; b\, z_0^m\, (a x_0 z_0^l + b y_0 z_0^m)^k \quad \text{and} \\ \lambda\, y_0 \frac{k+km}{1-k} &= \lambda\, y_0 \;+\; a\, z_0^l\, (a x_0 z_0^l + b y_0 z_0^m)^k , \end{aligned}$$

then $\alpha(t)$ *is an unbounded solution. Since these conditions are equivalent to the conditions*

$$\begin{aligned} x_0 &= -\frac{b}{a}\, \frac{l+km-1+k}{m+kl-1+k}\, y_0\, z_0^{m-l} \quad \text{and} \\ y_0^{k-1} &= \frac{l+km-1+k}{ab^k\,(1-k)} \left(\frac{m+kl-1+k}{(m-l)(1-k)} \right)^k z_0^{l+km} , \end{aligned}$$

the proof of assertion (3) of Theorem 1.5.1 follows.

The following, which is the main result of this section, gives an extension of the family (1.5.7) and shows that assertion (3) of Theorem 1.5.1 may also hold for k odd.

Theorem 1.5.2. *For each* $a, b, \lambda \in \mathbb{R}$, *each* $k, l, m \in \mathbb{N}$, *and each polynomial map* $f : \mathbb{R} \to \mathbb{R}$, *the vector field*

$$X(x,y,z) \;=\; \lambda\,(x,y,z) \;+\; f(axz^l + byz^m)\,(-bz^m, az^l, 0) \tag{1.5.8}$$

satisfies the following two properties:

(1) For all $\lambda \in \mathbb{R}$, *with* $\lambda < 0$, *the vector field* $X \in \mathcal{N}(\lambda, 3)$, *and*

(2) for all $\lambda \in \mathbb{R}$, *with* $\lambda < 0$, *all* $l, k, l-m \in \mathbb{N}$ *other than zero, and all* $a, b \in \mathbb{R} - \{0\}$ *such that either* $m+l$ *is odd, or* $m+l$ *is even and* $(m-l)\, ab\, A_k < 0$, *the differential system* $\dot{x} = X(x)$ *has unbounded orbits.*

Proof. The proof of (1) is straightforward. Concerning assertion (2), since there cannot be any unbounded orbits in the plane $z = 0$, outside of this plane we consider the change of coordinates

$$(u, v, w) = (axz^l + byz^m, \lambda(m-l)byz^m, \lambda(m-l)abz^{m+l}).$$

The system $\dot{x} = X(x)$ then becomes

$$(\dot{u}, \dot{v}, \dot{w}) \;=\; (\lambda(l+1)u + v, \lambda(m+1)v + wf(u), \lambda(m+l)w). \tag{1.5.9}$$

In order to analyze the behavior of the solutions of (1.5.9) near infinity, we consider the change of coordinates

$$(s, p, r) \;=\; (u^{-1}\,,\; v\,u^{-1}\,,\; wu^{k+1})$$

obtaining the system

$$\begin{aligned} \dot{s} &= -s\,[\lambda(l+1) + t]\,, \\ \dot{p} &= p\,[\lambda(m-l) - p] \;+\; r[A_0 s^k + \cdots + A_k)]\,, \\ \dot{r} &= r\,[\lambda(lk + m + k - l) + (k-1)p] \end{aligned} \tag{1.5.10}$$

where $f(u) = A_0 + A_1\,u + \cdots A_k\,u^k$, with $A_k \neq 0$. The singularities in the plane $s = 0$ are $(0,0,0)$, $(0, \lambda(m-l), 0)$ and $(0, p_0, r_0)$, where

$$p_0 = \frac{-\lambda(lk + m + k - 1)}{k-1} \qquad \text{and}$$

$$r_0 = \frac{\lambda^2(mk + l + k - l)(lk + m + k - 1)}{(k-1)^2 A_k}\,.$$

Note that $p_0 > 0$ and $r_0\,A_k > 0$. Moreover, the Jacobian matrix of the vector field at $(0, p_0, r_0)$ has determinant $-\lambda\,(l+m)\,r_0\,A_k > 0$, and eigenvalues $\mu_1 = \frac{\lambda\,(l+m)}{k-1} \;<\; 0$, and μ_2, μ_3 with $\mu_2\,\mu_3 < 0$. Therefore, the singularity $(0, p_0, r_0)$ has a stable manifold of dimension 2 and an unstable manifold of dimension 1, which is contained in the plane $s = 0$.

Therefore, system (1.5.10) has solutions $(s(t), p(t), r(t))$ such that $s(t) > 0$, $p(t) \;>\; 0$, and $\lim_{t\to+\infty}(s(t), p(t), r(t)) \;=\; (0, p_0, r_0)$. Hence system (1.5.9) has solutions $(u(t), v(t), w(t))$, with $u(t)$ and $v(t)$ positive, so that $\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} v(t) = +\infty$. Further, $\lim_{t\to+\infty} w(t) = 0^+$ (resp. 0^-) if $A_k > 0$ (resp. $A_k < 0$).

In order to obtain unbounded solutions $(x(t), y(t), z(t))$ of our system $\dot{x} = X(x)$, we must solve the system

$$\left\{\begin{array}{rcl} u(t) & = & a\, x(t)\, z(t)^l + b\, y(t)\, z(t)^m\,, \\ v(t) & = & \lambda\,(m-l)\, b\, y(t)\, z(t)^m\,, \\ w(t) & = & \lambda\,(m-l)\, ab\, z(t)^{m+l}\,. \end{array}\right. \tag{1.5.11}$$

If $w(0) = w_0$, the third equation is reduced to finding a z_0 so that

$$w_0 \;=\; \lambda\,(m-l)\, ab\, z_0^{m+l}\,.$$

Indeed, the conditions imposed on this theorem guarantee the existence of such a z_0. With this z_0, we obtain

$$\begin{array}{rcl} x(t) & = & \dfrac{u(t)\, e^{-\lambda\, l\, t}}{a\, z_0^l} - \dfrac{v(t)}{\lambda\,(m-l)}\,, \\ y(t) & = & \dfrac{v(t)\, e^{-\lambda\, m\, t}}{\lambda\,(m-l)\, b\, z_0^m}\,, \\ z(t) & = & z_0\, e^{\lambda\, t}\,. \end{array}$$

Thus $\lim_{t\to+\infty} |y(t)| = \infty$, and we have the result. $\square$

Remark 1.5.2. *Setting $A_0 = A_1 = \cdots = A_{k-1} = 0$ and $A_k = 1$, we obtain the vector field (1.5.7).*

1.6 The Positive Case

In this section, given $\lambda < 0$, we consider vector fields $X = \lambda\, I + H_k + H_m \in \mathcal{N}(\lambda, 3)$, where H_k and H_m are homogeneous of degree k and m, respectively, with $1 \le k < m$. We will show that these vector fields are linearly triangularizable and, therefore, they have the origin as a global attractor. By definition, a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is *triangular* if it has the form

$$F(x_1, x_2, \cdots, x_n) \;=\; (F_1(x_1), F_2(x_1, x_2), \cdots, F_n(x_1, x_2, \cdots, x_n))\,.$$

In addition, F is said to be *linearly triangularizable* if there exists a linear change of coordinates which makes F triangular. Recall that L. Markus and H. Yamabe show that for Hurwitz–triangular vector fields which vanish at the origin, the origin is a global attractor (see [C1–17]). Therefore, any linearly triangularizable Hurwitz–vector field X, with $X(0) = 0$, has the origin as a global attractor.

Before giving our main result, we recall some definitions and preliminary results.

Let κ be an arbitrary field of characteristic zero and let $\kappa[x] = \kappa[x_1, \cdots, x_n]$ denote the polynomial ring in the variables $x_1, \cdots, x_n$ over κ. Associated to each polynomial $f \in \kappa[x]$ and $i = 1, \cdots, n$, we denote by $\frac{\partial f}{\partial x_i}$ the polynomial which is the formal derivative of f with respect to x_i.

A *polynomial map* is a map $F = (F_1, \cdots, F_n) : \kappa^n \to \kappa^n$ of the form

$$(x_1, \cdots, x_n) \to (F_1(x_1, \cdots, x_n), \cdots, F_n(x_1, \cdots, x_n))$$

where each F_i belong to $\kappa[x]$. Given a polynomial map $F = (F_1, \cdots, F_n)$, we denote by JF, the *Jacobian of* F; that is the map which associate to each $x \in \kappa^n$ the $n \times n-$matrix $JF(x)$ whose $(i, j)-$entries is $\frac{\partial F_i}{\partial x_j}(x)$. Further, we said that JF is *nilpotent* if the matrix $JF(x)$ is nilpotent at every point $x \in \kappa^n$. Finally, we say that a polynomial map $F : \kappa^n \to \kappa^n$ is *homegeneous of degree* k if $F(tx) = t^k F(x)$ for all $x \in \kappa^n$ and all $t \in \kappa$.

Our principal tools are Theorems 1.6.1 and 1.6.2, whose proofs are contained in [C1–6, Theorem 1.1] and [C1–7, Theorem 1.1], respectively. Recall that κ is a field of characteristic zero. The set consisting of all the linear isomorphisms $T : \kappa^n \to \kappa^n$ is denoted $Gl_n(\kappa)$.

Theorem 1.6.1. *Let $H = (H_1, H_2, H_3) : \kappa^3 \to \kappa^3$ be a homogeneous polynomial map of degree $d \geq 2$. If JH is nilpotent, then there exists a $T \in Gl_3(\kappa)$ such that $THT^{-1}(x, y, z) = (0, h_2(x), h_3(x, y))$, where the h_i are homogeneous of degree d.*

Theorem 1.6.2. *Let A be a unique factorization domain of characteristic zero, and let $H = (H_1, H_2) \in A[x_1, x_2]^2$. Then $J_{x_1,x_2}(H)$ is nilpotent if and only if*

$$H(x_1, x_2) = f(a_1\, x_1 + a_2\, x_2)(a_2, -a_1)\ +\ (c_1, c_2)$$

for some $a_1, a_2, c_1, c_2 \in A$ *and* $f(t) \in A[t]$.

Remark 1.6.1. *1) A consequence of Theorem 1.6.1 is that if* $H = (H_1, H_2, H_3) : \kappa^3 \to \kappa^3$ *is a homogeneous polynomial map of degree* $d \geq 2$*, then* H_1, H_2, H_3 *are linearly dependent over* κ.

2) If $\lambda < 0$*, then the origin is a global attractor for any vector field* $X = \lambda I + H \in \mathcal{N}(\lambda, 3)$*, with* H *homogeneous of degree* $d \geq 2$.

3) Let $P(x, y, z), Q(x, y, z)$ *be homogeneous polynomials of degree* k *such that* $J_{x,y}(P, Q)$ *is nilpotent at every point* $(x, y, z) \in \mathbb{R}^3$*. Then Theorem 1.6.2 implies that*

$$\begin{aligned} P(x,y,z) &= -b(ax+by)(\alpha_1 z^{k-1} + \alpha_2 z^{k-2}(ax+by) + \\ &\quad \cdots + \alpha_k (ax+by)^{k-1}) + c_1 z^k \quad \text{and} \\ Q(x,y,z) &= a(ax+by)(\alpha_1 z^{k-1} + \alpha_2 z^{k-2}(ax+by) + \\ &\quad \cdots + \alpha_k (ax+by)^{k-1}) + c_2 z^k \,. \end{aligned}$$

Now let

$$X = \lambda I + H_k + H_m : \mathbb{R}^3 \to \mathbb{R}^3 \in \mathcal{N}(\lambda, 3) \tag{1.6.12}$$

be a polynomial vector field where H_k and H_m are homogeneous of degree k and m, respectively, with $2 \leq k < m$. Since JH_k and JH_m are necessarily nilpotent, we have that, modulus a linear change of coordinates, the vector field X has the form

$$X(x,y,z) \;=\; \lambda\,(x,y,z) \;+\; (P,Q,R)(x,y,z) + (0, Ax^m, S(x,y)) \tag{1.6.13}$$

where P, Q, R are homogeneous polynomials of degree k and $S(x, y)$ is a homogeneous polynomial of degree m, with $2 \leq k < m$. Moreover, there exists a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3 - \{(0, 0, 0\}$ such that

$$\alpha\,P \;+\; \beta\,Q \;+\; \gamma\,R \equiv 0\,.$$

In addition, we must have

1) $mAx^{m-1}P_y + Q_zS_y + P_zS_x = 0$,

2) $-P_x Q_z S_y - P_y(mAx^{m-1}R_z - Q_z S_x) + P_z(mAx^{m-1}R_y + Q_x S_y - Q_y S_x) = 0,$

3) $mAx^{m-1}P_z S_y = 0.$

In order to prove that the origin is a global attractor of the vector field (1.6.12), we will assume that X has the form (1.6.13). First we consider the cases where P, Q, or R vanish (see Lemmas 1.6.1, 1.6.2, and 1.6.3). We next show that the general case can be reduced to one of the preceding cases.

Lemma 1.6.1. *For $\lambda < 0$, consider the polynomial vector field*

$$X(x, y, z) = \lambda\,(x, y, z) \;+\; (0, Q, R)(x, y, z) + (0, Ax^m, S(x, y)) \in \mathcal{N}(\lambda, 3)$$

where $Q(x, y, z)$ and $R(x, y, z)$ are homogeneous polynomials of degree k, and $S(x, y)$ is a homogeneous polynomial of degree m, with $1 \le k < m$. Then X is linearly triangularizable.

Proof. The condition $DX(p) - \lambda I$ nilpotent at every point $p \in \mathbb{R}^3$ implies

$$Q_y + R_z \equiv Q_y R_z - Q_z R_y \equiv Q_z S_y \equiv 0\,.$$

Hence $J_{y,z}(Q, R)$ is nilpotent at every point $(x, y, z) \in \mathbb{R}^3$ and we have

$$\begin{aligned}
Q(x, y, z) &= -c(by + cz)(\alpha_1 x^{k-1} + \alpha_2 x^{k-2}(by + cz) + \\
&\quad \cdots + \alpha_k (by + cz)^{k-1}) + a_1 x^k\,, \\
R(x, y, z) &= b(by + cz)(\alpha_1 x^{k-1} + \alpha_2 x^{k-2}(by + cz) + \\
&\quad \cdots + \alpha_k (by + cz)^{k-1}) + a_2 x^k \quad \text{and} \\
S(x, y) &= Bx^m + Cy\,T(x, y)\,,
\end{aligned}$$

with $c\,C\,(\alpha_1^2 + \cdots + \alpha_k^2) = 0$ and $T(x, y)$ a homogeneous polynomial of degree $m - 1$. Observe that X is triangular if $c(\alpha_1^2 + \cdots + \alpha_k^2) = 0$, and if $c \neq 0$ and $b = C = 0$. When $c \neq 0$ and $b \neq 0$, since $C = 0$, the vector field X is linearly triangularizable through the change of coordinates $(u, v, w) = (x, by + cz, z)$, which completes the proof. $\square$

Lemma 1.6.2. *For $\lambda < 0$, consider the polynomial vector field*

$$X(x,y,z) = \lambda\,(x,y,z)\;+\;(P,0,R)(x,y,z) + (0, Ax^m, S(x,y)) \in \mathcal{N}(\lambda,3)$$

where $P(x,y,z), R(x,y,z)$ are homogeneous polynomials of degree k, and $S(x,y)$ is a homogeneous polynomial of degree m, with $1 \le k < m$. Then X is linearly triangularizable.

Proof. The condition $DX(p) - \lambda I$ nilpotent at every point $p \in \mathbb{R}^3$ implies that

$$P_x + R_z \equiv P_xR_z - P_zR_x \equiv 0 \quad \text{and}$$

$$mAx^{m-1}P_y + P_zS_x \equiv A(P_yR_z - P_zR_y) \equiv AP_zS_y \equiv 0\,.$$

We suppose $P \neq 0$. The first two conditions imply

$$\begin{aligned} P(x,y,z) &= -c(ax+cz)(\alpha_1 y^{k-1} + \alpha_2 y^{k-2}(ax+cz) + \\ &\quad \cdots \alpha_k(ax+cz)^{k-1}) + b_1 y^k \quad \text{and} \\ R(x,y,z) &= a(ax+cz)(\alpha_1 y^{k-1} + \alpha_2 y^{k-2}(ax+cz) + \\ &\quad \cdots \alpha_k(ax+cz)^{k-1}) + b_2 y^k\,. \end{aligned}$$

If $c = 0$, then $Ab_1 = 0$ and the vector field X is triangular. If $c \neq 0$ and $A = 0$, then $S(x,y) = Cy^m$. Further, if $a = 0$, then the vector field X is triangular. If $a \neq 0$, then, performing the change of coordinates $(u,v,w) = (y, ax+cz, z)$, the vector field X is triangular in the new coordinates. Next if $c \neq 0$ and $A \neq 0$, then $S(x,y) = D\,x^m$ and $ab_1 + cb_2 = kAb_1 - \alpha_1 Dc^2 = (k-1)A\alpha_1 + 2cD\alpha_2 = \cdots = A\alpha_{k-1} + kcD\alpha_k = 0$. When $D = 0$, we have $b_1 = b_2 = \alpha_1 = \cdots = \alpha_{k-1} = 0$, and consequently the vector field X is triangular with the change of coordinates $(u,v,w) = (ax+cz, x, y)$. When $D \neq 0$, we have $P(x,y,z) = b_1(y - \frac{A}{cD}(ax+cz))^k, R(x,y,z) = -\frac{a}{c}P(x,y,z)$, and consequently the vector field X is triangular with the change of coordinates $(u,v,w) = (y - \frac{A}{cD}(ax+cz), x, z)$. The proof is now complete. □

Lemma 1.6.3. *For $\lambda < 0$, consider the polynomial vector field*

$$X(x,y,z) = \lambda\,(x,y,z)\;+\;(P,Q,0)(x,y,z) + (0, Ax^m, S(x,y)) \in \mathcal{N}(\lambda,3)$$

where $P(x,y,z), Q(x,y,z)$ are homogeneous polynomials of degree k, and $S(x,y)$ is a homogeneous polynomial of degree m, with $1 \leq k < m$. Then X is linearly triangularizable.

Proof. The condition $DX(p) - \lambda I$ nilpotent at every point $p \in \mathbb{R}^3$ implies that

$$P_x + Q_y \equiv P_x Q_y - P_y Q_x \equiv 0$$

and that

$$mAx^{m-1}P_y + Q_z S_y + P_z S_x \equiv (P_y Q_z - P_z Q_y)S_x + (P_z Q_x - P_x Q_z)S_y \equiv AP_z S_y \equiv 0\,.$$

We suppose $P \neq 0$ and $Q \neq 0$. The preceding conditions imply that

$$\begin{aligned} P(x,y,z) &= -b(ax+by)(\alpha_1 z^{k-1} + \alpha_2 z^{k-2}(ax+by) + \\ &\quad \cdots + \alpha_k (ax+by)^{k-1}) + c_1 z^k\,, \\ Q(x,y,z) &= a(ax+by)(\alpha_1 z^{k-1} + \alpha_2 z^{k-2}(ax+by) + \\ &\quad \cdots + \alpha_k (ax+by)^{k-1}) + c_2 z^k\,, \end{aligned}$$

$$\begin{aligned} &(ax+by)[-mb^2Ax^{m-1}(2\alpha_2 z^{k-2} + \cdots + k\alpha_k(ax+by)^{k-2}) + \\ &(aS_y - bS_x)((k-1)\alpha_1 z^{k-2} + (k-2)\alpha_2 z^{k-3}(ax+by) + \\ &\cdots + \alpha_{k-1}(ax+by)^{k-2})] + [-mb^2A\alpha_1 x^{m-1} + k(c_1 S_x + c_2 S_y)]z^{k-1} = 0\,, \end{aligned}$$

$$(ac_1 + bc_2)\,[\alpha_1 z^{k-1} + \alpha_2 z^{k-2}(ax+by) + \cdots + \alpha_k(ax+by)^{k-1}](aS_y - bS_x) = 0\,,$$

and

$$\begin{aligned} &A\,[-b(ax+by)((k-1)\alpha_1 z^{k-2} + (k-2)\alpha_2 z^{k-3}(ax+by) + \\ &\cdots + \alpha_{k-1}(ax+by)^{k-2}) + kc_1 z^{k-1}]\,S_y \;=\; 0\,. \end{aligned}$$

If $a = b = 0$, or $(a,b) \neq (0,0)$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$, then $c_1 c_2 \neq 0$, $S(x,y) = B\,(c_2 x - c_1 y)^m$, and $AB = 0$. When $B = 0$ the vector field X is triangular. When $B \neq 0$, we have $A = 0$ and consequently, performing the linear change of coordinates $(u,v,w) = (c_2x - c_1y, z, x)$, the vector field X is triangular in the new coordinates.

Suppose $(a,b) \neq (0,0)$ and $(\alpha_1, \alpha_2, \cdots, \alpha_k) \neq (0,0,\cdots,0)$. When $A = 0$ and $ac_1+bc_2 = 0$, we have that either $S(x,y) = B(ax+by)^m$, or $S(x,y) \neq B(ax+by)^m$ and $\alpha_1 = \cdots = \alpha_{k-1} = c_1 = c_2 = 0$. In the first case (resp. second case), if $a \neq 0$, then the vector field X becomes triangular with the change of coordinates $(u,v,w) = (ax+by, z, y)$ (resp. $(u,v,w) = (ax+by, y, z)$); if $b \neq 0$, then the vector field X becomes triangular with the change of coordinates $(u,v,w) = (ax+by, z, x)$ (resp. $(u,v,w) = (ax+by, x, z)$). When $A = 0$ and $ac_1 + bc_2 \neq 0$, we have $S(x,y) = 0$, and the result follows from Remark 1.6.1.

If $(a,b) \neq (0,0)$, $(\alpha_1, \alpha_2, \cdots, \alpha_k) \neq (0,0,\cdots,0)$, and $A \neq 0$, then $S_y = 0$. Therefore, $S(x,y) = Bx^n$, $bB(ac_1+bc_2) = 0$, and $b(2b\alpha_2A+(k-1)\alpha_1B) = b(3b\alpha_3A+(k-2)\alpha_2B) = \cdots = b(kb\alpha_kA+B\alpha_{k-1}) = -b^2\alpha_1A+kBc_1 = 0$. If $b = 0$, then $B = 0$, and consequently the vector field X is triangular. If $b \neq 0$, then $B \neq 0$ and $ac_1 + bc_2 = 0$, with $a \neq 0$. Moreover,

$$(P,Q)(x,y,z) = \frac{b\alpha_1}{2AB}\,[B(ax+by) - Abz]^k(b,-a)\,,$$

consequently the vector field X becomes triangular after the change of coordinates $(u,v,w) = (B(ax+by) - Abz, x, y)$. The proof is now complete. □

We conclude this section with our main result.

Theorem 1.6.3. *For $\lambda < 0$, consider a polynomial vector field $X = \lambda I + H_k + H_n \in \mathcal{N}(\lambda, 3)$, with H_k and H_m homogeneous polynomials of degree k and m, respectively, and with $1 \leq k < m$. Then X is linearly triangularizable, and therefore the origin is a global attractor of X.*

Proof. After a linear change of coordinates, we may suppose

$$H_m(x,y,z) \;=\; (0, A\,x^m, S(x,y)) \quad \text{and} \quad H_k = (P,Q,R)$$

such that $\alpha\,P + \beta\,Q + \gamma\,R \equiv 0$, for some $(\alpha,\beta,\gamma) \in \mathbb{R}^3 - \{(0,0,0)\}$.

When $\gamma \neq 0$, after the change of coordinates $(u, v, w) = (x, y, \alpha x + \beta y + \gamma z)$, the vector field X takes the form

$$\begin{aligned} Y(u, v, w) &= \lambda(u, v, w) \ + \ (P, Q, 0)(u, v, \frac{1}{\gamma}(w - \alpha u - \beta v)) \ + \\ & \quad (0, Au^m, \beta A u^m + \gamma S(u, v))\,. \end{aligned}$$

The theorem now follows from Lemma 1.6.3. Now when $\gamma = 0$ and $\beta \neq 0$, after the change of coordinates $(u, v, w) = (x, \alpha x + \beta y, z)$, the vector field X takes the form

$$Y(u, v, w) = \lambda(u, v, w) \ + \ (P, 0, R)(u, \frac{1}{\beta}(v - \alpha u)) \ + \ (0, \beta A u^m, \gamma S(u, \frac{1}{\beta}(v - \alpha u)))\,,$$

thus the theorem follows from Lemma 1.6.2. Finally, if $\gamma = \beta = 0$, then $P \equiv 0$, and the result follows from Lemma 1.6.1. $\square$

Chapter 2

Couples of Transversal Nets with Singularities

2.1 A Class of Quartic Differential Forms

Let M be a connected, oriented two–manifold of class C^∞. We let $\mathcal{Q}(M)$ denote the set consisting of all smooth quartic differential forms defined on M which have the following property. At each point p in M, there exist a local chart $(u,v) : U \subset M \longrightarrow \mathbb{R}^2$ and smooth maps $E, F, G : (u,v)(U) \to \mathbb{R}$, with $EG - F^2$ positive everywhere, such that if

$$(u,v)^*(\omega) \;=\; a_4 dv^4 \;+\; 4a_3 dv^3 du \;+\; 6a_2 dv^2 du^2 \;+\; 4a_1 dv du^3 \;+\; a_0 du^4 \tag{2.1.1}$$

then

$$G\,(a_0, a_1, a_2) \;-\; 2F\,(a_1, a_2, a_3) \;+\; E\,(a_2, a_3, a_4) \equiv 0\,. \tag{2.1.2}$$

A remarkable example of a quartic in $\mathcal{Q}(\mathbb{R}^2)$ is

$$\omega \;=\; 4a\,(du^2 - dv^2)dudv \;+\; b\,(du^4 - 6du^2dv^2 + dv^4) \tag{2.1.3}$$

where $E = G = 1$ and $F = 0$. We will show that, locally, any quartic in $\mathcal{Q}(M)$ may be written in the form (2.1.3) in an appropriate coordinate chart, which we will call **main chart**. (See Definition 2.2.1 and Proposition 2.2.3).

For any quartic ω in $\mathcal{Q}(M)$, we have that either $\omega(p) \equiv 0$ (in which case p is called a **singular point** of ω) or $\omega(p)^{-1}(0)$ is the union of four distinct lines $L_1(\omega)(p)$, $L_2(\omega)(p)$, $L_3(\omega)(p)$ and $L_4(\omega)(p)$ of the tangent space T_pM (in which case p is called a **regular point** of ω). In general, these line fields do not define foliations over the set of regular points of ω. Nevertheless, they can be grouped in pairs, say $\mathcal{N}_1(\omega) = \{L_1(\omega), L_2(\omega)\}$ and $\mathcal{N}_2(\omega) = \{L_3(\omega), L_4(\omega)\}$, so that each $\mathcal{N}_i(\omega)$, with $i = 1, 2$, defines a net.

Locally, each quartic $\omega \in \mathcal{Q}(M)$ is the product of two positive quadratic forms. In fact, if (2.1.1) is the local expression of ω in a chart (u, v), then

$$a_4 \cdot (u,v)^*(\omega) \;=\; \omega^+ \cdot \omega^-$$

with

$$\omega^{\pm} \;=\; a_4\, dv^2 \;+\; 2\left(a_3 \pm \sqrt{a_3^2 - a_2 a_4}\right) du\, dv \;+\; (a_2 \pm R)\, du^2$$

and

$$R(p) \;=\; \lim_{q \to p} \frac{a_3 a_2 - a_1 a_4}{\sqrt{a_3^2 - a_2 a_4}}(q) \;=\; \frac{2F(p)}{E(p)} \sqrt{\left(\frac{G}{E}\, a_0^2 - \frac{2F}{E}\, a_0 a_1 + a_1^2\right)(p)}\,.$$

The quadratic forms ω^- and ω^+ belong to a special class called *positive*. Recall that a quadratic form $\tau = a(u,v)dv^2 + 2b(u,v)dudv + c(u,v)du^2$ is positive if, at each point p, either $(b^2 - ac)(p) > 0$ or $(a,b,c)(p) = (0,0,0)$. In the former case, the point p is a regular point of τ, and in the latter case, p is a singular point of τ. Thus there is a triple $C(\tau) = \{f_1(\tau),\ f_2(\tau),\ \mathrm{Sing}(\tau)\}$ associated with τ which is called the *configuration* of τ, where $\mathrm{Sing}(\tau)$ is the set consisting of all singular points of τ, and where $f_1(\tau)$ and $f_2(\tau)$ are the transversal one–dimensional foliations defined over the set of regular points of τ which are, respectively, tangent to the vector fields $X_1(\tau) = (a, -b - \sqrt{b^2 - ac})$ and $X_2(\tau) = (a, -b + \sqrt{b^2 - ac})$. (See for example [C2–10], [C2–11].) Observe that the positive quadratic forms ω^- and ω^+ are not differentiable at the singular points of the quartic. Further generically, they have a semi–curve of singular points that are not singular points of the quartic. The net $\mathcal{N}_1(\omega)$ (resp. $\mathcal{N}_2(\omega)$) corresponds to the configuration of ω^- (resp. ω^+). To describe this correspondence, we consider a main chart (u, v) where ω takes the form

$$(u,v)^*(\omega) \ = \ 4a\,(du^2 - dv^2)dudv \ + \ b\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

We obtain

$$b \cdot (u,v)^*(\omega) \ = \ \omega^+ \cdot \omega^-$$

with

$$\omega^\pm \ = \ b\,(dv^2 - du^2) \ + \ 2\,(-a \pm \sqrt{a^2+b^2})\,du\,dv\,.$$

The set $\mathrm{Sing}(\omega^-)$ (resp. $\mathrm{Sing}(\omega^+)$) is the set consisting of all points where b vanishes and a is non–positive (resp. non–negative). We obtain the net $\mathcal{N}_1(\omega)$ from the configuration $C(\omega^-)$ as follows. Let $p \in \mathrm{Sing}(\omega^-) - \mathrm{Sing}(\omega)$, and let V be a small neighborhood of p. We let $V^+ = V \cap b^{-1}(]0,\infty[)$ and $V^- = V \cap b^{-1}(]-\infty,0[)$. For $i = 1,2$, we denote the leaf of $f_i(\omega^-)/V^+$ (resp. of $f_i(\omega^-)/V^-$) which converges to p by γ_i^+ (resp. γ_i^-). We have that γ_i^+ (resp. γ_i^-) converges to p with slope $(-1)^i$ (resp. $-(-1)^i$). Further, if α_1 and α_2 are the leaves of $\mathcal{N}_1(\omega)/V$ that contain p, then $\alpha_1 = \gamma_1^+ \cup \{p\} \cup \gamma_2^-$ and $\alpha_2 = \gamma_2^+ \cup \{p\} \cup \gamma_1^-$.

The type of the quartics under study is related to the principal curvature lines of surfaces immersed in $\mathbb{R}^4$. In fact, the principal directions at a point p are obtained by solving an equation $\omega(p) = 0$, with $\omega \in \mathcal{Q}(\mathbb{R}^2)$. (See [C2–8], [C2–6], [C2–7], [C2–3], [C2–9], [C2–19], [C2–20].) Observe that the converse is locally true in the analytic case. More precisely, given an $\omega \in \mathcal{Q}(U)$, with real analytic coefficients $a_0, \cdots, a_4$ defined in a neighborhood $U \subset \mathbb{R}^2$ of a point p, there exists an immersion $f : V \to \mathbb{R}^4$ where $V \subset U$ is some small open neighborhood of p such that the differential equation of the lines of curvature of f is given by $\omega = 0$. (See [C2–6, Theorem 2.1] or [C2–3, Proposition 2.4].) Thus all of the local results of [C2–6], [C2–7], [C2–3], [C2–9] hold for the nets associated to an $\omega \in \mathcal{Q}(M)$.

2.2 Preliminaries

In that follws, f_x will denote the partial derivative of a map f with respect to a variable x.

Proposition 2.2.1. *Let* $\omega \in \mathcal{Q}(M)$ *have only isolated singular points. Then, for any local chart* $(u,v) : U \subset M \longrightarrow \mathbb{R}^2$, *there exist smooth maps* $E, F, G : (u,v)(U) \to \mathbb{R}$, *with* $EG - F^2$ *positive everywhere, such that if* $(u,v)^*(\omega) = a_4 dv^4 + 4a_3 dv^3 du + 6a_2 dv^2 du^2 + 4a_1 dv du^3 + a_0 du^4$, *then* $G\,(a_0, a_1, a_2) - 2F\,(a_1, a_2, a_3) + E\,(a_2, a_3, a_4) \equiv 0$.

Proof. For simplicity, a chart which satisfies the conditions of Proposition 2.2.1 will be called a $g-$**chart**. The proof is consequence of the following.

1) Let (u,v) be a $g-$chart. If ϕ is the change of coordinates

$$(u,v) \;=\; \phi^{-1}(x,y) \;=\; (f(x,y), g(x,y))$$

then (x,y) is a $g-$chart.

In fact, if

$$(x,y)^*(\omega) \;=\; b_4 dy^4 \;+\; 4b_3 dy^3 dx \;+\; 6b_2 dy^2 dx^2 \;+\; 4b_1 dy dx^3 \;+\; b_0 dx^4\,,$$

then

$$\begin{aligned} b_0 \;=\;& \big[f_x^4 - (6f_x^2 G g_x^2)/E - (8F f_x G g_x^3)/E^2 + \\ & (G((-4F^2)/E^3 + G/E^2)g_x^4)\big]\, a_0 \;+\; \big[f_x^3 g_x + (3F f_x^2 g_x^2)/E + \\ & f_x((4F^2)/E^2 - G/E)g_x^3 + (F((2F^2)/E^3 - G/E^2)g_x^4)\big]\, a_1 \end{aligned}$$

and

$$\begin{aligned} b_1 \;=\;& \big[4f_x^3 f_y - (12 f_x f_y G g_x^2)/E - (8F f_y G g_x^3)/E^2 - (12 f_x^2 G g_x g_y)/E \\ & -(24F f_x G g_x^2 g_y)/E^2 + (4G((-4F^2)/E^2 + G/E)g_x^3 g_y)/E\big]\, a_0 \;+ \\ & \big[3f_x^2 f_y g_x + (6F f_x f_y g_x^2)/E + f_y((4F^2)/E^2 - G/E)g_x^3 + f_x^3 g_y + \\ & (6F f_x^2 g_x g_y)/E + 3f_x((4F^2)/E^2 - G/E)g_x^2 g_y + \\ & (4F((2F^2)/E^2 - G/E)g_x^3 g_y)/E\big]\, a_1\,. \end{aligned}$$

Setting

$$\begin{aligned} \tilde{E} &= E(f_x)^2 + 2F f_x g_x + G(g_x)^2 \,, \\ \tilde{F} &= E f_x f_y + F(f_x g_y + f_y g_x) + G g_x g_y \,, \\ \tilde{G} &= E(f_y)^2 + 2F f_y g_y + G(g_y)^2 \end{aligned}$$

we obtain

$$\tilde{E}\tilde{G} - \tilde{F}^2 = (EG - F^2)(f_x g_y - f_y g_x)^2 \,.$$

Thus we again obtain the relationships

$$\begin{aligned} \tilde{G}b_0 - 2\tilde{F}b_1 + \tilde{E}b_2 &= 0\,, \\ \tilde{G}b_1 - 2\tilde{F}b_2 + \tilde{E}b_3 &= 0\,, \\ \tilde{G}b_2 - 2\tilde{F}b_3 + \tilde{E}b_4 &= 0\,. \end{aligned}$$

2) Suppose that associated to each $g-$chart $(u,v) : U \subset M \to \mathbb{R}^2$, with $(u,v)^*(\omega) = a_4 dv^4 + 4a_3 dv^3 du + 6a_2 dv^2 du^2 + 4a_1 dv du^3 + a_0 du^4$, there are two triples of smooth maps, E, F, G and $\tilde{E}, \tilde{F}, \tilde{G}$, so that in $(u,v)(U)$, we have $EG - F^2 > 0$, $\tilde{E}\tilde{G} - \tilde{F}^2 > 0$, $G(a_0,a_1,a_2) - 2F(a_1,a_2,a_3) + E(a_2,a_3,a_4) = 0$, and $\tilde{G}(a_0,a_1,a_2) - 2\tilde{F}(a_1,a_2,a_3) + \tilde{E}(a_2,a_3,a_4) = 0$. Then there exists a smooth map $\lambda : (u,v)(U) \to \mathbb{R}$ such that $(\tilde{E},\tilde{F},\tilde{G}) = \lambda(E,F,G)$.

For this, it suffices to show that, for every regular point p of ω, we have $\frac{G}{E}(p) = \frac{\tilde{G}}{\tilde{E}}(p)$ and $\frac{F}{E}(p) = \frac{\tilde{F}}{\tilde{E}}(p)\cdot$

First, we assume that there exists a regular point p so that $\frac{G}{E}(p) - \frac{\tilde{G}}{\tilde{E}}(p) \neq 0$. Then

$$(a_0,a_1,a_2)(p) = H(p)\,(a_1,a_2,a_3)(p)$$

with $H = 2\,\frac{\frac{F}{E} - \frac{\tilde{F}}{\tilde{E}}}{\frac{G}{E} - \frac{\tilde{G}}{\tilde{E}}}\cdot$ This implies that a_0,a_1,a_2,a_3 are non–vanishing at p, and

$$a_1 = H\,a_2 \quad \text{and} \quad a_0 = H\,a_1 = H^2\,a_2\,.$$

Therefore,

$$\begin{aligned} a_2 &= -\frac{G}{E}\, a_0 \;+\; \frac{2F}{E}\, a_1 \\ &= -\frac{G}{E}\, H^2 a_2 \;+\; H\,\frac{2F}{E}\, a_2 \end{aligned}$$

implies that $x = H(p)$ is a real solution of the equation

$$\frac{G(p)}{E(p)}\, x^2 \;-\; \frac{2F(p)}{E(p)}\, x \;+\; 1 \;=\; 0\,,$$

which is impossible because $E(p)\,G(p) \;-\; F(p)^2 > 0\,.$

Similarly, if there exists a regular point p so that $\frac{F}{E}(p) \;-\; \frac{\tilde{F}}{\tilde{E}}(p) \neq 0\,$, then

$$(a_1, a_2, a_3)(p) \;=\; \tilde{H}(p)\,(a_0, a_1, a_2)(p)$$

with $\tilde{H} \;=\; \frac{1}{2}\,\frac{\frac{G}{E} - \frac{\tilde{G}}{\tilde{E}}}{\frac{F}{E} - \frac{\tilde{F}}{\tilde{E}}}\,$, and $x = \tilde{H}(p)$ is a real solution of the equation

$$\frac{E(p)}{G(p)}\, x^2 \;-\; \frac{2F(p)}{G(p)}\, x \;+\; 1 \;=\; 0\,,$$

which is impossible.

$\square$

We next show that the coefficients $a_0, ..., a_4$ of the local expression of $\omega \in \mathcal{Q}(M)$ satisfy relationships similar to those of the case of principal curvature lines of surfaces immersed in $\mathbb{R}^4$. (See [C2–9, Lemma 2.1].)

Proposition 2.2.2. *Let* $\omega \;=\; a_4 dv^4 \;+\; 4a_3 dv^3 du \;+\; 6a_2 dv^2 du^2 \;+\; 4a_1 dv du^3 \;+\; a_0 du^4$ *be a quartic in* $\mathcal{Q}(\mathbb{R}^2)$ *. Let* $E, F, G : \mathbb{R}^2 \to \mathbb{R}$ *be smooth maps such that, for all* $p \in \mathbb{R}^2$*, we have* $(EG - F^2)(p) > 0$ *and*

$$G(p)(a_0, a_1, a_2)(p) \;-\; 2F(p)(a_1, a_2, a_3)(p) \;+\; E(p)(a_2, a_3, a_4)(p) \;=\; 0\,.$$

Then

$$\begin{aligned} Ea_2 &= -Ga_0 + 2Fa_1 \,, \\ E^2a_3 &= -2FGa_0 + (4F^2 - EG)a_1 \,, \\ E^3a_4 &= G(EG - 4F^2)a_0 + 4F(2F^2 - EG)a_1 \,. \end{aligned} \tag{2.2.4}$$

Moreover, if p is a regular point of ω, we have

$$H(p) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} (p) < 0 \;\; \textit{and} \;\; I(p) = (a_4\, a_0 - 4\, a_1\, a_3 + 3\, a_2^2)(p) > 0\,. \tag{2.2.5}$$

Proof. For all $p \in \mathbb{R}^2$, we have

$$\begin{aligned} Ga_0 - 2Fa_1 + Ea_2 &= 0\,, \\ Ga_1 - 2Fa_2 + Ea_3 &= 0\,, \\ Ga_2 - 2Fa_3 + Ea_4 &= 0\,. \end{aligned} \tag{2.2.6}$$

Observe that the first relationship of (2.2.6) corresponds to the first relationship of (2.2.4), the second relationship of (2.2.6) multiplied by E corresponds to the second relationship of (2.2.4), and the last relationship of (2.2.6) multiplied by E^2 corresponds to the last relationship of (2.2.4).

We now work on the set of the regular points of the quartic. Using relationships (2.2.4), we find

$$E^3\,(H, I) = (Ea_1^2 - 2Fa_0a_1 + Ga_0^2)\,(-G^2, 4(EG - F^2))\,. \tag{2.2.7}$$

Therefore $EG - F^2 > 0$ imply $H < 0$ and $I > 0$. □

We now show the existence of such main charts. In the case of curvature lines they correspond to the isothermal coordinates.

Definition 2.2.1. *Let $\omega \in \mathcal{Q}(M)$ and $p \in M$. A local chart (u, v) at p will be called a* **main chart** *of ω at p if*

$$(u, v)^*(\omega) = 4a\,(du^2 - dv^2)dudv + b\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Proposition 2.2.3. *Given $\omega \in \mathcal{Q}(M)$ and $p \in M$, there exists a main chart of ω at p.*

Proof. Observe that taking $F = 0$ and $E = G$ in (2.2.6), the local expression (2.1.1) already has the desired expression. Hence, given a local chart (u, v) at p and associated maps E, F, G to the quartic differential form ω, it suffices to find a coordinate change

$$u = f(x, y)\,,\ v = g(x, y)$$

so that, in a neighborhood of the origin, we have

$$\begin{aligned} Ef_xf_y + F(f_xg_y + f_yg_x) + Gg_xg_y &= 0 \quad \text{and} \\ E(f_x)^2 + 2Ff_xg_x + G(g_x)^2 &= E(f_y)^2 + 2Ff_yg_y + G(g_y)^2\,. \end{aligned}$$

Therefore, the problem is equivalent to finding isothermal coordinates in a neighborhood of a point of a surface. (See [Spi, Vol. IV, Addendum 1 of Chapter 9].) The conclusion follows. □

Remark 2.2.1. *Main charts are preserved by changes of coordinates of the form $(u, v) = (f(x, y), g(x, y))$ which verify*

$$(f_x, f_y) = (g_y, -g_x) \quad or \quad (f_x, f_y) = (-g_y, g_x)\,.$$

2.3 Simple Singular Points

Let ω be a quartic differential form in $\mathcal{Q}(M)$, and let p be a singular point of ω. Assume that (2.1.1) is the local expression of ω in a chart $(u, v) : (M, p) \to (\mathbb{R}^2, 0)$ with coefficients a_0, a_1, a_2, a_3, a_4 satisfying relationships (2.2.6). The point p will be called a **simple singular point** of ω if the Jacobian matrix $D(a_0, a_1)(0, 0)$ is non–singular.

Proposition 2.3.1. *Let $\omega \in \mathcal{Q}(M)$ and let p be a singular point of ω. Then the following properties are equivalent:*

a) The point p is a simple singular point of ω.

b) Let $(u,v):(M,p)\to(\mathbb{R}^2,0)$ *be a local chart. If*

$$(u,v)^*(\omega) \;=\; b_4dv^4 \;+\; 4b_3dv^3du \;+\; 6b_2dv^2du^2 \;+\; 4b_1dvdu^3 \;+\; b_0du^4\,,$$

then the Jacobian matrix $D(b_0,b_1)(0,0)$ *is non–singular.*

c) Let $(u,v):(M,p)\to(\mathbb{R}^2,0)$ *be a local chart. If*

$$(u,v)^*(\omega) \;=\; b_4dv^4 \;+\; 4b_3dv^3du \;+\; 6b_2dv^2du^2 \;+\; 4b_1dvdu^3 \;+\; b_0du^4\,,$$

then the Jacobian matrix $D(b_i,b_{i+1})(0,0)$ *is non–singular, for* $i=0,1,2,3$.

Proof. Let $(x,y)^*(\omega) \;=\; a_4\,dy^4 \;+\; 4a_3\,dy^3dx \;+\; 6a_2\,dy^2dx^2 \;+\; a_1\,dydx^3 \;+\; a_0\,dx^4$ be the local expression of ω in a chart $(x,y):(U,p)\to(\mathbb{R}^2,(0,0))$. Let $i=0,1,2,3$. Using relationships (2.2.4), we see

$$\begin{pmatrix} a_i \\ a_{i+1}\end{pmatrix} = M_i\begin{pmatrix} a_0 \\ a_1\end{pmatrix}$$

where M_i is a square matrix with $\det M_i = \left(\frac{G}{E}\right)^i \neq 0$. Therefore, the curves $\{a_0=0\}$ and $\{a_1=0\}$ are regular, meeting each other transversally at the origin if and only if the curves $\{a_i=0\}$ and $\{a_{i+1}=0\}$ are regular meeting each other transversally at the origin.

Consider the change of coordinates $(x,y)=(f(u,v),g(u,v))$, with $f(0,0) \;=\; g(0,0) \;=\; 0$, and let

$$(u,v)^*(\omega) \;=\; b_4dv^4 \;+\; 4b_3dv^3du \;+\; 6b_2dv^2du^2 \;+\; 4b_1dvdu^3 \;+\; b_0du^4\,.$$

Again, using relationships (2.2.4), we obtain

$$\begin{pmatrix} b_0 \\ b_1\end{pmatrix} = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11}\end{pmatrix}\begin{pmatrix} a_0 \\ a_1\end{pmatrix}$$

with

$$\det\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11}\end{pmatrix} \;=\; (E\,f_u^2 \;+\; 2\,F\,f_u\,g_u \;+\; G\,g_u^2)\;\det D(f,g)\,.$$

The proof now follows. $\square$

The next result is [C2–6, Lemma 3.2], so we give the lemma without proof.

Lemma 2.3.1. *Let $\omega \in \mathcal{Q}(M)$, and let $p \in M$ be a simple singular point of ω. There is a main chart $(u,v) : (M,p) \to (\mathbb{R}^2, 0)$ such that the local expression of ω is*

$$\begin{aligned}(u,v)^*(\omega) = \quad & 4(Au + Bv + S(u,v))(du^2 - dv^2)dudv + \\ & (v + R(u,v))(du^4 - 6du^2dv^2 + dv^4)\end{aligned} \tag{2.3.8}$$

where $A \neq 0$ and B are real numbers, and S and R are real–valued functions which satisfy

$$S(0,0) = R(0,0) = \frac{\partial S}{\partial u}(0,0) = \frac{\partial S}{\partial v}(0,0) = \frac{\partial R}{\partial u}(0,0) = \frac{\partial R}{\partial v}(0,0) = 0\,.$$

For the rest of this article, we endow the set $\mathcal{Q}(M)$ with the smooth Whitney topology.

Simple singular points are persistent under perturbations of the quartic differential form in $\mathcal{Q}(M)$ because they are defined by transversal conditions. We explain this fact in the next Proposition.

Proposition 2.3.2. *Let p_0 be a simple singular point of a quartic differential form $\omega_0 \in \mathcal{Q}(M)$. Then there exist a neighborhood U of p in M, a neighborhood $\mathcal{V}$ of ω_0 in $\mathcal{Q}(M)$, and a smooth map $p : \mathcal{V} \to U$ which associates each $\omega \in \mathcal{V}$ with the unique singular point of ω in U. Moreover, the singular point $p(\omega)$ is simple.*

Proof. The local expression of a quartic differential form ω in $\mathcal{Q}(M)$ associated to an arbitrary chart (u,v) is given by

$$\begin{aligned}(u,v)*(\omega) = \quad & (A_{40}a_0 + A_{41}a_1)dv^4 + (A_{30}a_0 + A_{31}a_1)dv^3du \\ & +(A_{20}a_0 + A_{21}a_1)dv^2du^2 + \; a_1dvdu^3 + a_0du^4\end{aligned} \tag{2.3.9}$$

where $a_0 = a_0(u,v)$, $a_1 = a_1(u,v)$, and $A_{ij} = A_{ij}(u,v)$, for $i = 2,3,4$ and $j = 0,1$, are smooth functions. Moreover, the singular points of ω are given by the equations $a_0 = a_1 = 0$. Consider a local chart $(u,v) : (M,p) \longrightarrow (\mathbb{R}^2, 0)$ such that the local expression of ω_0 has form (2.3.8). Therefore, $A \neq 0$. For ω in a neighborhood $\tilde{\mathcal{V}}$ of

ω_0 in $\mathcal{Q}(M)$, the local expression in the same coordinates is of the form (2.3.9), where $a_0(\omega_0)(u,v) = v + R(u,v)$ and $a_1(\omega_0)(u,v) = 4(Au + Bv + S(u,v))$. Next consider the smooth map $F : \tilde{\mathcal{V}} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$F(\omega, (u,v)) = (a_0(\omega)(u,v),\, a_1(\omega)(u,v))\,.$$

Since $F(\omega_0,\, (0,0)) = (0,0)$, and since the matrix

$$D_2\, F(\omega_0, (0,0)) = \begin{pmatrix} 4A & 4B \\ 0 & 1 \end{pmatrix}$$

is non–singular, there exist a neighborhood $\tilde{U}$ of $(0,0)$ in $\mathbb{R}^2$, a neighborhood $\mathcal{V} \subset \tilde{\mathcal{V}}$ of ω_0 in $\mathcal{Q}(M)$, and a smooth map $q : \mathcal{V} \to \tilde{U}$ such that $q(\omega_0) = (0,0)$ and $F(\omega, q(\omega)) = (0,0)$, for all $\omega \in \mathcal{V}$. The proof now follows. $\square$

The next two results are contained in [C2–3, Theorem 1.1]. We do not give their proofs.

Theorem 2.3.1. *Let $\omega \in \mathcal{Q}(M)$, and let $p \in M$ be a simple singular point of ω. Let $(u,v) : (M,p) \longrightarrow (\mathbb{R}^2, 0)$ be a local chart such that*

$$(u,v) * (\omega) = 4(Au + Bv + S)(du^2 - dv^2)dudv + (v + R))(du^4 - 6du^2dv^2 + dv^4)$$

where $A \neq 0$ and B are real numbers, and $S = S(u,v)$ and $R = R(u,v)$ are real–valued functions which satisfy

$$S(0,0) = R(0,0) = \frac{\partial S}{\partial u}(0,0) = \frac{\partial S}{\partial v}(0,0) = \frac{\partial R}{\partial u}(0,0) = \frac{\partial R}{\partial v}(0,0) = 0.$$

Then, under each of the conditions (a) through (e), the corresponding phase portrait is obtained by making into one, through a rigid translation, the pair of pictures (that is, nets) of the indicated figure.

(a) Condition H_3 : $\Delta < 0$. *(Figure 1)*

(b) Condition H_4 : $\Delta > 0$, $A < 0$ *and* $A \neq -1/4$. *(Figure 2)*

(c) Condition H_5 *:* $\Delta > 0,\ A > 0$. *(Figure 3)*

(d) Condition H_{34}*:* $\Delta > 0$ *and* $A = -1/4$ *and* $B \neq 0$. *(Figure 4)*

(e) Condition $\tilde{H}_3$*:* $A = -1/4$ *and* $B = 0$. *(Figure 5)*

Here

$$\begin{aligned} \Delta &= 4(1+B^2)^3 + 24(1+B^2)^2 A + 8(5-B^2)(1+B^2)A^2 + \\ &\quad 4(9+B^2)A^3 + (17+4B^2)A^4 + 4A^5\,. \end{aligned} \tag{2.3.10}$$

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Definition 2.3.1. *Let p be a singular point of a quartic differential form $\omega \in \mathcal{Q}(M)$. We will say that ω is* **locally topologically stable** *at p if both nets, $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$, are locally topologically stable at p.*

Theorem 2.3.2. *Let $\omega \in \mathcal{Q}(M)$, and let $p \in M$ be a simple singular point of ω. Consider a local chart $(u,v) : (M,p) \longrightarrow (\mathbb{R}^2, 0)$ as in Theorem 2.3.1. Then ω is locally topologically stable at p if and only if either condition H_3, or condition H_4, or condition H_5 holds.*

The next two results will be used in Subsection 2.5.1 to obtain versal unfoldings of the singular points H_{34} and $\tilde{H}_3$, thus showing that the former is of codimension one, and the latter is of codimension two.

Proposition 2.3.3. *Let p be a simple singular point of $\omega \in \mathcal{Q}(M)$. Consider a main local chart $(u,v) : (M,p) \to (\mathbb{R}^2, 0)$ such that the local expression of ω at p is of the form*

$$\begin{aligned}(u,v)*(\omega) &= 4(Au + Bv + S(u,v))(du^2 - dv^2)dudv + \\ &\quad (v + R(u,v))(du^4 - 6du^2dv^2 + dv^4)\,. \end{aligned} \tag{2.3.11}$$

Consider also the separatrix polynomial

$$g(s) = -s\,Q(s) \tag{2.3.12}$$

where

$$Q(s) \;=\; s^4 - 4Bs^3 - 2(3+2A)s^2 + 4Bs + 1 + 4A\,.$$

Then the point p is:

a) a locally stable singular point if the separatrix polynomial (2.3.12) only has simple roots;

b) an $H_{34}-$singular point if the separatrix polynomial (2.3.12) has a root of multiplicity two;

c) an $\tilde{H}_3-$singular point if the separatrix polynomial (2.3.12) has a root of multiplicity three.

Proof. This is a direct consequence of Theorems 2.3.1 and 2.3.2 , and the result [C2–3, Theorem 5.3]. □

Our next result gives a characterization of the H_3, H_4 and H_5 singularities better suited for our needs.

Let $\omega \in \mathcal{Q}(M)$. Given a main chart $(x,y) : U \to \mathbb{R}^2$, if $(x,y)^*(\omega) = 4a(x,y)(dx^2 - dy^2)dxdy + b(x,y)(dx^4 - 6dx^2dy^2 + dy^4)$, consider the maps $g : (x,y)(U) \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $\Delta, H : (x,y)(U) \subset \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

1) $g = (4a, b)$

2) $\Delta(x,y)$ is the discriminant of the homogeneous degree five polynomial $4Da_{(x,y)}(u,v)(u^2 - v^2)dudv + Db_{(x,y)}(u,v)(u^4 - 6u^2v^2 + v^4)$.

3) $H(x,y)$ is the determinant of the Jacobian matrix of the map $g = (4a, b)$.

Proposition 2.3.4. *Let p be a simple singular point of $\omega \in \mathcal{Q}(M)$, and let $(x,y) : (U,p) \to (\mathbb{R}^2, (0,0))$ be a main chart. If $\Delta(0,0) \neq 0$, then:*

1) p is of type H_3 if and only if $\Delta(0,0) < 0$.

2) p is of type H_4 if and only if $\Delta(0,0) > 0$ and $H(0,0) < 0$.

3) p is of type H_5 if and only if $\Delta(0,0) > 0$ and $H(0,0) > 0$.

Proof. In the case a main chart $(u,v) : (U,p) \to (\mathbb{R}^2,(0,0))$ is such that

$$(u,v) * (\omega) \;=\; 4(Au + Bv + S(u,v))(du^2 - dv^2)dudv + \\ (v + R(u,v))(du^4 - 6du^2dv^2 + dv^4)$$

we have

$$\Delta(0,0) \;=\; (1+4A)^2\,[4(1+B^2)^3 + 24(1+B^2)^2A + 8(5-B^2)(1+B^2)A^2 \\ +4(9+B^2)A^3 + (17+4B^2)A^4 + 4A^5]$$

and

$$H(0,0) \;=\; 4\,A\,.$$

The proof now follows from Theorem 2.3.1.

Consider now an arbitrary main chart (u,v) with

$$(u,v)^*(\omega) \;=\; 4a(u,v)(du^2 - dv^2)dudv \;+\; b(u,v)(du^4 - 6du^2dv^2 + dv^4)\,.$$

Since the roots of the equation

$$4Da_{(0,0)}(u,v)(u^2 - v^2)uv + Db_{(0,0)}(u,v)(u^4 - 6u^2v^2 + v^4) \;=\; 0$$

correspond to the possible directions of asymptotic convergence to the singular point for the leaves of the nets, the sign of $\Delta(0,0)$ is invariant by coordinate changes. On the other hand, $H(0,0)$ is negative (resp. positive) if and only if the Poincaré index of the singular point is $\frac{1}{4}$ (resp. $-\frac{1}{4}$). Therefore, the sign of $H(0,0)$ is also invariant by coordinate changes. From these considerations the proof follows. $\square$

2.4 A Non–simple Case

Definition 2.4.1. *Let $\omega \in \mathcal{Q}(M)$ and let p be a singular point of ω. We will say p is a* **rank–k singular point** *of ω, with $k = 0, 1, 2$, if there exists a main chart $(x, y) : (U, p) \to (\mathbb{R}^2, (0, 0)$ such that*

$$(x, y) * (\omega) \;=\; 4a(dx^2 - dy^2)dxdy \;+\; b(dx^4 - 6dx^2dy^2 + dy^4)$$

and the Jacobian matrix $D(a, b)(0, 0)$ has rank–k.

Proposition 2.4.1. *Let p be a rank–1 singular point of $\omega \in \mathcal{Q}(M)$. Then there exists a main chart (x, y) such that*

$$(x, y) * (\omega) \;=\; 4a(dx^2 - dy^2)dxdy \;+\; b(dx^4 - 6dx^2dy^2 + dy^4)$$

and $j_1(a, b)(0, 0) \;=\; (By, y)$, with $B > 0$.

Proof. Without loss of generality, we may suppose that there is a main chart (x, y) so that

$$(x, y) * (\omega) \;=\; 4a(dx^2 - dy^2)dxdy \;+\; b(dx^4 - 6dx^2dy^2 + dy^4)$$

with

$$j_1(a, b)(0, 0) \;=\; (Ax + By)(\lambda, 1)\,,$$

where $\lambda\,(A^2 + B^2) \neq 0$. For $\alpha, \beta \in \mathbb{R}$, with $\alpha^2 + \beta^2 \neq 0$, we consider $(x, y) = \phi(u, v) = (\alpha u - \beta v, \beta u + \alpha v)$. Then

$$\phi^*\omega = 4\tilde{a}(u, v)(du^2 - dv^2)dudv + \tilde{b}(u, v)(du^4 - 6du^2dv^2 + dv^4)$$

where

$$j_1(\tilde{a}, \tilde{b})(0, 0) \;=\; (A_{10}, B_{10})\,u \;+\; (A_{01}, B_{01})\,v$$

and

$$\begin{aligned}
A_{10} &= (A\alpha + B\beta)\,(-4\alpha^3\beta + 4\alpha\beta^3 + \alpha^4\lambda - 6\alpha^2\beta^2\lambda + \beta^4\lambda)\,,\\
A_{01} &= (\alpha B - A\beta)\,(-4\alpha^3\beta + 4\alpha\beta^3 + \alpha^4\lambda - 6\alpha^2\beta^2\lambda + \beta^4\lambda)\,,\\
B_{10} &= (A\alpha + B\beta)\,(\alpha^4 - 6\alpha^2\beta^2 + \beta^4 + 4\alpha^3\beta\lambda - 4\alpha\beta^3\lambda)\,,\\
B_{01} &= (\alpha B - A\beta)\,(\alpha^4 - 6\alpha^2\beta^2 + \beta^4 + 4\alpha^3\beta\lambda - 4\alpha\beta^3\lambda).
\end{aligned}$$

If $A = 0$ (resp. $B = 0$), then $B \neq 0$ (resp. $A \neq 0$). We set $\beta = 0$ and $\alpha = \frac{1}{\sqrt[5]{B}}$ (resp. $\alpha = 0$ and $\beta = -\frac{1}{\sqrt[5]{A}}$), and we obtain $A_{10} = B_{10} = 0$, $B_{01} = 1$ and $A_{01} \neq 0$. If $A \neq 0$ and $B \neq 0$, we set $\alpha = m\beta$ with $m = -\frac{A}{B}$ to obtain $B_{10} = 0$, and we are under the conditions of the preceding case. Finally, if the coefficient A_{01} obtained is negative, it suffices to consider the change of coordinates $(u, v) = (-s, t)$. □

Proposition 2.4.2. *Let p be an isolated rank–1 singular point of $\omega \in \mathcal{Q}(M)$. Let $(x, y) : (U, p) \to (\mathbb{R}^2, (0, 0)$ be a local chart. Assume that*

$$(x, y)^*(\omega) \;=\; a_4\, dy^4 \;+\; 4a_3\, dy^3 dx \;+\; 6a_2\, dy^2 dx^2 \;+\; 4a_1\, dy dx^3 \;+\; a_0\, dx^4\,.$$

Then there exist a pair $(\beta_1, \beta_2) \neq (0, 0)$ and real constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$, with $(\alpha_0, \alpha_1) \neq (0, 0)$, such that

$$j_1(a_i)(0, 0) \;=\; \alpha_i\,(\beta_1\, x + \beta_2\, y)\,, \quad \textit{for} \quad i = 0, 1, 2, 3, 4\,.$$

Proof. Let $(u, v) : (V, p) \to (\mathbb{R}^2, (0, 0))$ be a main chart such that $V \subset U$ and $(u, v)^*(\omega) = 4a(du^2 - dv^2)dudv + b(du^4 - 6du^2dv^2 + dv^4)$, with $j_1(a, b)(0, 0) = (Bv, v)$. Consider the changes of coordinates $(u, v) = (f(x, y), g(x, y))$. Then

$$\begin{aligned}
a_0(x, y) &= 4(f_x^3\, g_x - f_x\, g_x^3)\, a(u, v) \;+\; (f_x^4 - 6\, f_x^2\, g_x^2 + g_x^4)\, b(u, v)\,,\\
a_1(x, y) &= (3f_x^2 f_y g_x - f_y g_x^3 + f_x^3 g_y - 3f_x g_x^2 g_y) a(u, v) \;+\\
&\quad\; (f_x^3 f_y - 3f_x f_y g_x^2 - 3f_x^2 g_x g_y + g_x^3 g_y) b(u, v)\,,\\
a_2(x, y) &= 2(f_x f_y^2 g_x + f_x^2 f_y g_y - f_y g_x^2 g_y - f_x g_x g_y^2)\, a(u, v) +\\
&\quad\; (f_x^2 f_y^2 - f_y^2 g_x^2 - 4f_x f_y g_x g_y - f_x^2 g_y^2 + g_x^2 g_y^2)\, b(u, v)\,,\\
a_3(x, y) &= (f_y^3 g_x + 3f_x f_y^2 g_y - 3f_y g_x g_y^2 - f_x g_y^3)\, a(u, v) +\\
&\quad\; (f_x f_y^3 - 3f_y^2 g_x g_y - 3f_x f_y g_y^2 + g_x g_y^3)\, b(u, v)\,,\\
a_4(x, y) &= 4(f_y^3 g_y - f_y g_y^3)\, a(u, v) \;+\; (f_y^4 - 6f_y^2 g_y^2 + g_y^4)\, b(u, v)\,.
\end{aligned}$$

Therefore, $j_1(a_i)(0, 0) = \alpha_i\,(\beta_1\, x + \beta_2\, y)$ where $\beta_1\, x + \beta_2\, y = j_1(g)(0, 0)$. Setting $\alpha_i = B\,\sigma_i + \tau_i$, for $i = 0, 1$, we have

$$M \;=\; \begin{pmatrix} \sigma_0 & \sigma_1 \\ \tau_0 & \tau_1 \end{pmatrix} \;=\; \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} Q & P \\ P & -Q \end{pmatrix}$$

with $P = (f_x^2 - g_x^2)(0,0)$, $Q = 2(f_x g_x)(0,0)$, $R = (f_x f_y - g_x g_y)(0,0)$, and $S = (f_y g_x + f_x g_y)(0,0)$. Since

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = (f_x^2 + g_x^2)(0,0) \det D(f,g)(0,0) \neq 0$$

we have $\det M \neq 0$, and therefore $(\alpha_0, \alpha_1) \neq (0,0)$. □

Let p be a singular point of $\omega \in \mathcal{Q}(M)$. At p, consider a main chart (u,v) such that

$$(u,v)^*(\omega) = 4a(du^2 - dv^2)dudv + b(du^4 - 6du^2dv^2 + dv^4).$$

Observe that a simple singular point corresponds to the case where $\{a = 0\}$ and $\{b = 0\}$ are regular curves meeting each other transversally at the origin. In this section, we weaken this condition in the mildest way by considering the case where the curves $\{a = 0\}$ and $\{b = 0\}$ have quadratic contact at the origin. More precisely,

Definition 2.4.2. *A rank–1 singular point p of $\omega \in \mathcal{Q}(M)$ will be called an $H_{45}-$**singular point** if there exists a main chart $(x,y) : (U,p) \to (\mathbb{R}^2, (0,0)$ such that*

$$(x,y)^*(\omega) = 4a(dx^2 - dy^2)dxdy + b(dx^4 - 6dx^2dy^2 + dy^4)$$

and the curves $\{a = 0\}$ and $\{b = 0\}$ have quadratic contact at the origin.

Proposition 2.4.3. *Let p be an $H_{45}-$singular point of $\omega \in \mathcal{Q}(M)$. Then there exists a main chart (x,y) such that*

$$(x,y)^*(\omega) = 4a(dx^2 - dy^2)dxdy + b(dx^4 - 6dx^2dy^2 + dy^4)$$

where

$$\begin{aligned} a(x,y) &= By + a_{20}x^2 + a_{11}xy + a_{02}y^2 + R(x,y), \\ b(x,y) &= y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + S(x,y) \end{aligned}$$

with $B > 0$, $a_{20} - Bb_{20} \neq 0$, and $j_2(R,S)(0,0) = (0,0)$. Here $j_k(f)(q)$ denotes the k–jet of the map f at the point q.

Proof. This is a direct consequence of Proposition 2.4.1. □

Our next result determines the phase portrait of the nets around an $H_{45}-$singular point. In the study of the phase portrait of the nets around a simple singular point p (see [C2–6, Lemma 5.1] and [C2–3, Theorem 5.3]), we consider the surface LM defined on the projective line bundle PM over M by the solutions of equation $4a(dx^2-dy^2)dxdy + b(dx^4 - 6dx^2dy^2 + dy^4) = 0$. This surface is regular in $P^{-1}(p)$ (where P denotes the projection of PM onto M) if and only if p is a simple singular point (see [C2–6, Lemma 4.1]). Thus, this procedure cannot be applied to obtain of the phase portrait of the nets around an $H_{45}-$singular point. Therefore, we use the decomposition

$$b\,\omega \;=\; \omega^+\,\omega^-$$

where

$$\omega^{\pm} \;=\; b\,(dy^2 - dx^2) \;+\; 2\,(-a \pm \sqrt{a^2+b^2})\,dx\,dy\,.$$

We subsequently study the foliations $f_i(\omega^+)$ associated with ω^+, and the foliations $f_i(\omega^-)$ associated with ω^-, with $i = 1, 2$, by considering the relationships between the leaves of $f_i(\omega^-)$ (resp. $f_i(\omega^+)$) and the leaves of the net $\mathcal{N}_1(\omega)$ (resp. $\mathcal{N}_2(\omega)$).

Theorem 2.4.1. *Let p be an $H_{45}-$singular point of $\omega \in \mathcal{Q}(M)$. The phase portraits of the nets $\mathcal{N}_1(\omega)$ and $\mathcal{N}_2(\omega)$ around p are homeomorphic to those shown in Figure 6.*

$\mathcal{N}_1(\omega)$ $\mathcal{N}_2(\omega)$

Figure 6

Proof. Let (x, y) be a main chart such that

$$(x,y)^*(\omega) \;=\; 4a(dx^2-dy^2)dxdy \;+\; b(dx^4 - 6dx^2dy^2 + dy^4)$$

where

$$a(x,y) \;=\; By \;+\; R(x,y)\,, \quad b(x,y) \;=\; y + S(x,y)$$

with $B > 0$, $j_2(R,S)(0,0) = (a_{20}, b_{20})\,x^2 + (a_{11}, b_{11})\,xy + (a_{02}, b_{02})\,y^2$, and $a_{20} - Bb_{20} \neq 0$. First, we suppose $a_{20} - Bb_{20} > 0$.

We first study the configuration around the origin of the positive quadratic differential form $\omega^+ \;=\; b\,(dy^2 - dx^2) \;+\; 2\,(-a + \sqrt{a^2+b^2})\,dx\,dy$. Since $a(x,y) \;=\; (a_{20} - Bb_{20})\,x^2 \;+\; \cdots \;>\; 0$, for $(x,y) \in b^{-1}(0) - \{(0,0)\}$ sufficiently close to the origin, the set $\mathrm{Sing}(\omega^+)$ is a regular curve through the origin. Moreover, the roots of the separatrix equation at the origin

$$S(\omega^+)(x,y) = y(y^2 - x^2) + 2(-By + \sqrt{1+B^2}\mid y \mid)xy\,,$$

which are the possible directions of asymptotic convergence to the origin for the leaves of the foliations $f_1(\omega^+)$ and $f_2(\omega^+)$, are the following. The line $y = 0$ and the segments $y = v_i\,x$, with $(-1)^i\,x \geq 0$, for $i = 1,2,3,4$, where

$$\begin{aligned} v_i &= B - \sqrt{B^2+1} \;+\; (-1)^i\sqrt{(B - \sqrt{B^2+1})^2 + 1} \quad \text{if} \quad i = 1,2, \\ v_i &= B + \sqrt{B^2+1} \;-\; (-1)^i\sqrt{(B + \sqrt{B^2+1})^2 + 1} \quad \text{if} \quad i = 3,4\,. \end{aligned} \tag{2.4.13}$$

Observe that $v_1 < -1 < v_4 < 0 < v_2 < 1 < v_3$. Consider the blowing–up

$$(x,y) \;=\; (u, uv)\,.$$

If $(S,R)(u,uv) \;=\; u^2\,(S_1, R_1)(u,v)$, then $(u,v)^*(\omega^+) \;=\; u\,\omega_1^+$, with

$$\omega_1^+ \;=\; u^2 A_1(u,v)dv^2 + 2uA_2(u,v)dudv + A_3(u,v)du^2\,.$$

For $u \neq 0$, we have

$$\begin{aligned} A_1(u,v) &= v + u\,S_1(u,v), \\ A_2(u,v) &= v\,A_1(u,v) - \left[Bv + u\,R_1(u,v) - \frac{\mid u \mid}{u}\sqrt{H(u,v)}\right], \\ A_3(u,v) &= (v^2 - 1)\,A_1(u,v) - 2v\left[Bv + u\,R_1(u,v) - \frac{\mid u \mid}{u}\sqrt{H(u,v)}\right] \end{aligned}$$

with

$$H(u,v) \;=\; (Bv + u\,R_1(u,v))^2 + (v + u\,S_1(u,v))^2\,.$$

The singular points of ω_1^+ on $u = 0$ correspond to the solutions of $A_3(0,v) \;=\; 0$. Hence the origin is a singular point; the other singular points are the following. For $u \geq 0$ (resp. $u \leq 0$), they are $(0, v_2)$ and $(0, v_4)$ (resp. $(0, v_1)$ and $(0, v_3)$).

In order to obtain the local configuration of ω_1^+ around these singular points, we consider the vector fields $X_i = X_i(\omega_1^+)$, $Y_i = Y_i(\omega_1^+)$ and $Z_i = Z_i(\omega_1^+)$, with $i = 1, 2$, defined by

$$\begin{aligned}
X_i(u,v) &= \left(u^2 A_1(u,v), -uA_2(u,v) + (-1)^i\sqrt{u^2(A_2^2 - A_1\,A_3)(u,v)}\right)\,, \\
Y_i(u,v) &= \left(uA_1(u,v), -A_2(u,v) + (-1)^i\sqrt{(A_2^2 - A_1\,A_3)(u,v)}\right)\,, \\
Z_i(u,v) &= \left(u\left[A_2(u,v) + (-1)^i\sqrt{u^2(A_2^2 - A_1\,A_3)(u,v)}\right], -A_3(u,v)\right)\,.
\end{aligned}$$

We know that X_i is tangent to the foliation $f_i(\omega_1^+)$, and that Y_i is tangent to Z_i. Now for u positive (resp. u negative), Y_i is also tangent to X_i (resp. to X_{3-i}). (See for example [C2–12, Section 4].)

Since $A_2(0, v_i) \;=\; \frac{1}{2}\,(1 + v_i^2)$ for $i = 1, 2, 3, 4$, the point $(0, v_i)$ is a regular point of Y_1 and a singular point of Y_2. Moreover, since $\frac{\partial A_3}{\partial v}(0, v_i) > 0$, the point $(0, v_i)$ is a hyperbolic saddle of Z_2. Since $A_2(0,0) \;=\; 0$, in order to obtain the local configuration

around the origin, we consider the blowing–up

$$x \;=\; s\,, \quad y \;=\; s^2\,t\,.$$

Let

$$a(x,y) \;=\; By \;+\; R(x,y) \;=\; By \;+\; a_{20}\,x^2 \;+\; a_{11}\,xy \;+\; a_{02}\,y^2 \;+\; R_2(x,y)$$

and

$$b(x,y) \;=\; y + S(x,y) \;=\; y \;+\; b_{20}\,x^2 \;+\; b_{11}\,xy \;+\; b_{02}\,y^2 \;+\; S_2(x,y)\,,$$

with $j_2(R_2, S_2)(0,0) = (0,0)$, and $(R_2, S_2)(s, s^2\,t) \;=\; s^3\,(R_3, S_3)(s,t)$.

Then $(s,t)^*(\omega^+) \;=\; s^2\,\omega_2^+$, with

$$\omega_2^+ \;=\; s^4\,N_1(s,t)\,dt^2 \;+\; 2\,s^2\,N_2(s,t)\,ds\,dt \;+\; N_3(s,t)\,ds^2\,,$$

where

$$\begin{aligned} N_1(s,t) &= b_{20}+t+s(b_{11}t+b_{02}st^2+R_3(s,t))\,,\\ N_2(s,t) &= 2stN_1-M_1(s,t)+\sqrt{H_2(s,t)}\,,\\ N_3(s,t) &= (4s^2t^2-1)N_1+4st\left[-M_1(s,t)+\sqrt{H_2(s,t)}\right] \end{aligned}$$

and

$$\begin{aligned} M_1(s,t) &= a_{20}+B\,t+s(a_{11}t+a_{02}st^2+S_3(s,t))\,,\\ H_2(s,t) &= N_1(s,t)^2 \;+\; M_1(s,t)^2\,. \end{aligned}$$

The unique singular point of ω_2^+ on the line $s=0$ is the point $(0,-b_{20})$. For $i=1,2$, consider the vector fields

$$X_i(s,t) \;=\; \left(s^4\,N_1(s,t), -s^2\,N_2(s,t)+(-1)^i\,s^2\,\sqrt{N_2(s,t)^2-N_1(s,t)N_3(s,t)}\right)$$

and

$$Y_i(s,t) \;=\; \frac{s^2\,N_1(s,t)}{M_1(s,t)+\sqrt{H_2(s,t)}}\,(P(s,t),Q_i(s,t))$$

where

$$\begin{aligned} P(s,t) &= s^2\left(M_1(s,t)+\sqrt{H_2(s,t)}\right)\,,\\ Q_i(s,t) &= -2st\left(M_1(s,t)+\sqrt{H_2(s,t)}\right)-N_1(s,t)+\\ &\quad (-1)^i\sqrt{2}\,\sqrt{H_2(s,t)+M_1(s,t)\,\sqrt{H_2(s,t)}}\;. \end{aligned}$$

Then X_i is tangent to the foliation $f_i(\omega_2^+)$, for $i=1,2$. Since $M_1(0,-b_{20})>0$, the vector fields Y_i are well defined in a neighborhood of the point $(0,-b_{20})$. Further, we have $Y_i=X_i$ (resp. $Y_i=X_{3-i}$) for $N_1(s,t)$ positive (resp. negative). Observe that the vector fields $(P(s,t),Q_i(s,t))$, with $i=1,2$, are non–vanishing at the point $(0,-b_{20})$. To complete our analysis of the local configuration of ω^+, we consider the blowing–up

$$(x, y) \;=\; (st, st^2)\,.$$

If $(R,S)(st,st^2) \;=\; s^2t^2\,(R_4,S_4)(s,t)$, then $(s,t)^*(\omega^+) \;=\; st^2\,\omega_3^+$, with

$$\omega_3^+ \;=\; s^2\,A_3(s,t)\,dt^2 \;+\; 2\,s\,t\,B_3(s,t)\,ds\,dt \;+\; t^2\,C_3(s,t)\,ds^2$$

where, for $s \neq 0$, we have

$$\begin{aligned}
A_3(s,t) &= -1 - 4\,B\,t + 4t^2 + 4\,\frac{|\,s\,|}{s}\,t\,\sqrt{H_3(s,t)} \;- \\
&\quad s\,[S_4(s,t) + 4\,t\,R_4(s,t) - 4\,t^2\,S_4(s,t)]\,, \\
B_3(s,t) &= -1 - 3\,B\,t + 2\,t^2 + 3\,\frac{|\,s\,|}{s}\,t\,\sqrt{H_3(s,t)} \;- \\
&\quad s\,[S_4(s,t) + 3\,t\,R_4(s,t) - 2\,t^2\,S_4(s,t)]\,, \\
C_3(s,t) &= -1 - 2\,B\,t + t^2 + 2\,t\,\frac{|\,s\,|}{s}\,\sqrt{H_3(s,t)} \;- \\
&\quad s\,[S_4(s,t) + 2\,t\,R_4(s,t) - t^2\,S_4(s,t)]
\end{aligned}$$

with

$$H_3(s,t) \;=\; (B + s\,R_4(s,t))^2 \;+\; (1 + s\,S_4(s,t))^2\,.$$

The vector fields associated with ω_3 are

$$X_i(\omega_3^+)(s,t) \;=\; (s^2\,A_3(s,t), -st\,B_3(s,t) + (-1)^i\,\sqrt{s^2t^2(B_3^2 - A_3C_3)(s,t)})\,,$$

and

$$Y_i(\omega_3^+)(s,t) \;=\; (s\,A_3(s,t),\; t[-B_3(s,t) + (-1)^i\,\sqrt{(B_3^2 - A_3C_3)(s,t)}])$$

with $i = 1, 2$. As usual, for $i = 1, 2$, the vector field $X_i(\omega_3^+)$ is tangent to the foliation $f_i(\omega_3^+)$, and the vector field $Y_i(\omega_3^+)$ is tangent to $X_i(\omega_3^+)$ (resp. X_{3-i}) for st positive (resp. negative). Since $A_3(0,0) = B_3(0,0) = -1$ and $(B_3^2 - A_3C_3)(s,t) = t^2\,F(s,t)$ with $F(0,0) > 0$, we conclude that the origin is a saddle singular point for $Y_i(\omega_3^+)$, with $i = 1, 2$.

Therefore, the configuration of ω_1^+ (resp. ω^+) around the line $u = 0$ (resp. the origin) is the one shown in Figure 7 (resp. Figure 8), which proves that the phase portrait of the net $\mathcal{N}_1(\omega)$ is homeomorphic to the one shown in Figure 6.

$f_1(\omega_1^+)$ $f_2(\omega_1^+)$

Figure 7

$f_1(\omega^+)$ $f_2(\omega^+)$

Figure 8

We now study the configuration around the origin of $\omega^- = b\,(dy^2 - dx^2) + 2\,(-a - \sqrt{a^2+b^2})\,dx\,dy$. Since $a(x,y) = (a_{20} - Bb_{20})\,x^2 + \cdots > 0$ for $(x,y) \in b^{-1}(0) - \{(0,0)\}$ sufficiently close to the origin, the set $\mathrm{Sing}(\omega^-)$ is reduced only to the origin. The roots of the separatrix equation at the origin

$$S(\omega^-)(x,y) = y(y^2 - x^2) + 2(-By - \sqrt{1+B^2} \mid y \mid)xy$$

are the line $y = 0$ and the segments $y = v_i\,x$, with $(-1)^i\,x \leq 0$, for $i = 1,2,3,4$, where the $v_i's$ are given by (2.4.13). Performing the blowing–up

$$(x,y) \; = \; (u, uv)$$

we obtain $(u,v)^*(\omega^-) = u\,\omega_1^-$. Similarly, the origin is a singular point of ω_1^-; the other singular points on $u = 0$ are $(0,v_1)$ and $(0,v_3)$, for $u \geq 0$, and $(0,v_2)$ and $(0,v_4)$, for $u \leq 0$.

We now study the corresponding vector fields $X_i = X_i(\omega_1^-)$, $Y_i = Y_i(\omega_1^-)$ and $Z_i = Z_i(\omega_1^-)$, with $i = 1,2$, obtaining the following. The points $(0,v_i)$, with $i = 1,2,3,4$, are regular points of Y_1, and are singular points of Y_2. Moreover, the points $(0,v_i)$ are hyperbolic saddles of Z_2.

As before, to determine the local configuration around the origin, we perform the blowing–up

$$x = s\,, \quad y = s^2\, t\,.$$

Let

$$a(x,y) = By + R(x,y) = By + a_{20}\, x^2 + a_{11}\, xy + a_{02}\, y^2 + R_2(x,y)$$

and

$$b(x,y) = y + S(x,y) = y + b_{20}\, x^2 + b_{11}\, xy + b_{02}\, y^2 + S_2(x,y)\,,$$

with $j_2(R_2, S_2)(0,0) = (0,0)$ and $(R_2, S_2)(s, s^2\, t) = s^3\, (R_3, S_3)(s,t)\,.$
Then $(s,t)^*(\omega^-) = s^2\, \omega_2^-$, with

$$\omega_2^- = s^4\, N_1(s,t)\, dt^2 + 2\, s^2\, N_2(s,t)\, ds\, dt + N_3(s,t)\, ds^2$$

where

$$\begin{aligned} N_1(s,t) &= b_{20} + t + s(b_{11} t + b_{02} s t^2 + S_3(s,t)), \\ N_2(s,t) &= 2stN_1 - M_1(s,t) - \sqrt{H_2(s,t)}\,, \\ N_3(s,t) &= (4s^2t^2 - 1)N_1 + 4st\left[-M_1(s,t) - \sqrt{H_2(s,t)}\right] \end{aligned}$$

and

$$\begin{aligned} M_1(s,t) &= a_{20} + Bt + s(a_{11} t + a_{02} s t^2 R_3(s,t))\,, \\ H_2(s,t) &= N_1(s,t)^2 + M_1(s,t)^2\,. \end{aligned}$$

The unique singular point of ω_2^- on the line $s = 0$ is the point $(0, -b_{20})$. Observe that $N_2(0, -b_{20}) = -2M_1(0, -b_{20}) < 0$ and $N_1(0, -b_{20}) = N_3(0, -b_{20}) = 0$. For $i = 1, 2$, consider the vector fields $X_i(s,t) = s^2\, Y_i(s,t)$, with

$$Y_i(s,t) = \left(s^2\, N_1(s,t), -N_2(s,t) + (-1)^i \sqrt{N_2(s,t)^2 - N_1(s,t) N_3(s,t)}\right)\,.$$

Then the point $(0, -b_{20})$ is a singular point for Y_1, and is a regular point for Y_2. Moreover, the vector field Y_1 is tangent to the vector field

$$Z_1(s,t) = \left(s^2 \left(-N_2(s,t) + \sqrt{N_2(s,t)^2 - N_1(s,t) N_3(s,t)}\right), N_3(s,t)\right)$$

which has a saddle node singular point at the point $(0, -b_{20})$ with parabolic sector in $s \leq 0$.

In order to complete our analysis of the local configuration of ω^-, we consider the blowing–up

$$(x, y) \;=\; (st, st^2)\,.$$

If $(R, S)(st, st^2) \;=\; s^2t^2\,(R_4, S_4)(s, t)$, then $(s, t)^*(\omega^-) \;=\; st^2\,\omega_3^-$, with

$$\omega_3^- \;=\; s^2\,A_3(s,t)\,dt^2 \;+\; 2\,s\,t\,B_3(s,t)\,ds\,dt \;+\; t^2\,C_3(s,t)\,ds^2$$

where, for $s \neq 0$ we have

$$\begin{aligned}
A_3(s,t) \;&=\; -1 - 4\,B\,t + 4t^2 - 4\,\frac{|\,s\,|}{s}\,t\,\sqrt{H_3(s,t)} \;- \\
&\quad\; s\,[S_4(s,t) + 4\,t\,R_4(s,t) - 4\,t^2\,S_4(s,t)]\,, \\
B_3(s,t) \;&=\; -1 - 3\,B\,t + 2\,t^2 - 3\,\frac{|\,s\,|}{s}\,t\,\sqrt{H_3(s,t)} \;- \\
&\quad\; s\,[S_4(s,t) + 3\,t\,R_4(s,t) - 2\,t^2\,S_4(s,t)]\,, \\
C_3(s,t) \;&=\; -1 - 2\,B\,t + t^2 - 2\,t\,\frac{|\,s\,|}{s}\,\sqrt{H_3(s,t)} \;- \\
&\quad\; s\,[S_4(s,t) + 2\,t\,R_4(s,t) - t^2\,S_4(s,t)]
\end{aligned}$$

with

$$H_3(s,t) \;=\; (B + s\,R_4(s,t))^2 \;+\; (1 + s\,S_4(s,t))^2\,.$$

The vector fields associated with ω_3^- are

$$X_i(\omega_3^-)(s,t) \;=\; (s^2\,A_3(s,t), -st\,B_3(s,t) + (-1)^i\,\sqrt{s^2t^2(B_3^2 - A_3C_3)(s,t)})$$

and

$$Y_i(\omega_3^-)(s,t) \;=\; (s\,A_3(s,t),\, t[-B_3(s,t) + (-1)^i\,\sqrt{(B_3^2 - A_3C_3)(s,t)}])$$

with $i = 1, 2$. As usual, for $i = 1, 2$, the vector field $X_i(\omega_3^-)$ is tangent to the foliation $f_i(\omega_3^-)$, and the vector field $Y_i(\omega_3^-)$ is tangent to $X_i(\omega_3^-)$ (resp. X_{3-i}) for st positive (resp. negative). Since $A_3(0,0) = B_3(0,0) = -1$ and $(B_3^2 - A_3C_3)(s,t) = t^2\,F(s,t)$ with $F(0,0) > 0$, we conclude that the origin is a saddle singular point for $Y_i(\omega_3^-)$, with $i = 1, 2$.

Therefore, the configuration of ω_1^- (resp. ω^-) around the line $u = 0$ (resp. the origin) is the one shown in Figure 9 (resp. Figure 10), which proves that the phase portrait of the net $\mathcal{N}_2(\omega)$ is homeomorphic to the one shown in Figure 6.

$f_1(\omega_1^-)$ $f_2(\omega_1^-)$

Figure 9

$f_1(\omega^-)$ $f_2(\omega^-)$

Figure 10

Finally, in the case $a_{20} - Bb_{20} < 0$, the configuration obtained is the same as the one already obtained, though with the nets interchanged. The proof of the theorem is now complete. $\square$

2.5 Smooth families in $\mathcal{Q}(\mathbb{R}^2)$

In this section we deal with local problems around isolated singular points of rank greater or equal than one of quartics in $\mathcal{Q}(M)$. Such problems are normal forms, finite determinacy and versal unfoldings. Thus we will work with quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$.

The notion of equivalence of families of quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$ used in this article is the following.

Definition 2.5.1. *Consider two smooth families* (ω_μ) *and* (v_μ) *in* $\mathcal{Q}(\mathbb{R}^2)$ *with (the same) parameter* $\mu \in \mathbb{R}^k$*. Let* $\mathcal{N}_1(\omega_\mu)$ *and* $\mathcal{N}_2(\omega_\mu)$ *(resp.* $\mathcal{N}_1(v_\mu)$ *and* $\mathcal{N}_2(v_\mu)$*) be the nets associated to* ω_μ *(resp.* v_μ*). The families* (ω_μ) *and* (v_μ) *are called* $\mathbf{C^0}$***−equivalent*** *(over the identity) if there exist homeomorphisms* $h^i_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ *such that, for each* $\mu \in \mathbb{R}^k$*, we have that* h^i_μ *is a* C^0-*equivalence between the nets* $\mathcal{N}_i(\omega_\mu)$ *and* $\mathcal{N}_i(v_\mu)$*, with* $i = 1, 2$*.*

Remark 2.5.1. *For local families around the origin of* $\mathbb{R}^2 \times \mathbb{R}^k$ *, we impose for* $i = 1, 2$ *, the conditions that* $h^i_{\bar{0}}(0,0) = 0$ *, that* h^i_μ *only be defined for* $((x,y),\mu)$ *which belongs to a neighborhood* $V \times W$ *of* $((0,0),\bar{0})$ *in* $\mathbb{R}^2 \times \mathbb{R}^k$ *, and that* $\{(h^i_\mu(x,y),\mu) \in V \times W$ *be a neighborhood of* $((0,0),\bar{0})$ *.*

Definition 2.5.2. *Let* $U \subset \mathbb{R}^k$ *and* $V \subset \mathbb{R}^l$ *be neighborhoods of the origin. If* $\phi : (V,\bar{0}) \to (U,\bar{0})$ *is a smooth map and* (ω_μ) *is a smooth family of quartics in* $\mathcal{Q}(\mathbb{R}^2)$ *with parameter* $\mu \in U$ *, the family* $(v_\alpha) = \omega_{\phi(\alpha)}$ *, with parameter* $\alpha \in V$ *, is called a* ***family*** $\mathbf{C^\infty}$***−induced*** *by* ϕ *.*

Recall that an **unfolding of** a quartic $\omega \in \mathcal{Q}(\mathbb{R}^2)$ is any smooth family (ω_μ) in $\mathcal{Q}(\mathbb{R}^2)$ with $\omega_{\bar{0}} = \omega$; thus we have the following Definition.

Definition 2.5.3. *An unfolding* (ω_μ) *of* ω_0 *is called a* ***versal unfolding*** *of* ω_0 *if all unfoldings of* ω_0 *are* C^0-*equivalent to an unfolding* $C^\infty-$*induced from* (ω_μ)*.*

Our principal tool is the following result, similar to Proposition 2.2.3, which assert the existence of main charts for families of quartics in $\mathcal{Q}(\mathbb{R}^2)$. The proof is an adaptation of the one presented in [C2–21, Addendum 1] for the existence of smooth isothermal coordinates and it is presented in Section 2.6.

Proposition 2.5.1. *Let* $(\omega(\mu))$*, with parameter* $\mu \in \mathbb{R}^k$*, be an arbitrary smooth family in* $\mathcal{Q}(\mathbb{R}^2)$*. Given* $p_0 \in \mathbb{R}^2$*, there exits a local chart* $\phi : (U \times V, (p_0, \bar{0})) \to (\mathbb{R}^2 \times \mathbb{R}^k, ((0,0),\bar{0}))$ *of the form* $\phi(p,\mu) = (u(p,\mu), v(p,\mu), \mu)$*, with* $\phi(p,\bar{0}) = (u(p), v(p), \bar{0})$ *for all* $p \in U_0$*, such that in the chart* $\phi_\mu : (U_0, p(\mu)) \to (\mathbb{R}^2, (0,0))$ *defined by* $\phi_\mu(p) =$

$\phi(p,\mu)$ for all $\mu \in V$, the local expression of $\omega(\mu)$ is

$$\phi_\mu^*(\omega(\mu)) \;=\; 4\,a(\mu)\,(du^2 - dv^2)\,dudv \;+\; b(\mu)\,(du^4 - 6\,du^2dv^2 + dv^4)\,. \tag{2.5.14}$$

2.5.1 Simple singular points

Our next result asserts that for a smooth family $\omega(\mu)$ in $\mathcal{Q}(\mathbb{R}^2)$ such that $\omega(0)$ has a simple singular point at the origin, without loss of generality we may assume that the origin is a singular point of $\omega(\mu)$, for small $|\mu|$. Here is a precise statement.

Lemma 2.5.1. *Let $\omega(\mu)$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family of quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$ such that $\omega(0)$ has a simple singular point at the origin. Then there exists a change of coordinates of the form $(x,y,\mu) = (h(u,v,\mu),\mu)$ such that, for each μ with small $|\mu|$, the origin is a singular point of the quartic*

$$(x,y)^*(\omega(\mu))\,.$$

Proof. We may assume that

$$\omega(\mu) \;=\; 4\,a(u,v,\mu)\,(du^2 - dv^2)\,dudv \;+\; b(u,v,\mu)\,(du^4 - 6\,du^2dv^2 + dv^4)\,.$$

By hypothesis we have that $(a(0,0,\bar{0}), b(0,0,\bar{0}))((0,0)) = (0,0)$, and that

$$D_1(a(0,0,\bar{0}), b(0,0,\bar{0}))\,((0,0)) = \begin{pmatrix} a_{01} & a_{02} \\ a_{11} & a_{12} \end{pmatrix}$$

is non–singular. Since the map $(a,b) : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^2$ is smooth, it follows from the implicit function theorem that there exists a smooth map S defined on a small neighborhood of $\bar{0} \in \mathbb{R}^k$ so that $S(\bar{0}) = (0,0)$ and

$$(a_0, a_1)(S(\mu),\mu) = (0,0)$$

for all μ in such a neighborhood. Using the change of coordinates

$$(x,y,\mu) = (u,v,\mu) - (S(\mu),\bar{0})$$

we obtain the Lemma. $\square$

Our next result shows that the normal form (2.3.8) also holds for families in $\mathcal{Q}(\mathbb{R}^2)$ which pass through a quartic having a simple singular point.

Lemma 2.5.2. *Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$, such that $\omega(0)$ has a simple singular point at the origin. Then there exits a local chart $\phi : (U \times V, ((0,0),\bar{0})) \to (\mathbb{R}^2 \times \mathbb{R}^k, ((0,0),\bar{0}))$ of the form $\phi(p,\mu) = (u(p,\mu), v(p,\mu), \mu)$, with $\phi(p,\bar{0}) = (u(p), v(p), \bar{0})$ for all $p \in U_0$, such that in the chart $\phi_\mu : (U_0, p(\mu)) \to (\mathbb{R}^2, (0,0))$ defined by $\phi_\mu(p) = \phi(p,\mu)$ for all $\mu \in V$, the local expression of $\omega(\mu)$ is*

$$\begin{aligned}\phi_\mu^*(\omega(\mu)) \;&=\; 4(A(\mu)u + B(\mu)v \;+\; R(\mu))(du^2 - dv^2)dudv \;+ \\ &(v + S(\mu))(du^4 - 6du^2dv^2 + dv^4)\,,\end{aligned}$$

with $A(\mu) \neq 0$ and $j_1(R(\mu), S(\mu))(0,0) = (0,0)$.

Proof. For μ in a neighborhood V of the origin in $\mathbb{R}^k$, there exists local chart (s,t,μ) such that the local expression of $\omega(\mu)$ is

$$\begin{aligned}(s,t)^*(\omega(\mu)) \;&=\; 4(\tilde{A}(\mu)s + \tilde{B}(\mu)t + \tilde{R}(\mu))(ds^2 - dt^2)dsdt \;+ \\ &(\tilde{C}(\mu)s + \tilde{D}(\mu)t + \tilde{S}(\mu))(ds^4 - 6ds^2dt^2 + dt^4)\end{aligned}$$

with $j_1(R(\mu), S(\mu))(0,0) = (0,0)$ and $A(\mu)D(\mu) - B(\mu)C(\mu) \neq 0$.

Let $L_{(\alpha,\beta)} : \mathbb{R}^3 \to \mathbb{R}^3$, with parameter $(\alpha,\beta) \in \mathbb{R}^2$, be the family of linear isomorphisms such that the inverse of $L = L_{(\alpha,\beta)}$ is given by

$$L^{-1}(s,t,\mu) = ((1+\alpha)s - \beta t, \beta s + (1+\alpha)t, \mu)\,.$$

Observe that for all $(\alpha,\beta) \in \mathbb{R}^2$, the map $L_{(\alpha,\beta)}$ is a linear rotation at the first two coordinates. Therefore, in the chart

$$(s,t,\mu) = ((1+\alpha)u - \beta v, \beta u + (1+\alpha)v, \mu)$$

the local expression of $\omega(\mu)$ is given by

$$\begin{aligned}(u,v)^*(\omega(\mu)) \;&=\; 4(A(\mu)u + B(\mu)v + \tilde{R})(du^2 - dv^2)dudv \\ &+(C(\mu)u + D(\mu)v + \tilde{S})(du^4 - 6du^2dv^2 + dv^4)\,.\end{aligned}$$

To complete the proof, it suffices to show that there exists $(\alpha, \beta) = (\alpha(\mu), \beta(\mu))$ so that $(C(\mu), D(\mu)) \equiv (0,1)$, for $\mu \in V$ sufficiently close to $\bar{0}$. In fact,

$$\begin{aligned} C(\mu) &= 4(1+\alpha)^4\tilde{A}(\mu)\beta + 4(1+\alpha)^3\tilde{B}(\mu)\beta^2 - 4(1+\alpha)^2\tilde{A}(\mu)\beta^3 - \\ & \quad 4(1+\alpha)\tilde{B}(\mu)\beta^4 + (1+\alpha)^4\beta\tilde{D}(\mu) - 6(1+\alpha)^2\beta^3\tilde{D}(\mu) + \beta^5\tilde{D}(\mu) + \\ & \quad (1+\alpha)^5\tilde{C}(\mu) - 6(1+\alpha)^3\beta^2\tilde{C}(\mu) + (1+\alpha)\beta^4\tilde{C}(\mu) \end{aligned}$$

and

$$\begin{aligned} D(\mu) &= 4(1+\alpha)^4\tilde{B}(\mu)\beta - 4(1+\alpha)^3\tilde{A}(\mu)\beta^2 - 4(1+\alpha)^2\tilde{B}(\mu)\beta^3 + 4(1+ \\ & \quad \alpha)\tilde{A}(\mu)\beta^4 + (1+\alpha)^5\tilde{D}(\mu) - 6(1+\alpha)^3\beta^2\tilde{D}(\mu) + (1+\alpha)\beta^4\tilde{D}(\mu) - \\ & \quad (1+\alpha)^4\beta\tilde{C}(\mu) + 6(1+\alpha)^2\beta^3\tilde{C}(\mu) - \beta^5\tilde{C}(\mu)\,. \end{aligned}$$

If $\tilde{C}(\mu) = 0$, then $\tilde{D}(\mu) \neq 0$. We may set $\beta = 0$ and $1+\alpha = \frac{1}{\tilde{D}(\mu)^{\frac{1}{5}}}$ $\cdot$ Then $C(\mu) = 0$ and $D(\mu) = 1$. If $\tilde{C}(\mu) \neq 0$, we set $1+\alpha = m\,\beta$, with m a real root of the equation

$$\begin{gathered} \tilde{C}(\mu)x^5 + 2\,(2\tilde{B}(\mu) - 3\tilde{C}(\mu))x^4 + 2\,(2\tilde{B}(\mu) - 3\tilde{C}(\mu))x^3 - \\ 2\,(2\tilde{A}(\mu) + 3\tilde{D}(\mu))x^2 + (\tilde{C}(\mu) - 4\tilde{B}(\mu))x + \tilde{D}(\mu) = 0\,. \end{gathered}$$

Then $C(\mu) = 0$, and we are under the condition of the first case. The proof now follows. $\square$

To obtain a versal unfolding for a simple singular point, we will need the following.

Lemma 2.5.3. *Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$ such that $\omega(0)$ has a simple singular point at the origin. Consider a local chart (u, v, μ) such that*

$$\begin{aligned} \omega(\mu) &= 4(A(\mu)u + B(\mu)v + R(\mu)(u,v))(du^2 - dv^2)dudv + \\ & \quad (v + S(\mu)(u,v))(du^4 - 6du^2dv^2 + dv^4) \end{aligned}$$

with $j_1(R(\mu), S(\mu))(0,0) = (0,0)$. Then, for small $\mid \mu \mid$, the family $(\omega(\mu))$ is equivalent to the family

$$\tilde{\omega}(\mu) = 4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4)\,.$$

Proof. The Lemma is clear from the fact that both families have the same linear part at the origin. □

We next give a versal unfolding for singular points of type H_{34}.

Theorem 2.5.1. *A versal unfolding of an $H_{34}-$singular point is the family of quartic $\upsilon(\lambda)$, with $\lambda \in \mathbb{R}$, given by*

$$\upsilon(\lambda) \;=\; 4\left(\left(\lambda - \frac{125}{32}\right) u + \frac{51}{32}\, v\right)(du^2 - dv^2)du dv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Proof. Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$ so that $\omega(0)$ has an $H_{34}-$singular point at the origin. By Lemma 2.5.3 we may suppose that

$$\omega(\mu) \;=\; 4(A(\mu)u + B(\mu)v)(du^2 - dv^2)du dv + v(du^4 - 6du^2dv^2 + dv^4) \qquad (2.5.15)$$

with $A(\mu) \neq 0$.

First we claim that we may suppose $A(0) \neq -\frac{1}{4}\cdot$ In fact $A(0) = -\dfrac{1}{4}$ if and only if the root of multiplicity two of the separatrix polynomial $g(s)$ is $s = 0$. Now if $A = -\dfrac{1}{4}$ and s_0 is a simple root of $g(s)$, then we make a rotation that in $\mu = 0$ sends s_0 on $s = 0$. In the resulting chart, the local expression of ω is also of the form (2.5.15). Hence, the corresponding coefficient $A(0) \neq -\dfrac{1}{4}$, which completes the proof of the claim.

Consider the real–valued function ψ defined on a neighborhood of the origin of $\mathbb{R}^k$ by

$$\begin{aligned}\psi(\mu) \;=\;& 16\,[4(1 + B(\mu)^2)^3 + 24(1 + B(\mu)^2)^2 A(\mu) + \\ & 8(5 - B(\mu)^2)(1 + B(\mu)^2)A(\mu)^2 + 4(9 + B(\mu)^2)A(\mu)^3 + \\ & (17 + 4B(\mu)^2)A(\mu)^4 + 4A(\mu)^5]\,.\end{aligned}$$

Then the unfolding induced by ψ from the family $(\upsilon(\lambda))_{\lambda\in\mathbb{R}}$ is

$$\tilde{\upsilon}(\mu) \;=\; 4\left(\left(\psi(\mu) - \frac{125}{32}\right) u + \frac{51}{32}\, v\right)(du^2 - dv^2)du dv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Since the discriminant (2.3.10) associated to the family $(\omega(\mu))$ is $\Delta(\mu) = \psi(\mu)$, and since the discriminant associated to the family $(\tilde{\upsilon}(\mu))$ is of the form

$$\tilde{\Delta}(\mu) \;=\; \psi(\mu)\, h(\psi(\mu))$$

where $h(x)$ is a degree 4 polynomial with $h(0) > 0$, both families are equivalent for small $|\mu|$. The proof is now complete. □

We now consider the singular points of type $\tilde{H}_3$.

Theorem 2.5.2. *A versal unfolding of an $\tilde{H}_3-$singular point is the family of quartic $\upsilon(\lambda)$, with $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, given by*

$$\upsilon(\lambda) \;=\; 4\left(\left(\lambda_1 - \frac{1}{4}\right) u + \lambda_2\, v\right)\,(du^2 - dv^2)dudv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Proof. Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$ so that $\omega(0)$ has an $\tilde{H}_3-$singular point at the origin. By Lemma 2.5.3 we may suppose

$$\omega(\mu) \;=\; 4(A(\mu)u + B(\mu)v)(du^2 - dv^2)dudv + v(du^4 - 6du^2dv^2 + dv^4)$$

with $A(0) = -\frac{1}{4}$ and $B(0) = 0$. Let us consider the real bi–valued function ψ defined on a neighborhood of the origin of $\mathbb{R}^k$ by

$$\psi(\mu) \;=\; (A(\mu), B(\mu))\,.$$

Then the unfolding induced by ψ from the family $(\upsilon(\lambda))_{\lambda\in\mathbb{R}^2}$ is

$$\tilde{\upsilon}(\mu) \;=\; 4\left(\left(A(\mu) - \frac{1}{4}\right) u + B(\mu)\, v\right)\,(du^2 - dv^2)dudv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Since the discriminant (2.3.10) associated to the family $(\omega(\mu))$ is equal to that associated to the family $(\tilde{\upsilon}(\mu))$, for every μ, we conclude that both families are equivalent. The proof is now complete. □

The next two theorems give the bifurcation diagrams of these types of singular points.

Theorem 2.5.3. *Consider the one–parameter family of quartic $\omega(\lambda)$ given by*

$$\omega(\lambda) \;=\; 4\left(\left(\lambda - \frac{125}{32}\right)u + \frac{51}{32}\,v\right)\,(du^2 - dv^2)dudv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Then, for all values of λ, the origin is a singular point of $\omega(\lambda)$. Moreover, for small $\mid \lambda \mid$, the origin is of type H_3 for $\lambda < 0$, of type H_{34} for $\lambda = 0$, and of type H_4 for $\lambda > 0$.

Proof. Since the associated discriminant is

$$\Lambda = \frac{\lambda}{8192}(6640625 - 48348750\lambda + 30426304\lambda^2 - 6680064\lambda^3 + 524288\lambda^4)\,,$$

the proof follows. $\square$

Theorem 2.5.4. *Consider the two–parameter family of quartic $\omega(\lambda)$, with $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, given by*

$$\omega(\lambda) \;=\; 4\left(\left(\lambda_1 - \frac{1}{4}\right)u + \lambda_2\,v\right)\,(du^2 - dv^2)dudv + v\,(du^4 - 6du^2dv^2 + dv^4)\,.$$

Then the origin is a singular point for all values of $\lambda = (\lambda_1, \lambda_2)$. Moreover, for small $\mid \lambda \mid$, we have that:

i) The origin is of type H_3 if $\Lambda < 0$.

ii) The origin is of type H_{34} if $\Lambda = 0$ and $\lambda_1 \neq 0$, or if $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

iii) The origin is of type H_4 for $\Lambda > 0$ and $\lambda_1 \neq 0$.

iv) The origin is of type $\tilde{H}_3$ for $\lambda = (\lambda_1, \lambda_2) = (0, 0)$.

Here

$$\begin{aligned}\Lambda \;=\; & \frac{1}{4}(625\,\lambda_1 + 1200\,\lambda_2 + 1376\,\lambda_1^3 + 768\,\lambda_1^4 + 256\,\lambda_1^5 + 125\,\lambda_2^2 + \\ & 2080\,\lambda_1\,\lambda_2^2 + 1952\,\lambda_1^2\,\lambda_2^2 + 256\,\lambda_1^4\,\lambda_2^2 + 352\,\lambda_2^4 + 1792\,\lambda_1\,\lambda_2^4 \\ & -512\,\lambda_1^2\,\lambda_2^4 + 256\,\lambda_2^6)\,.\end{aligned}$$

$\Lambda = 0$ λ_2

$\Lambda > 0$

$\Lambda < 0$ λ_1

Figure 11

Proof. For the proof, it suffices to observe that the corresponding values of A and B are

$$A \;=\; \lambda_1 \;-\; \frac{1}{4} \quad \text{and} \quad B \;=\; \lambda_2\,.$$

$\square$

2.5.2 A non–simple case

Lemma 2.5.4. *Let $\omega(\mu)$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family of quartic differential forms in $\mathcal{Q}(\mathbb{R}^2)$ such that $\omega(0)$ has a $H_{45}-$singular point at the origin. Then there exits a local chart $\phi : (U \times V, ((0,0),\bar{0})) \to (\mathbb{R}^2 \times \mathbb{R}^k, ((0,0),\bar{0}))$ of the form $\phi(p,\mu) = (u(p,\mu), v(p,\mu), \mu)$, with $\phi(p,\bar{0}) = (u(p), v(p), \bar{0})$ for all $p \in U_0$, such that in the chart $\phi_\mu : (U_0, p(\mu)) \to (\mathbb{R}^2, (0,0))$ defined by $\phi_\mu(p) = \phi(p,\mu)$ for all $\mu \in V$, the local expression of $\omega(\mu)$ is*

$$\phi_\mu^*(\omega(\mu)) \;=\; 4\,a(\mu)\,(du^2 - dv^2)\,dudv \;+\; b(\mu)\,(du^4 - 6\,du^2dv^2 + dv^4)\,, \tag{2.5.16}$$

where

$$\begin{aligned} a(\mu)(u,v) &= A_1(\mu)\,u + A_2(\mu)\,v + R(u,v,\mu) \\ b(\mu)(u,v) &= n_0(\mu) + n_1(\mu)(A_1(\mu)\,u + A_2(\mu)\,v) + S(u,v,\mu)\,, \end{aligned}$$

with $A_1(0) = n_0(0) = 0$ and $n_1(0)\,A_2(0) = 1$.

Proof. Making a rotation if necessary we may assume (see Proposition ...)

$$(x,y)*(\omega(\mu)) \;=\; 4\,a(\mu)\,(dx^2-dy^2)dxdy \;+\; b(\mu)\,(dx^4-6\,dx^2dy^2+dy^4)\,,$$

with

$$\begin{aligned} a(0)(x,y) &= B\,y \;+\; a_{20}\,x^2 \;+\; a_{11}\,xy \;+\; a_{02}\,y^2 \;+\; R(x,y)\,, \\ b(0)(x,y) &= y \;+\; b_{20}\,x^2 \;+\; b_{11}\,xy \;+\; b_{02}\,y^2 \;+\; R(x,y)\,, \end{aligned}$$

and $B>0$, $a_{20}-Bb_{20}\neq 0$. Consider the map $S:\mathbb{R}^k\times(u,v)(U)\to\mathbb{R}^2$ defined by

$$S(\mu,q)=(a(\mu)(q),\det D(a(\mu),b(\mu))(q))\,.$$

Thus S is smooth, $S(\bar{0},(0,0))=(0,0)$, and

$$D_2S(\bar{0},(0,0))=\begin{pmatrix} 0 & B \\ 2(a_{20}-B\,b_{20}) & * \end{pmatrix}.$$

According to the Implicit Function Theorem, there exist neighborhoods V of $\bar{0}$ in $\mathbb{R}^k$ and $U_1\subset(u,v)(U)$ of the origin and a smooth map $s:V\to U_1$ such that $s(\bar{0})=(0,0)$ and $S(\mu,s(\mu))=(0,0)$ for all $\mu\in V$. Using the change of coordinates

$$(u,v,\mu)=(x,y,\mu)-(s(\mu),\bar{0})$$

we obtain the Lemma. □

Lemma 2.5.5. *Let $(\omega(\mu))$, with parameter $\mu\in\mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$, such that $\omega(0)$ has a $H_{45}-$singular point at the origin. Then there exits a local chart $\phi:(U\times V,((0,0),\bar{0}))\to(\mathbb{R}^2\times\mathbb{R}^k,((0,0),\bar{0}))$ of the form $\phi(p,\mu)=(u(p,\mu),v(p,\mu),\mu)$, with $\phi(p,\bar{0})=(u(p),v(p),\bar{0})$ for all $p\in U_0$, such that in the chart $\phi_\mu:(U_0,p(\mu))\to(\mathbb{R}^2,(0,0))$ defined by $\phi_\mu(p)=\phi(p,\mu)$ for all $\mu\in V$, the local expression of $\omega(\mu)$ is*

$$\phi_\mu^*(\omega(\mu)) \;=\; 4\,a(\mu)\,(du^2-dv^2)\,dudv \;+\; b(\mu)\,(du^4-6\,du^2dv^2+dv^4)\,, \qquad (2.5.17)$$

with

$$a(\mu)(u,v) \;=\; B(\mu)\,v \;+\; R(u,v,\mu)$$
$$b(\mu)(u,v) \;=\; n_0(\mu) \;+\; v \;+\; S(u,v,\mu)\,,$$

and $n_0(0) = 0,\ B(0) = B > 0.$

Proof. Without loss of generality we may suppose

$$(x,y)^*(\omega(\mu)) \;=\; 4\,a(\mu)\,(dx^2 - dy^2)\,dxdy \;+\; b(\mu)\,(dx^4 - 6\,dx^2dy^2 + dy^4)\,, \qquad (2.5.18)$$

where

$$a(\mu)(x,y) \;=\; A_1(\mu)\,x + A_2(\mu)\,y + R(x,y,\mu)\,,$$
$$b(\mu)(x,y) \;=\; n_0(\mu) + n_1(\mu)(A_1(\mu)\,x + A_2(\mu)\,y) + S(x,y,\mu)\,,$$

with $A_1(0) = n_0(0) = 0$ and $n_1(0)\,A_2(0) = 1$.

Let $L_{(\alpha,\beta)} : \mathbb{R}^3 \to \mathbb{R}^3$, with parameter $(\alpha,\beta) \in \mathbb{R}^2$, be the family of linear isomorphisms such that the inverse of $L = L_{(\alpha,\beta)}$ is given by

$$L^{-1}(x,y,\mu) = ((1+\alpha)x - \beta y, \beta x + (1+\alpha)y, \mu)\,.$$

Observe that for all $(\alpha,\beta) \in \mathbb{R}^2$, the map $L_{(\alpha,\beta)}$ is a linear rotation at the first two coordinates. Putting $\beta = -A_1(1+\alpha)/A_2$, in the chart

$$(x,y,\mu) = ((1+\alpha)u - \beta v, \beta u + (1+\alpha)v, \mu)$$

the local expression of $\omega(\mu)$ is given by

$$(u,v)^*(\omega(\mu)) \;=\; 4[m_0(\mu) + B(\mu)v + N(u,v,\mu)](du^2 - dv^2)dudv$$
$$+\,[n_0(\mu) + D(\mu)v + M(u,v,\mu)](du^4 - 6du^2dv^2 + dv^4)\,,$$

where

$$m_0(\mu) \;=\; -4n_0(\mu)\,A_1(\mu)\,(A_1(\mu)^2 - A_2(\mu)^2)\,(1+\alpha(\mu))^4/(A_2(\mu))^3\,,$$
$$B(\mu) \;=\; P(\mu)\,(1+\alpha(\mu))^5\,,$$
$$n_0(\mu) \;=\; n_0(\mu)\,[A_1(\mu)^2 - A_2(\mu)^2]^2\,(1+\alpha(\mu))^4/(A_2(\mu))^4$$
$$D(\mu) \;=\; Q(\mu)\,(1+\alpha(\mu))^5\,,$$

with $P(0) = A_2(0) \neq 0$ and $Q(0) = n_1(0)\, A_2(0) = 1$.

Therefore, there exist α defined at a small neighborhood of the origin of $\mathbb{R}^k$ such that

$$\begin{aligned}(u,v)^*(\omega(\mu)) \;&=\; 4[m_0(\mu) + B(\mu)v + N(u,v,\mu)](du^2 - dv^2)dudv \\ &+ [n_0(\mu) + v + M(u,v,\mu)](du^4 - 6du^2dv^2 + dv^4)\,.\end{aligned}$$

Finally, with a second change of coordinates of the form

$$(u,v,\mu) \;=\; (s, v_0(\mu) + t, \mu)\,,$$

with $v_0(\mu)$ a suitable map, we obtain the desire result. $\square$

Proposition 2.5.2. *Let $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, be an arbitrary smooth family in $\mathcal{Q}(\mathbb{R}^2)$, such that $\omega(0)$ has a $H_{45}-$singular point at the origin. Then there exist $r > 0$, a neighborhood U of the origin in $\mathbb{R}^2$ and a smooth map $f : B(0,r) \subset \mathbb{R}^k \to \mathbb{R}$, such that for every $\mu \in B(0,r)$, the following three properties are satisfied.*

a) $f(\mu) = 0$ if and only if $\omega(\mu)$ has a unique singular point in U, which is a $H_{45}-$singular point.

b) $f(\mu) > 0$ if and only if $\omega(\mu)$ has only two singular points in U, one is a H_4-singular point and the other a H_5-singular point.

c) $f(\mu) < 0$ if and only if $\omega(\mu)$ has no singular points in U.

Proof. Without loss of generality we may suppose

$$(x,y)^*(\omega(\mu)) \;=\; 4\,a(\mu)\,(dx^2 - dy^2)\,dxdy \;+\; b(\mu)\,(dx^4 - 6\,dx^2dy^2 + dy^4)\,,$$

where

$$\begin{aligned} a(\mu)(x,y) \;&=\; B(\mu)\,y \;+\; R(\mu)(x,y)\,, \\ b(\mu)(x,y) \;&=\; n_0(\mu) \;+\; y \;+\; S(\mu)(x,y)\,, \\ j_2(R(\mu), S(\mu))(0,0) \;&=\; (a_{20}(\mu), b_{20}(\mu))\,x^2 \;+\; (a_{11}(\mu), b_{11}(\mu))\,xy \\ &+\; (a_{02}(\mu), b_{02}(\mu))\,y^2 \end{aligned}$$

with $n_0(0)=0$, $B(\mu)>0$ and $a_{20}(\mu)-B(\mu)b_{20}(\mu)\neq 0$.

Without loss of generality, assume that $a_{20}(\mu)-B(\mu)b_{20}(\mu)>0$.

Consider a neighborhood $U_0 \subset (x,y)(U)$ of the origin and a square $R_\delta = I_\delta \times I_\delta$ where $I_\delta = [-\delta,\delta]$, with $\delta>0$ such that $R_\delta \subset U_0$. Taking δ sufficiently small if necessary, we assume the following three properties for $\|\mu\|$ small:

1) $\Delta(\mu)(x,y)\neq 0$ in R_δ, where $\Delta(\mu)(x,y)$ is the discriminant of the homogeneous degree five polynomial $4\,Da(\mu)_{(x,y)}(u,v)(u^2-v^2)uv+Db(\mu)_{(x,y)}(u,v)(u^4-6u^2v^2+v^4)$.

2) There exists a smooth map $h(\mu): I_\delta \to \mathbb{R}$ such that $a(\mu)(x,y) = B(\mu)(y-h(\mu)(x))M_1(x,y,\mu))$, with $M_1(0,0,\bar{0})=1$.

3) The curves $y=h(\mu)(x)$ and $H(\mu)(x,y)=0$, where $H(\mu)(x,y)$ is the determinant of the Jacobian matrix of the map $(a(\mu),b(\mu))$ at (x,y), have the point $(x,y)(p(\mu))=(0,0)$ as the unique common point in R_δ; furthermore, the intersection is transversal.

We next show that the map f given by $f(\mu)=b_\mu(0,0)$ satisfies our Proposition. In effect, the map f is smooth. For μ fixed, we set $m(x)=b_\mu(x,h(\mu)(x))$. Then

$$m'(x) = (\frac{\partial a(\mu)}{\partial y}(x,h(\mu)(x)))^{-1}H(\mu)(x,h(\mu)(x))\,.$$

Since $a_{20}(\mu)-B(\mu)b_{20}(\mu)$ is positive, the map $H_2(\mu)(x,h(\mu)(x))$ is positive (resp. negative) for $-\delta<x<0$ (resp. $0<x<\delta$). This implies that $m(x)$ decreases strictly in $]-\delta,0)[$ and increases strictly in $]0,\delta[$. Assertions a), b) and c) now follow from Proposition 2.3.4. $\square$

Lemma 2.5.6. *Consider the family of quartics*

$$\omega(\mu) \;=\; 4\,a(\mu)(du^2-dv^2)dudv \;+\; b(\mu)\,(du^4-6du^2dv^2+dv^4)\,,$$

where

$$\begin{aligned} a(\mu)(u,v) &= B(\mu)\,v \;+\; a_{20}(\mu)\,u^2 + a_{11}(\mu)\,u\,v + a_{02}(\mu)\,v^2 + R_2(\mu)(u,v) \\ b(\mu)(u,v) &= n_0(\mu) \;+\; v \;+\; b_{20}(\mu)\,u^2 + b_{11}(\mu)\,u\,v + b_{02}(\mu)\,v^2 + S_2(\mu)(u,v)\,, \end{aligned}$$

with $n_0(0) = 0\,,\ B(0) = B > 0,\ a_{20}(0) - B\, b_{20}(0) \neq 0$ *and* $j_2(R_2, S_2)(\mu)(0,0) = (0,0)$. *Then, for small* $\mid \mu \mid$, *the family* $(\omega(\mu))$ *is equivalent to the family*

$$\begin{aligned} \tilde{\omega}(\mu) \;\; &= \;\; 4\,[B(\mu)v \;+\; a_{20}(\mu)\,u^2]\,(du^2 \;-\; dv^2)\,du\,dv \\ &\qquad + [n_0(\mu) \;+\; v \;+\; b_{20}(\mu)\,u^2]\,(du^4 \;-\; 6\,du^2\,dv^2 \;+\; dv^4)\,. \end{aligned}$$

Proof. The map $f(\mu)$ of Proposition 2.5.2 associated to this family is $f(\mu) = n_0(\mu)\,$. Therefore there exists a neighborhood U of $(0,0)$ and $\delta > 0$ such that for all $\|\mu\| < \delta$ we have

a) $n_0(\mu) = 0$ imply that $\omega(\mu)$ and $\tilde{\omega}(\mu)$ has a unique singular point in U, which is a $H_{45}-$singular point.

b) $n_0(\mu) > 0$ imply that $\omega(\mu)$ and $\tilde{\omega}(\mu)$ has only two singular points in U, one is a H_4-singular point and the other a H_5-singular point.

c) $n_0(\mu) < 0$ imply that $\omega(\mu)$ and $\tilde{\omega}(\mu)$ has no singular points in U.

Furthermore, the local configuration of $\mathcal{N}_1(\omega)$ and $\mathcal{N}_1(\tilde{\omega})$ (resp. $\mathcal{N}_2(\omega)$ and $\mathcal{N}_2(\tilde{\omega})$) at the origin is homeomorphic to the ones shown in Figure 12 (resp. Figure 13).

$n_0(\mu) > 0$ $\qquad$ $n_0(\mu) = 0$ $\qquad$ $n_0(\mu) < 0$

Figure 12

$n_0(\mu) > 0$ $\qquad$ $n_0(\mu) = 0$ $\qquad$ $n_0(\mu) < 0$

Figure 13

From these considerations the proof follows. □

Theorem 2.5.5. *A versal unfolding of an $H_{45}-$singular point is the family of quartic $\upsilon(\lambda)$, with $\lambda \in \mathbb{R}$, given by*

$$\upsilon(\lambda) \;=\; 4\,v\,[du^2 - dv^2]\,du\,dv \;+\; (\lambda + v - u^2)\,[du^4 - 6du^2dv^2 + dv^4]\,.$$

Proof. The proof follows from Lemma 2.5.6 because any pair of family of quartics $(\omega(\mu))$, with parameter $\mu \in \mathbb{R}^k$, of the form

$$\begin{aligned}\omega(\mu) \;&=\; 4\,[B(\mu)v \;+\; a_{20}(\mu)\,u^2]\,(du^2 \;-\; dv^2)\,du\,dv \\ &\quad + [n_0(\mu) \;+\; v \;+\; b_{20}(\mu)\,u^2]\,(du^4 \;-\; 6\,du^2\,dv^2 \;+\; dv^4)\end{aligned}$$

with $n_0(0) = 0$, $B(\mu) > 0$, $a_{20}(\mu) - B(\mu)\,b_{20}(\mu) > 0$, are equivalent. □

2.6 Appendix

This section is devoted to prove the existence of main charts for families in $\mathcal{Q}(\mathbb{R}^2)$ (see Proposition 2.5.1.) The proof is inspired on the one presented by M. Spivak in [C2–21, Addendum 1] for the existence of smooth isothermal coordinates. The strategy is to prove the same sequence of results for families with parameters in $\mathbb{R}^k$.

As in the case with no parameters, given a local chart $(u, v) : (U, p_0) \to (\mathbb{R}^2, (0,0))$ and a family of smooth maps $E(\lambda), F(\lambda), G(\lambda)$ defined at a neighborhood $V \subset (u,v)(U)$ of the origin, with parameter λ in a neighborhood of the origin of $\mathbb{R}^k$, that verifies $E(\lambda)G(\lambda) - F(\lambda)^2$ positive in V, we must find a coordinate change

$$(u, v, \lambda) \;=\; (f(x,y,\lambda), g(x,y,\lambda), \lambda)$$

so that, in a neighborhood of the origin of $\mathbb{R}^2 \times \mathbb{R}^k$, we have

$$E(\lambda)f_xf_y + F(\lambda)(f_xg_y + f_yg_x) + G(\lambda)g_xg_y \;=\; 0 \quad \text{and}$$
$$E(\lambda)(f_x)^2 + 2F(\lambda)f_xg_x + G(\lambda)(g_x)^2 \;=\; E(\lambda)(f_y)^2 + 2F(\lambda)f_yg_y + G(\lambda)(g_y)^2\,.$$

Here, we introduce the notation of formal complex derivatives, identifying (x, y) with $z = x + \imath\, y$ and considering

$$\omega_z \;=\; \frac{1}{2}\,(\omega_x - \imath\,\omega_y) \quad \text{and} \quad \omega_{\bar z} \;=\; \frac{1}{2}\,(\omega_x + \imath\,\omega_y)\,.$$

As in the case with no parameters, to find a solution $(f(x, y, \lambda), g(x, y, \lambda))$ of equations above is equivalent to find a solution $\omega(z, \lambda)$ of the complex equation

$$\omega_{\bar z}(z, \lambda) \;=\; \mu(z, \lambda)\,\omega_z(z, \lambda)\,, \tag{2.6.19}$$

with

$$\mu(z, \lambda) \;=\; \frac{G(z, \lambda) - E(z, \lambda) - 2\imath F(z, \lambda)}{G(z, \lambda) + E(z, \lambda) + 2\sqrt{E(z, \lambda)G(z, \lambda) - F(z, \lambda)^2}}\,.$$

Therefore $|\mu(z, \lambda)| < 1$ and the equation (2.6.19) have the same class of differentiability that the maps $E(\lambda), F(\lambda), G(\lambda)$.

Also, instead of solving the equation (2.6.19), we will instead solve the more general equation

$$\omega_{\bar z}(z, \lambda) \;=\; \mu(z, \lambda)\,\omega_z(z, \lambda) \;+\; \gamma(z, \lambda)\,\omega(z, \lambda) \;+\; \delta(z, \lambda)\,, \tag{2.6.20}$$

where μ, γ, δ are C^α at z and $|\mu(0, \bar 0)| < 1$.

To be precise in the formulation of the results we introduce some definitions.

Definition 2.6.1. *Given $(z_0, \lambda_0) \in \mathcal{C} \times \mathbb{R}^k$, $R > 0$, $0 < \alpha < 1$ and an integer $n \geq 1$, we denote by $D(z_0, R)$ (resp. $B(\lambda_0, R)$) the open ball in $\mathcal{C}$ (resp. in $\mathbb{R}^k$) with center at z_0 (resp. λ_0) and radius R. Also we define $H_{(z_0,\lambda_0)}(\alpha, R)$ as the set consisting of the maps $f : D(z_0, R) \times B(\lambda_0, R) \to \mathcal{C}$ such that*

1) There exists $K > 0$ such that $|f(z_1, \lambda) - f(z_2, \lambda)| \leq K\, |z_1 - z_2|^\alpha$, for all $z_1, z_2 \in D(z_0, R)$ and for all $\lambda \in B(\lambda_0, R)$.

2) For every $z \in D(z_0, R)$, the map $\lambda \rightsquigarrow f(z, \lambda)$ is smooth in $B(\lambda_0, R)$.

Finally, we recursively define $H_{(z_0,\lambda_0)}(n + \alpha, R)$ as the set consisting of the maps $f : D(z_0, R) \times B(\lambda_0, R) \to \mathcal{C}$ that verify condition 2) above and such that the derivatives f_z and $f_{\bar z}$ exist and belong to $H_{(z_0,\lambda_0)}(n - 1 + \alpha, R)$.

For simplicity we denote $\tilde{0} = (0, \bar{0})$, where 0 is the origin of $\mathcal{C}$ and $\bar{0}$ is the origin of $\mathbb{R}^k$. Also we put $D(R) = D(0, R)$, $B(R) = B(\bar{0}, R)$ and $H(n+\alpha, R) = H_{\tilde{0}}(n+\alpha, R)$. We start establishing a proposition similar to [C2–21, Addendum 1, Proposition 24].

Proposition 2.6.1. *Let* $f \in H(\alpha, R)$ *such that*

i) $|f(z, \lambda)| \leq M$ *for all* $(z, \lambda) \in D(R) \times B(R)$.

ii) $|f(z_1, \lambda) - f(z_2, \lambda)| \leq K\,|z_1 - z_2|^\alpha$ *for all* $z_1, z_2 \in D(R)$ *and* $\lambda \in B(R)$.

Define

$$F(z_0, \lambda_0) \;=\; -\frac{1}{\pi} \int_{D(R)} \frac{f(z, \lambda_0)}{z - z_0}\, dxdy\ ,$$

for $(z_0, \lambda_0) \in D(R) \times B(R)$. *Then* $F \in H(1+\alpha, R)$ *and*

a) $F_{\bar{z}}(z_0, \lambda_0) \;=\; f(z_0, \lambda_0)$.

b) $F_z(z_0, \lambda_0) \;=\; -\frac{1}{\pi} \int_{D(R)} \frac{f(z,\lambda_0) - f(z_0,\lambda_0)}{(z-z_0)^2}\, dxdy$.

c) $|F(z_0, \lambda_0)| \leq 4\,R\,M$ *, for all* $(z_0, \lambda_0) \in D(R) \times B(R)$.

d) $|F_z(z_0, \lambda_0)| \leq \frac{2^{\alpha+1}}{\alpha}\, R^\alpha\, K$*, for all* $(z_0, \lambda_0) \in D(R) \times B(R)$.

e) $|F_z(z_1, \lambda_0) - F_z(z_2, \lambda_0)| \leq C\,K\,|z_1 - z_2|^\alpha$ *for all* $z_1, z_2 \in D(R)$ *and* $\lambda_0 \in B(R)$, *where* C *is a constant that does not depend on* R*, or on the function* f.

Proof. Similar to the one's in the case with no parameters. □

The next result show that there is no loss of generality in assuming that $\mu(\tilde{0}) = 0$ in equation (2.6.20).

Lemma 2.6.1 (Lemmachen). *Suppose that given maps* μ, γ, δ *in* $H(\alpha, R)$ *with* $\mu(\tilde{0}) = 0$*, and arbitrary complex numbers* a *and* b*, there exists* $0 < \tilde{R} \leq R$ *such that the equation (2.6.20) has a solution* $\omega \in H(1+\alpha, \tilde{R})$ *that verify* $\omega(\tilde{0}) = a$ *and* $\omega_z(\tilde{0}) = b$*. Then, given* μ, γ, δ *in* $H(\alpha, R)$ *with* $\left|\mu(\tilde{0})\right| < 1$*, and arbitrary complex numbers* a *and* b*, there exists* $0 < \tilde{R} \leq R$ *such that the equation (2.6.20) has a solution* $\omega \in H(1+\alpha, \tilde{R})$ *that verify* $\omega(\tilde{0}) = a$ *and* $\omega_z(\tilde{0}) = b$.

Proof. Let μ, γ, δ be maps in $H(\alpha, R)$ with $\left|\mu(\tilde{0})\right| < 1$ and let $a, b \in \mathcal{C}$. Associated to these maps we set

$$\begin{aligned}
\rho(z,\lambda) &= \frac{\mu(z-\mu(\tilde{0})\bar{z},\lambda)-\mu(\tilde{0})}{1-\mu(z-\mu(\tilde{0})\bar{z},\lambda)\,\overline{\mu(\tilde{0})}}\,,\\
\sigma(z,\lambda) &= \frac{\gamma(z-\mu(\tilde{0})\bar{z},\lambda)(1-\mu(\tilde{0})\,\overline{\mu(\tilde{0})}}{1-\mu(z-\mu(\tilde{0})\bar{z},\lambda)}\,,\\
\tau(z,\lambda) &= \frac{\delta(z-\mu(\tilde{0})\bar{z},\lambda)(1-\mu(\tilde{0})\,\overline{\mu(\tilde{0})}}{1-\mu(z-\mu(\tilde{0})\bar{z},\lambda)}\,.
\end{aligned}$$

Therefore $\rho, \sigma, \tau \in H(\alpha, R_1)$ with $R_1 = \frac{R}{1+\left|\mu(\tilde{0})\right|}$ and $\left|\rho(\tilde{0})\right| = 0$. Let $\tilde{w}(z,\lambda) \in H(1+\alpha, \tilde{R}_1)$ such that

$$\tilde{\omega}_{\bar{z}}(z,\lambda) \;=\; \rho(z,\lambda)\,\tilde{\omega}_z(z,\lambda) \;+\; \sigma(z,\lambda)\,\tilde{\omega}(z,\lambda) \;+\; \tau(z,\lambda)\,,$$

that verify $\tilde{\omega}(\tilde{0}) = a$ and $\tilde{\omega}_z(\tilde{0}) = (1-\mu(\tilde{0})\overline{\mu(\tilde{0})})\,b \;-\; \overline{\mu(\tilde{0})}\,(\sigma(\tilde{0})\,a + \tau(\tilde{0}))$.

Then, straightforward calculations show that

$$\omega(z,\lambda) \;=\; \tilde{\omega}\left(\frac{z+\mu(\tilde{0})\bar{z}}{1-\mu(\tilde{0})\overline{\mu(\tilde{0})}}\,,\lambda\right)$$

is a solution of equation (2.6.20) that belong to $H(1+\alpha, \tilde{R})$, with $\tilde{R} = \tilde{R}_1\,(1-\left|\mu(\tilde{0})\right|)$, and that verifies $\omega(\tilde{0}) = a$ and $\omega_z(\tilde{0}) = b$. $\square$

To find an integral equation equivalent to

$$\omega_{\bar{z}}(z,\lambda) \;=\; \mu(z,\lambda)\,\omega_z(z,\lambda) \;+\; \gamma(z,\lambda)\,\omega(z,\lambda) \;+\; \delta(z,\lambda)\,, \quad \mu(\tilde{0}) = 0\,, \tag{2.6.21}$$

we put

$$F(z_0,\lambda) \;=\; -\frac{1}{\pi}\int_{D(R)} \frac{\mu(z,\lambda)\omega_z(z,\lambda)+\gamma(z,\lambda)\omega(z,\lambda)+\delta(z,\lambda)}{z-z_0}\,dxdy\,\cdot$$

Proposition 2.6.1 gives

$$F_{\bar{z}} \;=\; \mu\,\omega_z + \gamma\,\omega + \delta \;=\; \omega_{\bar{z}}$$

if ω satisfies (2.6.21), and hence $(\omega - F)_{\bar{z}} = 0$, so that

$$\begin{aligned}\omega(z_0,\lambda) \quad = \quad & -\frac{1}{\pi}\int_{D(R)} \frac{\mu(z,\lambda)\omega_z(z,\lambda)}{z-z_0}\,dxdy \;- \\ & \frac{1}{\pi}\int_{D(R)} \frac{\gamma(z,\lambda)\omega(z,\lambda)}{z-z_0}\,dxdy \;- \\ & \frac{1}{\pi}\int_{D(R)} \frac{\delta(z,\lambda)}{z-z_0}\,dxdy \;+\; g(z_0,\lambda)\,,\end{aligned}$$

for some function g which is complex analytic in z_0.

By the same arguments used in the case with no parameters, we can see that it suffices to show that we can solve the following equation for functions $\mu, \gamma \in H(\alpha, R)$, with $\mu(\tilde{0}) = 0$, and any function $h \in H(1+\alpha, R)$:

$$\begin{aligned}\omega(z_0,\lambda) = & -\frac{1}{\pi}\int_{D(R)} \frac{\mu(z,\lambda)\omega_z(z,\lambda)}{z-z_0}\,dxdy - \frac{1}{\pi}\int_{D(R)} \frac{\gamma(z,\lambda)\omega(z,\lambda)}{z-z_0}\,dxdy \\ & +\frac{1}{\pi}\int_{D(R)} \frac{\mu(z,\lambda)\omega_z(z,\lambda)}{z}\,dxdy + \frac{1}{\pi}\int_{D(R)} \frac{\gamma(z,\lambda)\omega(z,\lambda)}{z}\,dxdy \\ & +z_0\left\{\frac{1}{\pi}\int_{D(R)} \frac{\mu(z,\lambda)\omega_z(z,\lambda)}{z^2}\,dxdy + \frac{1}{\pi}\int_{D(R)} \frac{\gamma(z,\lambda)\omega(z,\lambda)}{z^2}\,dxdy\right\} \\ & + h(z_0,\lambda) \end{aligned} \tag{2.6.22}$$

The integral equation (2.6.22) will be solved using the Contraction Lemma. On $H(\alpha, R)$ we consider the metric defined by the norm

$$\|\omega\|_R \;=\; \sup_{(z,\lambda)\in D(R)\times B(R)} |\omega(z,\lambda)| \;+\; R^{\alpha} \sup_{z_1,z_2\in D(R), z_1\neq z_2, \lambda\in B(R)} \frac{|\omega(z_1,\lambda)-\omega(z_2,\lambda)|}{|z_1-z_2|^{\alpha}}\;.$$

It is easy to see that $H(\alpha, R)$ is complete in this metric and that

$$\|\omega_1\omega_2\|_R \le \|\omega_1\|_R \cdot \|\omega_2\|_R\;.$$

On $H(1+\alpha, R)$ we consider the metric defined by the norm

$$\|\omega\|_R \;=\; \sup_{(z,\lambda)\in D(R)\times B(R)} |\omega(z,\lambda)| \;+\; R\cdot\|\omega_z\|_R \;+\; R\cdot\|\omega_{\bar{z}}\|_R\;.$$

Also, it easy to see that $H(1+\alpha, R)$ is complete in this metric and that there is an inequality of the form

$$\|\omega\|_R \le \text{constant} \cdot \|\omega\|_R\;, \tag{2.6.23}$$

for $\omega \in H(1+\alpha, R)$.

Proposition 2.6.2. *Let* $\mu, \gamma \in H(\alpha, R_0)$ *with* $\mu(\tilde{0}) = 0$, *and let* $h \in H(1+\alpha, R_0)$. *Then for sufficiently small* $R > 0$ *there is* $\omega \in H(1+\alpha, R)$ *satisfying the integral equation (2.6.22) for all* $(z_0, \lambda) \in D(R) \times B(R)$.

Proof. Since the proof is similar to the case with no parameter (see [C2–21, Addendum 1, Proposition 26]) we reproduce here only the more significant part. We suppose that $\mu, \gamma \in H(\alpha, R_0)$ for some $R_0 \leq 1$, and we will henceforth consider only $R \leq R_0$. For $\omega \in H(1+\alpha, R)$, define the function $S\omega$ on $D(R) \times B(R)$ by setting $(S\omega)(z_0, \lambda)$ equal to the right side of (2.6.22) without the $h(z_0, \lambda)$.

Claim: There is a constant C', depending only on α, and not on R, such that

$$\|S\omega\|_R \leq C' \cdot R^\alpha \cdot \|\omega\|_R \ , \tag{2.6.24}$$

for all $\omega \in H(1+\alpha, R)$.

Assuming this claim, the remainder of the proof goes as follows. Since $R^\alpha \to 0$ as $R \to 0$, there is R_* such that for all $R \leq R_*$ we have

$$\|S\omega\|_R \leq C'' \cdot \|\omega\|_R \ ,$$

where C'' is a constant with

$$C'' < \min\left\{1, \frac{\|h\|_R}{3}\right\} \ .$$

Define $T : H(1+\alpha, R) \to H(1+\alpha, R)$ by

$$T\omega \ = \ S\omega \ + \ h \, .$$

If $R \leq R_*$, then for all ω with

$$\|\omega\|_R \leq \frac{3}{2} \ \|h\|_R$$

we have

$$\begin{aligned} \|T\|_R \ = \ \|S\omega \ + \ h\|_R \ &\leq \ \|S\omega\|_R \ + \ \|h\|_R \\ &\leq \ \frac{\|h\|_R}{3} \cdot \|\omega\|_R \ + \ \|h\|_R \\ &\leq \ \frac{1}{2} \ \|h\|_R \ + \ \|h\|_R \\ &= \ \frac{3}{2} \ \|h\|_R \ . \end{aligned}$$

Thus, for $R \leq R_*$, the map T takes the complete metric space

$$M = \{\omega \in H(1+\alpha, R) : \|\omega\|_R \leq \frac{3}{2} \|h\|_R\}$$

into itself. Moreover, the map $T : M \to M$ is a contraction, for

$$\begin{aligned} \|T\omega_1 - T\omega_2\|_R &= \|S\omega_1 - S\omega_2\|_R \\ &= \|S(\omega_1 - \omega_2)\|_R \leq C'' \|\omega_1 - \omega_2\|_R . \end{aligned}$$

By the contraction Lemma, there is some $\omega \in M$ with

$$\omega = T\omega = S\omega + h ,$$

which is precisely the equation we want.

The proof of the Claim is omitted. □

Corollary 2.6.1. *Let $\mu, \gamma, \delta \in H(\alpha, R_0)$ with $|\mu(\tilde{0})| < 1$, and let $a, b \in \mathcal{C}$ be arbitrary complex numbers. Then, there are $0 < \tilde{R} \leq R$ and $\omega \in H(1+\alpha, R)$ such that*

$$\omega_{\bar{z}} = \mu\,\omega_z + \gamma\,\omega + \delta , \quad \omega(\tilde{0}) = a , \quad \omega_z(\tilde{0}) = b . \tag{2.6.25}$$

Proof. Consequence of Proposition 2.6.2 and Lemma 2.6.1. □

Now we want to prove that if μ, γ, δ in Corollary 2.6.1 belong to $H(n+\alpha, R)$, then there is a solution of (2.6.25) which is in $H(n+1+\alpha, \tilde{R})$, for some $0 < \tilde{R} \leq R$.

Lemma 2.6.2. *If $f \in H(n+\alpha, R)$ $(n \geq 1)$ and we define for $(z_0, \lambda) \in D(R) \times B(R)$*

$$F(z_0, \lambda) = -\frac{1}{\pi} \int_{D(R)} \frac{f(z,\lambda)}{z - z_0}\, dxdy ,$$

then $F \in H(n+1+\alpha, R)$.

Proof. The proof is similar for the case with no parameters (see [C2–21, Addendum 1, Lemma 28]) because clearly F is smooth with respect to the parameter λ. □

Proposition 2.6.3. *Let* $\mu, \gamma, \delta \in H(n+\alpha, R)$ *with* $\left|\mu(\tilde{0})\right| < 1$*, and let* $a, b \in \mathcal{C}$ *be arbitrary complex numbers. Then, there are* $0 < \tilde{R} \leq R$ *and* $\omega \in H(n+1+\alpha, \tilde{R})$ *such that*

$$\omega_{\bar{z}} \;=\; \mu\,\omega_z + \gamma\,\omega + \delta\,, \quad \omega(\tilde{0}) = a\,, \quad \omega_z(\tilde{0}) = b\,.$$

Proof. Induction on n. The case $n = 0$ is Corollary 2.6.2. Now suppose the result is true for n, and let $\mu, \gamma, \delta \in H(n+1+\alpha, R)$.

Case 1. $\gamma = 0$. Let $f \in H(n+1+\alpha, R_0)$ satisfying

$$f_{\bar{z}} \;=\; \mu\, f_z \;+\; \mu_z\, f \;+\; \delta_z\,, \quad f(\tilde{0}) \;=\; b\,, \quad f_z(\tilde{0}) \;=\; 0\,. \tag{2.6.26}$$

Define W by

$$\overline{W}(z_0, \lambda) \;=\; -\frac{1}{\pi}\int_{D(R_0)} \frac{\bar{f}(z,\lambda)}{z - z_0}\, dxdy\,.$$

Then $W \in H(n+2+\alpha, R_0)$ by Proposition 2.6.2 and by Proposition 2.6.1 we have

$$\bar{f}(z_0,\lambda) \;=\; \overline{W}_{\bar{z}}(z_0,\lambda) \;=\; \overline{W_z(z_0,\lambda)} \quad \Rightarrow \quad f(z_0,\lambda) \;=\; W_z(z_0,\lambda)\,.$$

So

$$\begin{aligned} (W_{\bar{z}})_z \;&=\; W_{z\bar{z}} \;=\; f_{\bar{z}} \;=\; \mu\, f_z \;+\; \mu_z\, f \;+\; \delta_z \qquad \text{by (2.6.26)} \\ &=\; (\mu\, f)_z \;+\; \delta_z \;=\; (\mu\, W_z)_z \;+\; \delta_z\,. \end{aligned}$$

Hence $(W_{\bar{z}} \;-\; \mu\, W_z \;-\; \delta)_z \;=\; 0$. This means that we can write

$$W_{\bar{z}}(z,\lambda) \;-\; \mu(z,\lambda)\, W_z(z,\lambda) \;-\; \delta(z,\lambda) \;=\; g(\bar{z},\lambda)\,, \tag{2.6.27}$$

where g is complex analytic in z and smooth in λ. Let G be a function which is complex analytic in z and smooth in λ with $G(\tilde{0}) \;=\; W((\tilde{0}) \;-\; a$ and such that $G_{\bar{z}}(\bar{z},\lambda) \;=\; g(\bar{z},\lambda)$, and let

$$\omega(z,\lambda) \;=\; W(z,\lambda) \;-\; G(\bar{z},\lambda)\,.$$

Then

$$\begin{aligned}
\omega_z &= W_z - 0 \\
\omega_{\bar z}(z,\lambda) &= W_{\bar z}(z,\lambda) - g(\bar z,\lambda) = \mu(z,\lambda)\,W_z(z,\lambda) + \delta(z,\lambda)\text{by (2.6.27)} \\
&= \mu(z,\lambda)\,\omega_z(z,\lambda) + \delta(z,\lambda)\,.
\end{aligned}$$

Thus ω is a solution of our equation which is in $H(n+2+\alpha,R_0)$. We also have

$$\begin{aligned}
\omega(\tilde 0) &= W(\tilde 0) - G(\tilde 0) = a \\
\omega_z(\tilde 0) &= W_z(\tilde 0) = f(\tilde 0) = b\,.
\end{aligned}$$

Case 2. **General case**. Let $\beta,\sigma \in H(n+2+\alpha,R_0)$ satisfying

$$\begin{aligned}
\beta_{\bar z} &= \mu\,\beta_z + \gamma\,; \qquad & \beta(\tilde 0) = 0\,, \quad \beta_z(\tilde 0) = 0 \\
\sigma_{\bar z} &= \mu\,\sigma_z + e^{-\beta}\,\delta\,; \qquad & \sigma(\tilde 0) = a\,, \quad \sigma_z(\tilde 0) = b\,.
\end{aligned}$$

Then $\omega = e^{\beta}\,\sigma \in H(n+2+\alpha,R_0)$ satisfies

$$\omega_{\bar z} = \mu\,\omega_z + \gamma\,\omega + \delta\,, \quad \omega(\tilde 0) = a\,, \quad \omega_z(\tilde 0) = b\,.$$

□

Proposition 2.6.4. *Let $\omega \in H(n+\alpha,R)$ be a solution of $\omega_{\bar z} = \mu\,\omega_z$. Consider the map $W(z,\lambda) = (w(z,\lambda),\lambda)$ and a C^1-complex valued map f defined at a neighborhood of the set $W(D(R)\times B(R))$. Then*

- *a) If f is analytic en z , then $\sigma = f\circ W$ is also a solution.*
- *b) Suppose that $\sigma = f\circ W$ is a solution. If $\omega_z \neq 0$ and $|\mu|<1$ on $D(R)\times B(R)$, then f is analytic in z .*

Proof. a) Since f is analytic en z , we have $f_{\bar z}=0$ and

$$\begin{aligned}
\sigma_z &= (f\circ W)_z = (f_z\circ W)\,W_z = (f_z\circ W)\,\omega_z\,, \\
\sigma_{\bar z} &= (f\circ W)_{\bar z} = (f_z\circ W)\,W_{\bar z} = (f_z\circ W)\,\omega_{\bar z}\,.
\end{aligned}$$

Hence

$$\sigma_{\bar z} \;=\; (f_z \circ W)\,\omega_{\bar z} \;=\; \mu\,(f_z \circ W)\,\omega_z \;=\; \mu\,\sigma_z\,.$$

b) We have

$$\begin{aligned}\sigma_z &= (f_z \circ W)\,\omega_z \;+\; (f_{\bar z} \circ W)\,(\bar\omega)_z\,,\\ \sigma_{\bar z} &= (f_z \circ W)\,\omega_{\bar z} \;+\; (f_{\bar z} \circ W)\,(\bar\omega)_{\bar z}\,.\end{aligned}$$

Since σ is a solution we have

$$(f_z \circ W)\,\omega_{\bar z} \;+\; (f_{\bar z} \circ W)\,(\bar\omega)_{\bar z} \;=\; \mu\,[(f_z \circ W)\,\omega_z \;+\; (f_{\bar z} \circ W)\,(\bar\omega)_z]\,.$$

Since ω is a solution, this leads to

$$(f_{\bar z} \circ W)\,[(\bar\omega)_{\bar z} \;-\; \mu\,(\bar\omega)_z] \;=\; 0\,. \tag{2.6.28}$$

Since $\omega_{\bar z} = \mu\,\omega_z$ implies that

$$(\bar\omega)_z \;=\; \overline{(\omega_{\bar z})} \;=\; \bar\mu\,\overline{(\omega_z)} \;=\; \bar\mu\,(\bar\omega)_{\bar z}\,,$$

we see that

$$(\bar\omega)_{\bar z} \;-\; \mu\,(\bar\omega)_z \;=\; (\bar\omega)_{\bar z} \;-\; \mu\,\bar\mu\,(\bar\omega)_{\bar z} \;=\; (\bar\omega)_{\bar z}\,(1 - |\mu|^2) \;=\; \overline{(\omega_z)}\,(1 - |\mu|^2).$$

Then, it follows from (2.6.28) that $f_{\bar z} = 0$, i.e. f is analytic in z. □

Proposition 2.6.5. *Let $\mu \in H(n+\alpha, R)$ with $|\mu| < 1$. Let ω be a solution of*

$$\omega_{\bar z} \;=\; \mu\,\omega_z\;, \tag{2.6.29}$$

defined in $D(R_0) \times B(R_0)$. If ω is smooth in λ then $\omega \in H(n+1+\alpha, R_0)$. So, if μ is smooth, any solution ω which is smooth in λ of (2.6.29) is also smooth.

Proof. Let ω be a solution smooth in λ of (2.6.29) defined in $D(R_0) \times B(R_0)$ and let (z_0, λ_0) be a point in $D(R_0) \times B(R_0)$. We must prove that there exists $r_0 > 0$ such that $\omega \in H(z_0, \lambda_0)(n+1+\alpha, r_0)$.

Let $r > 0$ such that $D(z_0, r) \times B(\lambda_0, r) \subset D(R_0) \times B(R_0)$. For $(z, \lambda) \in D(r) \times B(r)$ defines $\tilde{\mu}(z, \lambda) = \mu(z + z_0, \lambda + \lambda_0)$. Then $\tilde{\mu} \in H(n + \alpha, r)$, and taking $r > 0$ sufficiently small, we can suppose that there exists a solution $\tilde{\sigma} \in H(n + 1 + \alpha, r)$ of the equation $\tilde{\sigma}_{\bar{z}} = \tilde{\mu}\, \tilde{\sigma}_z$ with $\tilde{\sigma}_z \neq 0$. Then σ defined by $\sigma(z, \lambda) = \tilde{\sigma}(z - z_0, \lambda - \lambda_0)$ is in $H_{(z_0, \lambda_0)}(n + 1 + \alpha, r)$ and verifies $\sigma_{\bar{z}} = \mu\, \sigma_z$.

Since we also have $\sigma_z \neq 0$, taking $r > 0$ sufficiently small, we can suppose that $\Sigma(z, \lambda) = (\sigma(z, \lambda), \lambda)$ has an inverse $\Sigma^{-1} : \Sigma(D(z_0, r) \times B(\lambda_0, r)) \to D(z_0, r) \times B(\lambda_0, r)$ and that the set $\Sigma(D(z_0, r) \times B(\lambda_0, r))$ is open. Then, if we define $f : \Sigma(D(z_0, r) \times B(\lambda_0, r)) \to \mathcal{C}$ by $f(z, \lambda) = \omega(\Sigma^{-1}(z, \lambda))$, we have

$$\omega = f \circ \Sigma\, .$$

Since f is analytic in z (see Proposition 2.6.4,part b)) and smooth in λ, we have that $\omega \in H_{(z_0, \lambda_0)}(n + 1 + \alpha, r)$. $\square$

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