PROBABILISTIC AND ANALYTICAL ASPECTS OF NILPOTENT GROUP ACTIONS ON THE INTERVAL

Contents

INTRODUCTION

Much work has been done on centralizers of C^2-diffeomorphisms of the interval [5, 14, 23, 24]. This theory has been extensively used for studying the algebraic constraints of finitely-generated subgroups of $\mathrm{Diff}_+^2([0,1])$. For example, using the famous Kopell lemma [14], Plante and Thurston showed that nilpotent groups of C^2-diffeomorphisms of $[0,1[$ (resp. $]0,1[$) are Abelian (resp. metabelian); see [21].

As is well known, most of the rigidity properties are lost when we consider centralizers of C^1-diffeomorphisms. In relation to Plante-Thurston's theorem, this fact is corroborated by the work of Farb and Franks. In [6], they construct an embedding ϕ_{FF} of N_d into $\mathrm{Diff}_+^1([0,1])$, where N_d denotes the (nilpotent) group of $(d+1)\times(d+1)$ lower-triangular matrices whose entries are integers which equal 1 on the diagonal (see §1.0.1 for the details). Since every finitely-generated, torsion-free, nilpotent group embeds into N_d for some $d \geq 1$ (see [22]), one concludes that all these groups can be realized as groups of C^1-diffeomorphisms of the (closed) interval (compare [12]).

Major progress has been recently made in the understanding of the loss of rigidity for centralizers in intermediate differentiability classes, that is, between C^1 and C^2 (see [4, 13, 15]). Recall that, for $0 < \alpha < 1$, a diffeomorphism f is said to be of class $C^{1+\alpha}$ if its derivative is α-Hölder continuous. In other words, there exists a constant M such that for all x, y,

$$|f'(x) - f'(y)| \leq M|x-y|^{\alpha}. \tag{1}$$

We denote the group of $C^{1+\alpha}$-diffeomorphisms of $[0,1]$ by $\mathrm{Diff}_+^{1+\alpha}([0,1])$. The main object of this work is to establish the following theorem, a complete proof of which is given in Chapter 1 though an alternative (more conceptual) proof is given in Chapter 2.

Theorem A. *If $d \geq 3$ and $\alpha > \frac{2}{d(d-1)}$, then the action $\phi_{_{FF}}$ is not topologically conjugated to an action by $C^{1+\alpha}$-diffeomorphisms of* $[0,1]$.

Notice that for $d = 2$, this theorem still holds and follows from Plante-Thurston's theorem. Theorem A should be considered as a partial complement to [15, Theorem B] which establishes that, for all $0 < \alpha < 1$, every subgroup Γ of $\mathrm{Diff}_+^{1+\alpha}([0,1])$ without free subsemigroups is virtually nilpotent. (Although the last result still holds for the open interval $]0,1[$, Theorem A fails to be true in this context, but it extends –with the very same proof– to the case of the half-closed interval).

Recall that [4, Theorem B] deals with Abelian group actions that are dynamically very similar to $\phi_{_{FF}}$, and a direct application of it shows that $\phi_{_{FF}}$ is not conjugated to an action by $C^{1+\alpha}$-diffeomorphisms of $[0,1[$ for any $\alpha > \frac{1}{d-1}$. The fact that our critical regularity here is actually smaller relies on that compared to the Abelian actions of [4], the action $\phi_{_{FF}}$ has a more complicated combinatorial dynamics in that the growth of certain orbits is polynomial with degree precisely equal to $\frac{d(d-1)}{2}$. We should point out that similar combinatorial dynamics appear for the actions of the natural quotients of the Grigorchuk-Machi's group [7] for which the method of this article should also provide the best possible regularity (compare [15, Theorem A]). Moreover, it is worth mentioning that the very same arguments show that Theorem A above still applies to topological *semiconjugacies.*

As we pointed out, we provide two proofs of the theorem. In both, we attempt to obtain the same, namely, control of the distortion along suitable compositions of elements in any regularity larger than the critical one.

In the first proof (Chapter 1), the control of distrotion is obtained by a nontrivial modification of the probabilistic techniques of [4, 13]. These essentially consits in ramdom walk type arguments that require a complete knownledge of the combinatorial structure of the orbits.

The second proof (Chapter 2) is based on a clever remark of R. Tessera. He noticed that the random walk type arguments have a natural traslation into the framework of moduli of curves. While this method is less elementary than the first one, it has the advantage of providing a more general setting in which the proof of Theorem A becomes simpler and enlightened.

The converse of Theorem A was essentially established by Jorquera in his PhD thesis [11]. The proof is based on classical constructions of Denjoy and Pixton (a clever exposition of these techniques appears in [25]; see also [16]).

Theorem B. *For each* $d \geq 2$ *and* $\alpha < \frac{2}{d(d-1)}$, *the action* ϕ_{FF} *is topologically conjugated to an action by* $C^{1+\alpha}$*-diffeomorphisms of* $[0,1]$.

Both theorems A and B above are the core of the work [2], which is still in revision in a prestigious journal. At the time of writing [2], we were unable to settle the $C^{1+\frac{2}{d(d-1)}}$ case, though we conjectured that the rigidity (*i.e.* Theorem A) should still hold for this critical regularity. This has been confirmed in the recent work [17].

Theorems A and B strongly suggest that, attached to each finitely-generated, torsion-free nilpotent group Γ, there should be a positive exponent $\alpha(\Gamma) \leq 1$ that is critical for embedding Γ into $\mathrm{Diff}_+^{1+\alpha}([0,1])$. However, it is still unclear to us what should be the value of $\alpha(\Gamma)$. Here, it is important to point out that it seems hard to adapt the techniques of proof of [2] (i.e. those of Charper 1) to the general case. Nevertheless, the ideas exploited in Chapter 2 seem suitable for this, so that we hope they will lead in the near future to the complete solution of the problem of determining the optimal regularity for actions of arbitray finitely-generated, torsion-free, nilpotent groups on the interval.

Chapter 1

NON-EXISTENCE OF EMBEDDINGS FOR $\alpha > \frac{2}{d(d-1)}$

1.0.1 A reminder on Farb-Franks' action $\phi_{_{FF}}$

We deal with the group N_d of $(d+1) \times (d+1)$ lower-triangular matrices with integer entries, all of which are equal to 1 on the diagonal. Notice that N_2 corresponds to the Heisenberg group. In general, N_d is a nilpotent group of nilpotence degree d. A nice system of generators of N_d is $\{f_{2,1}, \ldots, f_{d+1,d}\}$, where $f_{i,j}$ is the elementary matrix whose unique nonzero entry outside the diagonal is the (i,j)-entry (with $i > j$).

The group N_d acts linearly on $\mathbb{Z}^{d+1}$ with the affine hyperplane $1 \times \mathbb{Z}^d$ remaining invariant. The thus-induced action on $\mathbb{Z}^d$ allows producing an action on the interval as follows. Let $\big\{ I_{i_1,\ldots,i_d} \colon \; (i_1, \ldots, i_d) \in \mathbb{Z}^d \big\}$ be a family of intervals such that the sum $\sum_{i_1,\ldots,i_d} |I_{i_1,\ldots,i_d}|$ is finite, say equal to 1 after normalization. We join these intervals lexicographically on the closed interval $[0,1]$, and we identify $f_{j+1,j}$ to a certain homeomorphism sending each interval $I = I_{i_1,\ldots,i_d}$ into the interval J given by:

- $J := I_{i_1+1, i_2, \ldots, i_{d-1}, i_d}$ for $j = 1$,
- $J := I_{i_1, \ldots, i_{j-1}, i_j + i_{j-1}, i_{j+1}, \ldots, i_d}$ for $2 \leq j \leq d$.

It is not hard to perform this procedure in a equivariant way (for instance, using piecewise-affine maps), thus preserving the group structure and hence obtaining an embedding of N_d into $\mathrm{Homeo}_+([0,1])$. (Much harder is to obtain an embedding into the group of diffeomorphisms.) For this action, an interval of the form $I_{i_1,\dots,i_d}$ is sent by $f \in N_d$ into $I_{j_1,\dots,j_d}$, where $f\big((1,i_1,\dots,i_d)^T\big) = (1,j_1,\dots,j_d)^T$. Notice that up to topological conjugacy, all the actions obtained by this procedure are equivalent. This includes Farb-Franks' action $\phi_{_{FF}}$, which is obtained via this method for a well-chosen family of diffeomorphisms between the intervals of type I, J above so that the resulting $f_{i,j}$'s are C^1-diffeomorphisms.

1.0.2 From control of distortion to the proof of Theorem A

Let us begin by stating a general principle from [4] in the form of the following

Proposition 1.0.1. *Let $f_1,\dots,f_k$ be C^1-diffeomorphisms of the interval $[0,1]$ that commute with a C^1-diffeomorphism g. Assume that g fixes a subinterval I of $[0,1]$ and its restriction to I is nontrivial. Assume moreover that for a certain $0<\alpha<1$ and a sequence of indexes $i_j \in \{1,\dots,k\}$, the sum*

$$L_\alpha := \sum_{j\geq 0} \big|f_{i_j}\cdots f_{i_1}(I)\big|^\alpha \tag{1.1}$$

is finite. Then $f_1,\dots,f_k$ cannot be all of class $C^{1+\alpha}$.

Proof. Let $x_0 \in I$ be such that $g(x_0) \neq x_0$. Denote by $[a,b]$ the shortest interval containing x_0 that is fixed by g. For each $j \geq 1, n \geq 1$ and $z \in [a,b]$, the equality $g^n = (f_{i_j}\cdots f_{i_1})^{-1} \circ g^n \circ (f_{i_j}\cdots f_{i_1})$ yields

$$\log(g^n)'(z) = \log(f_{i_j}\cdots f_{i_1})'(z) + \log(g^n)'(f_{i_j}\cdots f_{i_1}(z)) - \log(f_{i_j}\cdots f_{i_1})'(g^n(z)).$$

Fix a constant M such that (1) (see introduction) holds for all $f \in \{f_1,\dots,f_k\}$ and all x,y in $[0,1]$. Letting $z_n := g^n(z)$ and noticing that z_n belongs to $[a,b] \subset I$ for all $n \geq 1$, we obtain

$$\begin{aligned}
|\log(g^n)'(z)| &\leq |\log(g^n)'(f_{i_j}\cdots f_{i_1}(z))| + \sum_{m=1}^{j} \big|\log(f_{i_m})'(f_{i_{m-1}}\cdots f_{i_1}(z)) - \log(f_{i_m})'(f_{i_{m-1}}\cdots f_{i_1}(z_n))\big| \\
&\leq |\log(g^n)'(f_{i_j}\cdots f_{i_1}(z))| + \sum_{m=1}^{j} M\big|f_{i_{m-1}}\cdots f_{i_1}(z) - f_{i_{m-1}}\cdots f_{i_1}(z_n)\big|^\alpha \\
&\leq |\log(g^n)'(f_{i_j}\cdots f_{i_1}(z))| + M\sum_{m=1}^{j} |f_{i_{m-1}}\cdots f_{i_1}(I)|^\alpha \\
&\leq |\log(g^n)'(f_{i_j}\cdots f_{i_1}(z))| + ML_\alpha.
\end{aligned}$$

The length of the intervals $f_{i_j} \cdots f_{i_1}(I)$ must necessarily converge to zero as j goes to infinite. Moreover, since g^n fixes I and commutes with $f_1, \ldots, f_k$, on each of these intervals there must be a point at which its derivative equals 1. By the continuity of $(g^n)'$, we conclude that the value of $(g^n)'(f_{i_j} \cdots f_{i_1}(z))$ converges to 1 as j goes to infinite. Hence we obtain $(g^n)'(z) \leq e^{ML_\alpha}$ for all $n \geq 1$ and all $z \in [a,b]$, which certainly contradicts the fact that the restriction of g to $[a,b]$ is nontrivial. $\square$

Let us come back to the action $\phi_{_{FF}}$. Notice that the group N_{d-1} can be naturally viewed as the subgroup of N_d formed by the elements whose last row coincide with that of the identity. We will denote by N_{d-1}^* the copy of N_{d-1} inside N_d.

Notice that the element $g := f_{d+1,1} \in N_d$ is centralized by N_{d-1}^*. Under the action $\phi_{_{FF}}$, this element fixes the interval

$$I^* := \bigcup_{j \in \mathbb{Z}} I_{0,\ldots,0,j}. \tag{1.2}$$

Moreover, this interval is sent into a disjoint one by any nontrivial element of N_{d-1}^*. We are hence in a situation close to that of the preceding proposition. Thus, we need to ensure the existence of a systems of generators for N_{d-1}^* and a sequence of compositions for which the associated sum (1.1) is finite provided that $\alpha > \frac{2}{d(d-1)}$. To do this, we will use the system of generators $\{f_{2,1}, f_{3,1}, \ldots, f_{d,1}\} \cup \{f_{2,1}, f_{3,2}, \ldots, f_{d,d-1}\}$.

It is worth mentioning that this is an analogous problem to that of the $\mathbb{Z}^d$-actions on the interval considered in [4, Théorème B]. However, the $\mathbb{Z}^d$-case is easier in that the generators of the dynamics commute, hence the orbit graph of the associated interval I^* has a simpler structure. Indeed, the space of infinite paths of this graph can be endowed of a natural probability measure such that for appropriately large values of α (namely, for $\alpha > 1/d$), almost every path has a finite L_α-series. In order to establish this, besides the restriction on the exponent α, the main property of the underlying process is that the arrival probabilities up to time k are equidistributed along the sphere of radius k (centered at the origin) for every $k \geq 1$. Although in [4] this is modeled via a Polya urn like model that charges only the positive powers of the generators, an alternative model sharing this property that charges both positive and negative powers of the generators is the Markov process depicted in Figure 1 below for the case $d = 2$ (the reader will easily check the equidistribution property along spheres as well as the general rule for the transition probabilities; the generalization for higher values of d is not very hard).

Remark 1.0.2. It seems to be an interesting and nontrivial problem to determine general conditions for an infinite graph ensuring the existence of a Markov process satisfying the equidistribution property above.

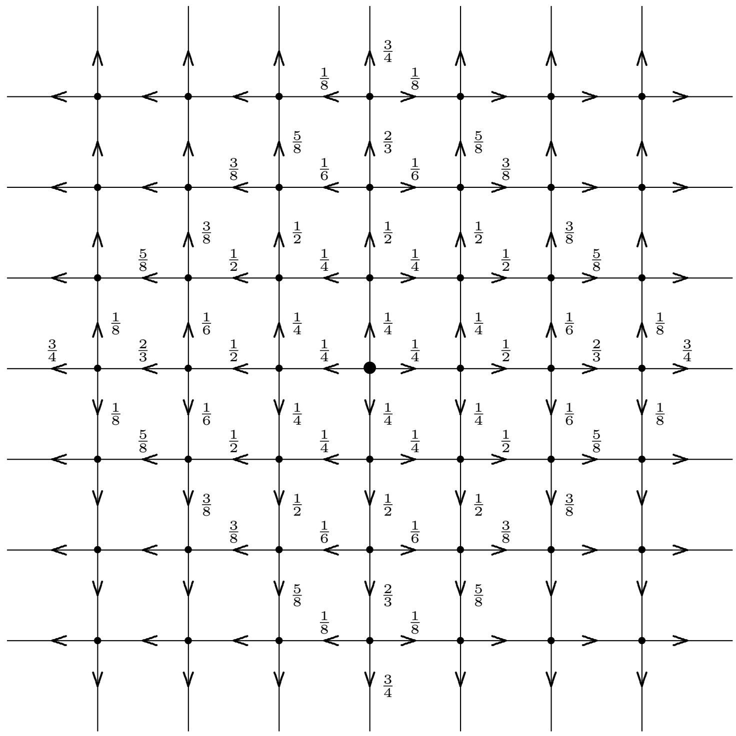


Figure 1

Let us now consider the orbit of the interval I^* defined by (1.2) for the action of N_{d-1}^*. For simplicity, let us first deal with the case $d = 3$. With respect to the generators $f_{2,1}, f_{3,1}, f_{3,2}$ of N_2^*, the orbit graph is depicted in Figure 2 below. Here, $f_{2,1}$ corresponds to the generator whose action on the the graph is moving to the right, whereas the action of both $f_{3,1}$ and $f_{3,2}$ consists in moving up, the former by one unit and the latter with an amplitude that depends on the position. (Notice that the directions of the arrows mean that we are only considering positive powers of the generators.)

Now, the difficulty comes from that, as the reader may easily check, it is impossible to put probability distributions on this graph yielding the equidistribution property along the spheres centered at the origin. (This is already impossible for the sphere of radius 4.) To overcome this problem, we will use the counting argument of (the first part of) [13], which actually corresponds to a deterministic counterpart of the random walk argument above. Indeed, this argument is more robust in that it does not need any equidistribution property, though it requires a certain extra argument to obtain our desired infinite path as a concatenation of finite paths that behave nicely for certain finite processes.

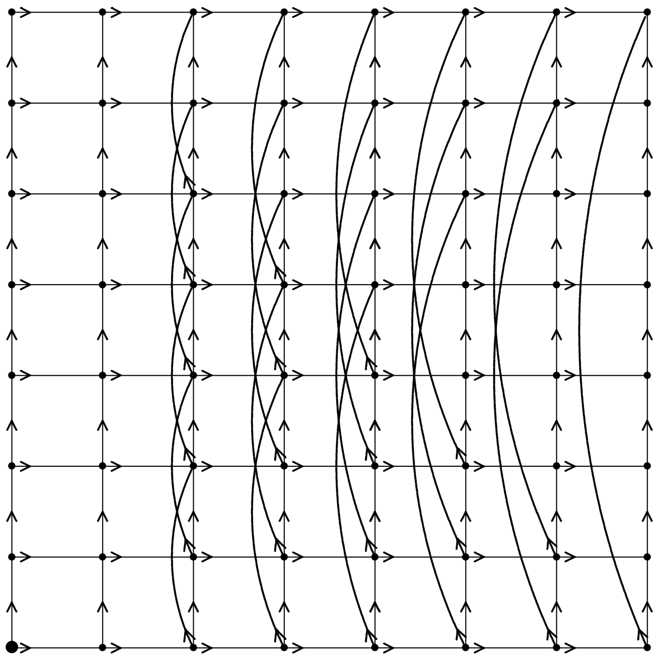


Figure 2

To close this section, let us finally explain why the exponent $\frac{2}{d(d-1)}$ is critical for the action ϕ_{FF}. For simplicity, let us first consider the case $d = 3$. Looking at the graph of Figure 2 above, one easily computes the growth of the balls. This appears to be cubic, in the sense that the number of points at distance $\leq n$ from the origin is $\frac{n^3+11n+6}{6} \sim n^3$. These points correspond to intervals in the orbit of I^* obtained up to $\leq n$ compositions of the generators. Since these intervals are disjoint, the length of a typical one should be of order $\sim 1/n^3$. Hence, along a generic sequence of compositions, the value of the corresponding sum L_α should be of order

$$\sum_{n \geq 1} \left(\frac{1}{n^3} \right)^{\alpha},$$

which is finite for $\alpha > \frac{1}{3} = \frac{2}{3(3-1)}$, as expected.

The case of a general $d \geq 3$ is similar. Indeed, the growth of the associated graph is polynomial with degree

$$1 + 2 + \ldots + (d-1) = \frac{d(d-1)}{2},$$

which allows to argue as before for any α larger than the critical exponent $\frac{2}{d(d-1)}$.

Remark 1.0.3. Notice that the polynomial degree above is in concordance with the Bass-Guivarch formula for the growth of N_{d-1}; see [8, Appendix]. This view will be pursued in Chapter 2.

1.0.3 Proof of Theorem A: the case $d = 3$

The proof of Theorem A is somewhat technical and requires hard notation. This is the reason why we have chosen to first give the proof for the case $d = 3$, where most of the ideas become more transparent and an important technical problem is overcomed by a trick consisting in the introduction of a small parameter $\varepsilon > 0$. For the general case, we use a slightly modified construction keeping essentially the same arguments. We begin with a lemma in the spirit of [13, Lemma 2.2].

Lemma 1.0.4. *Let $n \geq 1$ be an integer and let C_1, C_2, ε be positive constants. Let P be a set of $\leq C_1 n^{3+\varepsilon}$ pairs of non-negative integers (i,j) associated to which there is a number $\ell_{i,j} > 0$ such that $\sum_{(i,j)\in P} \ell_{i,j} \leq 1$. Suppose that P partitioned into $n' \geq n^2/C_2$ (resp. $n' \geq n^{2+\varepsilon}/C_2$) disjoint subsets $P_1, \ldots, P_{n'}$. Then, given $A > 1$ and $1 > \alpha > 0$, the proportion of indexes $m \in \{1, \ldots, n'\}$ for which*

$$\sum_{(i,j)\in P_m} \ell_{i,j}^{\alpha} \leq \frac{AC_1^{1-\alpha}C_2}{n^{3\alpha-1-\varepsilon(1-\alpha)}} \qquad \left(\text{resp. } \sum_{(i,j)\in P_m} \ell_{i,j}^{\alpha} \leq \frac{AC_1^{1-\alpha}C_2}{n^{3\alpha-1+\varepsilon\alpha}}\right)$$

is at least $1 - 1/A$.

Proof. Since $\sum_{(i,j)\in P} \ell_{i,j} \leq 1$ and P consists of at most $C_1 n^{3+\varepsilon}$ pairs, a direct application of Hölder's inequality yields

$$\sum_{(i,j)\in P} \ell_{i,j}^{\alpha} \leq (C_1 n^{3+\varepsilon})^{1-\alpha}.$$

Hence,

$$\frac{1}{n'} \sum_{m=1}^{n'} \sum_{(i,j)\in P_m} \ell_{i,j}^{\alpha} \leq \frac{C_1^{1-\alpha} n^{(3+\varepsilon)(1-\alpha)}}{n'},$$

and the latter expression is less than or equal to $C_1^{1-\alpha} C_2 n^{1-3\alpha+\varepsilon(1-\alpha)}$ (resp. $C_1^{1-\alpha} C_2 n^{1-3\alpha-\varepsilon\alpha}$). The lemma then follows as a direct application of Chebyshev's inequality. $\square$

Let us now come back to the graph associated to the action $\phi_{_{FF}}$ depicted in Figure 2, and let us set $\ell_{i,j} := |f_{2,1}^{i} f_{3,1}^{j}(I^*)|$. Fix positive constants α, ε such that

$$\alpha > \frac{1}{3} = \frac{2}{(3-1)(3-2)}, \qquad \varepsilon < \max\left\{\frac{3\alpha - 1}{1-\alpha}, 1\right\}. \tag{1.3}$$

For any real numbers $M \leq N$, we let $[[M, N]] := [M, N] \cap \mathbb{Z}$. Given an integer $n \geq 2$, we consider the set $P(n) := [[n, 8n-1]] \times [[0, \ldots, n^{2+\varepsilon}]]$. This set $P(n)$ consists of $7n([n^{2+\varepsilon}] + 1) \leq 10\, n^{3+\varepsilon}$ points (with $[\cdot]$ standing for the integer part), and is partitioned into the $n' = [n^{2+\varepsilon}] + 1 \geq n^{2+\varepsilon}$ disjoint sets (horizontal paths) $P(n,1), P(n,2), \ldots, P(n, n')$ given by

$$P(n,m) := \big\{(n,m), (n+1,m), \ldots, (8n-1,m)\big\}.$$

By the preceding lemma, for each $0 < A_n < 1$, the proportion of indexes $m \in \{1, \ldots, n'\}$ for which

$$\sum_{i=n}^{8n-1} \ell_{i,m}^{\alpha} \;=\; \sum_{(i,j) \in P(n,m)} \ell_{i,j}^{\alpha} \;\leq\; \frac{A_n 10^{1-\alpha}}{n^{3\alpha - 1 + \varepsilon\alpha}} \tag{1.4}$$

is at least $1 - 1/A_n$. Notice that each path $P(n,m)$ comes from the action of the generator $f_{2,1}$.

Similarly, for each integer $n \geq 2$, let us consider the set $Q(n) := [[n, 2n-1]] \cap [[0, \ldots, n^{2+\varepsilon}]]$ consisting of $n([n^{2+\varepsilon}]+1) \leq 2n^{3+\varepsilon}$ points. Although in general there is no partition of $Q(n)$ into paths induced by the action of $f_{3,1}, f_{3,2}$ all of them having the same number of points, a partition that almost satisfies this property (and that will be sufficient for our purposes) can be defined as follows. For each $n \leq m \leq 2n-1$ we divide the set $\{(m,0), (m,1), \ldots\}$ into n paths via the following rules:

– For each $0 \leq j \leq n-2$, there is a path starting at (m,j) jumping upwards of m units;

– The path starting at $(n-1, m)$ makes $m-n$ jumps upwards of 1 unit and then makes a jump of m units;

– The picture repeats "periodically", so that each infinite path is made of $n-1$ consecutive jumps of m units followed by $m-n$ jumps of 1 unit.

Figure 3 illustrates the case where $n = 3$ and $m = 5$ though the resulting paths are disposed horizontally instead of vertically by obvious reasons. Although one may give precise formulas for the points in each of these paths, this is not completely necessary. The main property that we will retain is the obvious fact that the number of points of each of them inside any rectangle $[[n, 2n-1]] \times [[0, K-1]]$ lies between $\frac{K}{n} - 2n$ and $\frac{K}{n} + 2n$. (An alternative construction leading to a much better -logarithmic- control of the deviation will be given in §1.0.4.) In particular, we have an induced partition of $Q(n)$ into $n'' = n^2$ paths $Q(n,1), Q(n,2), \ldots, Q(n,n'')$ for which the preceding lemma yields that for each $A_n > 0$, the proportion of indexes $m \in \{1, \ldots, n''\}$ satisfying

$$\sum_{(i,j) \in Q(n,m)} \ell_{i,j}^{\alpha} \leq \frac{A_n 2^{1-\alpha}}{n^{3\alpha - 1 - \varepsilon(1-\alpha)}} \tag{1.5}$$

is at least $1 - 1/A_n$. Notice again that each of these paths comes from the action of the generators $f_{3,1}$ and $f_{3,2}$ according to the amplitude of the jump.

Figure 3

We will apply the preceding construction for each integer $n = n_k := 4^k$, where $k \geq 1$. The choice of the constants A_{n_k} is as follows. First, we let r_k (resp. s_k) be the minimum (resp. maximum) number of points of a path of the form $Q(n_k, m)$ inside $Q(n_k)$. Similarly, we let r'_k (resp. s'_k) be the minimum (resp. maximum) number of points in a path of the form $Q(n_k, m)$ inside $P(n_{k-1}) \cap Q(n_k)$. Finally, we let

$$B := \prod_{k \geq 2} \frac{s_k}{r_k} \frac{s'_k}{r'_k}. \tag{1.6}$$

Notice that the value of B is finite. Indeed, by the discussion above, we have

$$4^{k(1+\varepsilon)} - 2 \cdot 4^k = n_k^{1+\varepsilon} - 2n_k \leq r_k \leq s_k \leq n_k^{1+\varepsilon} + 2n_k = 4^{k(1+\varepsilon)} + 2 \cdot 4^k$$

and

$$4^{k+k\varepsilon-1} - 2 \cdot 4^{k+1} = \frac{n_k^{2+\varepsilon}}{n_{k+1}} - 2n_{k+1} \leq r'_k \leq s'_k \leq \frac{n_k^{2+\varepsilon}}{n_{k+1}} + 2n_{k+1} = 4^{k+k\varepsilon-1} + 2 \cdot 4^{k+1},$$

which easily yield the convergence of the infinite product in the definition of B. We will also use the constant

$$C := 4 \sum_{k \geq 1} \frac{1}{2^{k(3\alpha - 1 - \varepsilon(1-\alpha))}}. \tag{1.7}$$

Notice again that since (1.3) implies that $3\alpha - 1 - \varepsilon(1-\alpha) > 0$, we have $C < \infty$.

We now fix $A_{n_1} \geq 2^{2+k(3\alpha-1-\varepsilon(1-\alpha))}BC$ such that (1.4) holds for $n = n_1$ and *every* m in the corresponding range. Finally, for $k \geq 2$, we set

$$A_{n_k} := BC\,2^{k(3\alpha-1-\varepsilon(1-\alpha))}.$$

We next state a key lemma whose proof is postponed in order to proceed immediately to the proof of Theorem A in the case $d = 3$.

Lemma 1.0.5. *There are two infinite sequences of paths $P(n_k, m'_k)$ and $Q(n_k, m''_k)$ such that (1.4) (resp. (1.5)) holds for $n = n_k$ and $m = m'_k$ (resp. $m = m''_k$) and such that $P(n_k, m'_k)$ intersects both $Q(n_k, m''_k)$ and $Q(n_{k+1}, m''_{k+1})$ for all $k \geq 1$.*

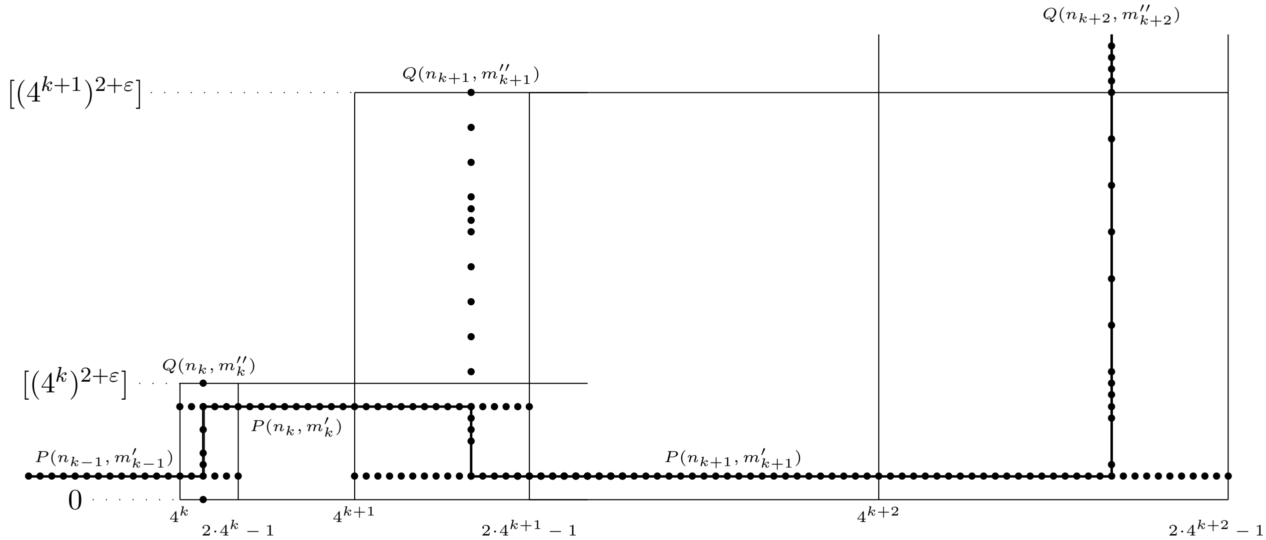


Figure 4

Assuming this lemma, the proof of Theorem A in the case $d = 3$ is at hand. Indeed, the concatenation of the sequence of finite paths provided by the lemma naturally yields an infinite path without loops which is in correspondence with a sequence of compositions by $f_{2,1}, f_{3,1}, f_{3,1}^{-1}, f_{3,2}, f_{3,2}^{-1}$ (see Figure 4). By construction, for this sequence of iterations, the value of the corresponding L_α-sum (1.1) for the interval I^{**} corresponding to the initial point of $Q(n_1, m'_1)$ is less than or equal to

$$\begin{aligned} 10^{1-\alpha} \sum_{k\geq 1} \frac{A_{n_k}}{n_k^{3\alpha-1+\varepsilon\alpha}} + 2^{1-\alpha} \sum_{k\geq 1} \frac{A_{n_k}}{n_k^{3\alpha-1-\varepsilon(1-\alpha)}} &\leq \frac{20A_{n_1}}{4^{3\alpha-1-\varepsilon(1-\alpha)}} + \sum_{k\geq 2} \frac{40BC}{2^{k(3\alpha-1-\varepsilon(1-\alpha))}} \\ &\leq 80A_{n_1}4^{\varepsilon(1-\alpha)} + 40BC^2. \end{aligned}$$

This interval I^{**} is in the orbit of I^*, from which it can be reached in no more than $(2\cdot 4^1 - 1)+4 = 11$ iterations of the generator $f_{2,1}$. By concatenating this finite path to the previous one, we obtain an infinite path associated to which the L_α-sum corresponding to I^* is finite, which allows to conclude the proof by the arguments developed in §1.0.2.

All that remains for completing the proof of Theorem A in the case $d=3$ is the

Proof of Lemma 1.0.5. The argument is similar to that of [13, Lemma 2.3], but it needs a slight modification. Namely, for each $k \geq 1$, we let D'_k be the density of indexes $m' \in \{1, \ldots, [n_k^{2+\varepsilon}]+1\}$ such that $P(n_k, m')$ is "reached" by a sequence of paths $Q(n_1, m''_1), P(n_1, m'_1), \ldots, Q(n_k, m''_k)$ satisfying:

– $P(n_i, m'_i)$ intersects both $Q(n_i, m''_i)$ and $Q(n_{i+1}, m''_{i+1})$ for all $1 \leq i \leq k-1$, whereas $P(n_k, m')$ intersects $Q(n_k, m''_k)$;

– Inequality (1.4) (resp. (1.5)) holds for $(n,m) = (n_i, m'_i)$ whenever $1 \leq i \leq k-1$ as well as for $(n,m) = (n_k, m')$ (resp. for $(n,m) = (n_i, m''_i)$ whenever $1 \leq i \leq k$).

Similarly, we denote by D''_k the density of indexes $m'' \in \{1, \ldots, n_k^2\}$ such that $Q(n_k, m'')$ is reached by a sequence of paths $Q(n_1, m''_1), P(n_1, m'_1), \ldots, P(n_{k-1}, m'_{k-1})$ satisfying:

– $P(n_i, m'_i)$ intersects both $Q(n_i, m''_i)$ and $Q(n_{i+1}, m''_{i+1})$ for all $1 \leq i \leq k-1$;

– Inequality (1.4) (resp. (1.5)) holds for $(n,m) = (n_i, m'_i)$ (resp. for $(n,m) = (n_i, m''_i)$) whenever $1 \leq i \leq k-1$ as well as for $(n,m) = (n_k, m'')$).

We claim that the following relations hold:

$$1 - D'_k \leq (1 - D''_k)\frac{s_k}{r_k} + \frac{1}{A_{n_k}}, \qquad 1 - D''_{k+1} \leq (1 - D'_k)\frac{s'_{k+1}}{r'_{k+1}} + \frac{1}{A_{n_{k+1}}}. \tag{1.8}$$

Assuming this for a while, we obtain for each $k \geq 1$,

$$1 - D'_k \leq (1 - D'_{k-1})\frac{s_k}{r_k}\frac{s'_k}{r'_k} + \frac{1}{A_{n_k}}\frac{s_k}{r_k} + \frac{1}{A_{n_k}}.$$

Using induction, this easily yields

$$1 - D'_k \leq (1 - D'_1)\prod_{i=2}^{k}\frac{s_i}{r_i}\frac{s'_i}{r'_i} + 2\sum_{i=2}^{k}\frac{1}{A_{n_i}}\prod_{j=2}^{i}\frac{s_j}{r_j}.$$

From the definition $n_i := 4^i$ and that of the constant B in (1.6), one concludes that for each

$k \geq 1$,

$$1 - D'_k \leq (1 - D'_1)B + 2B \sum_{i=1}^{k} \frac{1}{A_{n_i}}.$$

Now, the choice of A_{n_1} was made so that $D'_1 = 1$, hence

$$1 - D'_k \leq 2B \sum_{i \geq 1} \frac{1}{A_{n_i}} \leq \frac{1}{2}.$$

Thus, $D'_k \geq 1/2$ holds for all $k \geq 1$, which provides finite paths satisfying the desired properties of length as large as we want. The infinite path claimed to exist is obtained easily from this by means of a Cantor diagonal type argument.

Finally, it remains to show (1.8). The proof follows the same principle of that of [13, Lemma 2.3] but requires a little adjustment. First, we denote by $\hat{D}''_k$ the density of points in $Q(n_k)$ that are "well-attainable" in the sense that they belong to the last of a sequence of consecutively intersecting paths $Q(n_1, m''_1), P(n_1, m'_1), \ldots, P(n_{k-1}, m'_{k-1})$, $Q(n_k, m''_k)$ for which inequalities of type (1.4) or (1.5) hold according to the case. We have

$$(1 - D'_k) \leq (1 - \hat{D}''_k) + \frac{1}{A_{n_k}}. \tag{1.9}$$

Indeed, the term $1/A_{n_k}$ corresponds to the density of horizontal paths in $P(n_k)$ that are "bad by themselves" in the sense that the corresponding type (1.4) inequality does not hold for them. The term $(1 - \hat{D}''_k)$ corresponds to the density of paths in $P(n_k)$ that may be good by themselves but intersect $Q(n_k)$ at a set formed only by non-well-attainable points. (Notice that we are using the fact that all horizontal paths in $P(n_k)$ have the same number of points in $Q(n_k)$.) The left-side inequality in (1.8) then follows as a combination of (1.9) and the inequality

$$1 - \hat{D}''_k \leq (1 - D''_k) \frac{s_k}{r_k},$$

where the correction factor comes from the fact that although the number of points in each path of the form $Q(n_k, m)$ is not constant, it varies between r_k and s_k.

Similarly, in the right-side inequality, the term $1/A_{n_{k+1}}$ corresponds to the density of bad-by-themselves paths of the form $Q(n_{k+1}, m)$ in $Q(n_{k+1})$. The term $(1 - D'_k)$ corresponds to the "accumulated density of bad paths" up to $P(n_k)$, and equals the density of "non-well-attainable" points in $P(n_k) \cap Q(n_{k+1})$. Finally, the correction factor comes from the fact that the number of points in $P(n_k) \cap Q(n_{k+1})$ contained in each path of the form $Q(n_{k+1}, m)$ lies between r'_{k+1} and s'_{k+1}.

1.0.4 Proof of Theorem A: the general case

To deal with the general case we will follow a similar strategy, though most of the computations become more involved. We will now consider paths inside parallelepipeds of dimension $d-1$ having sides of length of (relative) order $k, k^2, \ldots, k^{d-1}$. This will make naturally appear the exponent $\frac{d(d-1)}{2}$ in relation to the total number of points in the parallelepiped. The most relevant difficulty will be related to the decomposition into paths. Indeed, the construction of the preceding section illustrated by Figure 3 is no longer satisfactory, and we will need to carry out a nontrivial modification of it. Since this is of independent interest and has potential applications in other contexts, the discussion of the new construction will be the subject of §1.0.5. Here we content ourselves in stating what we need for our purposes, which is summarized in the next

Lemma 1.0.6. *Let $M > N$ be positive integer numbers, with N of the form $1 + 2^k$. There exists a decomposition of $\mathbb{N}_0 := \{0, 1, \ldots\}$ into N subsets (paths) satisfying:*

(i) *The distance (jump) between two consecutive points of each path is either M or 1;*

(ii) *For all $0 \leq K_1 < K_2$, the maximal number of points of a path contained in $[[K_1, K_2]]$ differs from the minimal one by at most $4 + 2\frac{M-1}{N-1} + 4\log_2(N-1)$.*

We now proceed to the proof of Theorem A. Recall that the graph of the N^*_{d-1}-orbit of the interval I^* defined by (1.2) has $\mathbb{Z}^{d-1}$ as its set of vertices. We will hence inductively define parallelepipeds $Q(n) \subset \mathbb{Z}^{d-1}$. We start with $Q(0) := [[1, 1 + 4^{d+1}]]^{d-1}$. Assuming that $Q(n) := [[x_{1,n}, y_{1,n}]] \times \cdots \times [[x_{d-1,n}, y_{d-1,n}]]$ has been already defined, we let $i(n) \in \{1, \ldots, d-1\}$ be the residue class (mod. $d-1$) of n, and we set $Q(n+1) := \cdots \times [[1 + 4^{i(n)}(x_{i(n),n} - 1), y_{i(n),n}]] \times [[x_{i(n)+1,n}, 1 + 4^{i(n)+1}(y_{i(n)+1,n} - 1)]] \times \cdots$, where the dots mean that the corresponding factors remain untouched. (See Figure 5 for an illustration of the case $d = 4$, with $n \equiv 1$ (mod. 3).)

Notice that $x_{i,n}, y_{i,n}$ are of the form $1 + 2^k$ for all i, n. Although one may give precise formulas for $x_{i,n}, y_{i,n}$, we will only need to record the (easy to check) fact that for some universal constants C_1, C_2, C_3, C_4, we have the estimates

$$C_1 4^{\frac{in}{d-1}} \le y_{i,n} - x_{i,n} \le C_2 4^{\frac{in}{d-1}} \tag{1.10}$$

and

$$C_3 4^{\frac{in}{d-1}} \le x_{i,n} \le C_4 4^{\frac{in}{d-1}}. \tag{1.11}$$

In particular, the number of points in $Q(n)$ is

$$|Q(n)| = \prod_{j=1}^{d-1} (y_{j,n} - x_{j,n}) \le \prod_{j=1}^{d-1} C_2 4^{\frac{jn}{d-1}} = C_2^{d-1} 4^{\frac{n}{d-1}\sum_{j=1}^{d-1} j} = C_2^{d-1} 4^{\frac{nd}{2}}. \tag{1.12}$$

Each $Q(n)$ is decomposed into paths pointing in the $i(n)^{th}$-direction as follows. If $i(n) = 1$, then we decompose $Q(n)$ into "horizontal" paths of jump 1 at each step, so that the number of paths is

$$\prod_{j \neq 1} (y_{j,n} - x_{j,n}) \ge \prod_{j \neq 1} C_1 4^{\frac{jn}{d-1}} = C_1^{d-2} 4^{\frac{n}{d-1}\sum_{j\neq 1} j} = C_1^{d-2} 4^{n\left[\frac{d}{2} - \frac{1}{d-1}\right]}.$$

If $i(n) \neq 1$, then for each fixed coordinates $z_j \in [[x_{j,n}, y_{j,n}]]$, with $j \neq i(n)$, we identify

$$\{z_1\} \times \cdots \times \{z_{i(n)-1}\} \times [[x_{i(n),n}, y_{i(n),n}]] \times \{z_{i(n)+1}\} \times \cdots \times \{z_{d-1}\} \ \sim \ [[x_{i(n),n}, y_{i(n),n}]] \subset \mathbb{N}$$

and we decompose this set into $N := x_{i(n)-1,n}$ paths making jumps (in the $i(n)^{th}$-direction) of either 1 or $M := z_{i(n)-1,n}$ steps following the strategy of Lemma 1.0.6. The corresponding number of paths now equals

$$\begin{aligned} x_{i(n)-1,n} \prod_{j \neq i(n)} (y_{j,n} - x_{j,n}) \ &\ge \ C_3 4^{\frac{(i(n)-1)n}{d-1}} \prod_{j \neq i(n)} C_1 4^{\frac{jn}{d-1}} \\ &= \ C_3 4^{\frac{(i(n)-1)n}{d-1}} C_1^{d-2} 4^{\frac{n}{d-1}\sum_{j \neq i(n)} j} \ = \ C_3 C_1^{d-2} 4^{n\left[\frac{d}{2} - \frac{1}{d-1}\right]}. \end{aligned}$$

In either case, we denote by $Q(n,1), \ldots, Q(n, m_n)$ these paths, so that $m_n \ge C_5 4^{n\left[\frac{d}{2} - \frac{n}{d-1}\right]}$ for $C_5 := \min\{C_3 C_1^{d-2}, C_1^{d-2}\}$. What is important in the construction above is that each of these paths has a concrete dynamical meaning for the action of $N_{d-1}^* \subset N_d$. Namely, if $i(n) = 1$, they are induced by the action of the generator $f_{2,1}$, whereas for $i(n) \neq 1$, they are induced by the action of $f_{i(n),1}$ and $f_{i(n)-1,i(n)}$, where the first generator appears for 1-step jumps and the second one for jumps of amplitude $z_{i(n)-1,n}$.

Associated to each point $(i_1, \ldots, i_{d-1}) \in \mathbb{Z}^{d-1}$ there is a positive number $\ell_{i_1,\ldots,i_{d-1}}$, namely the length of the interval

$$I^*_{i_1,\ldots,i_{d-1}} := \bigcup_{j \in \mathbb{Z}} I_{i_1,\ldots,i_{d-1},j}.$$

Notice that the total sum of the $\ell_{i_1,\ldots,i_{d-1}}$'s equals 1. Moreover, all the intervals $I^*_{i_1,\ldots,i_{d-1}}$ are in the N^*_{d-1}-orbit of $I^* = I^*_{0,\ldots,0}$; see (1.2). Hence, as in the case $d = 3$, what we need to do is to ensure the existence of an infinite sequence of intersecting paths in $Q(1), Q(2), \ldots$ along which the total L_α-sum is finite provided that $\alpha > \frac{2}{d(d-1)}$. To do this, we start with the next

Lemma 1.0.7. *Given $0 < \alpha < 1$, there exists a constant $C_6 > 0$ such that for all $A > 0$ and all $n \geq 1$, the subset of indexes $m \in \{1, \ldots, m_n\}$ satisfying*

$$\sum_{(i_1,\ldots,i_{d-1}) \in Q(n,m)} \ell^\alpha_{i_1,\ldots,i_{d-1}} \leq \frac{A C_6}{4^{n\left[\frac{d\alpha}{2} - \frac{1}{d-1}\right]}} \tag{1.13}$$

has density at least $1 - 1/A$.

Proof. As in the case $d = 3$, by Hölder's inequality we have

$$\sum_{(i_1,\ldots,i_{d-1}) \in Q(n)} \ell^\alpha_{i_1,\ldots,i_{d-1}} \leq |Q(n)|^{1-\alpha} \leq C_2^{(d-1)(1-\alpha)} 4^{\frac{nd(1-\alpha)}{2}}.$$

Hence,

$$\frac{1}{m_n} \sum_{m=1}^{m_n} \sum_{(i_1,\ldots,i_{d-1}) \in Q(n,m)} \ell^\alpha_{i_1,\ldots,i_{d-1}} \leq \frac{C_2^{(d-1)(1-\alpha)} 4^{\frac{nd(1-\alpha)}{2}}}{C_5 4^{\frac{nd}{2} - \frac{n}{d-1}}} = \frac{C_2^{(d-1)(1-\alpha)}}{C_5 4^{\frac{nd\alpha}{2} - \frac{n}{d-1}}},$$

and the claim follows from Chebyshev's inequality for $C_6 := C_2^{(d-1)(1-\alpha)}/C_5$. $\square$

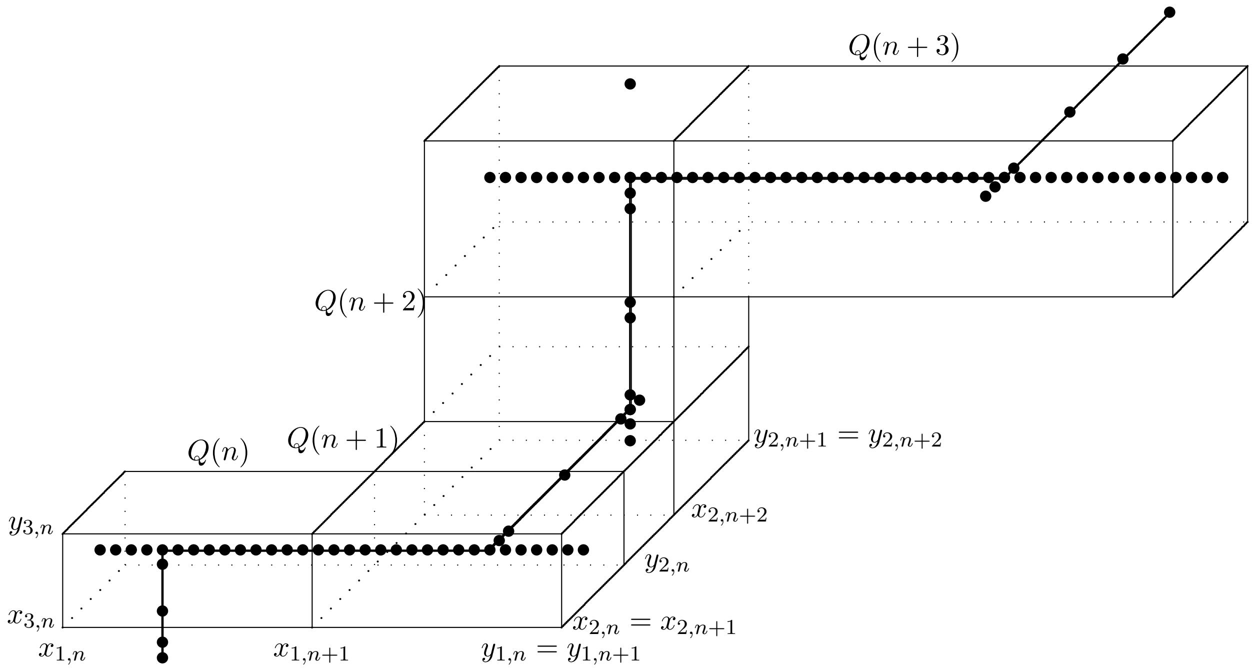


Figure 5

From now on, we fix $\alpha > \frac{d(d-1)}{2}$. We start by letting r_n (resp. s_n) be the minimum (resp. maximum) of points in a path of the form $Q(n,m)$ inside $Q(n) \cap Q(n+1)$. Similarly, we denote by r'_n (resp. s'_n) the minimum (resp. maximum) number of points of a path of the form $Q(n+1,m)$ inside $Q(n) \cap Q(n+1)$. Then we let

$$B := \prod_{n \geq 1} \frac{s_n}{r_n} \frac{s'_n}{r'_n}.$$

We claim that the value of B is finite. Indeed, we have $r_n = s_n$ whenever $i(n) = 1$, whereas $s'_n = r'_n$ whenever $i(n) = d-1$. For the other values of $i := i(n)$, the condition (ii) in Lemma 1.0.6 together with the inequalities $2\frac{y_{i-1,n}-1}{x_{i-1,n}-1} \leq 4^{d+2}$ and $2\frac{y_{i,n+1}-1}{x_{i,n+1}-1} \leq 4^{d+2}$ yield the estimates

$$\frac{y_{i,n+1} - x_{i,n+1}}{x_{i-1,n}} - 4 - 4^{d+2} - 4\log_2(x_{i-1,n}-1) \leq r_n \leq s_n \leq \frac{y_{i,n+1} - x_{i,n+1}}{x_{i-1,n}} + 4 + 4^{d+2} + 4\log_2(x_{i-1,n}-1)$$

and

$$\frac{y_{i+1,n} - x_{i+1,n}}{x_{i,n+1}} - 4 - 4^{d+2} - 4\log_2(x_{i,n+1}-1) \leq r'_n \leq s'_n \leq \frac{y_{i+1,n} - x_{i+1,n}}{x_{i,n+1}} + 4 + 4^{d+2} + 4\log_2(x_{i,n+1}-1),$$

which together with (1.10) and (1.11) easily imply the finiteness of B.

We will also use the (finite) constant

$$C := 2 \sum_{n \geq 1} \frac{1}{2^{n\left[\frac{d\alpha}{2} - \frac{1}{d-1}\right]}}.$$

Now we fix $A_1 \geq BC2^{\frac{d\alpha}{2}-\frac{1}{d-1}}$ such (1.13) holds for $n=1$ and *every* $m \in \{1,\dots,m_1\}$ when letting $A = A_1$. Finally, for $n \geq 2$, we set

$$A_n := BC2^{n\left[\frac{d\alpha}{2}-\frac{1}{d-1}\right]}.$$

Lemma 1.0.8. *There exists an infinite sequence of paths of the form $Q(n,m'_n)$ in $Q(n)$ such that, for all $n \geq 1$, the path $Q(n+1,m'_{n+1})$ intersects $Q(n,m'_n)$ and* (1.13) *holds for $m = m'_n$ and $A = A_n$.*

Proof. As in the case $d=3$, for each $n \geq 1$ we let D_n be the density of indexes $m \in \{1,\dots,m_n\}$ such that there exists a finite sequence of paths $Q(1,n'_1),\dots,Q(n,m'_n)$ satisfying:
– For each $1 \leq k \leq n-1$, the path $Q(k+1,m'_{k+1})$ intersects $Q(k,m'_k)$;
– Estimate (1.13) holds for each $m = m'_k$ and $A = A_k$.
Similar arguments to those leading to (1.8) yield

$$(1-D_{n+1}) \leq (1-D_n)\frac{s_n}{r_n}\frac{s'_n}{r'_n} + \frac{1}{A_n}.$$

Indeed, the product $\frac{s_n s'_n}{r_n r'_n}$ acts as a correction factor for the passage from $Q(n)$ to $Q(n+1)$ taking into account that the paths of the form $Q(n,m)$ do not have the same number of points in $Q(n) \cap Q(n+1)$, and similarly for those of the form $Q(n+1,m)$. By induction, the preceding inequality yields

$$1-D_n \leq (1-D_1)\prod_{k=1}^{n-1}\frac{s_k}{r_k}\frac{s'_k}{r'_k} + \sum_{k=1}^{n-1}\frac{1}{A_k}\prod_{j=1}^{k-1}\frac{s_j}{r_j}\frac{s'_j}{r'_j} \leq (1-D_1)B + B\sum_{k\geq 1}\frac{1}{A_k}.$$

The choice of A_1 was made so that $D_1 = 1$, hence

$$1-D_n \leq B\sum_{k\geq 1}\frac{1}{A_k} \leq \frac{1}{2}.$$

As a consequence, $D_n \geq 1/2$, which implies that for each n we may obtain a finite sequence of n paths with the desired properties. The infinite sequence is obtained via a Cantor diagonal type argument. $\square$

The proof of Theorem A is now at hand. Indeed, the concatenation of the paths provided by the preceding lemma yields an infinite sequence of points in $\mathbb{Z}^{d-1}$ along which the value of the L_α-sum is bounded from above by

$$\sum_{n\geq 1} \frac{A_n C_6}{4^{n\left[\frac{d\alpha}{2}-\frac{1}{d-1}\right]}} \leq \frac{A_1 C_6}{4^{\frac{d\alpha}{2}-\frac{1}{d-1}}} + \sum_{n\geq 2} \frac{B\,C\,C_6}{2^{n\left[\frac{d\alpha}{2}-\frac{1}{d-1}\right]}} \leq 2A_1C_6 + 2BC_6.$$

This is in correspondence to a sequence of intervals of the form $I_{i_1,\ldots,i_{d-1}}$ each of which is obtained from the preceding one by applying one of the generators in $\{f_{2,1}, f_{3,1}, \ldots, f_{d,1}\} \cup \{f_{2,1}, f_{3,2}\ldots, f_{d,d-1}\}$. Joining this infinite sequence to a finite one from the origin to a point in $Q(1,n_1')$, we obtain an infinite sequence of intervals in the N^*_{d-1}-orbit of the interval I^* for which the L_α-sum is finite, and hence the arguments of §1.0.2 may be applied. This concludes the proof.

1.0.5 An independent combinatorial lemma

The aim of this section is to give the proof of Lemma 1.0.6. We first give the details of the construction of the partition of $\mathbb{N}_0$ into N sets (paths) $P_1, ..., P_N$, and latter we check the desired properties. The construction is made in two steps, the former of which applies to arbitrary values of N, whereas the latter is restricted to integers of the form $1+2^k$.

Step 1. Let $M > N$ be positive integers. Assume that we are given a partition

$$[[0, M-1]] = R_0 \bigcup R_1 \bigcup \ldots \bigcup R_{N-1}$$

into "consecutive" sets, that is, such that $1 + \max R_i = \min R_{i+1}$ holds for all $0 \leq i \leq N-2$. Then this induces a partition of $\mathbb{N}_0$ as follows. Denoting $R \oplus k := \{n + k \colon n \in R\}$, we define

- $S_1 := \bigcup_{j=1}^{N-1} R_j \oplus j(M-1)$,
- $S_i := \bigcup_{j=i-1}^{N-1} R_j \oplus (j-i+1)(M-1) \,\cup\, \bigcup_{j=1}^{i-2} R_j \oplus (j-i+N)(M-1)$, for $2 \leq i \leq N$.

(Notice that, by definition, the second term in the definition of S_i above is empty for $i = 2$.) Now, what defines our partition of $\mathbb{N}_0$ is the "periodic repetition" of the sets $S_1, ..., S_N$. More precisely, we let

- $P_1 := R_0 \bigcup \bigcup_{k=0}^{\infty} S_1 \oplus kN(M-1)$,
- $P_i := \bigcup_{k=0}^{\infty} S_i \oplus kN(M-1)$, for $2 \leq i \leq N$.

To have a clearer view of this construction, the reader may easily check that for the particular choice $R_0 := \{0\}$, $R_1 := \{1\}, \ldots, R_{N-2} := \{N-2\}$ and $R_{N-1} := \{N-1, N, N+1, \ldots, M-1\}$, it yields to the paths constructed in §1.0.3 (see again Figure 3 for an illustration).

It is sometimes better to think on our paths as concatenations of "patches". In this view, for $2 \leq i \leq N$, the sequence representing S_i is $R_{i-1}R_i \ldots R_{N-1}R_1R_2 \ldots R_{i-2}$, which in notation modulo $N-1$ may be rewritten as $R_{i-1}R_i \ldots R_{i+N-2}$. This means that S_i is made of a copy of R_{i-1} followed by a copy of R_i translated by $M-1$ units, a copy of R_{i+1} translated by another $M-1$ units, and so on. Similarly, our paths P_i may be seen as infinite sequences of patches. Thinking on each S_i as a patch as well, for $2 \leq i \leq N$, the path P_i is represented by $S_iS_iS_i \ldots$. The sequence representing P_1 corresponds to $R_0S_1S_1S_1 \ldots$.

Step 2. Assuming that N has the form $1+2^k$, we will associate to it a particular choice of sets $R_1, ..., R_N$. Let $p \geq 1$ and $q \geq 0$ be the integers such that

$$M-1 = (N-1)p + q, \quad \text{with } q < N-1.$$

Let us consider the binary expansion of q:

$$q = 2^{r_1} + \ldots + 2^{r_l}, \quad \text{with } r_1 > .... > r_l \geq 0.$$

(Notice that since $q < N-1 = 2^k$, we have $k > r_1$.) Now, for $1 \leq i \leq N-1$, define s_i as being the largest integer s such that 2^{k-r_s} divides i whenever there is such an index s, and as being equal to zero otherwise. We claim that the following relation holds:

$$s_1 + s_2 + \ldots + s_{N-1} = q. \tag{1.14}$$

Indeed, by definition, s_i equals $s > 0$ if and only if i is a multiple of 2^{k-r_s} but not a multiple of $2^{k-r_{s+1}}$. Now, in $\{1, 2, \ldots, N-1\}$, there are exactly 2^{r_s} multiples of 2^{k-r_s}, namely the products of 2^{k-r_s} with the integers in $\{1, 2, 3, \ldots, 2^{r_s}\}$. Hence, the left-side expression in (1.14) equals

$$\sum_{s=1}^{l} s \left|\{i\colon\, s_i = s\}\right| = \sum_{s=1}^{l-1} s\left(2^{r_s} - 2^{r_{s+1}}\right) + l2^{r_l} = \sum_{s=1}^{l} 2^{r_s} = q. \tag{1.15}$$

Finally, let us inductively define:

- $R_0 := \{0\}$,
- $R_i := \{1 + \max R_{i-1}, ..., p + s_i + \max R_{i-1}\}$, where $1 \leq i \leq N-1$.

Notice that for $1 \leq i \leq N-1$, the number of points of R_i equals

$$p + s_i \leq p + l \leq p + k = p + \log_2(N-1). \tag{1.16}$$

Using (1.14), we conclude that the number of points contained in the union of the R_i's equals

$$1 + p(N-1) + s_1 + \ldots + s_{N-1} \;=\; 1 + p(N-1) + q \;=\; M.$$

Thus, the R_i's yield a partition of $[[0, M-1]]$ into consecutive sets. We claim that the corresponding partition of $\mathbb{N}_0$ into the paths $P_1, \ldots, P_N$ produced as in Step 1 satisfies the desired properties.

Step 3. We first notice that in order to prove property (ii) of Lemma 1.0.6, we may restrict ourselves to intervals of the form $[[0, K]]$ instead of general intervals $[[K_1, K_2]]$ provided we obtain the better bound $2 + \frac{M-1}{N-1} + 2\log_2(N)$ for the maximal difference of points in $[[0, K]]$ among our N paths. This is what we now proceed to do.

Let a, b be non-negative integers such that

$$K = aN(M-1) + b, \quad \text{with } b < N(M-1),$$

Let us first consider a path P_i such that $2 \leq i \leq N$. In terms of patch sequences, and using notation modulo $N-1$, the intersection of P_i with $[[0, K]]$ has the form

$$S_i \ldots S_i R_{i-1} R_i \ldots R_{i-1+t} T, \quad \text{with } t \leq N-1.$$

Here, the patch T is a starting part of the patch R_{i+t}. Moreover, the patch S_i appears precisely a times.

By construction, the number of points in the set represented above is a times the number of points in S_i plus the sum of the number of points in $R_{i-1} \ldots R_{i-1+t}$ plus the number of points in T. The former equals $a(M-1)$, hence it is independent of $i \in \{2, \ldots, N\}$, whereas the latter is smaller than or equal to $p + s_{i+t} \leq p + \log_2(N-1)$; see (1.16). As a consequence, the difference with respect to the number of points in $[[0, K]] \cap P_j$ (with $2 \leq j \leq N$) is at most $p + \log_2(N-1)$ plus the difference between the number of points in $R_{i-1} \ldots R_{i-1+t}$ and

$R_{j-1} \ldots R_{j-1+t}$. Since $p \leq 1 + \frac{M-1}{N-1}$, our task reduces to show that the last difference is at most $\log_2(N-1)$.

Now, the number of points in the first (resp. second) sequence above equals

$$(p + s_{i-1}) + (p + s_i) + \ldots + (p + s_{i-1+t}) \;=\; tp + s_{i-1} + \ldots + s_{i-1+t}$$

$$\big(\text{resp. } (p + s_{j-1}) + (p + s_j) + \ldots + (p + s_{j-1+t}) \;=\; tp + s_{j-1} + \ldots + s_{j-1+t}\big).$$

Define $\rho_{s,i}$ (resp. $\rho_{s,j}$) as being the number of indexes in $\{i-1, \ldots, i-1+t\}$ (resp. $\{j-1, \ldots, j-1+t\}$) that are multiples of 2^{k-r_s}. A similar argument to that leading to (1.15) yields

$$s_{i-1} + \ldots + s_{i-1+t} = \rho_{1,i} + \rho_{2,i} + \ldots + \rho_{l,i} \quad \big(\text{resp. } s_{j-1} + \ldots + s_{j-1+t} = \rho_{1,j} + \rho_{2,j} + \ldots + \rho_{l,j}\big).$$

Since

$$\frac{t}{2^{k-r_s}} \leq \rho_{s,i} \leq 1 + \frac{t}{2^{k-r_s}} \quad \big(\text{resp. } \frac{t}{2^{k-r_s}} \leq \rho_{s,j} \leq 1 + \frac{t}{2^{k-r_s}}\big),$$

we conclude that $|\rho_{s,i} - \rho_{s,j}|$ equals zero or 1. We thus conclude that

$$\big|s_{i-1} + \ldots + s_{i-1+t} - s_{j-1} - \ldots - s_{j-1+t}\big| \;\leq\; |\rho_{1,i} - \rho_{1,j}| + \ldots + |\rho_{l,i} - \rho_{l,j}| \;\leq\; l \;\leq\; k \;=\; \log_2(N-1),$$

as we wanted to show.

Actually, so far we have obtained the upper bound $1 + \frac{M-1}{N-1} + 2\log_2(N-1)$ for the difference between the number of points in $P_i \cap [[0, K]]$ and $P_j \cap [[0, K]]$. The extra 1 which lacks appears when making comparisons with the path P_1, taking into account that P_1 starts with $R_0 = \{0\}$. The proof of this follows the same ideas above. We leave the details to the reader.

Chapter 2

AN ALTERNATIVE PROOF OF THEOREM A

The aim of this charper is to give an alternative (and simpler) proof of Theorem A. As we saw in Chapter 1, the core of the proof consisted in finding "good paths" in a graph. Here, we will perform this task by appealing to the general framework of moduli of curves, as we next explain.

2.0.6 Some preliminaries and notation: moduli in metric spaces

A good general discussion of this theory can be found in [9, 10].

Let (X, μ) be a metric measured space. As usual, we denote by $B_r(x)$ the open ball in X of center x and radius r. For $C > 0$, we let $CB_r(x) := B_{Cr}(x)$. We use the generic notation B_r for a ball of radius r whenever the center is irrelevant for the discussion.

By a curve γ in X we mean a continuous function $\gamma\colon I = [a, b] \longrightarrow X$ for some closed interval I. The *length* of γ is defined as

$$l(\gamma) = \sup \left\{ \sum_{i=1}^{n} \left|\gamma(t_i) - \gamma(t_{i-1})\right| \right\},$$

where the supremum is taken over all finite sequences $a = t_0 < t_1 < \ldots < t_n = b$. If the number $l(\gamma)$ is finite, then we say that γ is *rectifiable*.

For $\gamma : I \longrightarrow X$ a rectifiable curve, there always exists a unique 1-Lipschitz continuous curve $\gamma' : [0, l(\gamma)] \longrightarrow X$ such that

$$\gamma = \gamma' \circ s,$$

where $s : I \longrightarrow [0, l(\gamma)]$ is the length function of γ. Given a nonnegative Borel function f, we define

$$\int_\gamma f := \int_{t=0}^{l(\gamma)} f \circ \gamma'(t) dt.$$

Given Γ a family of curves in X and a real number $p \geq 1$, the $p-modulus$ of Γ is defined as

$$mod_p(\Gamma) := \inf \int_X f^p d\mu,$$

where the infimum is taken over the Borel positive functions f that are *admissible* for Γ, that is

$$\int_\gamma f \geq 1$$

for all rectifiable curves γ in Γ.

Let U be an open set in X and u a real-valued function defined on U. We say that a Borel function f is a *very weak gradient* of u in U if for all x, y in U and for any rectifiable curve γ joining x and y, we have

$$\left|u(x) - u(y)\right| \leq \int_\gamma f.$$

Finally, we say that X satisfies a $(1, p)$-*weak-Poincaré's inequality* if there exist constants $C_p > 0$ and $C' \geq 1$ such that

$$\frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq C_p(diam(B)) \left(\frac{1}{\mu(C'B)} \int_{C'B} |f|^p d\mu \right)^{\frac{1}{p}}$$

whenever u is a bounded continuous function defined on $C'B$, f is a very weak gradient of u, B is an open ball in X, and u_B is the mean of u on B (with respect to μ). The term *weak* in this definition is because the ball in the right side is bigger than the ball in the left side.

We recall that a measure μ on X satisfies the *Q-lower bounded condition* (resp. *doubling condition*) if there exists a constant $C' > 0$ (resp. $C'' \geq 1$) such that $\mu(B_r) \geq C' r^Q$ (resp. $\mu(B_{2r}) \leq C'' \mu(B_r)$) for every ball B_r in X. Given E,F two closed sets in an open set U of X, we denote by (E, F, U) the set of all curves in U joinning E and F.

There exists another useful way to compute the moduli of families of curves, called the *capacity*. Given an open set U in X and E, F two closed subsets of U, we define the *p-capacity* of (E, F, U) as

$$cap_p(E, F, U) := \inf \int_X f^p d\mu,$$

where the infimum is taken over all weak gradients of all functions u such that $u|E \geq 1$ and $u|F \leq 0$.

A basic result from [10] states that

$$cap_p(E, F, U) = mod_p(E, F, U). \tag{2.1}$$

The idea of the alternative proof of Theorem A is to replace the upper estimates on the value of certain series appearing in Chapter 1 by a lower estimative for the modulus of certain families of curves in a suitable metric space. The following adapted version of [10, Theorem 5.9] is exactly what gives us the information what we need.

Theorem 2.0.9. *Let (X, μ) be a metric measured space with a locally finite, doubling measure μ satisfying the Q-lower bounded condition. Assume also that X satisfies a $(1, p)$-weak-Poincaré's inequality for some $1 \leq p \leq Q$. Then there exists a constant $C \geq 1$ such that for any two closed subsets E, F in a ball B_r of radius r satisfying*

$$\min\{diam(E), diam(F)\} \geq \lambda r^{1-Q} \mu(B_r)$$

for some $0 < \lambda \leq 1$, one has

$$cap_p(E, F, B_{Cr}) \geq C^{-1} \lambda \mu(B_r) r^{-p}.$$

Since we are interested on graphs, the next remark simplifies the discussion.

Remark 2.0.10. Suppose X is a locally finite graph with degree uniformly bounded by $K > 0$. In this framework, almost all previus definitions and results work except for the concept of the integral over a curve γ and other notions depending on this. Following [3], given a path $\gamma = (x_1, \ldots, x_n)$ in X (n can be infinite) and f a real positive function defined on the set of vertices of X, we define

$$\int_\gamma f := \sum_{i=1}^{n} f(x_i) \tag{2.2}$$

Let $\nabla u(x) := \sum_{d(x,y)=1} \big|u(x) - u(y)\big|$ be the *discrete gradient* of u, and put

$$cap'_p(E, F, B_r(x)) := \inf \left\{ \sum_{y \in B_r(x)} (\nabla u)^p(y) : u|E \geq 1, u|F \leq 0, u\colon B_r(x) \longrightarrow \mathbb{R} \right\}$$

Then $\quad cap'_p(E, F, B_{r-1}(x)) \leq C \cdot cap_p(E, F, B_r(x)), \quad$ for some constant C depending on K and p. Indeed, for any very weak gradient f of u, we have

$$\nabla u(x) \leq \sum_{y \in B_1(x)} f(y),$$

which implies

$$\sum_{y \in B_{r-1}(x)} (\nabla u)^p(y) \leq C \sum_{y \in B_r(x)} f^p(y),$$

as desired.

It is important to emphasize that equation (2.1) is true in the context above, that is, when integrals are given by (2.2). Also, Theorem 2.0.9 still holds for *cap'* instead of *cap*. Therefore, when dealing with graphs, in order to get a lower estimative of moduli, it suffices to work with *cap'* (and with the discrete gradient for the Poincaré inequality).

2.0.7 Schreier graphs as metric spaces

Let G be a finitely generated group, H a subgroup of G, and S a finite system of generators of G. We denote by $H\backslash G$ the *Schreier graph* associated to G, H and S. More precisely, the set of vertices of $H\backslash G$ is $V(H\backslash G) := \{Hg : g \in G\}$, and the set of edges is $E(H\backslash G) := \{(Hg, Hgs) : g \in G, s \in S\}$. Observe that if H is trivial, then $\{e\}\backslash G$ is the Cayley graph of G associated to S, which is denoted $Cayley(G, S)$.

Denote by $B_r^{H\backslash G}(Hg)$ (resp. $B_r^G(g)$) the ball of center Hg (resp. g) and radius r in $H\backslash G$ (resp. $Cayley(G, S)$)).

In this section, we will show that Schreier graphs satisfy the Poincaré inequality and, whenever they have polynomial growth, the doubling property. Although these facts are folklore, it is hard to find them in the literature, hence we will provide complete proofs. We begin with the Poincaré inequality.

Proposition 2.0.11. *The graph $H\backslash G$ satisfies a $(1,1)$-Poincaré's inequality. More precisely,*

$$\sum_{Hg_1 \in B_r^{H\backslash G}(Hg)} \left|u(Hg_1) - u_{B_r^{H\backslash G}(Hg)}\right| \leq 2r \sum_{Hg_1 \in B_r^{H\backslash G}(Hg)} |\nabla u|(Hg_1), \tag{2.3}$$

where u is any function defined on the set of vertices of $H\backslash G$, and ∇u is the discret gradient of u.

Proof. To simplify, we write $B_r(Hg)$ instead of $B_r^{H\backslash G}(Hg)$. Note that, by the triangle inequality,

$$|B_r(Hg)| \sum_{Hg_1 \in B_r(Hg)} \left|u(Hg_1) - u_{B_r(Hg)}\right| \leq \sum_{Hg_1, Hg_2 \in B_r(Hg)} \left|u(Hg_1) - u(Hg_2)\right|. \tag{2.4}$$

Hence, we just need to control $|u(Hg_1) - u(Hg_2)|$. Given Hg_1, Hg_2 in $B_r(Hg)$, for $i = 1, 2$, there exist $0 \leq d(Hg_i, Hg) := r_i \leq r$ and $\gamma_1^i, \ldots, \gamma_{r_i}^i$ in S such that the sequences

$$(Hg_1, Hg_1\gamma_1^1), (Hg_1\gamma_1^1, Hg_1\gamma_1^1\gamma_2^1), \ldots, (Hg_1\gamma_1^1\gamma_2^1\cdots\gamma_{r_1-1}^1, Hg_1\gamma_1^1\gamma_2^1\cdots\gamma_{r_1}^1),$$

$$(Hg, Hg\gamma_1^2), (Hg\gamma_1^2, Hg\gamma_1^2\gamma_2^2), \ldots, (Hg\gamma_1^2\gamma_2^2\cdots\gamma_{r_2-1}^2, Hg\gamma_1^2\gamma_2^2\cdots\gamma_{r_2}^2)$$

are paths that join Hg_1 with Hg and Hg with Hg_2, respectively, with the property that $Hg_1\gamma_1^1\gamma_2^1\cdots\gamma_i^1 \in \partial B_{r_1-i}(Hg)$ and $Hg\gamma_1^2\gamma_2^2\cdots\gamma_j^2 \in \partial B_j(Hg)$, where $i \in \{1, \ldots, r_1\}$ and $j \in \{1, \ldots, r_2\}$. Set

$$S_1(Hg_1) := |u(Hg_1) - u(Hg_1\gamma_1^1)| + |u(Hg_1\gamma_1^1) - u(Hg_1\gamma_1^1\gamma_2^1)| + \cdots + |u(Hg_1\gamma_1^1\gamma_2^1\cdots\gamma_{r_1-1}^1) - u(Hg)|,$$

$$S_2(Hg_2) := |u(Hg) - u(Hg\gamma_1^2)| + |u(Hg\gamma_1^2) - u(Hg\gamma_1^2\gamma_2^2)| + \cdots + |u(Hg\gamma_1^2\gamma_2^2\cdots\gamma_{r_2-1}^2) - u(Hg_2)|.$$

Then,

$$|u(Hg_1) - u(Hg_2)| \leq S_1(Hg_1) + S_2(Hg_2).$$

Now, we proceed to control S_1:

$$\sum_{r_1=0}^{r} \sum_{d(Hg_1,Hg)=r_1} S_1(Hg_1) \leq \sum_{r_1=0}^{r} \sum_{d(Hg_1,Hg)=r_1} \nabla u(Hg_1) + \nabla u(Hg_1\gamma_1^1) + \cdots + \nabla u(Hg_1\gamma_1^1\gamma_2^1\cdots\gamma_{r_1-1}^1).$$

Since $Hg_1\gamma_1^1\gamma_2^1\cdots\gamma_i^1 \in \partial B_{r_1-i}$, we get

$$\begin{aligned}\sum_{d(Hg_1,Hg)=r_1} \nabla u(Hg_1) + \nabla u(Hg_1\gamma_1^1) + \cdots + \nabla u(Hg_1\gamma_1^1\gamma_2^1\cdots\gamma_{r_1-1}^1) &\leq \sum_{j=0}^{r_1-1} \sum_{d(Hg_1,Hg)=r_1-j} \nabla u(Hg_1) \\ &= \sum_{d(Hg_1,Hg)\leq r_1} \nabla u(Hg_1) \\ &\leq \sum_{Hg_1 \in B_r(Hg)} \nabla u(Hg_1).\end{aligned}$$

Therefore,

$$\sum_{Hg_1 \in B_r(Hg)} S_1(Hg_1) \leq \sum_{i=0}^{r} \sum_{Hg_1 \in B_r(Hg)} S_1(Hg_1) = r \sum_{Hg_1 \in B_r(Hg)} \nabla u(Hg_1)$$

Similar computations for S_2 show that

$$\sum_{Hg_2 \in B_r(Hg)} S_2(Hg_2) \leq r \sum_{Hg_2 \in B_r(Hg)} \nabla u(Hg_2).$$

Thus,

$$\begin{aligned} \sum_{Hg_1 \in B_r(Hg)} \sum_{Hg_2 \in B_r(Hg)} \big|u(Hg_1) - u(Hg_2)\big| &\leq \sum_{Hg_1 \in B_r(Hg)} \sum_{Hg_2 \in B_r(Hg)} \big(S_1(Hg_1) + S_2(Hg_2)\big) \\ &\leq 2r|B_r(Hg)| \sum_{Hg_1 \in B_r(Hg)} \nabla u(Hg_1). \end{aligned}$$

Joining this inequality with (2.4), we finally obtain (2.3). $\square$

Remark 2.0.12. Inequality (2.3) combined with Jenhsen's inequality yield, for every real number $p > 1$,

$$\frac{1}{|B_r^{H\backslash G}(Hg)|} \sum_{Hg_1 \in B_r^{H\backslash G}(Hg)} \big|u(Hg_1) - u_{B_{H\backslash G}(Hg,r)}\big| \leq \frac{2r}{|B_r^{H\backslash G}(Hg)|^{\frac{1}{p}}} \left(\sum_{Hg_1 \in B_r^{H\backslash G}(Hg)} |\nabla u|^p(Hg_1) \right)^{\frac{1}{p}}.$$

Next, we verify the doubling property for the Schreier graph.

Proposition 2.0.13. For each $r > 0$ and $g \in G$,

$$\frac{|B_r^G(g)|}{|B_{2r}^G(g) \cap Hg|} \leq |B_r^{H\backslash G}(Hg)| \leq \frac{|B_{2r}^G(g)|}{|B_r^G(g) \cap Hg|}. \tag{2.5}$$

Moreover, if G has polynomial growth, then the cardinalities of the balls in $H\backslash G$ satisfy the doubling property.

Proof. We first prove our claim for $g \in H$, so we can assume $g = e$. Let $\pi \colon B_r^G(e) \longrightarrow B_r^{H\backslash G}(H)$ be the canonical projection (which is surjective). Then

$$\pi^{-1}(Hw) = \{w' \in B_r^G(e) : w' = hw, h \in H\} = B_r^G(e) \bigcap Hw. \tag{2.6}$$

We can assume $\|w\| = d(H, Hw) \leq r$, so $|\pi^{-1}(Hw)| = |(B_r^G(e)w^{-1}) \bigcap H| \leq |B_{2r}^G(e) \bigcap H|$. This means that π is at most $|B_{2r}^G(e) \cap H|$-injective. Therefore,

$$|B_r^G(e)| \leq |B_r^{H\backslash G}(H)| \cdot |B_{2r}^G(e) \cap H|,$$

which shows the left-side inequality of (2.5).

For the other inequality, note that if $\|w\| = d(Hw, H)| \le r$, then $B^G_{2r}(e)w^{-1}$ contains $B^G_r(e)$. Due to (2.6), this implies

$$\begin{aligned} |B^G_{2r}(e)| = \sum_{d(H,Hw)|\le 2r} |\pi^{-1}(Hw)| &\ge \sum_{d(H,Hw)|\le r} \left|B^G_r(e) \bigcap H\right| \\ &= \left|B^G_r(e) \cap H\right| \cdot \left|B^{H\backslash G}_r(H)\right|. \end{aligned}$$

This proves the right-side inequality of (2.5).

To deduce the inequality for an arbitrary $g \in G$, just notice that

$$|B^{H\backslash G}_r(Hg)| = |B^{g^{-1}Hg\backslash G}_r(H)|$$

and

$$|B^G_r(e) \cap g^{-1}Hg| = |B^G_r(g) \cap Hg|.$$

Finally, if G has polynomial growth, then the doubling property for $H\backslash G$ follows from

$$|B^{H\backslash G}_{2r}(Hg)| \le \frac{|B^G_{4r}(g)|}{|B^G_{2r}(g) \cap Hg|} \le C\frac{|B^G_r(g)|}{|B^G_{2r}(g) \cap Hg|} \le C|B^{H\backslash G}_r(Hg)|,$$

where $C = C(G)$ depends only on G. This proves the proposition. □

Remark 2.0.14. Note that if $d(Hg, H) = \|g\| \le Cr$, then by the doubling property, for some appropriate constant $C' = C'(G, C)$, we have

$$|B^{H\backslash G}_r(Hg)| \le |B^{H\backslash G}_{(1+C)r}(H)| \le C'|B^{H\backslash G}_r(H)|.$$

On the other hand, $|B^G_{2r}(g) \cap H| \le |B^G_{(2+C)r}(e) \cap H|$. Thus, the doubling property yields another constant $C'' = C(G, C)$ such that

$$|B^{H\backslash G}_r(Hg)| \ge \frac{|B^G_r(e)|}{|B^G_{(2+C)r}(e) \cap H|} \ge C''\frac{|B^G_{2(2+C)r}(e)|}{|B^G_{(2+C)r}(e) \cap H|} \ge C''|B^{H\backslash G}_r(H)|$$

This means that $|B^{H\backslash G}_r(Hg)| \sim |B^{H\backslash G}_r(H)|$.

2.0.8 Another proof of Theorem A

Recall from §1.0.2 that associated to the action $\phi_{_{FF}}$ there is a graph corresponding to the orbit of a certain interval. As we will next see, this is nothing but a Schreier graph, to which previous discussion may be applied.

Denote by G_{d-1} the group generated by the linear maps $g_i, h_j : \mathbb{Z}^{d-1} \longrightarrow \mathbb{Z}^{d-1}$ defined as:

$$g_i(x_1, \ldots, x_{d-1}) := (x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_{d-1}), \quad i = 1, \ldots, d-1,$$

$$h_j(x_1, \ldots, x_{d-1}) := (x_1, \ldots, x_j, x_{j+1} + x_j, x_{j+2}, \ldots, x_{d-1}), \quad j = 1, \ldots, d-2.$$

In the notation of §1.0.2, g_i and h_j correspond to the action of $f_{i+1,1}$ and $f_{j+2,j+1}$ over $1 \times \mathbb{Z}^{d-1}$, respectively.

The group G_{d-1} has polynomial growth of degree $\frac{d(d-1)}{2} + d - 2$. Although this may be directly obtained from the Bass-Guivarch formula [8, Appendix], an elementary computation proceeds as follows. First, observe that $g_{i+1}^{mn} = h_i^{-n} g_i^{-m} h_i^n g_i^m$ and $[g_i, g_j] = [f_i, f_j] = e$ for $j > i$. It is then easy to prove by inducction (on r) that each word $w \in B_r^{N_{d-1}}(e)$ can be written of the form $g_{d-1}^{q_{d-1}} h_{d-2}^{p_{d-2}} g_{d-2}^{q_{d-2}} \cdots g_2^{q_2} h_1^{p_1} g_1^{q_1}$, where $|p_j| \leq r$ and $|q_i| \leq r^i$, for $j = 1, \ldots, d-2$ and $i = 1, \ldots, d-1$. This gives us the desired upper bound for the cardinality of $B_r^{N_{d-1}}(e)$. For the lower bound, we proceed by induction on d. The estimate is clear for $d = 3$, since N_2 is the Heisenberg group. For larger values of f, notice that each word in $B_r^{N_{d-2}}(1)$ can be identified with a word in $B_r^{N_{d-1}}(e)$, so the number of words of length r with respect to $g_1, \ldots, g_{d-1}, f_1 \ldots, f_{d-2}$ is at least $Cr^{\frac{(d-1)(d-2)}{2} + d - 3}$ for a certain constant $C > 0$. Pick any word w in this set of $Cr^{\frac{(d-1)(d-2)}{2} + d - 3}$ elements and take a point $(x_1, \ldots, x_{d-1}) \in \mathbb{Z}^{d-1}$ such that the value of the $(d-2)$th-coordinate of $w(x_1, \ldots, x_{d-1})$ is greater than r^{d-1}. Take $|i| \leq r^{d-1}$ and $|j| \leq r$. Then the last coordinate of the points $g_{d-1}^i h_{d-2}^j w(x_1, \ldots, x_{d-1})$ are different. Since $g_{i+1}^{mn} = f_i^{-n} g_i^{-m} f_i^n g_i^m$, the points of the form g_{d-1}^i, with $|i| \leq r^{d-1}$, belong to $B_{C'r}^{N_{d-1}}(e)$ for a certain constant $C' > 0$. This easily yields the desired lower bound.

Define H_{d-1} as the subgroup of G_{d-1} generated by $h_1, \ldots, h_{d-2}$. Then the graph associated to the action $\phi_{_{FF}}$ can be interpreted as $H_{d-1} \backslash G_{d-1}$: We identified the coset H_{d-1} with the point $(0, \ldots, 0) \in \mathbb{Z}^{d-1}$; and then each coset $H_{d-1}g$ can be viewed as a point on $\mathbb{Z}^{d-1}$. For example, the the coset $H_{d-1}g_1^3h_1^6g_{d-2}^4$ corresponds to the point $(3, 18, 0, \dots, 0, 4, 0) \in \mathbb{Z}^{d-1}$.

The core of the proof of Thorem A given in Chapter 1 then reduces to the next proposition, for which we provide an alternative proof using the ideas introduced in the last two sections.

Proposition 2.0.15. *Given $u : V(H_{d-1}\backslash G_{d-1}) \longrightarrow \mathbb{R}^+$ a summable function and a real number $\alpha > \frac{2}{d(d-1)}$, there exists a path γ on $H_{d-1}\backslash G_{d-1}$ going to infinite, such that*

$$\sum_{H_{d-1}g\in\gamma} \big|u(Hg)\big|^\alpha < \infty.$$

Moreover, γ can be taken going to infinite in any prescribed direction.

To prove this, we will strongly use Theorem 2.0.9. As the Poincaré inequality and the doubling property are ensured by Propositions 2.0.11 and 2.0.13, respectively, what we need to check is the Q-lower bounded condition for $H_{d-1}\backslash G_{d-1}$ for some $Q > 0$.

We focus on the cardinalities of the balls in $H_{d-1}\backslash G_{d-1}$. Keeping in mind that each coset $H_{d-1}g$ can be viewed as a point in $\mathbb{Z}^{d-1}$, we take $(x_1, \dots, x_{d-1}) \in \mathbb{Z}^{d-1}$, and define $s_1 := \min\{\max\{r, |x_1|\}, r^2\}$ and $s_i := r\min\{\max\{s_{i-1}, |x_i|\}, r^i\}$, for $i = 2, \dots, d-2$. Then

$$\big|B_r^{H_{d-1}\backslash G_{d-1}}((x_1, \dots, x_{d-1}))\big| \sim \begin{cases} r^2 s_1 & \text{if } d = 3, \\ r^2 s_{d-2}\Pi_{j=2}^{d-2} s_j & \text{if } d \geq 4. \end{cases} \tag{2.7}$$

For example, for $d = 4$, we have

$$|B_r^{H_3\backslash G_3}((x_1, x_2, x_3))| \sim \begin{cases} r^6 & \text{if } \max\{|x_1|, |x_2|\} \leq r, \\ r^4 \max\{|x_1|, |x_2|\}^2 & \text{if } r < \max\{|x_1|, |x_2|\} \leq r^2, \\ r^8 & \text{if } r^2 < \max\{|x_1|, |x_2|\}. \end{cases}$$

Let us explain formula (2.7). In the case $d \geq 4$, for the first coordinate, we have a number of possibilities of order r. For the second one, the order is $r\min\{\max\{s_1, |x_2|\}, r^2\}$, as we use h_1^r to get r jumps of length $\max\{s_1, |x_2|\}$ in the second direction and then reach $\sim r^2$ of these points using the relation $g_2^{mn} = h_1^{-n}g_1^{-m}h_1^n g_1^m$. For the rest of the coordinates, using the relation $g_{i+1}^{mn} = h_i^{-n}g_i^{-m}h_i^n g_i^m$, the same procedure yields (2.7). For the case $d = 3$, these ideas still apply, but notice that we have a smaller number of coordinates.

Now it is clear that if $|x_i| \sim r$ for each i, then for all $(y_1, ..., y_{d-1}) \in \mathbb{Z}^{d-1}$ we have

$$\left|B_r^{H_{d-1}\backslash G_{d-1}}((x_1, \ldots, x_{d-1}))\right| \sim r^{\frac{d(d-1)}{2}} \gtrsim \left|B_r^{H_{d-1}\backslash G_{d-1}}((y_1, \ldots, y_{d-1}))\right|.$$

This establishes the desired Q-lower bound condition for $Q := \frac{d(d-1)}{2}$.

Proof of Proposition 2.0.15. To simplify, we use the notation G and H for N_{d-1} and G_{d-1}, respectively.

Without loss of generality, we can suppose that the sum of u over $V(H\backslash G)$ equals 1. Set $Q := \frac{(d-1)d}{2}$ and $p := \alpha^{-1}$. Let $\{w_0.w_1, \ldots\}$ be a sequence of points in G satisfying

$$d\big(B_{2^i}^{H\backslash G}(Hw_i), B_{2^{i+1}}^{H\backslash G}(Hw_{i+1})\big) = 2^{i+1}.$$

If $w_0 \in H$, then $2^i \leq d(Hw_i, H) \leq 1 + (3)\sum_{j=1}^{i-1} 2^j + 2^{i+1} \leq 6(2^i)$, and so we can assume $\|w_i\| = d(Hw_i, H) \leq 6(2^i)$. Associated to these points we consider the following collection of sets: $F_0 := \partial B_2^{H\backslash G}(Hw_1)$, $E_0 := \partial B_1^{H\backslash G}(Hw_0)$, $F_n := \partial B_{2^{n+1}}^{H\backslash G}(Hw_{n+1})$, $E_n := F_{n-1}$ and $K_n := B_{7(2)^n}^{H\backslash G}(Hw_n)$. By Remark 2.0.14,

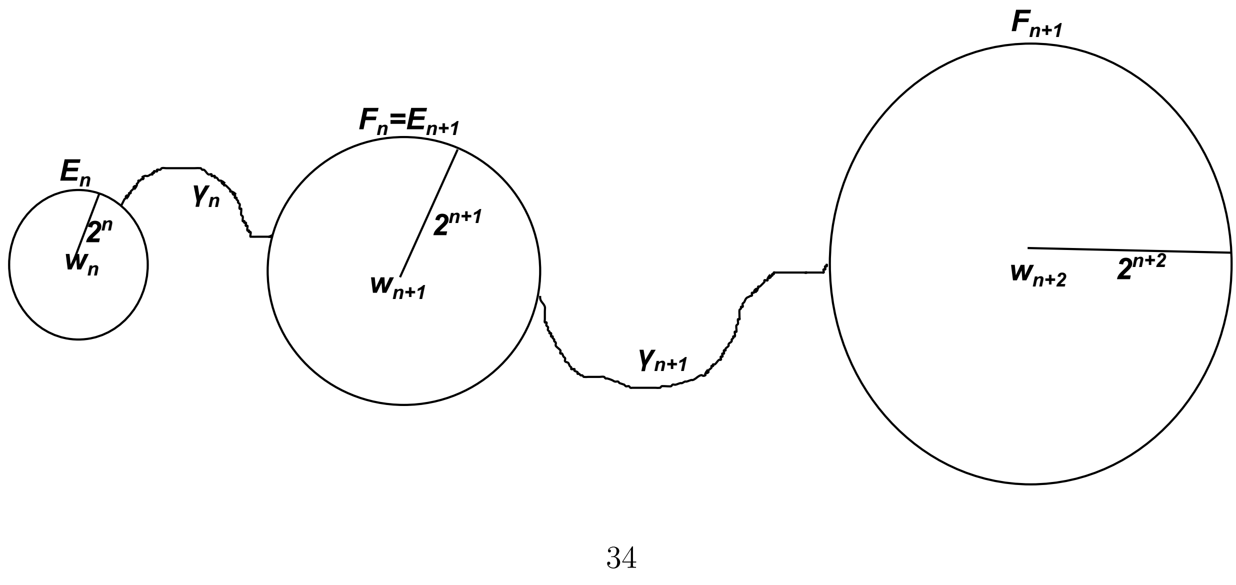
$$\min\{diam(F_n), diam(E_n)\} = 2^n = (7C_1)^{-1}(7(2)^n)^{1-Q}C_1(7(2)^n)^Q \geq \lambda(7(2)^n)^{1-Q}|K_n|,$$

with $\lambda = (7C_1)^{-1}$. So, if we denote by Λ_n the set of curves contained in CK_n that join E_n with F_n, the Theorem 2.0.9 and the remark 2.0.12, implies

$$mod_p(\Lambda_n) \geq C^{-1}\lambda(7\cdot(2^n))^{-p}|K_n| \geq \lambda C^{-1}C_1^{-1}(7\cdot(2^n))^{Q-p}.$$

Consequently, there must exist a curve γ_n in Λ_n such that

$$\sum_{Hg\in\gamma_n}(u(Hg))^\alpha \leq \{\lambda^{-1}CC_1(7\cdot(2^n))^{p-Q}\}^\alpha$$



If γ_n intersects γ_{n+1} for each n, then we are done. However, it is not clear that this happens. To solve this eventual problem, we define E'_n and F'_n as being the segments of γ_n and γ_{n+1}, respectively, which are contained in $B^{H\backslash G}_{2^{n+2}}(w_{n+1})$; then we consider Ω_n the family of curves in $B^{H\backslash G}_{2^{n+2}}(w_{n+1})$ that join these two sets. Since E'_n and F'_n join the boundaries of the the balls $B^{H\backslash G}_{2^{n+1}}(w_{n+1})$ and $B^{H\backslash G}_{2^{n+2}}(w_{n+1})$, Remark 2.0.14 yields

$$\begin{aligned}\min\{diamF'_n, diamE'_n\} &\geq 2^{n+2} - 2^{n+1} = 2^{n+1} = (2C_1)^{-1}C_1(2^{n+2})^Q(2^{n+2})^{1-Q} \\ &\geq \lambda'(2^{n+1})^{1-Q}|B_{2^{n+1}}(w_{n+1})|,\end{aligned}$$

where $\lambda' = (2C_1)^{-1}$. Thus, reasoning as above, we can find a curve $\sigma_n \in \Omega_n$ such that

$$\sum_{Hg\in\sigma_n} (u(Hg))^\alpha \leq \{(\lambda')^{-1}CC_1(2^n)^{p-Q}\}^\alpha$$

The concatenation of the paths γ_n and σ_n at intersecting points then provides us the desired path. Finally, note that we can choose the sequence of points $\{w_1, w_2, ...\}$ going to infinity in any prescribed direction.

Remark 2.0.16. The same argument in the critical case $\frac{d(d-1)}{2} = \frac{1}{\alpha} := p$ gives a curve β in the graph $H\backslash G$, starting at H, which can be written as

$$\beta_1 \cup \beta_2 \cup \beta_3 \cup \cdots ,$$

where β_m is a curve such that $d(H, \beta_m) \geq 2^{m+1}$ and $\sum_{Hg\in\beta_m} u(Hg)^d \leq C \cdot m$ for a constant C that depends only of the growth of G. This yields a path $(Hg_1, Hg_2, \ldots)$ in $H\backslash G$ such that

$$\sum_{i=1}^{m} u(Hg_i)^d \leq C\log_2(m+1) \leq C'\log(m).$$

Such paths plays a central role in [17], where their existence is established by constructive arguments similar to those of Chapter 1.

CONJECTURES

The Q-lower bounded condition for $H\backslash G$ can be stated as

$$r^Q \sim |B_r^{H\backslash G}(H)| \succsim |B_r^{H\backslash G}(g)|.$$

Under this assumption, the argument of proof of Proposition 2.0.15 would yield an affirmative answer to the the following

Conjecture 2.0.1. *Let G be a group of polynomial growth and H a subgroup of G. Given $u : V(H\backslash G) \longrightarrow \mathbb{R}^+$ a summable function and a real number $\alpha > \frac{1}{Q}$, there exists a path γ on $H\backslash G$ going to infinity, such that:*

$$\sum_{Hg\in\gamma} |u(Hg)|^\alpha < \infty.$$

Moreover, γ can be taken going to ininity in any prescribed direction.

The relevance of this conjecture in relation to nilpotent group actions on the interval lies in that each time we can associate a Schreier graph to the orbit of an interval that is fixed by a central element, this will provide us an upper bounded for the regularity of the action (at scale $C^{1+\alpha}$).

Note that for a group G of polynomial growth of degree d and $H = \{e\}$, the Q-lower bounded condition is trivially true with $Q = d$, so Conjecture 2.0.1 holds in this case. For the general case, in view of Proposition 2.0.13 , the problem reduces to the next

Conjecture 2.0.2. *Given a group G of polynomial growth and a subgroup H, then*

1. *There exist positive rational number Q' such that $|B_r^G(e) \cap H| \sim r^{Q'}$,*

2. *There exist a positive real number C such that for all $g \in G$ and $r < |g|$,*

$$|B_r^G(e) \cap H| \leq C|B_r^G(g) \cap Hg|.$$

There is a natural candidate for the value of Q' above. Indeed, suppose that G is nilpotent of degree n and denote by $G = G_1 \lhd G_2 \lhd \cdots \lhd G_{n+1} = \{e\}$ its decending central series. Define $H_i := H \cap G_i$, and let d_i be the rank of H_i/H_{i+1}. Then the candidate is

$$Q' = \sum_i i d_i.$$

Observe that for H trivial, this is nothing but the Bass-Guivarch formula for the exponent growth of G; see [8, Appendix].

Next we provide a proof that Q' is the right exponents for the case $n = 2$.

Lower bound: Since G_1/G_2 is abelian, $|B_r^{G_1/G_2}(G_2) \cap G_2H| \sim r^{rank(G_2H/G_2)}$. By Lemma 1 of [8, Appendix], $|G_2 \cap H \cap B_r^G(e)| \sim r^{2rank(G_2 \cap H)}$, so the lower bound follows.

Upper bound: It is clear that $|B_r^{G_1/G_2}(G_2) \cap G_2H| \sim r^{rank(G_2H/G_2)}$. Let $g_1, \ldots, g_d \in G$ be such that their projections generate $H/(G_2 \cap H)$. Then each word $w \in B_r^G(e) \cap H$ can be written in the form $(\Pi_{j=1}^d g_j^{l_j})w'$, where $\sum_i |l_i| \leq r$ and w' is an element in $G_2 \cap H$ whose length in G_2 is of order r^2, The upper bound easily follows from this.

For the second claim of Conjecture 2.0.2, we think that the distortion of H (see [18]) will play a role in the proof. Moreover, Breuillard has recently given a quite convincing hint of using Lie type methods; more precisely, a translation of the problem into the Lie algebra should yield an affirmative answer.

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