A Dynamical Systems Approach to Root-Finding Algorithms

Gerardo Honorato Gutiérrez

Agradecimientos

En primer lugar agradezco a mis padres Patricio e Hilda por su constante apoyo y cariño, además de Fabiola, Pato y Tatiana. A Felipe simplemente por su alegría.

A mi orientador Sergio Plaza, por proponer el tema de trabajo además de la confianza depositada a través de estos años.

Agradezco a los profesores que integran la comisión calificadora, Andrés Navas, Jan Kiwi y Rafael Labarca por las observaciones hechas al trabajo. Mi especial agradecimiento a Juan Rivera–Letelier por las estimulantes conversaciones en Matemática.

A los profesores y funcionarios del depto. de Matemática de la USACH quienes directa o indirectamente influyen en el buen desarrollo de cada uno de los estudiantes. Agradecimiento especial a Víctor Guiñez por su apoyo a lo largo de estos años.

A los profesores, ex–profesores y estudiantes del depto. de Matemática de la UCN por la agradable acogida el 2007. En especial a mis amigos Esteban y Luis por permitirme hacer de su casa, la mía. A mis amigos Juan, Francisco, Pancho, Felipe y Alvaro con quienes se comparte la vida gratamente más allá de las aulas.

Dedico este trabajo a Alicia, en un intento de compensación y reciprocidad al tiempo que falté y al amor que me dedica.

Finalmente agradezco el apoyo brindado a través de los proyectos: PBCT/CONYCIT, Anillo Sistemas dinámicos en baja dimensión, al proyecto Fondecyt ♯1095025 y becas VRID de la Universidad de Santiago de Chile. Gerardo.

Abstract

The main topic of this work is the dynamics of root–finding algorithms of complex polynomials. We study the dynamical aspects of König's method, Newton's method for multiple roots, σ–Schröder's method, Chebyshev's method, Whittaker's method and the super-Halley method.

If f is a complex polynomial, then the roots of f are (super) attracting fixed points of Newton's method N_f and infinity is the unique repelling fixed point of N_f. Using Newton's method, we show that the fixed points associated to a root of the methods under study are (super) attracting. Chebyshev's method, Whittaker's method and the super-Halley method have infinity as a repelling fixed point. Unlike Newton's method, these methods have other repelling fixed points.

It is well known that a useful tool for reducing the dimension of parameter space of Newton's method applied to cubic polynomials is the so–called Scaling Theorem. For this, we prove Scaling type theorems for a general family which include the methods we are studying.

A rational map of degree d whose fixed points are (super)attracting, with multiplier $1 - 1/n$ for $n \in \mathbb{N}$ and infinity as the unique repelling fixed point, is a Newton's method of a certain polynomial (see [30]). We show a similar result for Newton's method for multiple roots.

We give an example of a Julia set of Newton's method for multiple roots which is conjugated to the Julia set of the map $z^3 + \lambda/z$ for $\lambda = 3/16$.

In [13], J. Hubbard, D. Schleicher and S. Sutherland show that if f is a polynomial of degree d, then there is a finite set S_d, depending only on d, such that given any root α of f, there exists at least one point in S_d converging under iterations of N_f to α.

A natural question is if this result still works for other families of Root-Finding Algorithms of high order different to Newton's method. A first step to begin an study in that direction is to know if the immediate basins of attraction of these Root-Finding Algorithms are simply connected.

Our main results are negatives. We proof that for $\sigma \geq 3$ the Julia set of König's method is not always connected.

In a similar way, we exhibit an explicit example for Newton's method for multiple roots which have disconnected Julia set, and more generally, we prove that for $\sigma \geq 3$ the Julia set of σ–Schröder's method is not always connected. Thus, Przytycki's and Shishikura's results (see [27], [32]) are, in general, no longer true for König's method when $\sigma \geq 3$. Consequently, our result establishes restrictions for extending the main result of [13] to higher order root–finding algorithms.

As a remarkable consequence of these results, we prove that the Julia set of König's method applied to a rational map is not always connected. In particular the Julia set of Newton's method for rational maps is not always connected.

Contents

Introduction

A useful tool to approximate solutions of nonlinear equations $f(z) = 0$ consists in using some iterative method, namely T. An iterative method begins with an initial guess z_0, called the pivot, which then improved by means of an iteration $z_{n+1} = T(z_n)$.

Nevertheless, not all the choices of z_0 assure a convergence to a root of the equation $f(z) = 0$. Hence, conditions are imposed on z_0, and eventually, on f or T, or both, in order to ensure the convergence of $\{z_n\}_{n\geq 0}$ to a solution α of the equation $f(z) = 0$.

As an example of those iterative methods we have the Newton method, which is the most famous of them. For a polynomial equation $f(z) = 0$, we start with an initial point z_0, and we define the Newton iterative method by

$$N_f(z_n) = z_n - \frac{f(z_n)}{f'(z_n)}.$$

This method defines a rational map on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ on itself. Thus from this point of view it works naturally the Fatou-Julia theory for iterated holomorphic maps.

As a reference, we will mention books and surveys for the Fatou-Julia theory. Blanchard in [3] and also P. Blanchard and A. Chiu in [4] gives an extensive and comprehensive review of the theory of iteration of rational maps. Milnor in [21] gives a general treatment of the theory of iteration of rational maps. A more advanced approach on the subject is presented by S. Morosawa et al. [22] and by C. McMullen in [19]. As far as the history of complex dynamics is concerned, the book by D. Alexander [1] (and references therein) is a quite valuable source.

One way to overcome the difficult of the non convergence of the sequence generated by an iterative method, is to find the 'better' (which means generally convergent) method T. A *purely iterative algorithm* is a rational map $T: Poly_d \to Rat_k$, associating to each degree d complex monic polynomial $f \in Poly_d(\cong \mathbb{C}^d)$ a degree k rational function $T_f(z) \in Rat_k(\cong \mathbb{P}^{2k+1})$, such that the coefficients of T_f are themselves rational functions of the coefficients of f. The algorithm is said to be *generally convergent* if $T_f^n(z) \to \{a\ root\ of\ f\}$, for all (f, z) in a open dense set of full Lebesgue measure $U \subset Poly_d \times \mathbb{C}$.

In this context, Newton's method is generally convergent for quadratic polynomials and is no generally convergent for cubic polynomials. However, in [17] McMullen show an explicit generally convergent purely iterative algorithm for cubic polynomials. In that work, McMullen proved that there is no generally convergent purely iterative algorithm for finding the roots of a polynomial of degree greater than 4 or more. That is, there exist periodic attracting cycles of high order to T_f for certain polynomials f of degree greater or equal to 4, which are not associated to a root of f. In other words there no exists the 'better' method for polynomials of degree 4 or more. Moreover in a subsequent work, McMullen found the location of the failure of those algorithms, (see[18]).

An interesting problem is to determine how many of such periodic attracting cycles has an iterative root finding algorithms. For Newton's method, Hurley in [14], proved that for each degree d (of polynomials) the maximum number of those periodic attracting cycles no associated to a root is $d-2$ (observe that the number is consistent with the fact of Newton's method is generally convergent for quadratic polynomials). For König's method, Buff and Henriksen in [5], found bounds for maximum number of the periodic attracting cycles of high order, in terms of the degree d of the polynomials and the order σ of the König's method.

Another approach to the problem of the non convergence of the sequence generated by an iterative method, is to consider Newton's method and allow more than one starting point. This study has been developed by Hubbard, Schleicher and Sutherland in [13], where the following result is proved: *Denote by $\mathcal{P}_d$ the set of polynomials of degree d, the roots of which are contained in the unit disk. For every degree $d \geq 2$, one can construct a set S_d consisting of at most $1.11d \cdot \log^2 d$ points in the complex* *plane with the following property. Given any polynomial* $f \in \mathcal{P}_d$ *and any root* α *of* f *, there is a starting point* $z_0 \in S_d$ *such that the iterative sequence* $z_n = N_f^n(z_0)$ *, converges to* α *as* $n \mapsto \infty$ *.*

Chapter 1

Background.

1.1 Notations and generalities.

In this chapter we give notational conventions and facts about classic complex dynamical systems.

We let denote $\mathbb{C}$ the complex plane and $\overline{\mathbb{C}}$ be the Riemann sphere.

(*Big oh Notation*) Let U be an open set of $\mathbb{C}$ and let $f, g : U \to \mathbb{C}$ be two functions on the complex plane. We define O big and write

$$f = O(g) \quad \text{when} \quad z \to z_0,$$

provided there exists a constant C such that

$$|f(z)| \leq C|g(z)|,$$

for all z sufficiently close to z_0.

Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map on the Riemann sphere, that is, $R(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials with no common factors. The *degree* of R is defined as $\deg(R) = \max\{\deg(p), \deg(q)\}$. We shall consider only rational maps of degree greater than or equal to two in what follows.

For $z \in \overline{\mathbb{C}}$ we define its *orbit* as the set $\mathrm{orb}(z) = \{z, R(z), \ldots, R^k(z), \ldots\}$, where R^k means the k–fold iterate of R. A point z_0 is a *fixed point* of R if $R(z_0) = z_0$.

A *periodic point* of period n is a point z_0 such that $R^n(z_0) = z_0$ and $R^j(z_0) \neq z_0$ for $j = 1, \ldots, (n-1)$. If $z_0 \in \overline{\mathbb{C}}$ is a periodic point of period $n \geq 1$, then it is a fixed point of R^n. The derivative of the $k-$fold iterate R^k at a point of an orbit is a well defined complex number called the *multiplier* of the orbit.

A fixed point z_0 of R is respectively *attracting, repelling* or *indifferent* in case that the multiplier $|R'(z_0)|$ is less than, greater than or equal to 1. A *superattracting fixed point* of R is a fixed point which is also a zero of the derivative R'. A periodic point of period n is said to be attracting, superattracting, repelling or indifferent according its multiplier is less than, greater than or equal to 1, respectively.

Theorem 1.1.1. *A rational map of degree* $d \geq 1$ *has precisely* $d+1$ *fixed points in* $\overline{\mathbb{C}}$.

1.2 Julia and Fatou sets.

Let U be an open set of $\overline{\mathbb{C}}$ and $\mathcal{F} = \{f \mid f : U \to \mathbb{C}\}$ a family of functions. The family $\mathcal{F}$ is *normal* in U if every sequence $\{f_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ which converges uniformly on compact subsets of U.

The Montel's theorem is one of the basic tools used in the classical theory of iterate rational maps. The proof is based on the Schwarz's lemma, and the fact that the triple punctured sphere is covered by the unit disk.

Theorem 1.2.1. (Montel) *If* $\mathcal{F}$ *is a family of meromorphic functions on a domain* U, *each of one omits three fixed values of* $\overline{\mathbb{C}}$, *then* $\mathcal{F}$ *is a normal family.*

A point $z \in \overline{\mathbb{C}}$ is an element of *Fatou set*, $\mathcal{F}(R)$, of R if there exists a neighborhood U of z such that the family of iterates $\{R^n : U \to \overline{\mathbb{C}} \mid n = 1, 2, \ldots\}$ is a normal family. The Julia sets $\mathcal{J}(R)$ is the complement on $\overline{\mathbb{C}}$ of the Fatou set.

A remarkable characterization of the Julia set is given by the next theorem.

Theorem 1.2.2. *The Julia set of a rational map is the closure of the set of repelling periodic points.*

Now we give some general properties of the Julia and Fatou sets of a rational map (see [21]).

Theorem 1.2.3. *Let* $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *be a rational map, then*

1. $\mathcal{J}(R)$ *is not empty. The Fatou set can be empty, as shows us the Lattès example* $R(z) = \dfrac{(z^2+1)^2}{4z(z^2-1)}$.
2. $\mathcal{J}(R) = \mathcal{J}(R^n)$ *for all* $n \in \mathbb{N}$.
3. $R(\mathcal{J}(R)) = R^{-1}(\mathcal{J}(R)) = \mathcal{J}(R)$ *, that is,* $\mathcal{J}(R)$ *is completely invariant.*
4. $\mathcal{J}(R)$ *is closed.*
5. $\mathcal{J}(R)$ *contains no isolated points, that is,* $\mathcal{J}(R)$ *is perfect.*

Let z_0 be an attracting fixed point of R. We define the *basin of attraction* of z_0 as the set $B(z_0) = \{z \in \overline{\mathbb{C}} \;:\; R^n(z) \longrightarrow z_0 \text{ as } n \longrightarrow \infty\}$. The *immediate basin of attraction* of an attracting fixed point z_0 of R, denoted by $B^*(z_0)$, is the connected component of $B(z_0)$ containing z_0.

If z_0 is an attracting periodic point of period n of R, the basin of attraction of the orbit of z_0 is the set $B(\text{orb}(z_0)) = \cup_{j=0}^{n-1} R^j(B(z_0))$, where $B(z_0)$ is the attraction basin of z_0 as a fixed point of R^n, and its immediate basin of attraction is the set $B^*(\text{orb}(z_0)) = \cup_{j=0}^{n-1} R^j(B^*(z_0))$.

If R has an attracting periodic point z_0, then the basin of attraction is contained in the Fatou set and $\mathcal{J}(R) = \partial B(z_0)$, the boundary of $B(z_0)$. Therefore, the chaotic dynamics of R is contained in its Julia set.

A value v is a *critical value* of R if the equation $R(z) = v$ has a solution with multiplicity greater than one. Such a solution c is called a *critical point.* The next result is essentially due to Fatou.

Theorem 1.2.4. *Let* $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *a rational of degree* $d \geq 2$ *and* z_0 *a periodic attracting point, then the immediate basin of attraction of* z_0 *contains at least one critical point of* R.

We say that R is *hyperbolic* if the forward orbit of each critical point of R converges towards to some attracting periodic orbit.

Theorem 1.2.5. *A rational map* $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *of degree* $d \geq 2$ *has at most* $2d - 2$ *critical points in* $\overline{\mathbb{C}}$*.*

1.3 Topology of Julia and Fatou sets.

Let X a topological space. We say that X is *locally connected* in $z_0 \in K$ if every open subset of X is a union of connected open subsets of X.

Proposition 1.3.1. *The Fatou set* $\mathcal{F}(R)$ *has either,* 0, 1, 2 *or infinitely many components.*

Proposition 1.3.2. *Let* R *be a rational map. Then* $\mathcal{J}(R)$ *is connected if and only if each component of* $\mathcal{F}(R)$ *is simply connected.*

Theorem 1.3.1. *Let* R *be a hyperbolic rational map. If the Julia set* $\mathcal{J}(R)$ *is connected, then it is locally connected.*

1.4 Spherical metric.

Let z_1 and z_2 be two points in $\mathbb{C}$ corresponding to P_1 and P_2, respectively on $\overline{\mathbb{C}}$. If $P_i = (\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2$, then the euclidean distance between P_1 and P_2 is given by

$$|P_1 - P_2| = [(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_1)^2]^{1/2}.$$

We denote $|P_1 - P_2|$ by $\chi(z_1, z_2)$, the *chordal* distance between z_1 and z_2. It can be shown that if z_1 and z_2 are in the finite plane, then

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}.$$

On the other hand, the *spherical arc length ds* on the Rieman sphere $\overline{\mathbb{C}}$ works out to be

$$\frac{|ds|}{1 + |z|^2}.$$

The spherical length

$$L(\gamma) = \int_\gamma \frac{|ds|}{1+|z|^2}$$

of a curve γ on $\overline{\mathbb{C}}$ induces a metric in the following way. Given distinct points z_1, z_2 on the Riemann sphere, define

$$\sigma(z_1, z_2) = \inf\{L(\gamma)\},$$

where the infimum is taken over all differentiable curves on $\overline{\mathbb{C}}$ joining z_1 with z_2. Then, $\sigma(z_1, z_2)$ is the euclidean length of the shortest arc of the great circle on $\overline{\mathbb{C}}$ joining z_1 and z_2 and defines a metric on the sphere known as the *spherical metrics.* Indeed, $\chi(z_1, z_2) \le \sigma(z_1, z_2) \le \frac{\pi}{2}\chi(z_1, z_2)$, so that the two metrics are *uniformly equivalent* and generate the same open sets on $\overline{\mathbb{C}}$.

We say that a sequence of functions $\{f_n\}$ converges *spherically uniformly* to f on a set $E \subseteq \mathbb{C}$ if, for any $\varepsilon > 0$, there is a number n_0 such that $n \ge n_0$ implies

$$\chi(f(z), f_n(z)) < \varepsilon,$$

for all $z \in E$.

Theorem 1.4.1. *Let $\{f_n\}$ be a sequence of meromorphic functions defined on a open set U. Then $\{f_n\}$ converges spherically uniformly on compact subsets of U to f if and only if around each point $z_0 \in U$ there is a closed disk $D(z_0, r)$ in which*

$$|f_n - f| \to 0$$

or

$$\left|\frac{1}{f_n} - \frac{1}{f}\right| \to 0,$$

uniformly as $n \mapsto \infty$.

1.5 Iterative Root-Finding algorithms.

In what follows, we assume that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a polynomial.

Definition 1.5.1. *We say that a map $f \mapsto T_f$ carrying a complex–valued function f to a function $T_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an* iterative root-finding algorithm, *if $T_f(z)$ has a fixed point at every root of f, and given an initial guess z_0, the sequence of iterates $(z_k)_{k\geq 0}$ where $z_{k+1} = T_f(z_k)$ converges to a root $r \in \overline{\mathbb{C}}$ of f whenever z_0 is sufficiently close to r.*

Let $z_{n+1} = z_n - \phi(z_n)$ be an iterative root-finding algorithm such that for every simple root r of $f(z)$ if we have that $\phi'(r) = 1$, $\phi''(r) = \cdots = \phi^{(k-1)}(r) = 0$, and $\phi^{(k)}(r) \neq 0$ then we say that the root–finding algorithm is (at least) *order* k convergent.

A critical point of an iterative root-finding algorithm $T_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which does not coincide with a root of f is called a *free critical point* of T_f. A fixed point of an iterative root-finding algorithm T_f which does not coincide with a root of f is called a *extraneous fixed point* of T_f.

Now, we recall the Smale's definition of purely iterative algorithms, see [33].

Let $\mathcal{P}_d$ be the space of all polynomials of degree less than or equal to d and define

$$j : \mathbb{C} \times \mathcal{P}_d \to J_k$$

by

$$j(z, f) = (z, f(z), f'(z), \ldots, f^{[k]}(z)).$$

Here J_k (a jet space) is $\mathbb{C}^{k+2}$ representing the source and the first k derivatives. The *datum* is a rational map $F : J_k \to \mathbb{C}$, which will be written in the following form

$$F(z, \xi_0, \ldots, \xi_k) = z - \frac{P(z, \xi_0, \ldots, \xi_k)}{Q(z, \xi_0, \ldots, \xi_k)}, \tag{1.1}$$

where P and Q are polynomials in the $k+2$ variables with no common factors.

Definition 1.5.2. *A* purely iterative algorithm*(in Smale's sense), is a rational endomorphism $T_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ depending on $f \in \mathcal{P}_d$ and having the form*

$$T_f(z) = F(j(z, f(z))).$$

In [17], McMullen defines purely iterative algorithm as a mapping $T : Poly_d \to Rat_k$, associating to each degree d complex monic polynomial $f \in Poly_d (\cong \mathbb{C}^d)$ a degree k rational function $T_f(z) \in$

$Rat_k(\cong \mathbb{P}^{2k+1})$, such that the coefficients of T_f are themselves rational functions of the coefficients of f. Note that McMullen's definition of purely iterative algorithm is more general than Smale's definition. Smale's definition is more appropriate for our proposes.

Example 1.5.1. (1) If f is a polynomial, Newton's method given by the formula

$$N_f(z) = z - \frac{f(z)}{f'(z)}$$

is an iterative root-finding algorithm and a purely iterative root algorithm.

(2) The Halley method associated to f, is an iterative root finding of order 3 and a purely iterative root finding algorithm, given by

$$H_f(z) = z - \frac{2f'(z)f(z)}{2(f'(z))^2 - f(z)f''(z)}. \tag{1.2}$$

We consider a slight modifications in the definition of purely iterative algorithm (in Smale's sense). Let Rat_d be the space of rational maps of degree less than or equal to d. Define

$$\hat{\jmath} : \overline{\mathbb{C}} \times Rat_d \to J_k$$

by

$$\hat{\jmath}(z,\, R) = (z, R(z), R'(z),\, \dots,\, R^{[k]}(z))\,.$$

Here, $\hat{\jmath}$ is the string of derivatives of R up to order k and J_k can be viewed as the set of all possible values of $\hat{\jmath}$. In general, the *datum* is a rational map $F : J_k \to \mathbb{C}$, which has the following form:

$$F(z,\, \xi_0,\, \dots,\, \xi_k) = z - \frac{P(z,\, \xi_0,\, \dots,\, \xi_k)}{Q(z,\, \xi_0,\, \dots,\, \xi_k)},$$

where P and Q are polynomials in the $k+2$ variables with no common factors.

Definition 1.5.3. *We define a rational endomorphism $\widetilde{T}_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ depending on $f \in \mathcal{P}_d$ by*

$$\widetilde{T}_f(z) = F(\hat{\jmath}(z,\, N_f(z))\,,$$

where N_f *is the Newton map associated to* f *. The map* $\widetilde{T}$ *is called a* purely iterative algorithm for Newton's maps*. Henceforth, instead of* $\widetilde{T}$ *we writte* T *.*

Example 1.5.2. (1) If f is a polynomial, we define a purely iterative algorithm for Newton's maps as follow. Let

$$F(z,\, \xi_0,\, \xi_1) = z - \frac{P(z,\, \xi_0,\, \xi_1)}{Q(z,\, \xi_0,\, \xi_1)}\,,$$

where $P(z,\, \xi_0,\, \xi_1) = (z-\xi_0)(1-2\xi_1)$, $Q(z,\, \xi_0,\, \xi_1) = (1+\xi_1)$. Thus,

$$T_f(z) = F(j(z,\, N_f(z)) = z - (z - N_f(z))\frac{(1-2N_f'(z))}{(1+N_f'(z))}\,.$$

The map T_f is

$$T_f(z) = z - \frac{f(z)}{f'(z)}\left(\frac{f'(z)^2 - 2f(z)f''(z)}{f'(z)^2 + f(z)f''(z)}\right).$$

Notice that T_f is a purely iterative algorithm for Newton's maps which is not an iterative root-finding algorithm. For instance, take the polynomial $f(z) = (z-1)^2 z$. The fixed point associate to the root 1 is indifferent.

(2) The Halley method associated to $f(z)$ is a purely iterative root finding algorithm for Newton's maps. In fact,

$$H_f(z) = \quad z - \frac{2f'(z)f(z)}{2(f'(z))^2 - f(z)f''(z)}$$

$$= z - \frac{2(z - N_f(z))}{2 - N_f'(z)}$$

where N_f is the Newton method associated to f .

Chapter 2

Newton's Method.

In this chapter we give some results for Newton's method which are widely known, see for instance [24].

2.1 Definition and basic results.

Let $f(z)$ be a complex polynomial. Newton's iterative method associated to f is

$$N_f(z) = z - \frac{f(z)}{f'(z)}.$$

The function N_f defines a rational map on the Riemann sphere $\overline{\mathbb{C}}$ and, thus, it defines a discrete dynamical system $z_{n+1} = N_f(z_n)$. The next proposition describe the nature of fixed point for the rational map N_f.

Proposition 2.1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ a polynomial of degree $d \geq 2$. Denote by α_i its zeros and by n_i their multiplicities. Then, the fixed points of Newton's method $N_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are (super)attracting or repelling.*

a) The (super)attracting fixed points are exactly the zeros α_i, and their multipliers are $(n_i - 1)/n_i$. Thus, when $n_i = 1$, the local degree of N_f is at least 2.

b) The rational map N_f has a repelling fixed point at ∞ with multiplier $d/(d-1)$.

c) If f has d distinct roots, the degree of the rational map N_f is d.

If the initial guess z_0 is chosen "near" a simple root α of f, then the sequence $(z_n)_{n\in\mathbb{N}}$ converges quadratically to the root α, that is, $|z_{n+1}-\alpha| \leq c|z_n-\alpha|^2$ for some constant $c>0$, since α is a superattracting fixed point of $N_f(z)$, and generically we have that $N_f''(\alpha) \neq 0$.

The following theorem is proved by E. Schröder in 1870 and A. Cayley 1879, independently. We will discuss this theorem in Chapter 4. For instance see [1].

Theorem 2.1.1. *Let be a quadratic polynomial with distinct roots. Then Newton's method is globally, analytically conjugate to the polynomial $z \mapsto z^2$.*

2.2 Conjugacy classes

Definition 2.2.1. *Let $R_1, R_2 : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be two rational maps. We say that R_1 and R_2 are conjugated if there is a Möbius transformation $\phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that $R_2 \circ \phi(z) = \phi \circ R_1(z)$ for all z.*

In order to discuss the conjugacy classes we have mentioned the next useful result. This theorem is from the mathematical folklore. For a proof, see [25].

Proposition 2.2.1. (Scaling theorem) Let $f(z)$ be an analytic function on the Riemann sphere, and let $A(z) = \alpha z + \beta$ with $\alpha \neq 0$, be an affine map. If $g(z) = f \circ A(z)$, then $A \circ N_g \circ A^{-1}(z) = N_f(z)$, that is, N_f is analytically conjugated to N_g by A.

Example 2.2.1. For the polynomial $g(z) = (\alpha z + \beta)^d - 1$, where $d \geqslant 2$ and $\alpha, \delta \neq 0$, Newton's method has simply connected basin of attraction and is an hyperbolic map. By conjugation we can assume that $g(z) = z^d - 1$ with $d \geqslant 2$. A simple computation give us

$$N_f(z) = z - \frac{z^d - 1}{dz^{d-1}} = \frac{z^d(d-1)+1}{dz^{d-1}}$$

and

$$N'_f(z) = \frac{(d-1)(z^d-1)}{dz^d}\,.$$

Observe that there is no free critical points and consequently N_f is hyperbolic.

2.3 Characterization of Newton's method.

The following proposition make a reference to the rational maps which arise of Newton's method applied to generic polynomials.

Proposition 2.3.1. *A rational map* $R:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ *of degree* $d\geqslant 2$ *is the Newton map of a polynomial of degree at least two if and only if infinity is a unique repelling fixed points and for all other fixed points* $a_1, a_2, \ldots, a_d \in \mathbb{C}$ *there exists a number* $n_j \in \mathbb{N}$ *such that* $R'(a_j) = \dfrac{n_j - 1}{n_j} < 1$ *.*

This result have been proved by many authors. According to Smale [33], Greg Saunder proved this results in his thesis in 1984. For generic polynomials, Janet Head in his doctoral thesis, 1987. In 1992, Nishisama and Fujimura obtained a proof of this results, see [23]. Later, Buff and Henriksen in [6] proved this result for König's Root-Finding algorithms applied to a polynomial f with simple roots. In particular this result contain the Newton method. The list continues with an extension to the rational case, that is, when f is rational map, given by E. Crane in his thesis [7]. The case when f is an entire map is covered by J. Rückert in [30].

2.4 The topology of Julia and Fatou sets.

In this section we give a short recapitulation of known facts about the topology of Julia and Fatou sets for Newton's method applied to a polynomial.

In [27], F. Przytycki proved the following result, see also [13].

Theorem 2.4.1. *Let* α *an attracting fixed points for Newton's method* N_f *, associated to a complex polynomial* f *, and let* U *the immediate basin of attraction of* α *. Then,* U *is simply connected.*

A stronger fact is due for M. Shishikura in [32]. Using quasiconformal surgery he proved,

Theorem 2.4.2. *The Julia set of Newton's method of a polynomial is connected.*

The Newton method for cubic polynomials deserve a special attention, because surely has the most complete description.

First, in 1983 Curry, Garnet and Sullivan [8] studied Newton's Method for a generic cubic polynomial. With numerical experiments in the parameter space, they showed a surprising connection with Mandelbrot set.

Probably at the same time that F. Przytycki and M. Shishikura, H. Meier in [20] proved the connectedness of the Julia set for Newton's method applied to a cubic polynomials. The same result with different arguments is given by L. Tan in [34].

Regarding local connectedness of Julia set for Newton's method applied to a cubic polynomials, the first results comes from the work of P. Roesch in [28]. In general lines Roesch was able to show that in the most of cases Newton's method for cubic polynomials have locally connected Julia sets.

Chapter 3

König's Root-Finding Algorithm.

3.1 Definition and basic results.

The König's root finding algorithms initially appears as an element of the family defined by E. Schröder in [31].

Definition 3.1.1. *Let* $f : \mathbb{C} \to \mathbb{C}$ *a complex polynomial and let* $\sigma \geqslant 2$ *be an integer. König's method of order* σ *associated to* f *is the rational map* $K_{f,\sigma} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *defined by the formula:*

$$K_{f,\sigma} = \mathrm{Id} + (\sigma - 1)\frac{(1/f)^{[\sigma-2]}}{(1/f)^{[\sigma-1]}} \, ,$$

where $(1/f)^{[k]}$ *is the derivative of order* k *of* $1/f$ *.*

Note that the lowest order function coincides with Newton's method and for $\sigma = 3$ is the so called Halley's method given by the formula 1.2.

The following proposition is proved by X. Buff and C. Henriksen in [6].

Proposition 3.1.1. *Let* $f : \mathbb{C} \to \mathbb{C}$ *a polynomial. Denote by* α_i *its zeros and by* $n_i \geqslant 1$ *their multiplicities. Then, for any integer* $\sigma \geqslant 2$ *, the fixed points of König's method* $K_{f,\sigma} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *are either (super)attracting or repelling.*

a) *The (super)attracting fixed points are exactly the zeros* α_i *of* f *and their multipliers are* $1-(\sigma-1)/(n_i+\sigma-2)$ *. When* $n_i=1$ *, the local degree of* $K_{f,\sigma}$ *at* α_i *is at least equal to* σ *.*

b) *The extraneous fixed points of* $K_{f,\sigma}$ *are exactly the zeros of* $(1/f)^{[\sigma-2]}$ *. If* β_j *is a zero of* $(1/f)^{[\sigma-2]}$ *with multiplicity* m_j *, then it is a repelling fixed point of* $K_{f,\sigma}$ *with multiplier* $1+(\sigma-1)/m_j$ *.*

3.2 The topology of Julia and Fatou sets for König's method.

The main result in this chapter shows that for $\sigma \geq 3$, the Julia set of König's root finding algorithms is not always connected.

In his thesis [10], Elhasadi proved that the immediate basins of attraction of Halley's method ($\sigma = 3$) applied to polynomials with real coefficients, simple and real zeros, are simply connected.

Lemma 3.2.1. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *a sequence of polynomials converging uniformly to an analytic function* f *on a open set* $U \subset \mathbb{C}$ *. If* $K_{f,\sigma}$ *is a meromorphic function, then the sequences of rational maps* $K_{f_n,\sigma}$ *converges uniformly to* $K_{f,\sigma}$ *on every compact subset of* U *, for the spherical metric on* $\overline{\mathbb{C}}$ *.*

Proof. We consider the König's root finding algorithms proposed by N. Argiropoulos, V. Drakopoulos and A. Böhm in [2]. Let us define $h_1(z) = 1$. Thus we have

$$\left(\frac{h_1(z)}{f(z)}\right)' = \frac{h_1'(z)f(z) - h_1 f'(z)}{[f(z)]^2} = \frac{h_2(z)}{[f(z)]^2},$$

$$\left(\frac{h_2(z)}{[f(z)]^2}\right)' = \frac{h_2'(z)f(z) - 2h_2 f'(z)}{[f(z)]^3} = \frac{h_3(z)}{[f(z)]^3},$$

and in general

$$\left(\frac{1}{f(z)}\right)^{(k)} = \frac{h_{k+1}(z)}{[f(z)]^{k+1}},$$

where $h_{k+1}(z) = h_k'(z)f(z) - kh_k(z)f'(z)$ for $k = 1, 2, \ldots, \sigma-1$. Now, we rewrite $K_{f,\sigma}$ by letting

$$K_{f,\sigma}(z) = z + (\sigma-1)\frac{h_{\sigma-1}(z)f(z)}{h_\sigma(z)}.$$

Observe that the existence of h_k for $k = 2, 3, \ldots, \sigma$ requires a knowledge of the first $k-1$ derivatives of $f(z)$. This give us advantages in comparison with the definition 3.1.1, that requires a knowledge of the first $k-1$ derivatives of $1/f$.

We put $h_{f,\,k+1} := h_{k+1}$ to emphasize the dependence on f. Since f_n converge uniformly to f on a compact C contained on U, we see that, if $k = 1$ then

$$h_{f_n,\,2} = h'_{f_n,\,1} f_n - h_{f_n,\,1} f'_n = -f'_n$$

converge uniformly on C to $h_{f,\,2}$. Assume that $h_{f_n,\,k}$ converge uniformly on the compact C to $h_{f,\,k}$. Thus, by induction

$$h_{f_n,\,k+1} = h'_{f_n,\,k} f_n - k h_{f_n,\,k} f'_n$$

converge uniformly on the compact C to

$$h_{f,\,k+1} = h'_{f,\,k} f - k h_{f,\,k} f'\,.$$

Now, we will show that when $\sigma \geqslant 2$, $K_{f_n,\,\sigma}$ converges uniformly to $K_{f,\,\sigma}$ on every compact subset of U, for the spherical metric on $\overline{\mathbb{C}}$.

Define the expressions $P_{f,\,\sigma}(z) = z h_{f,\,\sigma}(z) + (\sigma - 1) h_{f,\,\sigma-1}(z) f(z)$ and $Q_{f,\,\sigma}(z) = h_\sigma(z)$ and write

$$K_{f,\,\sigma} = \frac{P_{f,\,\sigma}}{Q_{f,\,\sigma}}.$$

For an arbitrary $z_0 \in U$ choose a closed disk $D(z_0,\, r) \subset U$. Suppose that $Q_{f,\,\sigma}(z) \neq 0$ on $D(z_0,\, r)$. By Theorem 1.4.1

$$K_{f_n,\,\sigma} = \frac{P_{f_n,\,\sigma}}{Q_{f_n,\,\sigma}}$$

converge uniformly (for the euclidean metric) on $D(z_0,\, r)$ to

$$K_{f,\,\sigma} = \frac{P_{f,\,\sigma}}{Q_{f,\,\sigma}}.$$

On the other hand, suppose that $Q_{f,\,\sigma}(z_0) = 0$ and choose $r > 0$ small enough, so that $K_{f,\,\sigma}(z_0) \neq 0$ on $D(z_0,\, r)$. By Theorem 1.4.1,

$$\frac{1}{K_{f_n,\,\sigma}} = \frac{Q_{f_n,\,\sigma}}{P_{f_n,\,\sigma}}$$

converge uniformly on $D(z_0, r)$ to

$$\frac{1}{K_{f,\sigma}} = \frac{Q_{f,\sigma}}{P_{f,\sigma}}.$$

In view of Theorem 1.4.1, the sequences of rational maps $K_{f_n,\sigma}$ converges uniformly to $K_{f,\sigma}$ on every compact subset of U, for the spherical metric on $\overline{\mathbb{C}}$. □

We need the next classic theorem, whose importance lies in its interplay between interpolation an approximation, for a proof see [29].

Theorem 3.2.1. (Mergelyan Approximation) *Let C be a compact set in $\mathbb{C}$ such that the complement is connected, and suppose that f is continuous on C and analytic in the interior of C. To each $\epsilon > 0$ there exists a polynomial p such that $|f - p| < \epsilon$ on C.*

Theorem 3.2.2. *For all $\sigma \geqslant 3$, there exists a complex polynomial with the property that the Julia set of König root finding algorithms associated to this polynomial is not connected.*

Proof. Let $\varepsilon \in (0, \frac{1}{4})$ and $\sigma \geqslant 3$. Let us define

$$f_\sigma(z) = \frac{(-i\sqrt{\sigma-1})^{\sigma-1}\exp(i\sqrt{\sigma-1}z)}{[-i\sqrt{\sigma-1}z - (\sigma-1)]}.$$

Observe that f_σ is a meromorphic function and for every $\sigma \geqslant 3$, $z = i\sqrt{\sigma-1}$ is a simple pole of f_σ. In particular, f_σ is an holomorphic map on $A = \{z \in \mathbb{C} : |z| \leqslant 1 + \varepsilon/2\}$.

The König's method of order σ of f_σ is

$$K_{f_\sigma}(z) = z + \frac{1}{z} + i\sqrt{\sigma-1}.$$

Let $I_\sigma = i\sqrt{\sigma-1} + [-2, 2]$. The map K_{f_σ} carries the unit circle in a two to one manner onto I_σ. For $z_0 \notin I_\sigma$ the equation $K_{f_\sigma}(w) = z_0$ has two solutions, one of which lies inside the unit circle and one of which lies outside. Hence, K_{f_σ} maps the exterior of the closed unit disk isomorphically onto the complement $\mathbb{C} \setminus I_\sigma$.

Now we consider the cuadratic polynomial $g_\sigma(z) = (z - i\sqrt{\sigma-1})(z - (i\sqrt{\sigma-1} + R))$, where R is a large enough real number. The König's method of order σ apply to g_σ has two superattracting fixed points which are $i\sqrt{\sigma-1}$ and $i\sqrt{\sigma-1}+R$. Thus for any fixed real number $r<R$ we have $K_{g_\sigma}(i+\sqrt{\sigma-1}+r)=i+\sqrt{\sigma-1}$, as $R\to\infty$. Hence, there exists $\delta>0$ sufficiently small such that $B=\{z\in\mathbb{C}:d(z,\,I_\sigma)\leqslant\delta\}$ is completely contained on the basin of attraction of the fixed point $i\sqrt{\sigma-1}$. This shows that, $I_\sigma\subset\mathcal{F}(K_{g_\sigma})$.

Let $\delta<\varepsilon/2$ and $C=A\cup B$. Define the function

$$F(z)=\begin{cases} f_\sigma \ \text{ if } z\in A, \\ \\ g_\sigma(z) \ \text{ if } z\in B. \end{cases}$$

According to Mergelyan's theorem there exists a polynomial p_n such that $|p_n(z)-F(z)|<\varepsilon$ for all $z\in C$. By Lemma 3.2.1, it follows that $K_{p_n,\sigma}:=K_n$ converges uniformly to $K_{F,\sigma}$ on C, for the spherical metric on $\overline{\mathbb{C}}$.

Since F has a zero at $i\sqrt{\sigma-1}$, there exists a natural number n_1 such that p_n has a zero at $i\sqrt{\sigma-1}$ and consequently K_{n_1} has an attracting fixed point in $i\sqrt{\sigma-1}$. Since K_F has a pole in 0, by Hurwitz's theorem there exists n_2 and $R>0$ such that K_{n_2} has a pole q in $B(0,\,R)$. Thus by theorem 3.1.1, q is in the Julia set of K_{n_2}.

On the other hand, there exists n_3 such that the map K_{n_3} has a circle around the pole q, completely contained in the Fatou set of K_{n_3}. If we choose $N=\max\{n_1,\,n_2,\,n_3\}$, the conclusion is that there exists a polynomial p_N such that the Julia set of K_{p_N} has disconnected Julia set.

Remark 1. Observe that the sets I_σ and A are separated only if $\sigma>2$. For $\sigma=2$ we cannot disconnect those sets. This is consistent with the fact that Newton's method has connected Julia set.

Chapter 4

Newton's method for multiple roots

4.1 Definitions and basic results

The Newton method for multiple roots was defined by Erns Schröder in his study of iteration of Newton's method in 1871.

Let $f(z)$ be a complex polynomial. The *Newton method for multiple roots* associated to $f(z)$ is

$$M_f(z) = z - \frac{f(z)f'(z)}{[f'(z)]^2 - f(z)f''(z)}. \tag{4.1}$$

Notice that applying the classical Newton's method to $f(z)/f'(z)$ we obtain Newton's method for multiple roots. This fact makes this method more interesting because the term $f(z)/f'(z)$ has the effect of converting the multiple roots of $f(z)$ to simple ones.

Following [1], let explain briefly some of the interest of Schröder in this function.

In order to prove the theorem 2.1.1, Schröder has noted that for $f(z) = z^2 - 1$,

$$M_f(z) = \frac{1}{N_f(z)} = \frac{2z}{1+z^2} = -i\tan(2\arctan(iz)).$$

This implies that

$$M_f^n(z) = \frac{1}{N_f^n(z)} = -i\tan(2^n\arctan(iz)).$$

The last terms in the equation has the following property:

If z is on the half right plane

$$\lim_{n\to\infty} -i\tan(2^n \arctan(iz)) = 1,$$

and if z is on the half left plane

$$\lim_{n\to\infty} -i\tan(2^n \arctan(iz)) = -1.$$

We begins our study of Newton's method for multiple roots with the following remark.

Remark 2. Infinity is not a fixed point for $M_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

Proposition 4.1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ a polynomial. Denote by α_i its zeros (simple or multiple). Then, the fixed points of Newton's method for multiple roots $M_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are either superattracting or repelling.*

a) The super attracting fixed points are exactly the zeros α_i.

b) The extraneous fixed points of M_f are exactly the zeros of f' which are not zero of f. If β_j is a zero of f' with multiplicity m_j, then it is a repelling fixed point of M_f with multiplier $1 + 1/m_j$. If f has N distinct roots, then M_f has at most $N-1$ repelling fixed points in $\mathbb{C}$.

c) If f has N distinct roots, the degree of Newton's method for multiple roots $M_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, is at most $2N-2$. Generically, the degree of M_f is exactly $2d-2$.

Proof. Before proving this proposition, we rewrite the rational map M_f in a different way. Put

$$\begin{aligned} M_f(z) = & \quad z - \frac{f(z)f'(z)}{[f'(z)]^2 - f(z)f''(z)} \\ \\ = & \quad z - \frac{[z - N_f(z)]}{1 - N_f'(z)}, \end{aligned}$$

where N_f is Newton's method associated to the polynomial f, and N_f' its derivative. These expressions shows that M_f is a purely iterative algorithm for Newton's maps.

a) If f has a zero α of multiplicity n, then by Proposition 2.1.1, α is a (super)attracting fixed point of Newton's Method with multiplier $(n-1)/n$. Thus

$$N_f(z) = \alpha + (z-\alpha)\left(\frac{n-1}{n}\right) + O(z-\alpha)^2$$

and

$$N_f'(z) = \left(\frac{n-1}{n}\right) + O(z-\alpha)\,.$$

It follows that,

$$z - N_f(z) \quad = \frac{1}{n}(z-\alpha) + O(z-\alpha)^2$$

and

$$1 - N_f'(z) = \frac{1}{n} + O(z-\alpha).$$

As a consequence,

$$\begin{aligned} M_f(z) \quad &= z - \frac{[z - N_f(z)]}{1 - N_f'(z)} \\ &= z - \frac{(1/n)(z-\alpha) + O(z-\alpha)^2}{1/n + O(z-\alpha)} \\ &= \alpha + O(z-\alpha)^2. \end{aligned}$$

Therefore $M_f(\alpha) = \alpha$ and $M_f'(\alpha) = 0$.

b) Let β a zero of order m of f'. Therefore β is pole of order m of Newton's method N_f. Thus, there exists a constant $\lambda \in \mathbb{C}^*$, such that

$$z - N_f(z) = -\frac{\lambda}{(z-\beta)^m}[1 + O(z-\beta)]$$

and

$$1 - N_f'(z) = \frac{\lambda m}{(z-\beta)^{m+1}}[1 + O(z-\beta)].$$

This facts allow us to construct Newton's method for multiple roots around the fixed point β. It follows that,

$$M_f(z) = z - \frac{[z - N_f(z)]}{1 - N_f'(z)}$$

$$= z - \frac{(-\lambda)/(z-\beta)^m[1 + O(z-\beta)]}{(\lambda m)/(z-\beta)^{m+1}[1 + O(z-\beta)]}$$

$$= \beta + (z-\beta)(1 + \frac{1}{m}) + O(z-\alpha)^2.$$

The fixed points of M_f are the zeros of f and the zeros β of f' which are not simultaneously zeros of f. If f has N distinct roots then the zeros of f' are $N-1$. Thus, the numbers of repelling fixed points in $\mathbb{C}$ is at most $N-1$.

$c)$ We know that for any rational map, the number of fixed points counted with multiplicities is equal to the degree plus one. We have seen that the number of repelling fixed points in $\mathbb{C}$ is at most $N-1$, and generically is exactly $d-1$. Let

$D :=$ degree of M_f

$A := \#\{$superattracting fixed points of $M_f\} = N$

$R := \#\{$repelling fixed points of $M_f\} = N-1.$

Therefore, the degree of M_f is at most

$$D = A + R - 1 = N + N - 1 - 1 = 2N - 2\,.$$

This concludes the prove of $c)$.

4.2 Conjugacy classes.

Lemma 4.2.1. (Scaling for Newton's method for multiple roots) Let f and g two polynomials, and $T : \mathbb{C} \to \mathbb{C}$ defined by $T(z) = \alpha z + \beta$ an affine automorphism. Then $M_f \circ T = T \circ M_g$ if and only if there exists a constant $\lambda \in \mathbb{C}^*$ such that $g = \lambda f \circ T$.

Proof. The rational map M_f is a purely iterative algorithm for Newton's maps and satisfies the conditions of 6.2.1 in Chapter 6.

4.3 Characterization of Newton's method for multiple roots.

In this section we study under which conditions a rational maps is Newton's map for multiple roots of a generic complex polynomial.

Lemma 4.3.1. *Assume that two rational maps h_1 y h_2 have the same fixed points β_i with the same multipliers $\lambda_i \neq 1$. Then, the two rational maps are equal.*

Proof. Let us work in a coordinate system where ∞ is not one of the fixed points β_i. The rational function $1/(z-h_1)$ tends to 0 as z tends to ∞. Moreover, the poles of $1/(z-h_1)$ are the fixed points β_i. Since $\lambda_i \neq 1$ these are all simple poles, and the residues of $1/(z-h_1)$ at β_i is $1/(1-\lambda_i)$. The same is true for $1/(z-h_2)$. Thus, we see that the rational map $1/(z-h_1) - 1/(z-h_2)$ has no poles in $\mathbb{C}$ and tends to 0 as z tends to ∞. Thus, this rational map is equal to 0. This proves that the rational maps h_1 and h_2 are equal. □

Theorem 4.3.1. *Assume $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map, the fixed points of which are either superattracting or repelling. Denote its superattracting fixed points by $\alpha_1, \alpha_2, \ldots, \alpha_d$ and assume that any repelling fixed point $\beta_1, \beta_2, \ldots, \beta_{d-1}$ has a multiplier of the form $1 + 1/m_j$, with $m_j \in \mathbb{N}$. Then $h = M_f$, where $f(z) = \prod_{i=1}^{d}(z-\alpha_i)$.*

Proof. We define the rational map $h_1(z) = z - \dfrac{[z-R(z)]}{[1-R'(z)]}$, where $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map given by

$$R(z) = z - \frac{\prod_{i=1}^{d}(z-\alpha_i)}{\prod_{i=1}^{d-1}(z-\beta_i)^{m_i}} .$$

We claim that $h = h_1$. Notice that the rational maps h and h_1 have the same fixed points in $\mathbb{C}$, the same multipliers, and all of them differ from 1. Hence, from the lemma above we have $h = h_1$.

The map R has d super attracting fixed points in $\alpha_1, \alpha_2 \ldots, \alpha_d$. It has a repelling fixed point at infinity with multiplier $d/(d-1)$. Thus for the proposition 2.3.1, R is Newton's method of a polynomial $f(z) = \prod_{i=1}^{d}(z-\alpha_i)$. It follows that, $R(z) = N_f$.

4.4 The Julia and Fatou sets for Newton's method for multiple roots.

In this section we present two examples of the Julia set of Newton's method for multiple roots.

Example 4.4.1. Consider the polynomial $f(z) = z^4 - z^2$. We will show that Newton's method for multiple roots applied to the polynomial f, has not connected Julia set.

Evaluating the polynomial f in Newton's method for multiple roots, we have $M_f(z) = \dfrac{2z^3}{2z^4 - z^2 + 1}$. The superattracting fixed points are $0, 1, -1$ and the repelling fixed points are $\sqrt{2}/2$ and $-\sqrt{2}/2$. The other critical points are $i\sqrt{6}/2$ and $-i\sqrt{6}/2$, which are in the immediate basin of attraction of zero. Observe that the repelling fixed points are symmetric with respect the imaginary axis. We will show that the $i\mathbb{R} \cup \{\infty\}$ is in the immediate basin of attraction of 0 and, it follows that $\mathcal{J}(M_f)$ is disconnected, see Figure 1 (in green the basin of attraction of 1, in brown the basin of attraction of 0 and in white, the basin of attraction of -1). Since,

$$M_f(1/z) = \frac{2z}{z^4 - z^2 + 2},$$

∞ is a preimage of 0. As a consequence, ∞ is in the immediate basin of attraction of 0.

On the other hand

$$M_f(iz) = \frac{-2z^3}{2z^4 + z^2 + 1} i.$$

Hence, the imaginary axis is invariant under $M_f(z)$ and, our map is

$$y \mapsto \frac{-2y^3}{2y^4 + y^2 + 1}.$$

It is easy to see that

$$\left| \frac{-2y^3}{2y^4 + y^2 + 1} \right| < |y|.$$

These inequality above imply that all points on the imaginary axis are sent closer to the origin. Thus, the entire imaginary axis is in the immediate basin of attraction for 0.

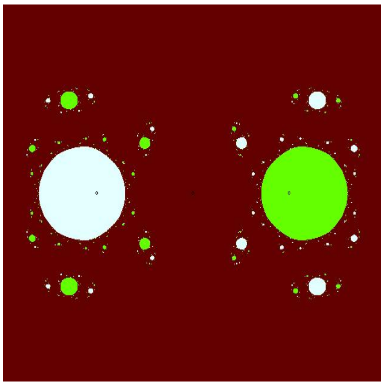


Figure 1

Example 4.4.2. We show an example of a Julia set of Newton's method for multiple roots which is conjugated by $1/2z$ to $F_\lambda(z) = z^3 + \dfrac{\lambda}{z}$ where $\lambda = 3/16$. We study the intersections of the immediate basins of attraction. Our proof uses a similar argument to the one of D. Look [16].

First, by Scaling theorem we can take a parametrization of the coefficients of cubic polynomials. Thus the family of one parameters of cubic polynomial $f_a(z) = z^3 + (a-1)z - a$ represent dynamically all of generic cubic polynomials applied to Newton's method for multiple roots. Now, we concentrate our attention in the case $a = 0$. The Newton method for multiple roots method is $M(z) := M_f(z) = \dfrac{4z^3}{3z^4+1}$.

The roots of f, $0, -1, 1,$ are super attracting fixed points of M and $\frac{1}{3}\sqrt{3}$, $-\frac{1}{3}\sqrt{3}$ are repelling. The remaining critical points i and $-i$, form a super attracting periodic orbit of period 2.

We let denote by $\mathcal{O}$, A_1, A_{-1}, A_i, A_{-i} the immediate basin of attraction of 0, 1 -1 i and $-i$, respectively.

Let ω such that $\omega^4 = 1$. There are a 4-symmetry

$$M(wz) = w^3 M(z).$$

This implies that if z_0 is attracted to a periodic cycle, then ωz_0, $\omega^2 z_0$ and $\omega^3 z_0$ are also attracted to periodic cycles, although they could be different cycles.

The poles of M are $\dfrac{1}{\sqrt[4]{3}} \exp\left(\dfrac{(2k+1)i\pi}{4}\right)$, for $k = 0,\ 1,\ 2,\ 3$. We will denote p the pole associated with $k = 0$. Thus the other poles are $\bar{p}, -p$ and $-\bar{p}$. An approximation of p is $0.5372849659 +$

$0.5372849659I$. We let define the *symmetric rays* as the sets $L_p := p\mathbb{R}$ and $L_{\bar{p}} := \bar{p}\mathbb{R}$. Notice that $-p \in L_p$ and $-\bar{p} \in L_{\bar{p}}$.

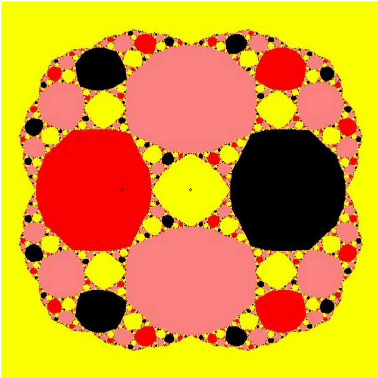


Figure 2

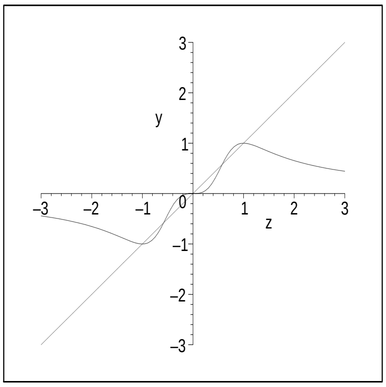


Figure 3

Lemma 4.4.1. *The Fatou set is the union of* $\mathcal{O}$, A_1 , $(-1)A_1 = A_{-1}$, $iA_1 = A_i$, $(-i)A_1 = A_{-i}$ *and all of their pre-images.*

Proof. The five critical points of M are in $\mathcal{O}$, A_1 , $(-1)A_1$, iA_1 and $(-i)A_1$. Hence we have all of the critical points for M accounted for and there can be no others attracting points. This implies that all components of the Fatou set eventually are iterated to one of $\mathcal{O}$, A_1 , $(-1)A_1$, iA_1 and $(-i)A_1$. This concludes the proof.

Lemma 4.4.2. *The basins of attraction* A_1 , A_{-1} , A_i , A_{-i} *does not intersect the symmetric rays .*

Proof. The map M leaves forward invariant the set $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ and maps reciprocally the symmetric rays L_p in $L_{\bar{p}}$. Therefore $L_p \cup L_{\bar{p}}$ is also forward invariant under M . Hence, non of the sets A_1 , A_{-1} , A_i , A_{-i} can meet the symmetric rays, because all points of these sets are attracted to 1 , -1 , i and $-i$ respectively, while the union of the symmetric axis is forward invariant.

Lemma 4.4.3. *The boundary of* A_1 , A_{-1} , A_i *and* A_{-i} *is respectively a simple closed curve.*

Proof. By symmetries, it is enough to show the result for A_1 . Let $\psi : \mathbb{D} \to A_1$ the Riemann map and $\overline{\psi} : \overline{\mathbb{D}} \to \overline{A_1}$ its extension given by the Caratheodory's theorem. Suppose that there exists

$\beta \in \partial A_1$ such that two internal rays R_{t_1} and R_{t_2} land on β. These rays together with the point β form a Jordan curve γ. Let Γ denote the bounded complement of the these curve. There must be other points of ∂A_1 in Γ, else we would have an entire sector of rays all landing at β and this gives a contradiction a contradiction because this set has measure zero. Since Γ is connected and simply connected, we have that Γ lies entirely within one of the two open components created by this Jordan curve γ. On the other hand, since Γ is bounded by γ we know by the maximum modulus theorem that $M(\Gamma)$ is bounded by $M(\gamma)$. Because γ lies inside A_1 we know that $M(\gamma - \beta)$ lies inside A_1. Therefore, the boundary of $M(\Gamma)$ lies inside A_1. Hence, $M(\Gamma)$ is either mapped to the unbounded complement of $M(\gamma)$ or to the bounded complement. It is know that any neighborhood of the Julia set for a rational map of degree $d \geq 2$ is eventually mapped by iterates of the map onto the entire Riemann sphere minus at most two points. Since $\Gamma \cap J(M) \neq \emptyset$ it cannot be the case that $M(\Gamma)$ is mapped to the bounded complement of $M(\gamma)$. However, if $M(\Gamma)$ is mapped to the unbounded complement then Γ must contain a pole. Thus the pole must lie in the boundary of A_1, wich is not possible. This contradictions concludes the proof.

Lemma 4.4.4. *$\partial\mathcal{O}$ meets ∂A_{-1} and ∂A_1.*

Proof. We claim that $(-1, -\frac{\sqrt{3}}{3}) \subset A_{-1}$ and $(-\frac{\sqrt{3}}{3}, 0) \subset \mathcal{O}$. Thus $-\frac{\sqrt{3}}{3} \in \partial A_{-1} \cap \partial\mathcal{O}$. In a similar way we claim that $(0, \frac{\sqrt{3}}{3}) \subset \mathcal{O}$ and $(\frac{\sqrt{3}}{3}, 1) \subset A_1$ which implies that $\frac{\sqrt{3}}{3} \in \partial A_1 \cap \partial\mathcal{O}$.

Now we proof the claims above. Since $M(x)$ is an odd function, it is enough to prove $(0, \frac{\sqrt{3}}{3}) \subset \mathcal{O}$ and $(\frac{\sqrt{3}}{3}, 1) \subset A_1$.

When $x \in (0, \frac{\sqrt{3}}{3})$, $0 < 3x^4 - 4x^2 + 1$. Since $x > 0$, we have $0 < x(3x^4 - 4x^2 + 1)$. Therefore, $M(x) < x$ for $x \in (0, \frac{\sqrt{3}}{3})$. This implies that $(0, \frac{\sqrt{3}}{3}) \subset \mathcal{O}$. On the other hand, it is not difficult to show that for $x \in (0, \frac{\sqrt{3}}{3})$, $x < M(x) < 1$. Thus $x < M(x) < M^2(x) < \cdots < 1$ which implies $(\frac{\sqrt{3}}{3}, 1) \subset A_1$.

Lemma 4.4.5. *$\partial\mathcal{O}$ meets ∂A_{-i} and ∂A_i.*

Proof. Since $\frac{1}{3}\sqrt{3}$ and $-\frac{1}{3}\sqrt{3}$ are repelling fixed points in ∂A_1 and ∂A_{-1} respectively, thus $\frac{i}{3}\sqrt{3}$

and $\frac{-i}{3}\sqrt{3}$ is a two repelling cycle with $\frac{i}{3}\sqrt{3} \in i\partial A_1 = \partial A_i$ and $\frac{-i}{3}\sqrt{3} \in -i\partial A_1 = \partial A_{-i}$. Note that

$$M_f(iz) = \frac{-4z^3}{3z^4+1}i.$$

Hence, the imaginary axis is invariant under $M_f(z)$ and, our map is

$$y \mapsto \frac{-4y^3}{3y^4+1}.$$

It is not difficult to see that

$$\left|\frac{-4y^3}{3y^4+1}\right| < |y|,$$

when $y \in (-\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3})$. These inequality imply that all points on the imaginary axis are sent closer to the origin. Therefore the imaginary interval $i(-\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3})$ is in the immediate basin of attraction for 0. We conclude that $-\frac{i}{3}\sqrt{3}$ and $\frac{i}{3}\sqrt{3}$ are in $\partial\mathcal{O}$. This shows that $-\frac{i}{3}\sqrt{3} \in \partial\mathcal{O} \cap A_{-i}$ and $\frac{i}{3}\sqrt{3} \in \partial\mathcal{O} \cap A_i$.

Lemma 4.4.6. *$\partial A_n \cap \partial A_m \neq \emptyset$ for $n \neq m$ and $n, m = 1, -1, i, -i$.*

Proof. By symmetries it is sufficient to proof that $\partial A_1 \cap \partial A_i \neq \emptyset$. Notice that A_1 is trapped in the region $S_1 = \{z \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$ and A_i is trapped in $S_i = \{z \in \mathbb{C} : \frac{\pi}{4} < \arg z < \frac{3\pi}{4}\}$. Since the Julia set is locally connected and A_1 and A_i are simply connected, the respective internal rays must be land in ∂A_1 and ∂A_i.

Since the degree of M is four there at most six distinct two cycles. One of the two cycle is $t = \frac{\sqrt{12+6\sqrt{7}}}{6} + i\frac{\sqrt{12+6\sqrt{7}}}{6}$ and its complex conjugated. Thus t is the landing point of at least two periodic ray of period two. This ray is completely contained in the basin of attraction A_1. Hence the point t is in ∂A_1. We have a similar result for the basin A_i. This show that t is in $\partial A_1 \cap \partial A_i$.

Chapter 5

σ-Schröder's Root-Finding Algorithm.

5.1 Definitions and basic results

In this chapter we study a generalization of Newton's method for multiples roots studied in Chapter 4. As far as we know, Ernst Schröder defined for the first time this generalization in [31]. Subsequently these method have been studied by T. Pomentale in [26].

Definition 5.1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ a complex polynomial and let $\sigma \geqslant 2$ be an integer. The σ-Schröder's method of order σ associated to f, is the rational map $S_{f,\sigma} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by the formula*

$$S_{f,\sigma} = \mathrm{Id} + (\sigma - 1)\frac{(f'/f)^{[\sigma-2]}}{(f'/f)^{[\sigma-1]}}\,,$$

where $(f'/f)^{[k]}$ is the derivative of order k of f'/f.

For $\sigma = 2$ the σ-Schröder's method coincides with Newton's method for multiples roots, already studied in Chapter 4. The obvious similarity with the König's root-finding algorithm of Chapter 3, leads to an identical computations to those developed by Buff and Henriksen in [6]. However this methods have substantial dynamical differences. For instance the fixed points associated to a root of σ-Schröder's method, are always superattracting.

Following [6], the next proposition is a straightforward computation.

Proposition 5.1.1. *Let $f : \mathbb{C} \to \mathbb{C}$ a polynomial. Denote by α_i its zeros and by $n_i \geqslant 1$ their multiplicities. Then, for any integer $\sigma \geqslant 2$, the fixed points of σ-Schröder's method of f are either (super)attracting or repelling.*

a) *The super attracting fixed points are exactly the zeros α_i.*

b) *The extraneous fixed points of $S_{f,\sigma}$ are exactly the zeros of $(f'/f)^{[\sigma-2]}$. If β_j is a zero of $(f'/f)^{[\sigma-2]}$ with multiplicity m_j, then it is a repelling fixed point of $S_{f,\sigma}$ with multiplier $1 + (\sigma - 1)/m_j$.*

5.2 Conjugacy classes.

Theorem 5.2.1. (Scaling theorem) *Let $f(z)$ be an analytic function on the Riemann sphere, and let $A(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map. If $g(z) = f \circ A(z)$, then for any integer $\sigma \geq 2$, we have*

$$S_{f,\,\sigma} \circ A = A \circ S_{g,\,\sigma},$$

that is, the σ-Schröder method of f is analytically conjugated to σ-Schröder's method of g.

Proof. Since $g(z) = f \circ A(z)$, we have

$$\left(\frac{g'}{g}\right)^{[\sigma]} = \alpha^{\sigma+1} \left(\frac{f'}{f}\right)^{[\sigma]} \circ A.$$

Thus,

$$A \circ S_{g,\sigma}(z) = \alpha z + (\sigma - 1)\frac{(f'/f)^{[\sigma-2]}}{(f'/f)^{[\sigma-1]}} \circ A(z) + \beta = S_{f,\,\sigma} \circ A,$$

this complete the proof.

5.3 The Topology of Julia and Fatou sets for σ-Schröder's method .

In this section we generalize the example 4.4.1.

Proposition 5.3.1. Consider the polynomial $f(z) = z^4 - z^2$. We will show that for every $\sigma \geq 2$, the σ-Schröder method applied to the polynomial f has not connected Julia set.

Proof. In general, if $f(z) = a_0 + a_1 z + \cdots,\ a_n z^n$ then

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{n} \frac{1}{z - \alpha_i},$$

where α_i are the zeros of $f(z)$ counted with multiplicities. Thus

$$\left(\frac{f'}{f}\right)^{[\sigma-1]}(z) = (-1)^{(\sigma-1)}(\sigma-1)!\sum_{i=1}^{n} \frac{1}{(z-\alpha_i)^{\sigma}}.$$

It follows that for $f(z) = z^4 - z^2$,

$$\left(\frac{f'}{f}\right)^{[\sigma-2]}(z) = (-1)^{(\sigma-2)}(\sigma-2)!\left(\frac{2}{z^{(\sigma-1)}} + \frac{1}{(z-1)^{(\sigma-1)}} + \frac{1}{(z+1)^{(\sigma-1)}}\right). \tag{5.1}$$

As a consequence,

$$S_{f,\sigma}(z) = \frac{z^{\sigma}((z+1)^{\sigma} - (z-1)^{\sigma})}{2(z+1)^{\sigma}(z-1)^{\sigma} + z^{\sigma}(z-1)^{\sigma} + z^{\sigma}(z+1)^{\sigma}}. \tag{5.2}$$

The repelling fixed point are the zeros of the equation 5.1. On the other hand, $S_{f,\sigma}$ is an odd function. Hence, there are two repelling fixed points which are symmetric with respect to the imaginary axis. In particular, when σ is an even natural number, the equation 5.1 has a real solution.

We will show that the $i\mathbb{R} \cup \{\infty\}$ is in the immediate basin of attraction of 0 and, it follows that $\mathcal{J}(S_{f,\sigma})$ is disconnected, see figure below. Since,

$$S_{f,\sigma}(1/z)\mid_{z=0} = 0,$$

∞ is a preimage of 0. As a consequence, ∞ is in the immediate basin of attraction of 0.

On the other hand

$$\begin{aligned} M_f(iz) &= -\frac{(zi)^\sigma(-(zi+1)^\sigma + (zi-1)^\sigma)}{((zi)^\sigma(zi+1)^\sigma + 2(zi-1)^\sigma(zi+1)^\sigma + (zi-1)^\sigma(zi)^\sigma)} \\ &= i\,\mathcal{M}_f(z), \end{aligned}$$

where $\mathcal{M}_f(z)$ is the map defined by,

$$\mathcal{M}_f(z) = \frac{(zi)^\sigma(-(zi+1)^\sigma + (zi-1)^\sigma)i}{((zi)^\sigma(zi+1)^\sigma + 2(zi-1)^\sigma(zi+1)^\sigma + (zi-1)^\sigma(zi)^\sigma)}.$$

Notice that $\mathcal{M}_f(z)$ restricted to the real line is a real map. Hence, the imaginary axis is invariant under $S_{f,\sigma}(z)$ and our map is

$$y \mapsto \frac{(yi)^\sigma(-(yi+1)^\sigma + (yi-1)^\sigma)i}{((yi)^\sigma(yi+1)^\sigma + 2(yi-1)^\sigma(yi+1)^\sigma + (yi-1)^\sigma(yi)^\sigma)}.$$

By induction on σ,

$$\left|\frac{(yi)^\sigma(-(yi+1)^\sigma + (yi-1)^\sigma)i}{((yi)^\sigma(yi+1)^\sigma + 2(yi-1)^\sigma(yi+1)^\sigma + (yi-1)^\sigma(yi)^\sigma)}\right| < |y|\,.$$

This inequality implies that all points on the imaginary axis are sent close to the origin. Thus the entire imaginary axis is in the immediate basin of attraction of 0. This concludes the proof.

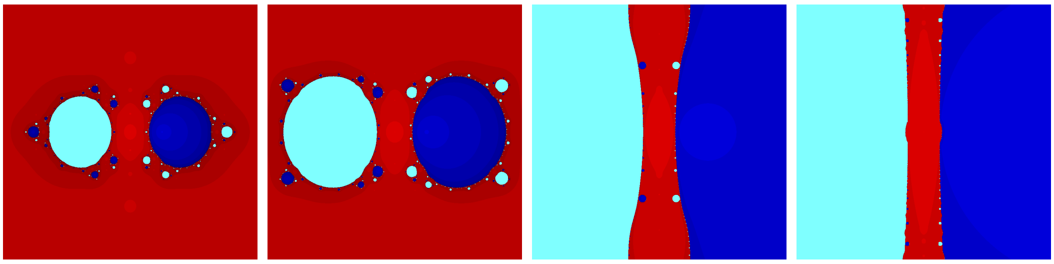


Figure 4. From left to right: The Julia set of $S_{f,\sigma}$ apply to the polynomial $f(z) = z^4 - z^2$, for $\sigma = 3, 4, 20$ and $\sigma = 100$ respectively.

Corollary 5.3.1. *For all $\sigma \geqslant 2$ there exists a rational map such that the Julia set of König's method is disconnected.*

Chapter 6

Whittaker's method, super-Halley method and Chebyshev's method

6.1 Definitions and basic results

The *Whittaker Root-Finding algorithms*, also known as convex acceleration method of Whittaker's method was introduced by E. T. Whittaker in [35]. It is an iterative map of order of convergence two, given by

$$W_f(z) = z - \frac{f(z)}{2f'(z)}\left(2 - \frac{f(z)f''(z)}{(f'(z))^2}\right). \tag{6.1}$$

In [11] J. A. Ezquerro and M.A. Hernández presented the so called *super–Halley Root-Finding algorithms* through a convex acceleration of Newton's method. This is an order three iterative map, given by

$$SH_f(z) = z - \frac{f(z)}{2f'(z)}\frac{\left(2 - \dfrac{f(z)f''(z)}{[f'(z)]^2}\right)}{\left(1 - \dfrac{f(z)f''(z)}{[f'(z)]^2}\right)}. \tag{6.2}$$

The *Chebyshev Root-Finding algorithms* is also called the super-Newton method and Householder's method is given by the formula

$$S_f(z) = z - \frac{f(z)}{f'(z)} - \frac{[f(z)]^2 f''(z)}{2[f'(z)]^3}. \tag{6.3}$$

Remark 3. In order to verify that Whittaker's method it is a purely iterative algorithm for Newton's maps it is enough to note that we can rewrite this method as

$$W_f(z) \quad = z - \frac{1}{2}[z - N_f(z)]\left[2 - N_f'(z)\right].$$

In a similar way, super–Halley's method is a purely iterative algorithm for Newton's maps, since it may be write as

$$SH_f(z) = z - \frac{1}{2}\left(z - N_f(z)\right)\left(\frac{2 - N_f'(z)}{1 - N_f'(z)}\right),$$

and for Chebyshev' method we can write

$$S_f(z) = z - (z - N_f(z))(1 + \frac{1}{2}N_f'(z)).$$

According to the definition 1.5.2, of purely iterative algorithm for Newton's maps we must define the polynomials P and Q in three variables. Hence, for Whittaker's method we consider $P(z, \xi_0, \xi_1) = (z - \xi_0)(1 - \frac{\xi_1}{2})$ and $Q(z, \xi_0, \xi_1) \equiv 1$. On the other hand, for the super–Halley method we take $P(z, \xi_0, \xi_1) = (z - \xi_0)(1 - \frac{\xi_1}{2})$ and $Q(z, \xi_0, \xi_1) = 1 - \xi_1$. Finally, for Chebyshev's method we define $P(z, \xi_0, \xi_1) = (z - \xi_0)(1 + \frac{\xi_1}{2})$ and $Q(z, \xi_0, \xi_1) = 1$

In the next figures we show some examples of Julia sets of the methods under study.

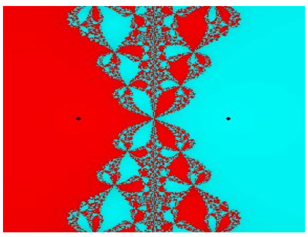


Figure 5

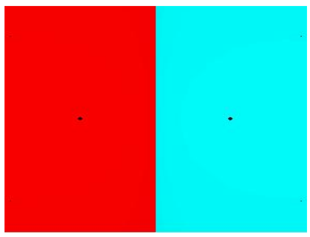
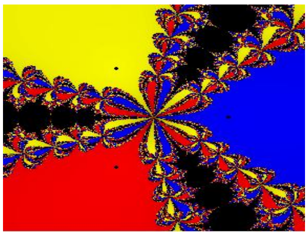


Figure 6

Figures 5 and 6 above show, respectively, the basins of attraction corresponding to the roots of $p(z) = z^2 - 1$ (black dots) for the iterative methods $W_p(z)$, and for the iterative method $SH_p(z)$. respectively.



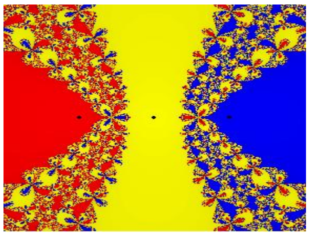
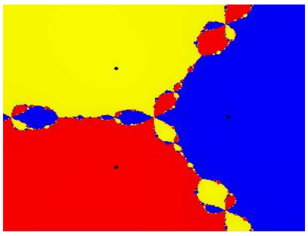


Figure 7 Figure 8

Figures 7 and 8 above show, respectively, the basins of attraction corresponding to the roots of $p(z) = z^3 - 1$ and of $p(z) = z^3 - z$ (black dots) for the iterative method $W_p(z)$.



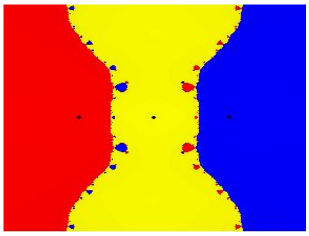


Figure 9 Figure 10

Figures 9 and 10 above show, respectively, the basins of attraction corresponding to the roots of $p(z) = z^3 - 1$ and of $p(z) = z^3 - z$ for the iterative methods $SH_p(z)$.

6.2 Conjugacy classes.

Now, we proof the Scaling theorem for a family of purely iterative algorithm for Newton's maps that includes the methods of this chapter.

Theorem 6.2.1. (Scaling theorem). *Assume that* $k = 1$, *and* T_f *a purely iterative algorithm for Newton maps where*

$$F(z, \xi_0, \xi_1) = z - \frac{P(z, \xi_0, \xi_1)}{Q(z, \xi_0, \xi_1)}, \tag{6.4}$$

and $P(z, \xi_0, \xi_1) = (z - \xi_0)(a_1 + b_1\xi_1)$, $Q(z, \xi_0, \xi_1) = (a_2 + b_2\xi_1)$.

Let f *and* g *two polynomials, and let* A *de an affine* $A(z) = \alpha z + \beta$, *with* $\alpha \neq 0$. *Consider*

$$T_f(z) = F(\hat{\jmath}(z,\, N_f(z)) \ \textit{and}\ T_g(z) = F(\hat{\jmath}(z,\, N_g(z))$$

If $g(z) = f \circ A(z)$ *, then* $T_f \circ A = A \circ T_g$ *, that is,* T_f *is analytically conjugate to* T_g *by* A *.*

Proof. Assume that there exists a constant $\lambda \in \mathbb{C}^*$ such that $g(z) = \lambda f \circ A(z)$. Using the Scaling theorem for Newton's method (see theorem 2.2.1), we have $A(N_g(z)) = \alpha N_g(z) + \beta = N_f(A(z))$. Hence

$$N_g'(z) = N_f'(\alpha z + \beta) = N_f'(A(z)).$$

This gives,

$$\begin{aligned} T_f(A(z)) &= F(j(A(z),\, N_f(A(z)))) \\ &= A(z) - \frac{P(A(z),\, N_f(A(z)),\, N_f'(A(z)))}{Q(A(z),\, N_f(A(z)),\, N_f'(A(z)))} \\ &= \alpha\left(z - \frac{(z - N_g(z))(a_1 + b_1 N_g'(z))}{(a_2 + b_2 N_g'(z))}\right) + \beta \\ &= A(T_g(z)). \end{aligned}$$

This ends the proof.

Theorem 6.2.2. (The Scaling theorem) *Let* $f(z)$ *be an analytic function on the Riemann sphere, and let* $A(z) = \alpha z + \beta$ *, with* $\alpha \neq 0$ *, be an affine map. If* $g(z) = f \circ A(z)$ *, then* $A \circ W_g \circ A^{-1}(z) = W_f(z)$ *,* $A \circ SH_g \circ A^{-1}(z) = SH_f(z)$ *and* $A \circ S_g \circ A^{-1}(z) = S_f(z)$ *. That is,* W_f *is analytically conjugated to* W_g *by* A *and* SH_f *is analytically conjugated to* SH_g *and* S_f *is analytically conjugated to* S_g *by* T *.*

Proof. The remark above show us how to rewrite the methods as the formula 6.4.

6.3 The nature of fixed points.

For Whittaker's method we have the following result.

Theorem 6.3.1. *Let* $f : \mathbb{C} \to \mathbb{C}$ *a polynomial of degree* $d \geq 2$ *. Denote by* α_i *its zeros and by* n_i *their multiplicities. Then,*

a) The roots α_i *, are (super)attracting fixed points of* W_f *and their multipliers are* $1 - (n_i + 1)/2n_i^2$ *.*

b) The map $W_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *has a repelling fixed point at* ∞ *with multiplier* $1 + (d+1)/(2d^2 - d - 1)$ *.*

c) The extraneous fixed points (that is, fixed points of W_f *that are not root of* f *) of* W_f *are exactly the zeros of* $(1/f)''$ *. If* f *has* k *distinct roots, then the degree of* W_f *is at most* $3k - 2$ *.*

Proof.

a) First, we use an appropriate change of Witthaker's mehod

$$\begin{aligned} W_f(z) = \quad & z - \frac{f(z)}{2f'(z)}\left(2 - \frac{f(z)f''(z)}{[f'(z)]^2}\right) \\ & = z - \frac{1}{2}[z - N_f(z)]\left[2 - N_f'(z)\right]. \end{aligned}$$

If f has a zero α of multiplicity n, then by theorem 2.1.1, α is a (super)attracting fixed point of Newton's Method with multiplier $(n-1)/n$. Thus

$$N_f(z) = \alpha + (z - \alpha)\left(\frac{n-1}{n}\right) + O(z-\alpha)^2.$$

It follows that,

$$z - N_f(z) \quad = \frac{1}{n}(z - \alpha) + O(z - \alpha)^2$$

and

$$2 - N_f'(z) \quad = \left(\frac{n+1}{n}\right) + O(z - \alpha).$$

As a consequence, we have

$$\begin{aligned} W_f(z) \quad & = \alpha + (z-\alpha) - \frac{(z-\alpha)(n+1)}{2n^2} + O(z-\alpha)^2 \\ & = \alpha + \left[1 - \left(\frac{n+1}{2n^2}\right)\right](z-\alpha) + O(z-\alpha)^2. \end{aligned}$$

b) When $|z|$ tends to ∞, we can write Newton's method apply f [[6] p. 994] as

$$N_f(z) = \left(\frac{d-1}{d}\right)z + O(1) \text{ and } N_f'(z) = \left(\frac{d-1}{d}\right) + O\left(\frac{1}{z}\right).$$

Thus,

$$W_f(z) = \left(1 - \frac{d+1}{2d^2}\right)z + O(1).$$

Hence, ∞ is a fixed point of W_f with multiplier $2d^2/(2d^2-d-1) = 1+(d+1)/(2d^2-d-1)$.

c) Again, the formula for W_f is

$$W_f(z) = z - \frac{1}{2}[z - N_f(z)]\left[2 - N_f'(z)\right].$$

The extraneous fixed points of W_f are the zeros of $2 - N_f'(z)$ in $\mathbb{C}$, which are the same of $g = (1/f)''$.

Let α_i, $i = 1, \ldots, k$, be the zeros of f and n_i are their multiplicities. Since f is a polynomial of degree d, we have

$$\sum_{i=1}^{k} n_i = d.$$

For any rational map, the number of zeros in $\overline{\mathbb{C}}$ is equal to the number of poles in $\overline{\mathbb{C}}$. The poles of $g = (1/f)''$ are the points α_i, with multiplicity $n_i + 2$, and g has a zero of order $d+2$ at ∞. Thus, g has

$$\sum_{i=1}^{k}(n_i + 2) = d + 2k$$

poles counted with multiplicities. It follows that g has

$$(d + 2k) - (d + 2) = 2k - 2$$

zeros in $\mathbb{C}$, counted with multiplicities. Consequently, W_f has at most $2k-2$ extraneous fixed points.

For any rational map, the number of fixed points counted with multiplicities is equal to the degree of the rational map plus one. The fixed points of W_f are simple (there is no indifferent fixed points). There are k (super)attracting fixed points (from the roots), one repelling fixed point at infinity and at most $2k-2$ extraneous fixed points in $\mathbb{C}$. Therefore, the degree of W_f is at most

$$[k + 1 + 2k - 2] - 1 = 3k - 2.$$

This concludes the proof of $c)$.

In a similar way, we have the following result for the super–Halley method.

Theorem 6.3.2. *Let* $f : U \subset \mathbb{C} \to \mathbb{C}$ *a polynomial of degree* $d \geq 2$. *Denote by* α_i *its zeros and by* n_i *their multiplicities. Then,*

$a)$ *The roots* α_i, *are (super)attracting fixed points of* SH_f *and their multipliers are* $(n_i - 1)/2n_i$.

$b)$ *The map* $SH_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *has a repelling fixed point at* ∞ *with multiplier* $1 + (d+1)/(d-1)$.

$c)$ *The extraneous fixed points of* SH_f *are exactly the zeros of* $(1/f)''$. *If* f *has* k *distinct roots, then the degree of* SH_f *is at most* $3k - 2$.

Proof.

$a)$ It is easily to check that the super-Halley method can be writted as

$$SH_f(z) = z - \frac{z - W_f(z)}{1 - N'_f(z)}$$

where N'_f is the derivative of Newton's method and W_f is Whittaker's method.

By an argument similar to used to prove the theorem 6.3.1, and using the fact that

$$z - W_f(z) = (z - \alpha)\left(\frac{n+1}{2n^2}\right) + O(z-\alpha)^2.$$

As a consequence,

$$\begin{aligned} SH_f(z) &= z - \frac{(z-\alpha)\left(\dfrac{n+1}{2n^2}\right) + O(z-\alpha)^2}{\left(\dfrac{1}{n}\right) + O(z-\alpha)} \\ &= \alpha + (z-\alpha) - (z-\alpha)n\frac{(n+1)}{2n^2} + O(z-\alpha)^2 \\ &= \alpha + \left(\frac{n-1}{2n}\right)(z-\alpha) + O(z-\alpha)^2. \end{aligned}$$

b) When $|z|$ tends to ∞, we can write Newton's method apply to the polynomial f [[6] p. 994] as

$$N_f(z) = \left(\frac{d-1}{d}\right) z + O(1) \text{ and } N_f'(z) = \left(\frac{d-1}{d}\right) + O(1).$$

Thus,

$$SH_f(z) = \left(1 - \frac{d+1}{2d}\right) z + O(1)\,.$$

Hence, ∞ is a fixed point of SH_f with multiplier $2d/(d-1) = 1 + (d+1)/(d-1)$.

c) Again, we can change the formula for SH_f, conveniently as

$$SH_f(z) = z - \frac{z - W_f(z)}{1 - N_f'(z)}$$

where N_f' is the derivative of Newton's method and W_f is Whittaker's method.

The fixed points of SH_f are the zeros of Whittaker's method $z - W_f(z)$, which have already been studied in 6.3.1. So, we can repeat the argument of theorem 6.3.1 part *c*), to conclude that the degree of SH_f is at most $3k-2$.

Finally, we have the same results for Chebyshev's method. For the polynomial $f(z) = z^3 + \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{2}$ Chebyshev's method has a superattracting fixed point no associated to a root of f, see K. Kneisl [15].

With the same arguments used above, we obtain a similar result for Chebyshev's method.

Proposition 6.3.1. *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ a polynomial. Denote by α_i its zeros and by n_i their multiplicities. Then each of one zero α_i of polynomial f is (super)attracting fixed point of Chebyshev's method $S_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.*

a) *The multipliers in α_i are equal to $1 - (3n-1)/2n^2$.*

b) *The map $S_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ has a repelling fixed point at ∞ with multiplier $2d^2/(2d^2 - 3d + 1)$.*

c) *El grado de la aplicación $S_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ es a lo más $3d-2$.*

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