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Introduction

This work is on some cohomological aspects of the Lie algebras of formal pseudodifferential operators in one and several independent variables. We are motivated by several previous works on similar algebras such as the ”Lie algebra of vectors fields on the circle” or the ”Lie algebra of differential operators” of importance for the theory of integrable systems, symplectic geometry and abstract infinite dimensional Lie algebras, see for example [GF, DK, D, KP].

One of the first infinite-dimensional Lie algebras of interest for physics is the Virasoro algebra [V]. This algebra is a central extension of the Lie algebra of vector fields on the circle (also called centerless Virasoro algebra). The cohomology of centerless Virasoro was computed by Gelfand and Fuchs in [GF]. We note that this algebra appears naturally in physics. For example, it has been observed in [HHR, GR] that the centerless Virasoro algebra appears as algebra of nonlocal symmetries for the Camassa-Holm and Hunter-Saxton equations.

The Lie algebra of vector fields on the circle is included naturally in the Lie algebra of differential operators on the circle equipped with the Lie bracket given by the standard commutator of differential operators. This algebra, in turn, is included in the Lie algebra of formal pseudodifferential operators on the circle which has been very carefully studied for example by L. Dickey [D] and M. Adler [A] in connection with the algebraic and geometric theory of the famous Korteweg-de Vires equation and other integrable systems. We also stress that there exist some nontrivial generalizations of these works in connection with “twisted” and “quantum” analogs of classical integrable systems [KLR, FM, PSTS, PSTS1]

Thus, it is a natural problem to study the cohomology of the algebras of pseudodifferential operators as an extension of the study of the cohomology of the centerless Virasoro algebra. We mention that 2-cocycle of the algebra of differential operators on the circle was conrstucted in [KP] and a 2-cocycle of the algebra of pseudodifferential operators on the circle was constructed by Kravchenko and Khesin using logarithms in [KK, KW]. Also, a 2-cocycle for the quantum analog of the algebra of pseudodifferential operators was considered in [KLR].

In this Thesis we generalize some of these results and we also present classifications of central extensions in a purely algebraic context. We have divided our work in three chapters. The first chapter is on algebras of formal pseudodifferential operators in one variable. (Hereafter we will speak of *pseudodifferential symbols* instead of "formal pseudodifferential operators" as we will not consider them from an analytic point of view). The second chapter is on the generalization of the theory in Chapter 1 to the case of several independent variables and, finally, our third chapter is on some applications to the contruction of integrable systems using r-matrices and Manin triples [KW, KZ, RSTS, STS, STS1].

Let us describe in more detail what we do in each chapter:

Chapter 1 We introduce the main objects we will be using in this work. We define pseudodifferential symbols in one variable on an arbitrary associative and commutative algebra, we study its associative and Lie algebra structures, and we construct the Kravchenko-Khesin logarithmic 2-cocycle. Motivated by a recent thesis by Donin (see [DK]) we investigate the relevance of cosidering logarithms of general pseudodifferential symbols for the construction of central extensions. We prove that no new central extensions are obtained using this method. We also construct a hierarchy defined by a chain of Lie algebras of differential operators which admit nontrivial central extensions via the logarithmic 2-cocycle. This construction generalizes a theorem by Khesin [K] on hierachies of Lie algebras of differential operators on the circle. Finally, motivated by [KLR], we consider twisted pseudodifferential symbols on arbitrary associative and commutative algebras following [FM], and we construct central extensions and hierachies.

Chapter 2 We consider pseudodifferential symbols in several independent variables on an arbitrary associative and commutative algebra. We have been able to find only two papers directly related to our work, one by Parshin, [P], in which he considers a "generalized KP hierachy" and one by Dzhumadildaev, [D], in which he studies central extensions in this general context. We construct central extensions using logarithmic cocycles and we also exhibit full hierachies of differential operators in several independent variables admiting central extensions. Finally, we present a new proof of the main result in [D] on the classification of central extensions.

Chapter 3 We apply our results to the construction of integrable systems roughly following the techniques of [KLR, DK, P]. We introduce Manin triples, we define double extensions for the algebras of (twisted) pseudodifferential symbols in one and several independent variables and, using a purely algebraic theorem [B], we construct Manin triples for these algebras. This result then allows us to introduce hierachies of integrable equations in our algebraic context following [RSTS,STS1].

We close our work with a chapter on conclusions and open problems, and with an appendix containing the some the technical details of our proof of the Dzhumadildaev classification theorem.

Chapter 1

The Algebra of Formal Pseudodifferential Symbols in One Variable

1.1 Basic Definitions and Preliminary Results

In this section, we introduce some fundamental properties of the algebra of formal pseudodifferential symbols and set up our notation.

Let $\mathcal{A}$ be an associative and commutative algebra, $\delta : \mathcal{A} \to \mathcal{A}$ a derivation on $\mathcal{A}$, this is, a linear map such that $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in \mathcal{A}$. *The algebra of formal differential symbols DO* is generated by $\mathcal{A}$ and a symbol ξ with the relation

$$\xi \circ a = a\xi + \delta(a) \tag{1.1}$$

for all $a \in \mathcal{A}$. $\mathcal{A}$ become a subalgebra of DO and inductively, we can prove that

$$\xi^n \circ a = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \delta^j(a)\xi^{n-j} \tag{1.2}$$

for all $a \in \mathcal{A}$ and $n \geq 0$.

We can extend the algebra DO to obtain the *algebra of formal pseudodifferential symbols* ΨDO by introducing the differentiations with negative exponents;

$$\xi^{-1} \circ a = \sum_{i=0}^{\infty} (-1)^i \delta^i(a)\xi^{-1-i}$$

so that a general element of ΨDO is a formal series D of the form

$$D = \sum_{-\infty}^{n} a_i \xi_i \quad \text{, where } a_i \in \mathcal{A}.$$

The multiplication on ΨDO is generalized using (1.2) to all $n \in \mathbb{Z}$ by

$$\xi^n \circ a = \sum_{j=0}^{\infty} \left(\begin{array}{c} n \\ j \end{array} \right) \delta^j(a) \xi^{n-j}. \tag{1.3}$$

Here the binomial coefficient is defined by

$$\left(\begin{array}{c} n \\ j \end{array} \right) = \begin{cases} \dfrac{n(n-1)\cdots(n-j+1)}{j!} & \text{if } n \in \mathbb{Z}^+, j \in \mathbb{Z} \\ \delta_{j,0} (\text{the Kronecker delta}) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $D_1 = \sum\limits_{i=-\infty}^{M} a_i \xi^i$ and $D_2 = \sum\limits_{j=-\infty}^{N} b_j \xi^j$, we have that:

$$\begin{aligned} D_1 \circ D_2 &= \left(\sum_{i=-\infty}^{M} a_i \xi^i \right) \left(\sum_{j=-\infty}^{N} b_j \xi^j \right) \\ &= \sum_{i=-\infty}^{M} \sum_{j=-\infty}^{N} a_i \xi^i b_j \xi^j \\ &= \sum_{i=-\infty}^{M} \sum_{j=-\infty}^{N} \sum_{k=0}^{\infty} \frac{i\,(i-1)\ldots(i-k+1)}{k!} a \delta^k\left(b_j\right) \xi^{i-k} \xi^j \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial \xi^k} \left(\sum_{i=-\infty}^{M} a_i \xi^i \right) \left(\sum_{j=-\infty}^{N} \delta^k\left(b_j\right) \xi^j \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial \xi^k} \left(D_1\right) \delta^k\left(D_2\right). \end{aligned} \tag{1.4}$$

The Lie algebra structure on ΨDO is given by the usual commutator $[A, B] = A \circ B - B \circ A$, so that, for instance

$$[\xi^\alpha, a\xi^n] = \sum_{j=1}^{\infty} \left(\begin{array}{c} n \\ j \end{array} \right) \delta^j(a) \xi^{\alpha+n-j} \tag{1.5}$$

Lemma 1. *For any non-negative integer m, and $a, b \in \mathcal{A}$, we have*

$$\delta^m(ab) = \sum_{j=0}^{m} \binom{m}{j} \delta^{m-j}(a)\delta^j(b). \tag{1.6}$$

Proof. By induction. □

Let $\tau : \mathcal{A} \longrightarrow \mathbb{C}$ be a δ-invariant trace on $\mathcal{A}$, this is, τ is a linear map, $\tau(ab) = \tau(ba)$ and $\tau(\delta(a)) = 0$ for all $a, b \in \mathcal{A}$.
The Adler-Manin nonconmutative residue [A, M], $res : \Psi DO \longrightarrow \mathbb{C}$ is defined by

$$res\left(\sum_{i=-\infty}^{n} a_i \xi_i\right) = \tau(a_{-1})$$

We can prove that the res is a trace, i.e, $res[A, B] = 0$ for all $A, B \in \Psi DO$. For example if $\mathcal{A} = C^\infty(S^1), \delta = d/dx$, the linear functional $\tau : C^\infty(S^1) \longrightarrow \mathbb{C}$ given by $\tau(f) = \int_0^1 f(x)dx$ is a δ-invariant trace.

Lemma 2. *Let A be an algebra, δ a derivation on $\mathcal{A}$, τ a δ-invariant trace on $\mathcal{A}$ y m a positive integer. Then for all $a, b \in \mathcal{A}$, we have*

$$\tau\left(b\delta^m\left(a\right)\right) = \left(-1\right)^m \tau\left(a\delta^m\left(b\right)\right). \tag{1.7}$$

Proof. By definition, we have that $\delta\left(ab\right) = a\delta\left(b\right) + \delta\left(a\right)b$, then, $0 = \tau\left(\delta\left(ab\right)\right) = \tau\left(a\delta\left(b\right)\right) + \tau\left(\delta\left(a\right)b\right)$,then $-\tau\left(a\delta\left(b\right)\right) = \tau\left(\delta\left(a\right)b\right)$. We proceed by induction on m. Let us assume that (1.7) is true for $m = k$,i.e , $\tau\left(b\delta^k\left(a\right)\right) = \left(-1\right)^k \tau\left(a\delta^k\left(b\right)\right)$. Then

$$\begin{aligned}
\tau\left(b\delta^{k+1}\left(a\right)\right) &= \tau\left(b\delta^k\left(\delta\left(a\right)\right)\right) \\
&= \left(-1\right)^k \tau\left(\delta\left(a\right)\delta^k\left(b\right)\right) \\
&= \left(-1\right)^k \tau\left(\delta\left(a\right)\delta^k\left(b\right)\right) \\
&= \left(-1\right)^{k+1} \tau\left(a\delta^{k+1}\left(b\right)\right).
\end{aligned}$$

□

Remark 3. *If $a < 0, b \geq 0$ integer ,then*

$$\left(\begin{array}{c} a \\ b \end{array}\right) = \left(-1\right)^b \left(\begin{array}{c} b-a-1 \\ b \end{array}\right) \tag{1.8}$$

Proposition 4. *Let $\mathcal{A}$ be an algebra, δ a derivation on $\mathcal{A}$ and τ a δ-invariant trace. Then the linear map $res : \Psi DO \longrightarrow \mathbb{C}$ defined by*

$$res\left(\sum_{i=-\infty}^{M} a_i \xi^i\right) = \tau(a_{-1})$$

is a trace, that is, res is linear and it satisfies $res(AB) = res(BA)$ for all $A, B \in \Psi DO$. This is the Adler-Manin noncommutative residue [A, M].

Proof. Since res is clearly linear, it is sufficient to show that for any $a, b \in A$; $m, n \in \mathbb{Z}$

$$res\left(a\xi^n \circ b\xi^m\right) = res\left(b\xi^m \circ a\xi^n\right)$$

We consider several cases:
Let $m.n \geqslant 0$

$$\begin{aligned} res\left(a\xi^n \circ b\xi^m\right) &= res\left(\sum_{j=0}^{\infty}\left(\begin{array}{c} n \\ j \end{array}\right) a\delta^j\left(b\right)\xi^{m+n-j}\right) \\ &= \sum_{j=0}^{\infty} res\left(\left(\begin{array}{c} n \\ j \end{array}\right) a\delta^j\left(b\right)\xi^{m+n-j}\right) \\ &= \tau\left(\left(\begin{array}{c} n \\ m+n+1 \end{array}\right) a\delta^{m+n+1}\left(b\right)\right) \\ &= 0 \end{aligned}$$

since we obtain the coefficient of ξ^{-1} when $j = m+n+1$, and $\left(\begin{array}{c} n \\ m+n+1 \end{array}\right) = 0$. On the other hand,

$$\begin{aligned} res\left(b\xi^m \circ a\xi^n\right) &= res\left(\sum_{j=0}^{\infty}\left(\begin{array}{c} m \\ j \end{array}\right) b\delta^j\left(a\right)\xi^{m+n-j}\right) \\ &= \sum_{j=0}^{\infty} res\left(\left(\begin{array}{c} m \\ j \end{array}\right) a\delta^j\left(b\right)\xi^{m+n-j}\right) \\ &= \tau\left(\left(\begin{array}{c} m \\ m+n+1 \end{array}\right) a\delta^{m+n+1}\left(b\right)\right) \\ &= 0 \end{aligned}$$

If $m, n < 0$, then $m+n-j \leq n+m < -1$ (j a positive integer), and

$$res\left(a\xi^n b\xi^m\right) = res\left(\sum_{j=0}^{\infty}\begin{pmatrix} n \\ j \end{pmatrix} a\delta^j\left(b\right)\xi^{m+n-j}\right).$$

But, as $m+n-j < -1$, the coefficient of ξ^{-1} is 0,and then $res\left(a\xi^n b\xi^m\right) = 0$. Analogously $res\left(b\xi^m a\xi^n\right) = 0$. Now assume that $n \geqslant 0, m < 0$. If $n+m < -1$ then

$$res\left(a\xi^n b\xi^m\right) = res\left(\sum_{j=0}^{\infty}\begin{pmatrix} n \\ j \end{pmatrix} a\delta^j\left(b\right)\xi^{m+n-j}\right) = 0$$

because $n+m-j < -1-j \leq -1$ and then $n+m-j < -1$. Analogously $res\left(b\xi^m a\xi^n\right) = 0$. Finally, if $n+m \geqslant -1$, we let $k = m+n$. Then,

$$\begin{aligned} res\left(a\xi^n \circ b\xi^m\right) &= \sum_{j=0}^{\infty} res\left(\begin{pmatrix} n \\ j \end{pmatrix} a\delta^j\left(b\right)\xi^{m+n-j}\right) \\ &= \tau\left(\begin{pmatrix} n \\ k+1 \end{pmatrix} a\delta^{k+1}\left(b\right)\right) \\ &= \begin{pmatrix} n \\ k+1 \end{pmatrix}\tau\left(a\delta^{k+1}\left(b\right)\right) \end{aligned}$$

and on the other hand,

$$\begin{aligned} res\left(b\xi^m \circ a\xi^n\right) &= \sum_{j=0}^{\infty} res\left(\begin{pmatrix} m \\ j \end{pmatrix} b\delta^j\left(a\right)\xi^{m+n-j}\right) \\ &= \tau\left(\begin{pmatrix} m \\ k+1 \end{pmatrix} b\delta^{k+1}\left(a\right)\right) \\ &= \begin{pmatrix} k-n \\ k+1 \end{pmatrix}\tau\left(b\delta^{k+1}\left(a\right)\right) \\ &= \left(-1\right)^{k+1}\begin{pmatrix} n \\ k+1 \end{pmatrix}\tau\left(b\delta^{k+1}\left(a\right)\right) \qquad \text{by (1.8)} \\ &= \left(-1\right)^{k+1}\left(-1\right)^{k+1}\begin{pmatrix} n \\ k+1 \end{pmatrix}\tau\left(a\delta^{k+1}\left(b\right)\right) \qquad \text{by (1.7)} \\ &= \begin{pmatrix} n \\ k+1 \end{pmatrix}\tau\left(a\delta^{k+1}\left(b\right)\right). \end{aligned}$$

$\square$

Remark 5. *The algebra ΨDO as vector space, has a direct sum decomposition $\Psi DO = DO \oplus INT$ where*

$$DO = \left\{ \sum_{i=0}^{n} a_i \xi^i | a_i \in \mathcal{A} \right\} \text{ and } INT = \left\{ \sum_{i=-\infty}^{-1} a_i \xi^i | a_i \in \mathcal{A} \right\}$$

are Lie subalgebras of ΨDO.

Proposition 6. *$res([A, B]) = 0$ for all $A, B \in \Psi DO$. This implies that the bilinear form $\langle A, B\rangle = res(A \circ B)$ is invariant, i.e it satisfies $\langle [A, B], C\rangle = \langle A, [B, C]\rangle$. Also the subalgebras DO and INT are isotropic subspaces of ΨDO, i.e, the restrictions of this form to both DO and INT vanish.*

Proof. Let $A, B \in \Psi DO$, $res([A, B]) = res(A \circ B) - res(B \circ A) = 0$ because res is a trace. On the other hand,

$$\begin{aligned}
\langle [A, B], C\rangle &= res([A, B] \circ C) = res((A \circ B) \circ C) - res((B \circ A) \circ C) \\
&= res(A \circ B \circ C) - res(B \circ A \circ C) \\
&= res(A \circ B \circ C) - res(A \circ C \circ B) \\
&= res(A \circ (B \circ C - C \circ B)) \\
&= res(A \circ [B, C]) = \langle A, [B, C]\rangle
\end{aligned}$$

Also, if $a\xi^n, b\xi^m \in DO$ then

$$a\xi^n \circ b\xi^m = \sum_{k=0}^{n} \binom{n}{k} a\delta^k(b)\xi^{n+m-k}$$

and therefore $a\xi^n \circ b\xi^m$ does not have a ξ^{-1} term because $m+n-k > 0$. Thus $res(a\xi^n \circ b\xi^m) = 0$. Analogously we prove that if $a\xi^n, b\xi^m \in INT$ then the ξ^{-1} term in $a\xi^n \circ b\xi^m$ is zero. □

1.2 Topics on Lie Algebras

This section is devoted to studying some concepts of cohomology of Lie algebras.

1.2.1 Cohomology of Lie Algebras

Suppose $\mathfrak{g}$ is a Lie algebra and A is a module over $\mathfrak{g}$. Then a *q-dimensional cochain* of the algebra $\mathfrak{g}$ with coefficients in A is a skew-symmetric q-linear functional on $\mathfrak{g}$ with values in A; the space of all such cochains is denoted by $C^q(\mathfrak{g}, A)$. The differential $d = d_q : C^q(\mathfrak{g}, A) \longrightarrow C^{q+1}(\mathfrak{g}, A)$ is defined by the formula

$$dc(g_1, ..., g_{q+1}) = \sum_{1 \le s < t \le q+1} (-1)^{s+t-1} c([g_s, g_t], g_1, ..., \hat{g}_s, ..., \hat{g}_t, ..., g_{q+1})$$
$$+ \sum_{1 \le s \le q+1} (-1)^s g_s c(g_1, ..., \hat{g}_s, ... g_{q+1}) \tag{1.9}$$

where $c \in C(\mathfrak{g}, A)$, $g_1, ..., g_{q+1} \in \mathfrak{g}$. We complete the definition by putting $C^q(\mathfrak{g}, A) = 0$ for $q < 0$, and $d_q = 0$ for $q < 0$. We can check that $d_{q+1} \circ d_q = 0$ for all q and therefore $\{C^q(\mathfrak{g}, A), d\}$ is an algebraic complex. This complex is denoted by $C^\bullet(\mathfrak{g}, A)$, while $H^q(\mathfrak{g}, A)$ denotes the q-cohomology space of the algebra $\mathfrak{g}$ with coefficients in A. If A is a field and a trivial $\mathfrak{g}$-module, then the second sum of in the right-hand side of formula (1.9) vanishes and may be ignored. Usually in this case, the notation for $C^q(\mathfrak{g}, A), H^q(\mathfrak{g}, A)$ are abbreviated to $C^q(\mathfrak{g}), H^q(\mathfrak{g})$.

Summarizing, if A is a module over a Lie algebra $\mathfrak{g}$, then the space of cochains $C^q(\mathfrak{g}, A)$ with $q > 0$ consists of multilinear skewsymmetric maps in q arguments and $C^0(\mathfrak{g}, A) = A$. For small values of q the coboundary operator $d : C^q(\mathfrak{g}, A) \longrightarrow C^{q+1}(\mathfrak{g}, A)$ acts as follows:

(α) $q = 0$, $dm(x) = x(m)$, $m \in C^0(\mathfrak{g}, A) = A$

(β) $q = 1$, $df(x, y) = -f([x, y]) + xf(y) - yf(x)$

(γ) $q = 2$, $d\psi(x, y, z) = -\psi([x, y], z) - \psi([y, z], x) - \psi([z, x], y) + x\psi(y, z) + y\psi(z, x) + z\psi(x, y)$.

We consider the following sets:

$$\begin{aligned} Z^q(\mathfrak{g}, A) &= \{\psi \in C^q(\mathfrak{g}, A) : d\psi = 0\} \qquad \text{(cocycles)} \\ B^q(\mathfrak{g}, A) &= \left\{dr : r \in C^{q-1}\right\} \qquad \text{(coboundaries)} \\ H^q(\mathfrak{g}, A) &= Z^q(\mathfrak{g}, A)/B^q(\mathfrak{g}, A) \qquad \text{(group cohomology)} \end{aligned}$$

We say that two 2-cocycles $c, \widehat{c}$ are equivalent or cohomologous if $c - \widehat{c}$ is a 2-coboundary. In general, we say that two n-cocycles $c, \widehat{c}$ are equivalent if $c - \widehat{c}$ is a n-coboundary.

1.2.2 Algebraic Interpretations of Cohomology

In this section we recall some algebraic interpretations of cohomology that are important for this work. The proofs of the following proposition are in [FD], chapter 1, section 4.

A *derivation* δ of the Lie algebra $\mathfrak{g}$ is a linear map $\delta : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$. A derivation is *inner* if $\delta = \delta_x(\cdot) = [x, \cdot]$ where $x \in \mathfrak{g}$ is a fixed element. *Exterior derivations* are by definition elements of the quotient space of all derivations module the subspace of inner derivations.

Proposition 7. *$H^1(\mathfrak{g}, \mathfrak{g}) = Z^1(\mathfrak{g}, \mathfrak{g})/B^1(\mathfrak{g}, \mathfrak{g})$ can be interpreted as the space of exterior derivations of the algebra $\mathfrak{g}$.*

Definition 8. *A central extension of a Lie algebra $\mathfrak{g}$ by a vector space $\mathfrak{n}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ whose underlying vector space $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{n}$ is equipped with the following Lie bracket:*

$$[(X, u), (Y, v)] = ([X, Y], c(X, Y))$$

for some bilinear map $c : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{n}$.

Note that c depend only on X and Y, but not on u and v. This implies that the extension $\mathfrak{g} \oplus \mathfrak{n}$ is central, i.e $\mathfrak{n}$ is the center of the Lie algebra $\widetilde{\mathfrak{g}}$.

The skewsymmetry, bilinearity and the Jacoby identity on Lie algebra $\widetilde{\mathfrak{g}}$ are equivalent to the antisymmetry, bilinearity and the following cocycle identity for the map c:

$$c([X, Y], Z) + c([Z, X], Y) + c([Y, Z], X) = 0. \tag{1.10}$$

for any $X, Y, Z \in \mathfrak{g}$. This identity corresponds to identity in (γ) in the case that $\mathfrak{g}$ is a trivial $\mathcal{A}$-module.

A 2-cocycle c on $\mathfrak{g}$ with values in $\mathfrak{n}$ is called 2-coboundary if there exist a linear map $\alpha : \mathfrak{g} \longrightarrow \mathfrak{n}$ such that $c(X, Y) = \alpha([X, Y])$ for all $X, Y \in \mathfrak{g}$. Therefore in describing different central extension we are interested only in the 2-cocycles modulo 2-coboundaries, i.e., in the *second cohomology* $H^2(\mathfrak{g}, \mathfrak{n})$ of the Lie algebra $\mathfrak{g}$ with values in $\mathfrak{n}$: $H^2(\mathfrak{g}, \mathfrak{n}) = Z^2(\mathfrak{g}, \mathfrak{n})/B^2(\mathfrak{g}, \mathfrak{n})$, where $Z^2(\mathfrak{g}, \mathfrak{n})$ if the vector space of all 2-cocycles on $\mathfrak{g}$ with values on $\mathfrak{n}$, and $B(\mathfrak{g}, \mathfrak{n})$ is the subspace of 2-coboundaries.

Remark 9. *In general, a central extension of a Lie algebra $\mathfrak{g}$ by a vector space $\mathfrak{n}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ whose underlying vector space $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{n}$ is equipped with the following Lie bracket:*

$$[(X, u), (Y, v)] = ([X, Y], c(X, Y) + \rho(X)v - \rho(Y)u)$$

for some bilinear map $c : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{n}$ and a function $\rho : \mathfrak{g} \longrightarrow End(\mathfrak{n})$. In our case, definition 8, the action ρ is trivial.

Proposition 10. *There is a one-to-one correspondence between the equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{n}$ and the elements of $H^2(\mathfrak{g}, \mathfrak{n})$.*

Proof. See [FD] page 33 or [DI] page 236. □

The proof of the principal theorem of Section 2.4 is based in the following result, the proof is in [D1].

Proposition 11. *Let $\overline{H^1(\mathfrak{g})}$ be the subspace of $Z^1(\mathfrak{g}, \mathfrak{g})$ generated by of cocycles $\phi \in Z^1(\mathfrak{g}, \mathfrak{g}')$ satisfying $(\phi(x), x) = 0$ for all $x \in \mathfrak{g}$, where $(,) : \mathfrak{g}' \times \mathfrak{g} \longrightarrow \mathbb{C}$ is the natural pairing. Then there is an isomorphism $\overline{H^1(\mathfrak{g})} \cong H^2(\mathfrak{g}, \mathbb{C})$. In particular, if $\mathfrak{g}$ possesses a nondegenerate form, then the space of central extensions $H^2(\mathfrak{g}, \mathbb{C})$ is a direct summand in the space of outer differentiations.*

1.3 Outer Derivations and central extensions of ΨDO

In this section we go back to the algebra ΨDO considered in the section 1.1. We recall that ξ is a symbol and the multiplication on ΨDO is given by (1.4).

Following [KK], we can write formally the identity $\xi^t = e^{tlog\xi}$. This implies that

$$\frac{d}{dt}\Big|_{t=0}\xi^t = log\xi.$$

Hence, setting $\alpha = t$ and differentiating both sides of equation (1.5) at $t = 0$, we obtain

$$[\log\xi, a\xi^n] = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\delta^j(a)\xi^{n-j}, \tag{1.11}$$

that is, an element of ΨDO.

We will use the following combinatorial identity (see [FM]):

Lemma 12. *Let $s \geq 1$ and $k \geq 0$ be integers and $\beta \in \mathbb{Z}$. Then*

$$\frac{(-1)^{s-1}}{s}\binom{\beta - s}{k} = \sum_{j=s}^{s+k} \frac{(-1)^{j-1}}{j}\binom{\beta}{s+k-j}\binom{j}{j-s} \tag{1.12}$$

Proposition 13. *$[log\xi, \cdot]$ defines a derivation of the (both associative and Lie) algebra ΨDO.*

Proof. In this proof, we use the combinatorial identity 1.12. It sufficient to prove that for any $a, b \in A$, $m, n \in \mathbb{Z}$.

$$[\log\xi, a\xi^n \circ b\xi^m] = [\log\xi, a\xi^n] \circ b\xi^m + a\xi^n \circ [\log\xi, b\xi^m] \tag{1.13}$$

For the left side of (1.13) we have:

$$\begin{aligned}
[\log\xi, a\xi^n \circ b\xi^m] &= \left[\log\xi, \sum_{k=o}^{\infty}\left(\begin{array}{c} n \\ k \end{array}\right) a\delta^j(b)\xi^{n+m-k}\right] \\
&= \sum_{k=o}^{\infty}\sum_{j=1}^{\infty}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(-1)^{j+1}}{j}\delta^j\left(a\delta^k(b)\right)\xi^{n+m-k-j}.
\end{aligned} \tag{1.14}$$

For the right side of (1.13)we have:

$$[\log\xi, a\xi^n]\circ b\xi^m + a\xi^n\circ[\log\xi, b\xi^m] = \left(\sum_{j=o}^{\infty}\frac{(-1)^{j+1}}{j}\delta^j\,(a)\,\xi^{n-j}\right)b\xi^m + a\xi^n\left(\sum_{j=o}^{\infty}\frac{(-1)^{j+1}}{j}\delta^j\,(b)\,\xi^{m-j}\right)$$

$$= \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(\begin{array}{c} n-j \\ k \end{array}\right)\frac{(-1)^{j+1}}{j}\delta^{j}\left(a\right)\delta^{k}(b)\xi^{n+m-k-j}+\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(-1)^{j+1}}{j}a\delta^{k}(\delta^{j}\left(b\right))\xi^{n+m-k-j}$$

$$= \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(\begin{array}{c} n-j \\ k \end{array}\right)\frac{(-1)^{j+1}}{j}\delta^{j}\left(a\right)\delta^{k}(b)\xi^{n+m-k-j}+\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(-1)^{j+1}}{j}a\delta^{k+j}\left(b\right)\xi^{n+m-k-j}. \tag{1.15}$$

For any integer $r \geqslant 1$, the coefficient of ξ^{n+k-r} in (1.14) is

$$\sum\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(-1)^{j+1}}{j}\delta^{j}\left(a\delta^{k}(b)\right)$$

where the summation is over all integers $j \geqslant 1, k \geqslant 0$ such that $j+k=r$. Using (1.6) we have

$$\sum\sum_{i=o}^{j}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} i \\ j \end{array}\right)\frac{(-1)^{j+1}}{j}\delta^{j-i}(a)\delta^{i+k}(b)$$

$$= \sum\sum_{i=o}^{j-1}\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} i \\ j \end{array}\right)\frac{(-1)^{j+1}}{j}\delta^{j-i}(a)\delta^{i+k}(b)+\sum\left(\begin{array}{c} n \\ k \end{array}\right)\frac{(-1)^{j+1}}{j}a\delta^{r}(b) \tag{1.16}$$

where both summations are over all integers $j \geqslant 1, k \geqslant 0$ such that $j+k=r$. On the other hand, for $r \geqslant 1$, the coefficient ξ^{n+k-r} of (1.15) is

$$\sum\left(\begin{array}{c} n-s \\ l \end{array}\right)\frac{(-1)^{s+1}}{s}\delta^{s}(a)\delta^{l}(b)+\sum\left(\begin{array}{c} n \\ l \end{array}\right)\frac{(-1)^{s+1}}{s}a\delta^{r}(b) \tag{1.17}$$

where the sum is over all integers $s \geqslant 1, l \geqslant 0$ such that $s+l=r$. Therefore, in order for (1.16) and (1.17) to be equal it is enough to show that for fixed integers $s \geqslant 1, l \geqslant 0$ such that $s+l=r$, we have

$$\sum\left(\begin{array}{c} n \\ k \end{array}\right)\left(\begin{array}{c} j \\ i \end{array}\right)\frac{(-1)^{j+1}}{j}=\left(\begin{array}{c} n-s \\ l \end{array}\right)\frac{(-1)^{s+1}}{s},$$

where the sum is over all integers $j \geqslant 1, k \geqslant 0, i \geqslant 0$ such that $i=j-s, i+k=l$. This amounts to showing that

$$\sum_{j=s}^{l+s}\left(\begin{array}{c} n \\ l+s-j \end{array}\right)\left(\begin{array}{c} j \\ j-s \end{array}\right)\frac{(-1)^{j+1}}{j}=\left(\begin{array}{c} n-s \\ l \end{array}\right)\frac{(-1)^{s+1}}{s}$$

and this is consequence of (1.12). □

Theorem 14. *The map $c : \Psi DO \times \Psi DO \longrightarrow \mathbb{C}$ given by*

$$c(A, B) = res([\log \xi, A] \circ B)$$

defines a Lie algebra 2-cocycle on ΨDO.

Proof. We have to prove the cocycle identity

$$c([A, B]C) + c([C, A]B) + c([B, C]A) = 0 \tag{1.18}$$

This a direct calculation using that $[\log \xi, \cdot$ is a Lie algebra derivation:

$$\begin{aligned} c([A, B]C) &= res([\log \xi, [A, B]] \circ C]) \\ &= res([[\log \xi, A], B] \circ C) + res([A, [\log \xi, B]] \circ C) \\ &= res([\log \xi, A] \circ [B, C]) + res([\log \xi, B] \circ [A, C]) \end{aligned}$$

Analogously, $c([C, A] \circ B) = res([\log \xi, C] \circ [A, B]) + res([\log \xi, A] \circ [C, B])$ and $c([B, C]A) = res([\log \xi, B] \circ [C, A]) + res([\log \xi, C] \circ [B, A])$, these equations imply (1.18) □

Remark 15. *If $\mathcal{A} = C^\infty(S^1)$, $\delta = \partial_x$, the linear functional $\tau : C^\infty(S^1) \longrightarrow \mathbb{C}$ given by $\int_{S^1} f(x)dx$ is a δ-invariant trace. If we use our notation, we have the following:*

$$res(A) = res\left(\sum_{i=-\infty}^{n} a_i(x)\partial^i\right) = \tau(a_{-1}(x)).$$

We remark that in the literature [KK, K, KW, DK] on infinite-dimensional groups a different notation is used:

$$res(A) = a_{-1}(x) \text{ and } \tau(a_{-1}) = tr(a_{-1}) = \int_{s^1} a_{-1}(x).$$

In this work we follow the conventions of [D].

Remark 16. *If we apply Theorem 14 on $\Psi DO(S^1)$, then $c(A, B) = res([\log \partial, A]B)$. We obtain the following 2-cocycle:*

$$c(A, B) = Tr\left(\sum_{k \geq l} \frac{(-1)^k}{k} A_x^(k)\partial^{-k} B\right). \tag{1.19}$$

gives a nontrivial central extension of the Lie algebra $\Psi DO(S^1)$ of pseudodifferential symbols on S^1. The restriction of this cocycle to $DO(S^1)$ gives a nontrivial central extension of $DO(S^1)$. The restriction of this cocycle to the subalgebra of vector fields is the Gelfand-Fuchs cocycle

$$c\left(f(x)\partial, g(x)\partial\right) = \frac{1}{6}\int f'(x)g''(x)dx$$

of the Virasoro algebra. It is know that the Gelfand-Fuchs cocycle is non trivial [GF] and therefore the central extension of $\Psi DO(S^1)$ and $DO(S^1)$ are non trivial [KK].

Remark 17. *The Lie algebra $DO(S^1)$ of differential symbols on S^1 has exactly one central extension [L], but the Lie algebra $\Psi DO(S^1)$ of pseudodifferential symbols on S^1 has two independent central extensions [W]. The second cocycle has the following form:*

$$c(A, B) = Tr([x, A]B)$$

We will reprove this result as a corollary of our classification of central extensions of formal pseudodifferential operators in several variables appearing in Section 2.4.

Remark 18. *The restriction of the 2-cocycle c given by (1.19) to the subalgebra of differential operator $DO \subset \Psi DO$ is a multiple of the Kac-Peterson cocycle [KP] :*

$$c(f\partial^n, g\partial^m) = \frac{n!m!}{(m+n+1)!} \int_{S^1} f^{(m+1)} g^{(n)} d\theta$$

Remark 19. *Let Σ be a compact Riemann surface and let $\mathcal{M}$ be the space of meromorphic functions on Σ. Fix a meromorphic vector field v on the surface Σ and denote by D_v the operator of Lie derivative L_v along the field v: locally, in a neighbourhood U with local coordinate x, the field v is given by $v(x) = f(x)\partial/\partial x$ and then $Dg = L_v(g(x)) = f(x)g'(x)$ where $g'(x)$ is the derivative of g with respect to x.*
The associative algebra of meromorphic pseudodifferential symbols is [DK]

$$M\Psi DS = \left\{ \sum_{i=-\infty}^{n} a_i {D_v}^i | a_i \in \mathcal{M} \right\}$$

The multiplication in $M\Psi DO$ is defined in the usually form (1.4). We can now consider the Lie algebra structure of $M\Psi DO$ and the residue map $res_D \left(\sum_{i=-\infty}^{n} a_i D_v^i \right) = a_{-1} D_v^{-1}$, where D_v^{-1} is understood as a meromorphic differential on Σ. Also we can define the trace associated to the point $P \in \Sigma$ by $TrA = res_P res_D(A)$. Then as a consequence of theorem 14 we have a nontrivial 2-cocycle on $M\Psi DO$ given by $c_v(A, B) = Tr([logD_v, A]B)$.

1.4 $\log X$ and Outer derivations

In this section, we generalize $[\log \xi, \cdot]$ to $[\log X, \cdot]$, where X is any elements of ΨDO. Motivated by [DK] we would like to investigate if this generalization yields further central extensions of ΨDO. We consider an algebra $\mathcal{A}$ such that for all $a \in \mathcal{A}, a \neq 0, a^{-1} \in \mathcal{A}$. This is very important because the inverse appears in the definition of $[\log X, A]$ of the pseudodifferential symbol X^{-1}. The next proposition justifies the existence of X^{-1}.

Let $A = a_n \xi^n + a_{n-1} \xi^{n-1} + ...$ be a pseudodifferential symbols. If $a_n \neq 0$ and $a_m = 0$ for all $m > n$, then n is called the *order* or *degree* of A.

Proposition 20. *1. The vector spaces $(\Psi DO)_n = \{A \in \Psi DO : ord(A) \leq n\}$ define a filtration on ΨDO.*

2. For any $X = x_n\xi^n + x_{n-1}\xi^{n-1} + ...$, there exists a unique inverse operator, $X^{-1} = a_n\xi^{-n} + a_{-n-1}\xi^{-n-1} + ...$ such that $XX^{-1} = X^{-1}X = 1$.

Proof. 1. From the definition, we have that $(\Psi DO)_n \subseteq (\Psi DO)_{n+1}$. Also a simple calculation show that $[(\Psi DO)_n, (\Psi DO)_m] \subseteq (\Psi DO)_{n+m}$.

2. Let $X = x_n\xi^n + x_{n-1}\xi^{n-1} + \cdots$ and $X^{-1} = a_n\xi^n + a_{n-1}\xi^{n-1} + \cdots$,we need to solve the equation $XX^{-1} = 1$ for the unknown coefficients $a_n, a_{n-1}, ...$. Expanding $XX^{-1} = 1$, we obtain

$$x_na_n = 1;\ x_na_{n-1} + x_{n-1}a_n + nx_n\delta(a_n) = 0\ ; ...$$

the first equation implies that $a_n = x_n^{-1}$, the second equation can be solved for a_{n-1} in term of x_n, etc,etc,.

□

Definition 21. *For any pseudodifferential symbol X we define $\log X : \Psi DO \longrightarrow \Psi DO$, where the bracket of $\log X$ with a pseudodifferential symbols A ,$[\log X, A]$ is defined by means of the formula (1.4).*

Recall that $[\log\xi, A] = \log\xi \circ A - A \circ \log\xi$ (here the multiplication is defined by the same formula 1.4). Similary, one can regard $[\log X, A] = \log X \circ A - A \circ \log X$, where, $\log(1 + X) = \sum_{j\geq 1} \frac{(-1)^j}{j} X^j$. We can check that is X is an integral symbols then $\log(1 + X)$ is well pseudodifferential symbols.

For example, to any $a \in \mathcal{A}$, we can associate the operator $\log a : \Psi DO \longrightarrow \Psi DO$ given by:

$$[\log a, b\xi^n] = nb\frac{\delta(a)}{a}\xi^{n-1} + n(n-1)b\frac{a(\delta^2(a)) - (\delta(a))^2}{a^2}\xi^{n-2} + \cdots$$

Remark 22. *Let $\mathfrak{g}$ be a Lie algebra, the Baker-Campbell-Hausdorff-Dynkin formula is*

$$e^xe^y = e^{x+y+\frac{1}{2}[x,y]+\frac{1}{12}([x,[x,y]]+[y,[y,x]])+\cdots} \qquad x, y \in \mathfrak{g} \tag{1.20}$$

Note that $x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots = x + y + \theta(x, y)$ where $\theta(x, y)$ is a series only of (iterated) commutators of x and y. In this formula, the exponentials are defined by the usual power series, and 1.20 is understood as an identity among formal power series. We also note that if G is a Lie group and $\mathfrak{g}$ is the Lie algebra of G, then the exponential is the standard exponential function $\exp : \mathfrak{g} \longleftrightarrow G$ and 1.20 is an equation on G. [H, Ha]

The next theorem is a generalization of the theorem 2.8 of [DK].

Theorem 23. *The map $X \longmapsto \log X \in H^1(\Psi DO, \Psi DO)$ is nontrivial and, the derivation $\log(XY)$ is equivalent to the sum of the derivations $\log X + \log Y$. Also, the derivation $\log(X+Y)$ is equivalent to the derivation $\log X$ if the degree of the symbols X if greater that the degree of Y.*

Proof. We can rewrite the product XY of two symbols as $XY = e^{\log X} e^{\log Y}$. Using the Baker-Campbell-Hausdorff formula (1.20) we have:

$$XY = e^{\log X + \log Y + \theta(\log X, \log Y)}$$

Note that the term $\theta(\log X, \log Y)$ is a pseudodifferential symbol, because only commutators of $\log X$ and $\log Y$ appear in it, but not the logarithms themselves. (note that the $[\log X, \log Y]$ defined by (1.4) is a pseudodiferential symbol) Then,

$$\log XY = \log X + \log Y + \theta(\log X, \log Y)$$

Thus the derivation $\log XY$ is cohomological to $\log X + \log Y$, since $\theta(\log X, \log Y)$, defines a inner derivation (Note that , $\theta(\log X, \log Y) = \frac{1}{12}([\log X, [\log X, \log Y]] + [\log Y, [\log Y, \log X]]) + \cdots$, this is, only commutators of $\log X$ and $\log Y$). On the other hand, let $X = \sum_{i=-\infty}^{m} a_i \xi^i$, then we can write $X = (a_m \xi^m) \circ (1 + Y)$ where Y is an integral symbol. Then, $\log X$ is cohomological to $\log(a_m \xi^m) + \log(1 + Y)$. But, the derivation $\log(1 + Y)$ is inner: indeed, for definition $\log(1 + Y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} Y^k$, the right-hand side is well defined integral symbol, because degree of $Y^k \leq -k$. Thus $\log X$ is cohomological to $\log(a_m \xi^m)$. □

Corollary 24. *For any pseudodifferential symbols X and Y, the logarithmic 2-cocycle c_{XY} is equivalent to the sum of the 2-cocycle $c_X + c_Y$*

Proof. This is a direct consequence of [D1] and theorem 23. □

This Theorem and its corollary imply that we do not obtain new central extensions by considering $\log X$ instead by of $log \xi$.

1.5 A Hierarchy of Centrally Extended Lie Algebras

In this section, we define a whole hierarchy of Lie algebras admitting nontrivial central extensions. This work is a generalization of [K]; also, we consider the bilinear form $<, >$ defined in proposition 6.

For any positive integer m, let ΨDO_m be the subalagebra of ΨDO consisting of differential operators of the form $\sum_{i=m}^{M} a_i \xi^i$ for some non-negative integer M and $a_i \in \mathcal{A}$.

Theorem 25. *Suppose that the bilinear form defined in the proposition 6 is nondegenerate on ΨDO. The restriction of the 2-cocycle $c(A,B) = res([\log\xi, A]B)$ to ΨDO_m defines a nontrivial central extension of this subalgebra.*

Proof. Using $\langle , \rangle$ defined in the proposition 6, we can to identify ΨDO with the dual space ΨDO^*. Then a pseudodifferential symbol $A \in \Psi DO$ can be considered as a linear functional in ΨDO^*. The dual space for the subalgebra ΨDO_m can be viewed as subspace in $\Psi DO^* \simeq \Psi DO$. $\Psi DO^*_m \simeq \left\{\sum_{i=-\infty}^{-m-1} a_i\xi^i : a_i \in \mathcal{A}\right\}$. If we assume that $c(A,B)$ is a coboundary then for $A = \sum_{i=m}^{M} a_i\xi^i, B = \sum_{i=m}^{M} b_i\xi^i \in \Psi DO_m$ and for some $L \in \Psi DO^*_m$, $L = \sum_{i=-\infty}^{-m-1} l_i\xi^i$, we have that

$$
\begin{aligned}
c(A,B) &= res([A,B]L) \\
&= res\left(\left[\sum_{i=m}^{M} a_i\xi^i, \sum_{j=m}^{M} b_j\xi^j\right] \sum_{k=m}^{-m-1} l_k\xi^k\right) \\
&= res\left(\sum_{\substack{i,j,t,k,s,u \\ =m,m,0,-\infty,0,0}} \Big((\text{coefficient }) \, \delta^{(s)}(a_i)\delta^{(t)}(b_j)\delta^{(u)}(l_k)\xi^{i+j+k-(t+u+s)}\Big)\right) \\
&= \tau(\text{coefficient }_{-1})
\end{aligned}
$$

then the coefficient of ξ^{-1} satisfy that

$$
i + j + k - (t+u+s) = -1 \tag{1.21}
$$

for $s, t, u \geq 0, k \leq -m-1$.
Also, we have that

$$
\begin{aligned}
c(A,B) &= res([\log\xi, A]\, B) \\
&= res\left(\left[\log\xi, \sum_{i=m}^{M} a_i\xi^i\right] \sum_{j=m}^{M} b_j\xi^j\right) \\
&= res\left(\sum_{\substack{i,j,t,s \\ =m,m,0,0}} (\text{coefficient})\delta^{(s)}(a_i)\delta^{(t)}(b_j)\xi^{i+j-(s+t)}\right) \\
&= \tau(\text{coefficient }_{-1})
\end{aligned}
$$

then the coefficient of ξ^{-1} satisfy that

$$
i + j - (s+t) = -1 \tag{1.22}
$$

comparing (1.21) and (1.22) for any A, B, we find $k = u$, and this is contradicts the condition for k and u. □

1.6 The Algebra of Formal Twisted Pseudodifferential Symbols in One Variable

In this section we introduce the algebra of twisted formal pseudodifferential symbols, the logarithmic cocycle and we prove theorems analogous to Theorem 23 and Theorem 25.

We note that examples of twisted formal pseudodifferential symbols have appeared in [FM] and in [KLR] in connection with noncommutative differential geometry.

Definition 26. *Let $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ be an automorphism of fixed algebra $\mathcal{A}$.*

1. *A σ-derivation on $\mathcal{A}$ is a linear map δ such that*

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

2. *A σ-trace on $\mathcal{A}$ is a linear map $\tau : \mathcal{A} \longrightarrow \mathbb{C}$ such that*

$$\tau(ab) = \tau(\sigma(b)a), \quad \textit{for all } a, b \in \mathcal{A}.$$

Given a triple $(\mathcal{A}, \delta, \sigma)$ as in the above definition, we define the algebra of twisted formal pseudodifferential symbols, denoted by ΨDO_σ, as the set of all formal Laurent series in ξ with coefficients in $\mathcal{A}$:

$$\Psi DO_\sigma = \left\{ \sum_{i=-\infty}^{N} a_i \xi^i : N \in \mathbb{Z}, a_n \in \mathcal{A} \right\}.$$

Thus, as a module, ΨDO_σ is equal to ΨDO. However, we define multiplication in a different way. We impose the relation

$$\xi a = \sigma(a)\xi + \delta(a)$$

Using the above multiplication rule, we have for all $n \geq 0$,

$$\xi^n a = \sum_{i=0}^{n} P_{i,n}(\sigma, \delta)(a)\xi^i, \tag{1.23}$$

where $P_{i,n}(\sigma, \delta)$ is some noncommutative polynomial in σ and δ with $\binom{n}{i}$ terms of total degree n such that the degree of σ is i. For example, $\xi^2 a = \delta^2(a) + (\delta\sigma(a) + \sigma\delta(a))\xi + \sigma^2(a)\xi^2$.

We use induction to extend (1.23) for $n < 0$. We have then

$$\xi^n a = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} (-1)^{i_1+\cdots+i_n} \sigma^{-1}(\delta\sigma^{-1})^{i_n} \cdots \sigma^{-1}(\delta\sigma^{-1})^{i_1}(a)\xi^{n-i_1-\cdots-1_n} \tag{1.24}$$

Let $A = \sum_{n=-\infty}^{N} a_n \xi^n$ and $B = \sum_{m=-\infty}^{M} b_m \xi^m$, then we have the following formula for the multiplication:

$$\begin{aligned} AB = & \sum_{m=-\infty}^{M} \sum_{n<0} \sum_{i\geq 0} (-1)^{|i|} a - 1)^{i_1+\cdots+i_n} \sigma^{-1}(\delta\sigma^{-1})^{i_n} \cdots \sigma^{-1}(\delta\sigma^{-1})^{i_1}(b)\xi^{m+n+|i|} \\ & + \sum_{m=-\infty}^{M} \sum_{n<0} \sum_{j=0}^{n} a_n P_{j,n}(\sigma, \delta)(b_m)\xi^{m+j}, \end{aligned}$$

where $i = (i_1, \cdots, i_n)$ is an n-tuple of integers and $|i| = i_1 + \cdots + i_n$.

The next proposition and theorem are proved in [FM].

Proposition 27. *Let $\mathcal{A}$ an algebra, σ a automorphism of $\mathcal{A}$, τ a σ-trace on $\mathcal{A}$, δ a σ-derivation on $\mathcal{A}$. If $\tau \circ \delta = 0$, then for any $a, b \in \mathcal{A}$, and any m-tuple $i = (i_1, ..., i_m)$ of non-negative integers, we have:*

$$\tau(b\sigma^{-1}(\delta\sigma^{-1})^{i_1} \cdots \sigma^{-1}(\delta\sigma^{-1})^{i_m}(a)) = (-1)^{i_1+\cdots+1_m} \tau(a\delta^{i_m}\sigma\delta^{i_{m-1}} \cdots \sigma\delta^{i_1}(b))$$

Theorem 28. *Let $\mathcal{A}$ an algebra, σ a automorphism of $\mathcal{A}$, τ a σ-trace on $\mathcal{A}$, and δ a σ-derivation on $\mathcal{A}$. If $\tau \circ \delta = 0$, then the linear functional $res : \Psi DO_\sigma \longrightarrow \mathbb{C}$ defined by*

$$res\left(\sum_{i=-\infty}^{n} a_i \xi^i\right) = \tau(a_{-1})$$

is a trace.

Remark 29. *Note that if $\delta \circ \sigma = \sigma \circ \delta$, the multiplication formulas (1.23) and (1.24) in ΨDO_σ, imply that the multiplication is given by*

$$\xi^n a = \sum_{j=0}^{\infty} \binom{n}{j} \delta^j(\sigma^{n-j}(a))\xi^{n-j}. \tag{1.25}$$

A special case of (1.25) is the q-pseudodifferential symbols on the circle defined in [KLR]. *In this case, $\mathcal{A} = C^\infty(S^1)$, $\sigma(f) = f(qx)$ and $\delta(f) = \dfrac{f(qx) - f(x)}{q-1}$.*

Hereafter we assume that δ and σ commute. Let σ_t be a 1-parameter group of automorphisms of $\mathcal{A}$ with $\sigma_1 = \sigma$. We can define an algebra of twisted pseudodifferential symbols with elements of the form $\sum_{i=0}^{\infty} a_i \xi^{t-i}$, $t \in \mathbb{R}$. We can replace the integer n in (1.25) by $t \in \mathbb{R}$ and obtain

$$\xi^t a - \sigma_t(a)\xi^t = \sum_{j=1}^{\infty} \binom{t}{j} \delta^j(\sigma_{t-j})\xi^{t-j}.$$

By differentiating above formula in t at $t = 0$, and using the identity $\xi^t = e^{t \log \xi}$ and derive at $t = 0$, we obtain

$$\frac{d}{dt}\Big|_{t=0} \xi^t = (\log \xi)\xi^t\Big|_{t=0} = \log \xi,$$

and the commutation relation

$$[\log \xi, a] = \frac{d}{dt}\Big|_{t=0} \sigma_t(a) + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sigma^{-j}\delta^j(a)\xi^{-j}. \tag{1.26}$$

We note that $[\log \xi, \xi = 0]$.

Proposition 30. *The map* $[\log \xi, \cdot] : \Psi DO_\sigma \longrightarrow \Psi DO_\sigma$ *define by*

$$[\log \xi, a\xi^n] = \frac{d}{dt}\Big|_{t=0} \sigma_t(a)\xi^n + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sigma^{-j}\delta^j(a)\xi^{-j} \tag{1.27}$$

is a derivation.

Proof. See [FM]. □

Theorem 31. *The 2-cochain*

$$c(A, B) = res([\log \xi, A]B)$$

is a Lie algebra 2-cocycle.

Proof. See [FM]. □

Remark 32. *The algebra* Ψ_σ *as vector space, has a direct sum decomposition* $\Psi DO_\sigma = DO_\sigma \oplus INT_\sigma$*, where*

$$DO_\sigma = \left\{ \sum_{i=0}^{n} a_i \xi^i \Big| a_i \in \mathcal{A} \right\} \text{ and } INT_\sigma = \left\{ \sum_{i=-\infty}^{-1} a_i \xi^i \Big| a_i \in \mathcal{A} \right\}$$

are Lie subalgebras of Ψ_σ*:*

Indeed, if $a\xi^n, b\xi^m \in DO_\sigma$ then

$$a\xi^n b\xi^m = \sum_{k=0}^{n} \binom{n}{k} a\delta^k\left(\sigma^{n-k}(b)\right)\xi^{n+m-k}$$

but, $n+m-k \geq m \geq 0$. Similarly the case for $a\xi^n, b\xi^m \in INT_\sigma$.

Proposition 33. *$res([A,B]) = 0$ for all $A, B \in \Psi DO_\sigma$. This implies that the bilinear form $\langle A, B\rangle = res(AB)$ is invariant, i.e it satisfies $\langle [A,B], C\rangle = \langle A, [B,C]\rangle$. Also the subalgebras DO_σ and INT_σ are isotropic subspaces of ΨDO, i.e, the restrictions of this form to both DO_σ and INT_σ vanish.*

Proof. This is analogous to Proposition 6. □

Let $A = \sum_{i=0}^{n} a_i\xi^i \in \Psi DO_\sigma$,if $a_n \neq 0$, then n is called the order of the A.

Proposition 34. *1. The sets $(\Psi DO_\sigma)_n = \{A \in \Psi DO_\sigma : ord(A) \leq n\}$ define a filtration on ΨDO_σ.*

2. For any $X = x_n\xi^n + x_{n-1}\xi^{n-1} + \ldots$, such that x_n is invertible, there exist the inverse operator, $X^{-1} = a_n\xi^{-n} + a_{-n-1}\xi^{-n-1} + \ldots$ such that $XX^{-1} = X^{-1}X = 1$.

Proof. The proof is similar to nontwisted case . □

Similarly, to the ΨDO case, for $X \in \Psi DO_\sigma$ we can define $\log X : \Psi DO_\sigma \longrightarrow \Psi DO_\sigma$, where the bracket with a $A \in \Psi DO_\sigma$ is defined for the formula (1.25), and $\log(1+X) = \sum\limits_{j\geq 1} \frac{(-1)^j}{j} X^j$ is well defined.

Theorem 35. *The map $X \longmapsto \log X \in H^1(\Psi DO_\sigma, \Psi DO_\sigma)$ is a nontrivial cocycle and it is such that, the derivation $\log(XY)$ is equivalent to the sum of the derivations $\log X + \log Y$. Also, the derivation $log(X+Y)$ is equivalent to the derivation $logX$ if the degree of the symbols X is greater that the degree of Y.*

It is also possible define subalgebras $\Psi DO_{\sigma,m}$ of ΨDO_σ by

$$\Psi DO_{\sigma,m} = \left\{ \sum_{i=m}^{M} a_i\xi^i \middle| a_i \in \mathcal{A} \right\}.$$

Then, we have a analogous to Theorem 25:

Theorem 36. *Suppose that the bilinear form defined in proposition 33 is nondegenerate on ΨDO_σ. The restriction of the 2-cocycle $c(A,B) = res([\log\xi, A]B)$ to $\Psi DO_{\sigma,m}$ defines a nontrivial central extension of this subalgebra.*

Chapter 2

The Algebra of Formal Pseudodifferential Symbols in Several Variables

2.1 Preliminaries

In this section, we describe the algebra of formal pseudodifferential symbols in several variables, denote by $\Psi_n DO$.

We use the notation of [D]. Let $I = \{1, 2, ..., n\}$ and

$$\Gamma_n = \{\alpha = (..., \alpha_i, ...) : \alpha_i \in \mathbb{Z}, i \in I\};\ \Gamma_n^+ = \{\alpha \in \Gamma_n : \alpha_i \in \mathbb{Z}^+, i \in I\}$$

$$\varepsilon_i = (0, ..., 0, 1, 0, ..., 0);\ \vartheta = \sum_i \varepsilon_i \in \Gamma_n;\ \alpha! = \prod_i \alpha_i!, \alpha_i \in \Gamma_n^+$$

We also set $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$; $\delta^\gamma(b) = \delta_1^{\gamma_1}(b) \cdots \delta_n^{\gamma_n}(b)$ and

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \prod_i \begin{pmatrix} \alpha_i \\ \gamma_i \end{pmatrix}$$

and the binomial coefficient is defined as in Chapter 1.

Let $\mathcal{A}$ be an algebra on $\mathbb{C}$ and let δ_i with $i = 1, \cdots, n$, be derivations on $\mathcal{A}$. The algebra of formal differential symbols in several variables DO_n is, by definition, the algebra generated by $\mathcal{A}$ (as a subalgebra)and the symbols ξ_i, with the relations,

$$\xi_i a = a\xi_i + \delta_i(a), \tag{2.1}$$

for all $a \in \mathcal{A}$ and $i = 1, \cdots, n$. The elements of the DO_n are of the form

$$A = \sum_{\alpha \in \Gamma_n^+} a_\alpha \xi^\alpha$$

Using (2.1), we can prove that

$$a\xi^\alpha b\xi^\beta = \sum_{\gamma \in \Gamma_n^+} \left(\begin{array}{c} \alpha \\ \gamma \end{array} \right) a\delta^\gamma(b)\xi^{\alpha+\beta-\gamma}. \tag{2.2}$$

We extend the algebra DO_n to obtain the algebra of formal pseudodifferential operators ΨDO_n introducing the differentiations with negative exponent via

$$\xi_i^{-1} a = \sum_{j \geq 0} (-1)^j \delta_i^j(a) \xi_i^{-1-j}.$$

We define a the structure of Lie algebra on ΨDO_n by the usual commutator

$$[A, B] = AB - BA = \sum_{\gamma \in \Gamma_n^+} \left(\frac{1}{\gamma!} \right) \left(\frac{\partial^\gamma}{\partial \xi^\gamma}(A)\delta^\gamma(B) - \frac{\partial^\gamma}{\partial \xi^\gamma}(B)\delta^\gamma(A) \right) \tag{2.3}$$

Where for $A = a\xi^\alpha$,

$$\frac{\partial^\gamma}{\partial \xi^\gamma}(a\xi^\alpha) = \gamma! \binom{\alpha}{\gamma} a\xi^{\alpha-\gamma} \quad \text{and} \quad \delta^\gamma(a\xi^\alpha) = \delta_1^{\gamma_1}(a) \cdots \delta_n^{\gamma_n}(a)\xi^\alpha.$$

2.2 The Logarithm Of a Symbol and the Logarithmic 2-cocycle

We are interested in having a similar definition to 1.11, but in several variables, i.e, we want define $\log \xi$ for all $i = 1, ..., n$.

We know from Chapter 1 that $\log \xi$ does not belong to ΨDO, but $[\log \xi, X] \in \Psi DO$ for all X and that $[\log \xi, \cdot]$ is a derivation. We can define the action of $\log \xi_i$ on the algebra $\Psi_n DO$ using the commutator $[\log \xi_i, \cdot]$ (2.3) in the following form:

$$\begin{aligned} [\log \xi_i, X] &= \sum_{\gamma \in \Gamma_n^+} \frac{1}{\gamma!} \frac{\partial^\gamma}{\partial \xi^\gamma}(\log \xi_i)\delta^\gamma(X) \\ . &= \sum_{\gamma_i \geq 0} \frac{1}{\gamma_i!} \frac{\partial_i^\gamma}{\partial \xi_i^{\gamma_i}}(\log \xi_i)\delta^{\gamma_i}(X) \end{aligned} \tag{2.4}$$

for all $X \in \Psi_n DO$.

Proposition 37. *$[log\xi_i, \cdot]$ defines a derivation of the (both associative and Lie) algebra $\Psi_n DO$ for all $i = 1, ..., n$.*

Proof. One readily verifies,as in Chapter 1, that for any two symbols A and B in ΨDO

$$[\log \xi_i, AB] = [\log \xi_i, A]B + A[\log \xi_i, B]$$

so $\log \xi_i$ is a derivation of the associative algebra . This also implies that $\log \xi_i$ is a derivation of the Lie algebra structure. □

Let τ a trace such that, $\tau(a\delta^{\gamma}(b)) = (-1)^{\gamma}\tau(b\delta^{\gamma}(a))$ this is,

$$\tau(a\delta_1^{\gamma_1}(b)\cdots\delta_n^{\gamma_n}(b)) = (-1)^{\gamma}\tau(b\delta_1^{\gamma_1}(a)\cdots\delta_n^{\gamma_n}(a)). \tag{2.5}$$

Proposition 38. *Let $\mathcal{A}$ be an algebra, δ_i derivations on $\mathcal{A}$ and τ as in (2.5). Then the linear functional res:$\Psi_n DO \longrightarrow \mathbb{C}$ defined by*

$$res\left(\sum_{\alpha\in\Gamma_n} a_\alpha \xi^\alpha\right) = \tau(a_{-\vartheta})$$

is a trace.

Proof. It suffices to prove that for any $a, b \in \mathcal{A}$ and $\alpha, \beta \in \Gamma_n$,

$$res(a\xi^\alpha b\xi^\beta) = res(b\xi^\beta a\xi^\alpha) \tag{2.6}$$

Let $\alpha, \beta \in \Gamma_n$ such that for some α_i, β_j, we have $\alpha_i, \beta_j \geq 0$ o $\alpha_i, \beta_j < 0$, then verifies (2.6) for the proposition 4.
Without loss of generality, suppose that for all i, $\alpha_i \geq 0, \beta_i < 0$. If $\alpha_i + \beta_i < -1$,then using the proposition 4 we have the equality. Now, suppose that $\alpha_i + \beta_i \geq -1$ ans let $k_i = \alpha_i + \beta_i$. This case is similar to the case $n = 1$, then

$$res(a\xi^\alpha b\xi^\beta) = \begin{pmatrix} \alpha \\ k+\vartheta \end{pmatrix} \tau(a\delta^{k+\vartheta})$$

Also,

$$\begin{aligned} res\left(b\xi^{\beta}a\xi^{\alpha}\right) &= \tau\left(\left(\begin{array}{c}\beta\\ k+\vartheta\end{array}\right)b\delta^{k+\vartheta}(a)\right)\\ &= \left(\begin{array}{c}k-\alpha\\ k+\vartheta\end{array}\right)\tau\left(b\delta^{k+\vartheta}(a)\right)\\ &= (-1)^{k+\vartheta}\left(\begin{array}{c}\alpha\\ k+\vartheta\end{array}\right)\tau\left(b\delta^{k+\vartheta}(a)\right) \text{ aplicando (1.8) n-veces}\\ &= (-1)^{k+\vartheta}(-1)^{k+\vartheta}\left(\begin{array}{c}\alpha\\ k+\vartheta\end{array}\right)\tau\left(a\delta^{k+\vartheta}(b)\right) \text{ por (2.5)}\\ &= \left(\begin{array}{c}\alpha\\ k+\vartheta\end{array}\right)\tau\left(a\delta^{k+\vartheta}(b)\right)\end{aligned}$$

□

Theorem 39. *The 2-cochains*

$$c_i(A,B) = res([\log\xi_i, A]B)$$

are Lie algebra 2-cocycles of $\Psi_n DO$.

Proof. We have to verify the cocycle property

$$\begin{aligned} c_i([A,B],C) &= res([\log\xi_i,[A,B]]C)\\ &= res([[\log\xi_i,A],B]C) + res([A,[\log\xi_i,B]]C)\\ &= res([\log\xi_i,A][B,C]) + res([\log\xi_i,B][C,A]).\end{aligned}$$

Also, $res([\log\xi_i,A][B,C]) = res([\log\xi_i,C][A,B]) + res([\log\xi_i,B][A,C])$, these identities imply that $c_i([A,B],C) + c_i([C,A],B) + c_i([B,C],A) = 0$. □

2.3 A Hierarchy of Centrally Extended Lie Algebras

In this section, we define subalgebras of $\Psi_n DO$ and prove that they have nontrivial central extensions. Similar to the proposition 6, we consider that the bilinear form $< A,B >= res(AB)$ for $A,B \in \Psi_n DO$ is nondegenerate and we can prove that is symmetric and invariant.

In ΨDO, we constructed one hierarchy of subalgebras admitting nontrivial central extensions. But, in this case, $\Psi_n DO$, we have a hierarchy of subalgebras $\Psi_n DO_{\alpha_i}$ admitting nontrivial central extensions given by $\log\xi_i$ for a fixed i and we have another hierarchy of subalgebras $\Psi_n DO_{\alpha}$ admitting nontrivial central extension given by $\log\xi_i$ for each i.

Let $L = \sum_{\alpha\in\Gamma_n} a_\alpha \xi^\alpha = \sum_{(\alpha_1,\dots,\alpha_n)\in\Gamma_n} a_\alpha \xi_1^{\alpha_1}\dots\xi_n^{\alpha_n} \in \Psi_n DO$, then we define the subalgebra $\Psi_n DO_{\alpha_i}$ of $\Psi_n DO$ with element of the form $\sum_{\beta\in\Gamma_n} b_\beta \xi^\beta$, such that $\beta_i \geq \alpha_i \geq 0$.

Theorem 40. *The cocycle c_i defined by $\log \xi_i$ is a cocycle for the subalgebra $\Psi_n DO_{\alpha_i}$.*

Proof. The bilinear form $\langle,\rangle$ is symmetric, nondegenerate and invariant. Then we can identify $\Psi_n DO^*$ with $\Psi_n DO$. The dual space for the subalgebra $\Psi_n DO_{\alpha_i}$ can be viewed as subspace in $\Psi_n DO^*$. Then we have that $\Psi_n DO_{\alpha_i}{}^* \simeq \left\{\sum_{\theta\in\Gamma_n} l_\theta \xi^\theta : \theta_i \leq -\alpha_i - 1\right\}$. Let $A, B \in \Psi_n DO_{\alpha_i}$ and we assume that $c_i(A,B)$ is a coboundary. Then

$$\begin{aligned}
c(A,B) &= res([A,B]L)\\
&= res\left(\left[\sum_{\alpha\in\Gamma_n} a_\alpha\xi^\alpha, \sum_{\beta\in\Gamma_n} b_\beta\xi^\beta\right]\sum_{\theta\in\Gamma_n}^{-m-1} l_\theta\xi^\theta\right)\\
&= res\left(\sum\left((\text{coefficient })\,\delta^{(\varphi)}(a_\alpha)\delta^{(\tau)}(b_\beta)\delta^{(\mu)}(l_\theta)\xi^{\alpha+\beta+\theta-(\tau+\mu+\varphi)}\right)\right)\\
&= \tau(\text{coefficient }_{-\vartheta})
\end{aligned}$$

The coefficient of $\xi_{-\vartheta}$ verifies that

$$\alpha_i + \beta_i + \theta_i + (\tau_i + \mu_i + \varphi_i) = -1 \tag{2.7}$$

on the i-th position .
On the other hand, we have that

$$\begin{aligned}
c(A,B) &= res([\log\xi_i, A]\,B)\\
&= res\left(\left[\log\xi_i, \sum_{\alpha\in\Gamma_n} a_\alpha\xi^\alpha\right]\sum_{\beta\in\Gamma_n} b_\beta\xi^\beta\right)\\
&= res\left(\sum(\text{coefficient})\delta_i^{(\psi_i)}(a_\alpha)\delta^{(\tau_i)}(b_\beta)\xi^{\alpha+\beta-(\psi_i+\tau)}\right)\\
&= \tau(\text{coefficient }_{-\vartheta})
\end{aligned}$$

The coefficient of $\xi_{-\vartheta}$ satisfy that

$$\alpha_i + \beta_i + (\tau_i + \varphi_i) = -1 \tag{2.8}$$

on the i-th position. Comparing (2.7) with (2.8) for arbitrary A, B we have that $\mu_i = \theta_i$, and this is a contradiction, because $\theta_i \leq -\alpha_i - 1$ y $\mu_i \geq 0$. □

Now we say that $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i. Let $\alpha \in \Gamma_n^+$, $\Psi_n DO_\alpha$ and let us define as the subalgebra of $\Psi_n DO$ with elements of the form $\sum_{\beta\in\Gamma_n} b_\beta\xi^\beta$ such that $\beta \geq \alpha$.

Theorem 41. *The cocycle generated by* $\log \xi_i$ *for each* i *, is a cocycle in* $\Psi_n DO_\alpha$*.*

Proof. The proof is analogous to the proof of theorem 40. We see that if the cocycle generated by $\log \xi_i$ were trivial, then equations (2.7) and (2.8) would be are true in all n-components, and we would have $\mu_i = \theta_i$ for all i, this is impossible. $\square$

2.4 The Dzhumadil´daev Classification Theorem

In this section we present a new proof of the principal theorem of [D]. We use our notation from section [2.1].

Let $P_n^+ = \{\sum_\alpha \lambda_\alpha x^\alpha | \alpha \in \Gamma_n^+\}$ be the algebra of polynomials in variables $x_1, ..., x_n$, and let P_n be the algebra of Laurent power series of the form $\sum_{\alpha \in \Gamma_n} \lambda_\alpha x^\alpha$ such that the number of $\alpha \in \Gamma_n^+$ with nonzero λ_α is finite. The action of the derivation ∂_i in these algebra is $\partial_i(x^\alpha) = \alpha_i x^{\alpha - \epsilon_i}$, with $\alpha \in \Gamma_n$.

Let $\mathcal{U}$ be an other algebra with derivations $\partial_1, ..., \partial_n$. The tensor space $\mathcal{D}_n = \mathcal{U} \otimes P_n$ becomes an algebra if we endow it with the multiplication rule

$$(u \otimes f)(v \otimes g) = \sum_{\alpha \in \Gamma_n^+} \left(u \partial^\alpha(v) \otimes \partial^\alpha(f) g\right) / \alpha!$$

$\mathcal{D}_n$ is an associative algebra and it contain $\mathcal{D}_n^+ = \mathcal{U} \otimes P_n^+$ as a subalgebra.

Proposition 42. *There is a isomorphism between the associative algebra of formal pseudodifferential symbols* $\Psi_n DO$ *and* $\mathcal{D}_n$ *given by*

$$\begin{aligned} \Psi_n DO &\longrightarrow \mathcal{D}_n \\ u\xi^\alpha &\longmapsto u \otimes x^\alpha \end{aligned}$$

Corollary 43. *The Lie algebra* $\Psi_n DO$ *and* $\mathcal{D}_n$ *are isomorphic.*

Let $\mathcal{H}_n$ be the Lie algebra associated to $\mathcal{D}_n$ with $\mathcal{U} = P_n$. Then an element of $\mathcal{H}_n$ is of the form

$$\sum_{\alpha, \beta \in \Gamma_n} \lambda_{\alpha,\beta} x_+^\alpha x_-^\beta$$

(note that we are identifying $x_+^\alpha x_-^\beta \simeq x^\alpha \otimes x^\beta$) and the Lie bracket on $\mathcal{H}_n$ is given by

$$\left[x_+^\alpha x_-^\beta, x_+^{\overline{\alpha}} x_-^{\overline{\beta}}\right] = \sum_{\gamma \in \Gamma_n^+} \left(\frac{1}{\gamma!}\right) (\partial_- \wedge \partial_+)(x_+^\alpha x_-^\beta, x_+^{\overline{\alpha}} x_-^{\overline{\beta}})$$

where ∂_+, ∂_- are derivations acting on $\mathcal{H}_n$ as follows:

$$\partial_-^\gamma(x_+^\alpha x_-^\beta) = \gamma!\binom{\beta}{\gamma}x_+^\alpha x_-^{\beta-\gamma} \quad , \quad \partial_+^\gamma(x_+^\alpha x_-^\beta) = \gamma!\binom{\alpha}{\gamma}x_+^{\alpha-\gamma} x_-^{\beta}$$

Theorem 44. *Consider the linear map $res : \mathcal{H}_n \longrightarrow \mathbb{C}$ given by*

$$res\left(\sum_{\alpha,\beta\in\Gamma_n}\lambda_{\alpha,\beta}x_+^\alpha x_-^\beta\right) = \lambda_{-\vartheta,-\vartheta}$$

If we set $\langle D_1, D_2\rangle = res(D_1D_2)$, then $<,>$ is a symmetric bilinear nondegenerate invariant form.

Proof. See [D] □

Lemma 45. $x_+^{-\vartheta}x_-^{-\vartheta} \notin [\mathcal{H}_n, \mathcal{H}_n]$

Proof. Let $\lambda_{\alpha,\beta}x_+^\alpha x_-^\beta$ and $\lambda_{\bar{\alpha}\bar{\beta}}x_+^{\bar{\alpha}} x_-^{\bar{\beta}} \in \mathcal{H}_n$. We can prove that the coefficient of $x_+^{-\vartheta}x_+^{-\vartheta}$ is 0 in the bracket $[\lambda_{\alpha,\beta}x_+^\alpha x_-^\beta, \lambda_{\bar{\alpha},\bar{\beta}}x_+^{\bar{\alpha}} x_-^{\bar{\beta}}]$.Indeed,

$$\begin{aligned}\left[\lambda_{\alpha,\beta}x_+^\alpha x_-^\beta, \lambda_{\bar{\alpha},\bar{\beta}}x_+^{\bar{\alpha}} x_-^{\bar{\beta}}\right] &= \sum_{\alpha,\beta.\bar{\alpha},\bar{\beta}\in\Gamma_n^+}\frac{1}{\gamma!}\lambda_{\alpha,\beta}\lambda_{\bar{\alpha},\bar{\beta}}\gamma!\binom{\bar{\alpha}}{\gamma}\gamma!\binom{\beta}{\gamma}x_+^{\alpha+\bar{\alpha}-\gamma}x_-^{\beta+\bar{\beta}-\gamma} \\ &- \sum_{\alpha,\beta.\bar{\alpha},\bar{\beta}\in\Gamma_n^+}\lambda_{\alpha,\beta}\lambda_{\bar{\alpha},\bar{\beta}}\gamma!\binom{\alpha}{\gamma}\gamma!\binom{\bar{\beta}}{\gamma}x_+^{\alpha+\bar{\alpha}-\gamma}x_-^{\beta+\bar{\beta}-\gamma} \\ &= \sum_{\alpha,\beta.\bar{\alpha},\bar{\beta}\in\Gamma_n^+}\frac{\lambda_{\alpha,\beta}\lambda_{\bar{\alpha},\bar{\beta}}}{\gamma!}\left((\beta)_\gamma(\bar{\alpha})_\gamma - (\bar{\beta})_\gamma(\alpha)_\gamma\right).\end{aligned}$$

The coefficient of $x_+^{-\vartheta}x_-^{-\vartheta}$ in this commutator is equal to

$$\lambda = \sum_{\alpha,\beta.\bar{\alpha},\bar{\beta}\in\Gamma_n^+}\frac{\lambda_{\alpha,\beta}\lambda_{\bar{\alpha},\bar{\beta}}\delta_{\alpha+\bar{\alpha}+\vartheta}\delta_{\beta+\bar{\beta}+\vartheta}}{\gamma!}\left((\beta)_\gamma(\bar{\alpha})_\gamma - (\bar{\beta})_\gamma(\alpha)_\gamma\right)$$

where we set $(a)_r = a(a-1)...(a-r+1), \quad a, r \in \mathbb{R}, \quad r > 0$ and $(a)_0 = 0$. Let $a, b, \bar{a}, \bar{b} \in \mathbb{Z}$. We can prove that $a+\bar{a} = b+\bar{b}$ implies $(b)_{b+\bar{b}+1}(\bar{a})_{a+\bar{a}+1} = (\bar{b})_{b+\bar{b}+1}(a)_{a+\bar{a}+1}$. Thus, if $\alpha+\bar{\alpha} = \beta+\bar{\beta}$ then $(\beta)_{\beta+\bar{\beta}+\vartheta}(\bar{\alpha})_{\alpha+\bar{\alpha}+\vartheta} = (\bar{\beta})_{\beta+\bar{\beta}+\vartheta}(\alpha)_{\alpha+\bar{\alpha}+\vartheta}$. This implies that $\lambda = 0$. □

We shall consider the following Lie subalgebras of $\mathcal{H}_n$:

$$\mathcal{H}_n^+ = \left\{\sum_{\alpha,\beta\in\Gamma_n}\lambda_{\alpha,\beta}x_+^\alpha x_-^\beta \middle| \alpha\in\Gamma_n, \beta\in\Gamma_n^+\right\}$$

$$\mathcal{H}_{n,+} = \left\{ \sum_{\alpha,\beta\in\Gamma_n} \lambda_{\alpha,\beta} x_+^{\alpha} x_-^{\beta} \middle| \alpha \in \Gamma_n^+, \beta \in \Gamma_n \right\}$$

$$\mathcal{H}_{n,+}^+ = \left\{ \sum_{\alpha,\beta\in\Gamma_n} \lambda_{\alpha,\beta} x_+^{\alpha} x_-^{\beta} \middle| \alpha, \beta \in \Gamma_n^+ \right\}$$

Lemma 46. *The Lie algebra generated by $\mathcal{H}_{n,+}^+$ and the elements $x_1^{-1}, ..., x_n^{-1}$ coincides with $\mathcal{H}_n^+$. Similary, the algebra $\mathcal{H}_{n,+}$ is generated by the subalgebra $\mathcal{H}_{n,+}^+$ and the elements $x_{-1}^{-1}, ..., x_{-n}^{-1}$; finally, the Lie algebra $\mathcal{H}_{n,+}^+$, and the elements $x_{\pm 1}^{-1}, ..., x_{\pm n}^{-1}$ generate $[\mathcal{H}_n, \mathcal{H}_n]$.*

Proof. See [D] $\square$

Definition 47. *Let $f_0 : x^{-\vartheta} \mapsto 1$ and $f_0 : x^{\alpha} \mapsto 0$ if $\alpha \neq -\vartheta$. We can prove that f_0 is a derivation.*

The following result is the main theorem of this section. Its proof is long. We have divided it in three lemmas and we have left some import technical details in an appendix. We consider the case in one variable Ψ_1, this is, the basic element of the Lie algebra are $x_+ = x_{+1}$ and $x_- = x_{-1}$. We write $\mathcal{H}$ instead $\mathcal{H}_1$.

Theorem 48. *The first group of cohomology of the Lie algebra $\mathcal{H}_n$ with coefficients in $\mathcal{H}_n$, $H^1(\mathcal{H}_n, \mathcal{H}_n)$, is generated by $f_0, [\log x_{+i}, \cdot]$ and $[\log x_{-i}, \cdot]$ for $i = 1, ..., n$.*

Lemma 49. *$[\log x_+, \cdot]$ and $[\log x_-, \cdot]$ are outer derivations.*

Proof. We have that $[\log x_-, x_+] = x_-$.
If $[\log x_-, \cdot]$ is a inner derivation, then $[\log x_-, \cdot] = ad_{\sum d_{n,m} x_+^n x_-^m}(\cdot)$ for some coefficient $d_{n,m}$, then

$$\begin{aligned} [\log x_-, x_+] &= x_- \\ &= \left[\sum d_{n,m} x_+^n x_-^m, x_+\right] \\ &= \sum d_{n,m} m x_+^n x_-^{m-1} \end{aligned}$$

then, $x_- = \sum d_{n,m} m x_+^n x_-^{m-1}$ which is impossible. Analogously, we can prove that $[\log x_+, \cdot]$ is an outer derivation. $\square$

Remark 50. *The derivations $[\log x_+, \cdot], [\log x_-, \cdot]$ and f_0 are linearly independent.*

Lemma 51. *Let D be a derivation, then there is $\omega \in \mathcal{H}$ such that*

$$D = \lambda[\log x_+, \cdot] + \beta[\log x_-, \cdot] - [\omega, \cdot]$$

in x_+ and x_-, for some α, β.

Proof. we know that $[u, 1] = 0$ and then $[D(1), u] = 0$ for all $u \in \mathcal{H}$. This implies that $D(1) \in Z(\mathcal{H}) = \mathbb{C}$ thus we can write $D(1) = \lambda$.
Let $D(x_+) = \sum d^+_{n,m} x^n_+ x^m_-$ and $D(x_-) = \sum d^-_{n,m} x^n_+ x^m_-$. We have,

$$\begin{aligned} 1 = [x_-, x_+] &\implies \lambda = [D(x_-), x_+] + [x_-, D(x_+)] \\ &\implies \lambda = \sum d^-_{n,m}[x^n_+ x^m_-, x_+] + \sum d^+_{n,m}[x_-, x^n_+ x^m_-] \\ &\implies \lambda = \sum d^-_{n,m} m x^n_+ x^{m-1}_- + \sum d^+_{n,m} n x^{n-1}_+ x^m_- \end{aligned} \tag{2.9}$$

Using (2.9) we have that, $d^+_{n,m} = 0$ except when $n = 0$. Thus $d_{0,m} \neq 0$, this implies $D(x_+) = \sum d^+_{0,m} x^m_-$. Analogously, we have that $D(x_-) = \sum d^-_{n,0} x^n_+$.
Also, (2.9) implies that

$$d^-_{n,m} m x^n_+ x^{m-1}_- = -d_{n,m} n x^{n-1}_+ x^m_- \tag{2.10}$$

except when $n = -1, m = -1$ and $(n, m) = (0, 0)$. We rewrite (2.10) as

$$(m+1) d^-_{n,m+1} = -(n+1) d_{n+1,m}.$$

Then

$$D(x_+) = \sum_{m \neq 0,-1} -d^-_{0,m} x^m_- + d_{0,-1} x^{-1}_- + d^+_{0,0} \tag{2.11}$$

and

$$D(x_-) = \sum_{n \neq 0,-1} d^-_{n,0} x^n_+ + d_{-1,0} x^{-1}_+ + d^-_{0,0}. \tag{2.12}$$

Now let us write,

$$D(x_+) = \lambda[\log x_+, x_+] + \beta[\log x_-, x_+] - [\omega, x_+] = \beta x^{-1}_- - [\omega, x_+] \tag{2.13}$$

for some ω to be determined. Comparing (2.11) and (2.13), we have that $\beta = d^+_{0,-1}$ and

$$\begin{aligned} [\omega, x_+] &= \sum_{m \neq 0,-1} d^-_{0,m} x^m_- + d^+_{0,0} \\ &\implies \partial_-(\omega) = \sum_{m \neq 0,-1} d^-_{0,m} x^m_- + d^+_{0,0} \\ &\implies \omega = \sum_{m \neq 0,-1} \frac{d^-_{0,m}}{m+1} x^{m+1}_- + d^+_{0,0} x_- + \tilde{\omega} \end{aligned} \tag{2.14}$$

with $[\tilde{\omega}, x_+] = 0$. On the other hand,

$$D(x_-) = \lambda[\log x_+, x_-] + \beta[\log x_-, x_-] - [\omega, x_-] = \lambda x_+^{-1} - [\omega, x_-] \tag{2.15}$$

Comparing (2.12) and (2.15) and using (2.14), we have that $\lambda = d^-_{0,1}$ and

$$\begin{aligned}
[\omega, x_-] &= -\sum_{n\neq 0,-1} d^-_{n,0} x_+^n + -d^+_{0,0} \\
\implies \partial_-(\omega) &= -\sum_{n\neq 0,-1} d^-_{n,0} x_+^n - d^+_{0,0} \\
\implies \omega &= -\sum_{n\neq 0,-1} \frac{d^-_{n,0}}{n+1} x_+^{n+1} - d^-_{0,0} x_+ + \tilde{\tilde{\omega}}
\end{aligned} \tag{2.16}$$

with $[\tilde{\tilde{\omega}}, x_-] = 0$ Then, we can take ω as

$$\omega = -\sum_{n\neq 0,-1} \frac{d^-_{n,0}}{n+1} x_+^{n+1} + \sum_{m\neq 0,-1} \frac{d^-_{0,m}}{m+1} x_-^{m+1} - d^-_{0,0} x_+ - d^-_{0,0} x_-.$$

$\square$

Lemma 52. *If D is a derivation such that $D(x_\pm) = 0$, then D is a scalar multiple of f_0.*

Proof. We have that

$$1 = [x_-, x_+] \implies D(1) = [D(x_-), x_+] + [x_-, D(x_+)] \implies D(1) = 0$$

Now, there are several case to consider:

- $D(x_+x_-) = a$. Indeed:

$$\begin{aligned}
[x_-, x_+x_-] = x_- &\implies [D(x_-), x_+x_-] + [x_-, D(x_+x_-)] = D(x_-) \\
&\implies [x_-, D(x_+x_-)] = 0 \implies D(x_+x_-) = \sum d_s x_-^s
\end{aligned}$$

  Analogously $[x_+x_-, x_+] = x_+$, implies that $D(x_+) = \sum d_r x_+^r$. Thus $D(x_+x_-) = a$

- $D(x_+^n) = D(x_-^n) = 0$ for all $n \geq 2$. Indeed, let $n \geq 2$. Then

$$\begin{aligned}
[x_+, x_+^n] = 0 &\implies [D(x_+), x_+^n] + [x_+, D(x_+^n)] = 0 \\
&\implies [x_+, D(x_+^n)] = 0 \implies D(x_+^n) = \sum d_k x_+^k
\end{aligned}$$

Also, $[x_+x_-, x_+^n] = nx_+^n$ implies $[x_+x_-, D(x_+^n)] = nD(x_+^n)$. Thus, $\left[x_+x_-, \sum d_k x_+^k\right] = n\sum d_k x_+^k$ implies $D(x_+^n) = \lambda_n x_+^n$. On the other hand,

$$\begin{aligned}
[x_-, x_+^n] = nx_+^{n-1} &\Longrightarrow [x_-, D(x_+^n)] = nD(x_-^{n-1}) \\
&\Longrightarrow [x_-, \lambda_n x_+^n] = n\lambda_{n-1}x_+^{n-1} \\
&\Longrightarrow n\lambda_n x_+^{n-1} = n\lambda_{n-1}x_+^{n-1}
\end{aligned}$$

This implies that $\lambda_n = \lambda_{n-1}$ for all $n \geq 2$, but $\lambda_1 = 0$, and thus $\lambda_n = 0$ for all $n \geq 2$, which proves that $D(x_+^n) = 0$.
Analogously $D(x_-^n) = 0$ for all $n \geq 2$.

- $D(x_+^n x_-) = D(x_+ x_-^n) = 0$ for all $n \geq 0$. Indeed

$$\begin{aligned}
[x_-^2, x_+] = 4x_+x_- - 2 &\Longrightarrow 4D(x_+x_-) - 2D(1) = [D(x_-^2), x_+^2] + [x_-^2, D(x_+^2)] \\
&\Longrightarrow D(x_+x_-) = 0.
\end{aligned}$$

Also, we compute

$$\begin{aligned}
[x_+^n x_-, x_+] = x_+^n &\Longrightarrow D(x_+^n) = [D(x_+^n x_-), x_+] + [x_+^n x_-, D(x_+)] \\
&\Longrightarrow D(x_+^n x_-) = \sum d_s x_+^s
\end{aligned}$$

and

$$\begin{aligned}
[x_+x_-, x_+^n x_-] = (n-1)x_+^n x_- &\Longrightarrow [D(x_+x_-), x_+^n x_-] + [x_+x_-, D(x_+^n x_-)] = (n-1)D(x_+^n x_-) \\
&\Longrightarrow \left[x_+x_-, \sum d_s x_+^s\right] = (n-1)\sum d_s x_+^s \\
&\Longrightarrow \sum d_s x_+^s = (n-1)\sum d_s x_+^s \\
&\Longrightarrow D(x_+^n x_-) = \lambda_{n-1}x_+^{n-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[x_-, x_+^n x_-] = nx_+^{n-1}x_- &\Longrightarrow \lambda_{n-1}(n-1)x_+^{n-2} = \lambda_{n-2}nx_+^{n-2} \\
&\Longrightarrow \lambda_{n-1} = \frac{n}{n-1}\lambda_{n-2}
\end{aligned}$$

for all $n \geq 2$. But $\lambda_0 = 0$, and therefore this implies $\lambda_n = 0$ for all $n \geq 2$. Thus $D(x_+^n x_-) = 0$ for all $n \geq 2$.

- $D(x_+^n x_-^n) = 0$ for all $n, m \geq 2$.
  Suppose this is not so. Then, there exits an element $x_+^m x_-^n$, such that $D(x_+^m x_-^n) \neq 0$ and m, n are minimal with this property. We compute

$$[x_+^n x_-, x_+ x_-^m] = (1 - mn)x_+^n x_-^m - \frac{1}{2}mn(n-1)(m-1)x_+^{n-1}x_+^{m-1} + ...$$

  On the other hand,

$$D([x_+^n x_-, x_+ x_-^m]) = [D(x_+^n x_-), x_+ x_-^m] + [x_+^n x_-, D(x_+ x_-^m)] = 0$$

  This implies that

$$0 = D([x_+^n x_-, x_+ x_-^m]) = (1 - mn)D(x_+^n x_-^m) - \frac{1}{2}mn(n-1)(m-1)D(x_+^{n-1}x_+^{m-1}) + ...$$

  This implies that $(1 - mn)D(x_+^n x_-^m) = 0$, but $(1 - mn) \neq 0$ because $m, n \geq 2$ then $D(x_+^n x_-^m) = 0$, this contradicts the minimality of m, n. Thus $D(x_+^n x_-^m) = 0$ for all $m, n \geq 2$. But $m, n \geq 2$ are minimal with the property $D(x_+^m x_-^n) \neq 0$, thus $D(x_+^n x_-^n) = 0$.

- $D(x_+^{-1}) = D(x_+^{-1}) = 0$. Indeed, $[x_+, x_+^{-1}] = 0$ implies that, $D(x_+^{-1} = \sum d_s x_+^s)$ and $[x_+ x_-, x_+^{-1}] = -x_+$ implies $D(x_+^{-1}) = \lambda x_+^{-1}$. Also, $[x_+^2 x_-, x_+^{-1}] = -1$, implies $[x_+^2 x_-, D(x_+^{-1})] = 0$ and $[x_+^2 x_-, \lambda x_+^{-1}] = -\lambda$,implies $\lambda = 0$. Thus $D(x_+^{-1}) = 0$. Analogously $D(x_-^{-1}) = 0$.

We have checked that $D(u) = 0$ for all $u \in [H_1, H_1]$, and this implies that D is a scalar multiple of f_0. □

Remark 53. *We see that $\Psi_n DO$ can be written as $\Psi_{n-1}DO\widetilde{\otimes}\Psi DO$ as an associative algebra. The corresponding associative algebra structure is a consequence of the Künneth formula. We can compute the Hochschild cohomology of $\Psi_n DO$. We can prove (see Appendix) that this cohomology is isomorphic to $\bigwedge^\bullet D_n$ where D_n is the 2n-dimensional vector subspace of $Der(\Psi_n DO)$ generate by $[\log \xi_{\pm i}, \cdot]$, $i = 1, ..., n$.*

This observation, Lemma 51 and Lemma 52 imply Theorem 48. In fact, each $[\log x_{\pm i}, \cdot]$ is a derivation of $\Psi_n DO$ as a associative algebra, and therefore it is a derivation of $\Psi_n DO$ as a Lie algebra. Moreover, we already know that f_0 is a Lie algebra derivation and, we can check that an n-variables version of Lemma 51 holds. It follows that the space $H^1(\mathcal{H}_n, \mathcal{H}_n)$ is generated by $[\log x_{\pm i}, \cdot]$ and f_0.

Chapter 3

Applications

3.1 The Manin Triple of Pseudodifferential Symbols

In this section we assume that the bilinear forms $< \cdot, \cdot >$ on ΨDO and ΨDO_σ are nondegenerate.we take as our main reference for definition and basic properties of the book [KW].

3.1.1 Manin Triple and Double Extension of (Twisted) Pseudodifferential Symbols

Definition 54. *Three Lie algebras $\overline{\mathfrak{g}}, \mathfrak{g}_-$ and $\mathfrak{g}_+$ form a Manin triple if the following conditions are satisfied:*

1. *The Lie algebras $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are Lie subalgebras of $\overline{\mathfrak{g}}$ such that $\overline{\mathfrak{g}} = \mathfrak{g}_- \oplus \mathfrak{g}_-$ as vector spaces.*
2. *There exists a nondegenerate invariant bilinear form on $\overline{\mathfrak{g}}$ such that $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subspaces, i.e, the restrictions of this form to both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ vanish.*

Proposition 55. *The algebras $\Psi DO, DO$ and INT form a Manin triple with respect to the bilinear form $\langle A, B\rangle = res(AB)$.*

Proof. This is proposition 6. $\square$

In the same way, we have a analogous proposition for twisted pseudodifferential symbols, and the proof is the twisted version of the proposition 6 (i.e proposition 33).

Proposition 56. *The algebras $\Psi DO_\sigma, DO_\sigma$ and INT_σ form a Manin triple with respect to the bilinear form $\langle A, B\rangle = res(AB)$.*

The 2-cocycle c defined in (1.18) defines a central extension $\widehat{\Psi DO}$ of ΨDO. In this section we consider a bigger Lie algebra, obtained by extending ΨDO by adding the central extension c and the derivation $[\log\xi, \cdot]$.

Definition 57. *The double extended Lie algebra of ΨDO is the semidirect product of the Lie algebra $\widehat{\Psi DO}$ and the space of derivations $\{\alpha \log\xi | \alpha \in \mathbb{R}\}$, where $\widehat{\Psi DO}$ is the central extension $\widehat{\Psi DO} = \Psi DO \oplus \mathbb{R}c$ with Lie bracket $[(L_1, \alpha), (L_2, \beta)] = ([L_1, L_2], c(L_1, L_2))$. This Lie algebra can be written as a vector space as*

$$\widetilde{\Psi DO} = \widehat{\Psi DO} \oplus \mathbb{R}\log\xi = \Psi DO \oplus (\mathbb{R}\cdot c) \oplus (\mathbb{R}\cdot \log\xi)$$

with the Lie bracket given by

$$[(L_1, \alpha_1, \beta_1), (L_2, \alpha_2, \beta_2)] = ([L_1, L_2] + [\beta_2 \log\xi, L_1] - [\beta_1 \log\xi, L_2], c(L_2, L_1), 0).$$

This construction is a special case of a very general theorem appearing in [B].

Note that, as vector spaces, the Lie algebras $\widetilde{\Psi DO}$ has a direct sum decomposition as $\widetilde{\Psi DO} = \widehat{DO} \oplus \widetilde{INT}$, where

$$\widehat{DO} = \{\alpha c + \sum_{i=0}^{n} a_i \xi^i | a_i \in \mathcal{A}, \alpha \in \mathbb{R}\} = \text{ and } \widetilde{INT} = \{\beta \log\xi + \sum_{i=-\infty}^{-1} a_i \xi^i | a_i \in \mathcal{A}, \beta \in \mathbb{R}\}$$

We denote the elements of the form $\alpha c + \sum_{i=0}^{n} a_i \xi^i \equiv \left(\sum_{i=0}^{n} a_i \xi^i, \alpha, 0\right)$ and the elements $\beta \log\xi + \sum_{i=-\infty}^{-1} a_i \xi^i \equiv \left(\sum_{i=-\infty}^{-1} a_i \xi^i, 0, \beta\right)$

Remark 58. *We can give a twisted version of the above definition, if instead of $\widehat{\Psi DO}, \widetilde{\Psi DO}, \widehat{DO}$ and $\widetilde{INT}$ we consider $\widehat{\Psi DO_\sigma}, \widetilde{\Psi DO_\sigma}, \widehat{DO_\sigma}$ and $\widetilde{INT_\sigma}$.*

Theorem 59. *The bilinear form*

$$\langle (A, \alpha_1, \beta_1), (B, \alpha_2, \beta_2) \rangle := res(AB) + \alpha_1\beta_2 + \beta_1\alpha_2$$

on $\widetilde{\Psi DO}$(or $\widetilde{\Psi DO_\sigma}$) is invariant. Also, the subalgebras $\widehat{DO}$(or $\widehat{DO_\sigma}$) and $\widetilde{INT}$(or $\widetilde{INT_\sigma}$) are isotropic subspaces of $\widetilde{\Psi DO}$(or $\widetilde{\Psi DO_\sigma}$) with respect to this form.

Proof. We prove invariance. Indeed,

$$\begin{aligned}
&\langle [(L_1, \alpha_1, \beta_1), (L_2, \alpha_2, \beta_2)], (L_3, \alpha_3, \beta_3) \rangle \\
&= \langle ([L_1, L_2] + [\beta_2 \log\xi, L_1] - [\beta_1 \log\xi, L_2], c(L_2, L_1), 0), (L_3, \alpha_3, \beta_3) \rangle \\
&= \langle [L_1, L_2], L_3 \rangle + \langle [\beta_2 \log\xi, L_1], L_3 \rangle - \langle [\beta_1 \log\xi, L_2], L_3 \rangle + \beta_3 c(L_2, L_1) \\
&= \langle [L_1, L_2], L_3 \rangle + \beta_2 c(L_1, L_3) - \beta_1 c(L_2, L_3) + \beta_3 c(L_2, L_1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\langle (L_1,\alpha_1,\beta_1)\,,[(L_2,\alpha_2,\beta_2)\,,(L_3,\alpha_3,\beta_3)]\rangle\\
&=\langle (L_1,\alpha_1,\beta_1)\,,([L_2,L_3]+[\beta_3\log\xi,L_2]-[\beta_2\log\xi,L_3]\,,c(L_3,L_2),0)\rangle\\
&=\langle L_1,[L_2,L_3]\,,\rangle+\langle L_1,[\beta_3\log\xi,L_2],\rangle-\langle L_1,[\beta_2\log\xi,L_3],\rangle+\beta_1 c(L_3,L_2)\\
&=\langle L_1,[L_2,L_3]\,,\rangle+\beta_3 c(L_2,L_1)-\beta_2 c(L_3,L_1)+\beta_1 c(L_3,L_2).
\end{aligned}$$

Thus, using the invariance of the bilinear product on ΨDO and the antisymmetry of the 2-cocycle c we obtain the result. □

Corollary 60. *The bilinear form $<,>$ on $\widetilde{\Psi DO}$(or $\widetilde{\Psi DO_\sigma}$) identifies the subalgebra $\widetilde{INT}$(or $\widetilde{INT_\sigma}$) with the dual of the subalgebra $\widehat{DO}$(or $\widehat{DO_\sigma}$) and viceversa.*

The importance of Manin triples comes from the fact that there is a one to one correspondence between Manin triples and Lie bialgebras:

Definition 61. *A Lie algebra $\mathfrak{g}$ is called a Lie bialgebra if its dual space $\mathfrak{g}^*$ comes equipped with a Lie algebra structure $[,]^*$ such that the map $\alpha:\mathfrak{g}\longrightarrow\mathfrak{g}\wedge\mathfrak{g}$ dual to the commutator map $[,]^*:\mathfrak{g}^*\longrightarrow\mathfrak{g}^*\wedge\mathfrak{g}^*$ satisfies*

$$ad_X\alpha(Y)=\alpha([X,Y])$$

Here ad denotes the adjoint action $ad_X(Z\wedge W)=ad_X(Z)\wedge W+Z\wedge ad_x(W)$ of $\mathfrak{g}$ on $\mathfrak{g}\wedge\mathfrak{g}$.

Theorem 62. *Let $(\overline{\mathfrak{g}},\mathfrak{g}_+,\mathfrak{g}_-)$ be a Manin triple. Then the Lie algebra $\mathfrak{g}_-$ is naturally identified with $\mathfrak{g}_+^*$ with the structure of a Lie bialgebra. On the other hand, for any Lie bialgebra $\mathfrak{g}$ there is a natural Lie algebra structure on $\overline{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{g}^*$ such that $(\overline{\mathfrak{g}},\mathfrak{g},\mathfrak{g}^*)$ is a Manin triple.*

For a full proof the theorem 62, see [LW] and [STS]. Summarizing we have to the following result:

Corollary 63. *If the bilinear form defined in the theorem 59 is nondegenerate then the Lie algebras $\widetilde{\Psi}$(or $\widetilde{\Psi_\sigma}$), $\widehat{DO}$(or $\widehat{DO_\sigma}$) and $\widetilde{INT}$(or $\widetilde{INT_\sigma}$) form a Manin triple. In particular, $\widetilde{INT}$ is a Lie bialgebra.*

3.1.2 Manin Triple of Pseudodifferential Symbols on Several Variables

We say that the order of the pseudodifferential operator $L=\sum_{\alpha\in\Gamma_n}a_\alpha\xi^\alpha$ is N, if there is a $\alpha=(\alpha_1,...\alpha_n)$ with $\alpha_n=N$ such that $a_{\alpha_n}\neq 0$ and for all $\beta=(\beta_1,...,\beta_n)$ such that $\beta_n>N$ we have $a_\beta=0$.
The Lie algebra $\Psi_n DO$ has two natural subalgebras

$$INT_n=\{L\in\Psi|ord(L)<0\}\text{ and }DO_n=\{L\in\Psi|ord(L)\geq 0\}$$

As a vector space, the Lie algebra $\Psi_n DO$ is a direct sum of these algebras: $\Psi_n DO = INT_n \oplus DO_n$.
We assume that the bilinear form defined in the following theorem is nondegenerate.

Proposition 64. *The algebras* $(\Psi_n DO, INT_n, DO_n)$ *form a Manin triple with respect to the bilinear form* $\langle A, B\rangle = res(AB)$.

Proof. The proof is analogous to Proposition 6. $\square$

We can also define the order changing the coordinate α_N for a_I with $I \neq N$. (For example, a_{N-1}) Then the Lie algebra Ψ have two natural subalgebras

$$INT_i = \left\{ \sum_{(\ldots,-\infty,\ldots)\in\Gamma_n}^{(\ldots,-1,\ldots)} a_\alpha \xi^\alpha | a_\alpha \in A \right\} \text{ and } DO_i = \left\{ \sum_{(\ldots,0,\ldots)\in\Gamma_n}^{(\ldots,n,\ldots)} a_\alpha \xi^\alpha | a_\alpha \in A \right\}$$

where i indicates the coordinate on which the order is taken. Analogously we have that, as a vector space, $\Psi = INT_n \oplus DO_n$.

Proposition 65. *The algebras* $(\Psi_n DO, INT_i, DO_i)$ *Form a Manin triple with respect to the bilinear form* $\langle A, B\rangle = res(AB)$.

Proof. This is again analogous to Proposition 6. $\square$

Now we consider the doubly extended algebra $\widetilde{\Psi_n DO}$ of ΨDO, this is, the semidirect product of the Lie algebra $\widehat{\Psi_n DO} = \Psi DO \oplus \mathbb{R}\cdot c_i$ and the space of derivations $\{\beta \log \xi | \beta \in \mathbb{R}\}$. As a vector space, we have the descomposition

$$\widetilde{\Psi_n DO} = \widetilde{INT_i} \oplus \widehat{DO_i}$$

where

$$\widetilde{INT_i} = \left\{ \beta \log \xi_i + \sum_{(\ldots,-\infty,\ldots)\in\Gamma_n}^{(\ldots,-1,\ldots)} a_\theta \xi^\theta | a_\theta \in A \right\} \text{ and } \widehat{DO_i} = \left\{ \alpha c_i + \sum_{(\ldots,0,\ldots)\in\Gamma_n}^{(\ldots,n,\ldots)} a_\theta \xi^\theta | a_\theta \in A \right\}$$

Theorem 66. *The algebras* $(\widetilde{\Psi_n DO}, \widetilde{INT_i}, \widehat{DO_i})$ *form a Manin triple with respect to the bilinear form* $\langle A + b_1 c_i + \alpha_1 \log \xi_i, B + b_2 c_i + \alpha_2 \log \xi_i\rangle = res(AB) + b_2\alpha_1 + b_1\alpha_2$ *and Lie bracket given by* $[(A, b_1, \alpha_1), (B, b_2, \alpha_2)] = ([A, B] + [\alpha_2 \log \xi_i, A] - [\alpha_1 \log \xi_i, B], c_i(A, B), 0)$

Proof. The proof is similarity to Theorem 59. $\square$

Since it allows us construct Manin triple in several cases, we review it here:

Remark 67. *Let $\mathcal{A}$ a non-associative algebra over a field $\mathbb{K}$ and assume that f is a bilinear functional with the following properties:*

1. $f(ab, c) = f(a, bc)$ *for all* $a, b, c \in \mathcal{A}$.
2. $f(a, b) = 0$ *for all* $b \in \mathcal{A}$ *then* $a = 0$, *and* $f(a, b) = 0$ *for all* $a \in \mathcal{A}$ *then* $b = 0$

The pair $(\mathcal{A}, f)$ is called a pseudometrised algebra(or metrised algebra if f is symmetric).

Theorem 68. *[B] Let $(\mathcal{A}, f)$ be a metrised algebra over a field $\mathbb{K}$. Furthermore let B be another finite-dimensional Lie algebra over $\mathbb{K}$ and suppose that there is a Lie homomorphism $\phi : B \longrightarrow Der_f(\mathcal{A})$, where $Der_f(\mathcal{A})$ denotes the space of all f-antisymmetric derivations of $\mathcal{A}$ (i.e the derivations d of $\mathcal{A}$ for which $f(da, \tilde{a}) + f(a, d\tilde{a}) = 0$ for all $a, \tilde{a} \in \mathcal{A}$). Let B^* denote the dual space of B. Let $w : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}^*$ be the bilinear antisymmetric map $(a, \tilde{a}) \longmapsto (b \longmapsto f(\phi(b)a, \tilde{a}))$ and for $b \in B$ and $\beta \in B^*$ denote by $b \cdot \beta$ the coadjoint representation (i.e $(b \cdot \beta(\tilde{b})) = -\beta(b\tilde{b})$). Take the vector space direct sum $\mathcal{A}_B = B \oplus A \oplus B^*$ and define the following multiplication for $b, \tilde{b} \in B, a, \tilde{a} \in \mathcal{A}$ and $\beta, \tilde{\beta} \in B^*$:*

$$(b + a + \beta)(\tilde{b} + \tilde{a} + \tilde{\beta}) = b\tilde{b} + \phi(b)\tilde{a} - \phi(\tilde{b})a + a\tilde{a} + w(\tilde{a}, a) + b \cdot \tilde{\beta} - \tilde{b} \cdot \beta$$

Moreover, define the following symmetric bilinear form f_B on $\mathcal{A}_B$:

$$f_B(b + a + \beta, \tilde{b} + \tilde{a} + \tilde{\beta}) = \beta(\tilde{b}) + \tilde{\beta}(b) + f(a, \tilde{a})$$

The pair $(\mathcal{A}_B, f_B)$ is a metrised algebra over $\mathbb{K}$ called the double extension of $\mathcal{A}$ by (β, ψ).

Remark 69. *Theorem 68 gives us the possibility of defining a double extension of $\widetilde{\Psi_n DO}$ but, now we consider of n- central extension $\mathbb{R} \cdot c_1 \oplus \cdots \mathbb{R} \cdot c_n$ and the n-$\log \xi_i$́s. As vector space*

$$\widetilde{\Psi_n DO} = \Psi_n DO \oplus (\mathbb{R} \cdot c_1) \oplus \cdots \oplus (\mathbb{R} \cdot c_n) \oplus (\mathbb{R} \cdot \log \xi_1) \oplus \cdots \oplus (\mathbb{R} \cdot \log \xi_n),$$

and the Lie bracket is given by:

$$[(A, \alpha, \beta), (B, \bar{\alpha}, \bar{\beta})] = ([A, B] + [\bar{\beta} \cdot \log \xi, A] - [\beta \cdot \log \xi, B], c(A, B), 0)$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $\bar{\alpha} = (\bar{\alpha_1}, \bar{\alpha_2}, ..., \bar{\alpha_n})$, $\bar{\beta} = (\bar{\beta_1}, \bar{\beta_2}, ..., \bar{\beta_n})$, $\log \xi = (\log \xi_1, \log \xi_2, ..., \log \xi_n)$, $0 = (0, ..., 0) \in \mathbb{R}^n$ and $c(A, B) = (c_1(A, B), ..., c_n(A, B))$. The operation "$\cdot$" is the usual dot product in $\mathbb{R}^n$, for example, $\beta \cdot \log \xi = \beta_1 \log \xi + ... + \beta_n \log \xi_n$. Also, the bilinear form is given by:

$$< A, B >= res(A, B) + \alpha \cdot \bar{\beta} + \beta \cdot \bar{\alpha}$$

3.2 The KdV and KP hierarchies

The objective of this section is to define a Poisson structure on ΨDO^*. Remember that on ΨDO, we have a bilinear nondegenerate, symmetric form and it satisfies

$$\langle [P,Q]\,, S\rangle = \langle [S,P]\,, Q\rangle \quad S, P, Q \in \Psi DO.$$

Then we can identify the dual space of the Lie algebra ΨDO with ΨDO itself.

Definition 70. *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}^*$ is its dual space, the functional derivative of a function $f : \mathfrak{g}^* \longrightarrow \mathbb{R}$ at $\mu \in \mathfrak{g}^*$ is the unique element $\dfrac{\delta f}{\delta \mu}$ of $\mathfrak{g}$ determined by*

$$\left\langle \nu, \frac{\delta f}{\delta \mu} \right\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} f\left(\mu + \epsilon\nu\right)$$

for all $\nu \in \mathfrak{g}^*$, in which $\langle , \rangle$ denotes a natural paring between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Definition 71. *The Lie-Poisson bracket on the dual space $\mathfrak{g}^*$ is defined as follows, for all function $F, G : \mathfrak{g}^* \longrightarrow \mathbb{R}$ and $\mu \in \mathfrak{g}^*$:*

$$\{F, G\}\left(\mu\right) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle$$

The decomposition $\Psi DO = INT \oplus DO = \Psi DO_- \oplus \Psi DO_+$ allows us to define a new structure of Lie algebra $[,]_R$ on ΨDO using by R-matrices:

Definition 72. *Let $(\mathfrak{g}, [,])$ be a Lie algebra and $R : \mathfrak{g} \longrightarrow \mathfrak{g}$ a linear operator on $\mathfrak{g}$. We define a bilinear antisymmetric bracket $[,]_R$ on $\mathfrak{g}$ by*

$$[P, Q]_R = [R(P), Q] + [P, R(Q)] \quad Q, P \in \mathfrak{g}.$$

The linear operator R is called a classical R-matrix if the bracket $[,]_R$ satisfies the Jacobi identity.

If $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ as vector space, then we can choose $R = 1/2\left(\pi_+ - \pi_-\right)$, where $\pi_\pm$ are the projection operators on $\mathfrak{g}_\pm$. Following standard notice(see for instance [KW, STS1]), we use $[,]_0$ and $\{,\}_0$ instead of $[,]_R$ and $\{,\}_R$ if $R = 1/2\left(\pi_+ - \pi_-\right)$.

Proposition 73. *If $F_M \in \Psi DO^*$ is defined by $F_M(L) = \langle L, M\rangle$, then $\dfrac{\delta F_M}{\delta L} = M$.*

Proof. Fix $M \in \Psi DO^*$ and take the functional F_M as F, i.e $F = \langle \cdot, M\rangle$. Then,

$$\left\langle P, \frac{\delta F_M}{\delta L} \right\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F_M\left(L + \epsilon P\right) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \langle L + \epsilon P, M\rangle = \langle P, M\rangle\,.$$

This implies that $\dfrac{\delta F_M}{\delta L} = M$ $\square$

Proposition 74. *1. Let $H_k(L) = res(L^k)$, $k = 1, 2, ...$ for $L \in \Psi DO$.*
Then $\dfrac{\delta H_k}{\delta L} = kL^{k-1}$

2. For any functional F on ΨDO^*; $\{H_k, F\}(L) = 0$ and $\{H_k, H_l\}_R(L) = 0$. In particular if $R = (1/2)(\pi_+ - \pi_-)$.we have $\{H_k, H_l\}_0(L) = 0$

Proof. Let $M \in \Psi DO$. We have,

$$\begin{aligned}
\left\langle M, \frac{\delta H_k}{\delta L} \right\rangle &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} H_k(L + \epsilon M) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} res(L + \epsilon M)^k \\
&= res\left(ML^{k-1} + LML^{k-2} + \cdots + L^{k-1}M\right) \\
&= res\left(kL^{k-1}\right)
\end{aligned}$$

Thus, for all $M \in \Psi DO$,

$$\left\langle M, \frac{\delta H_k}{\delta L} \right\rangle = \left\langle M, kL^{k-1} \right\rangle$$

and 1 follows. Moreover,

$$\{H_k, F\}(L) = \left\langle L, \left[\frac{\delta H_k}{\delta L}, \frac{\delta F}{\delta L}\right] \right\rangle = \left\langle L, \left[kL^{k-1}, \frac{\delta F}{\delta L}\right] \right\rangle = \left\langle \left[L, kL^{k-1}\right], \frac{\delta F}{\delta L} \right\rangle = 0$$

for all $L \in \Psi DO$.
Finally, we have:

$$\begin{aligned}
\{H_k, H_l\}_R(L) &= \left\langle L, \left[\frac{\delta H_k}{\delta L}, \frac{\delta H_l}{\delta L}\right]_R \right\rangle \\
&= \left\langle L, \left[R\left(\frac{\delta H_k}{\delta L}\right), \frac{\delta H_l}{\delta L}\right] \right\rangle + \left\langle L, \left[\frac{\delta H_k}{\delta L}, R\left(\frac{\delta H_l}{\delta L}\right)\right] \right\rangle \\
&= -\left\langle L, \left[\frac{\delta H_l}{\delta L}, R\left(\frac{\delta H_k}{\delta L}\right)\right] \right\rangle + \left\langle R\left(\frac{\delta H_l}{\delta L}\right), \left[L, \frac{\delta H_k}{\delta L}\right] \right\rangle \\
&= -\left\langle R\left(\frac{\delta H_k}{\delta L}\right), \left[L, \frac{\delta H_l}{\delta L}\right] \right\rangle + \left\langle R\left(\frac{\delta H_l}{\delta L}\right), \left[L, \frac{\delta H_k}{\delta L}\right] \right\rangle \\
&= -\left\langle R\left(\frac{\delta H_k}{\delta L}\right), \left[L, kL^{k-1}\right] \right\rangle + \left\langle R\left(\frac{\delta H_l}{\delta L}\right), \left[L, kL^{k-1}\right] \right\rangle = 0.
\end{aligned}$$

In particular, if $R = 1/2(\pi_+ - \pi_-)$, we have $\{H_k, H_l\}_0(L) = 0$. $\square$

Theorem 75. *Let $H_k(L) = res(L^k)$ be a functional on ΨDO^*. The corresponding Hamiltonian equation of motion with respect $\{\cdot,\cdot\}_0$ are*

$$\frac{\partial L}{\partial t} = \left[\left(kL^{k-1}\right)_+, L\right]$$

Proof. The Hamiltonian equation of motion on $\Psi DO^*$with Hamiltonian H_k are:

$$\frac{\partial \widetilde{L}}{\partial t} = \left\{\widetilde{L}, H_k\right\}_0 = \left\langle L, \left[\cdot, \left(\frac{\delta H_k}{\delta \widetilde{L}}\right)\right]_0\right\rangle$$

If we set $\widetilde{L} =< L, \cdot >$ for some $L \in \Psi DO$, we have

$$\begin{aligned}
\left\langle \frac{\partial L}{\partial t}, \cdot \right\rangle &= \left\langle L, \left[\cdot, \frac{\delta H_k}{\delta L}\right]_0\right\rangle = \left\langle L, \left[\cdot, kL^{k-1}\right]_0\right\rangle \\
&= \left\langle L, \left[(\cdot)_+ \left(, kL^{k-1}\right)_+\right]\right\rangle - \left\langle L, \left[(\cdot)_-, \left(kL^{k-1}\right)_-\right]\right\rangle \\
&= -\left\langle (\cdot)_+, \left[L, \left(kL^{k-1}\right)_+\right]\right\rangle + \left\langle (\cdot)_-, \left[L, kL^{k-1} - \left(kL^{k-1}\right)_+\right]\right\rangle \\
&= -\left\langle (\cdot)_+, \left[L, \left(kL^{k-1}\right)_+\right]\right\rangle - \left\langle (\cdot)_-, \left[L, \left(kL^{k-1}\right)_+\right]\right\rangle \\
&= -\left\langle (\cdot), \left[L, \left(kL^{k-1}\right)_+\right]\right\rangle = \left\langle (\cdot), \left[\left(kL^{k-1}\right)_+, L\right]\right\rangle
\end{aligned}$$

□

Now, if we have the Lie algebra of pseudodifferential symbols ΨDO, we may ask: what happens if we change ΨDO for $\Psi DO \oplus \mathbb{R}$? i.e we have ΨDO and our central extension.

We consider the following subspaces of ΨDO

$$DO \oplus \mathbb{R} = \left\{\alpha + \sum_{i=0}^{N} a_i \xi^i : a_i \in \mathcal{A}\right\} \text{ and } INT \oplus \{0\} = \left\{0 + \sum_{i=-\infty}^{-1} a_i \xi^i : a_i \in \mathcal{A}\right\}.$$

Then as a vector space, $\Psi DO \oplus \mathbb{R} = (DO \oplus \mathbb{R}) \oplus (INT \oplus \{0\})$.

Proposition 76. *$DO \oplus \mathbb{R}$ and $INT \oplus \{0\}$ are Lie subalgebras of Lie algebra $\Psi DO \oplus \mathbb{R}$.*

Proof. Let $A, B \in INT$. Then $[A, B] \in INT$, because INT is a Lie subalgebra that ΨDO. Now, we can prove that $c(A, B) = 0$. Indeed, let $A = a\xi^n$ and $B = b\xi^n$ be basic element such that $n, m \leq -1$. Then

$$[\log \xi, a\xi^n] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \delta^k(a) \xi^{n-k} \Longrightarrow$$
$$[\log \xi, a\xi^n]\, b\xi^m = \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+1}}{k} \delta^k(a) \delta^j(b) \xi^{n+m-k-j}$$

as $n+m-k-j \leq -3$, we have $c(a\xi^n, b\xi^m) = 0$. Analogously, if $A, B \in DO$ then $[A, B] \in DO$ and, if $A = a\xi^n$ and $B = b\xi^n$ are basic element such that $n, m \geq 0$, implies that $n+m-k-j \geq -1$, then $c(a\xi^n, b\xi^m) \in \mathbb{R}$. □

Definition 77. *Let $(A, \alpha), (B, \beta) \in \Psi \oplus \mathbb{R}$. We define a bilinear form on $\Psi \oplus \mathbb{R}$ by*

$$\langle (A, \alpha), (B, \beta) \rangle = \langle A, B \rangle + \alpha\beta$$

We use the bilinear form to identify the dual space of the Lie algebra $\Psi DO \oplus \mathbb{R}$ with $\Psi DO \oplus \mathbb{R}$ itself.

Proposition 78. *Let $H_k(L, l) = \dfrac{l}{k+l} res\left(L^{(k+l)/l}\right)$. Then $\dfrac{\delta H_k}{\delta L} = L^{k/l}$ with $k, l = 1, 2, ...$*

Proof. Let $(M, m) \in \Psi \oplus \mathbb{R}$. We have

$$\begin{aligned}
\left\langle (M, m), \frac{\delta H_k}{\delta (L, l)} \right\rangle &= \left\langle (M, m), \left(\frac{\delta H_k}{\delta L}, \frac{\delta H_k}{\delta l} \right) \right\rangle \\
&= \left\langle M, \frac{\delta H_k}{\delta L} \right\rangle + m \frac{\delta H_k}{\delta l} \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} H_k(L + \epsilon M) + m \frac{\delta H_k}{\delta l} \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{l}{l+k} res\left(L^{(l+k)/l}\right) + m \frac{\delta H_k}{\delta l} \\
&= res(L^{k/l}) + m \frac{\delta H_k}{\delta l} \\
&= \left\langle M, L^{k/l} \right\rangle + m \frac{\delta H_k}{\delta l} \\
&= \left\langle (M, m), \left(L^{k/l}, \frac{\delta H_k}{\delta l} \right) \right\rangle
\end{aligned}$$

and the result is follows. □

Remember that $\Psi DO \oplus \mathbb{R} = (\Psi DO \oplus \mathbb{R})_+ \oplus (\Psi DO \oplus \mathbb{R})_-$, where $(\Psi DO \oplus \mathbb{R})_+ = DO \oplus \mathbb{R}$ and $(\Psi DO \oplus \mathbb{R})_- = INT \oplus \{0\}$. Then for $(L, l) \in \Psi DO \oplus \mathbb{R}$, we have that $\pi_+(L, l) = (L_+, l)$ and $\pi_-(L, l) = (L_-, 0)$ and we denote $\pi_\pm(L, l) = (L, l)_\pm$.

Theorem 79. *Let $H : (\Psi DO \oplus \mathbb{R})^* \longrightarrow \mathbb{R}$ be a Hamiltonian function. The corresponding Hamiltonian equation of motion with respect to the bracket $\{\cdot, \cdot\}$ is*

$$\frac{\partial L}{\partial t} = \left[\frac{\partial H_k}{\partial L}, L\right] + lc\left(\cdot, \frac{\delta H_k}{\delta L}\right)$$

Proof. The Hamiltonian equation of motion on $(\Psi DO \oplus \mathbb{R})^*$ with Hamiltonian H are

$$\frac{\partial(\tilde{L}, \tilde{l})}{\partial t} = \left\langle (\tilde{L}, \tilde{l}), \left[\cdot, \frac{\delta H}{\delta(\tilde{L}, \tilde{l})}\right]\right\rangle = \left\langle (\tilde{L}, \tilde{l}), \left[\cdot, \left(\frac{\delta H}{\delta \tilde{L}}, \frac{\delta H}{\delta \tilde{l}}\right)\right]\right\rangle$$

If we set $(\tilde{L}, \tilde{l}) = \langle (L, l), \cdot, \rangle$ for some $(l, l) \in \Psi DO \oplus \mathbb{R}$, we have

$$\left\langle \frac{\partial (L, l)}{\partial t}, \cdot, \right\rangle = \left\langle (L, l), \left(\left[\cdot, \frac{\delta H}{\delta L}\right], c\left(\cdot, \frac{\delta H}{\delta L}\right)\right)\right\rangle$$

then

$$\begin{aligned} \left\langle \left(\frac{\partial L}{\partial t}, \frac{\partial l}{\partial t}\right), \cdot \right\rangle &= \left\langle (L, l), \left(\left[\cdot, \frac{\delta H}{\delta L}\right], c\left(\cdot, \frac{\delta H}{\delta L}\right)\right)\right\rangle \\ \left\langle \frac{\partial L}{\partial t}, \cdot \right\rangle + \cdot \frac{\partial l}{\partial t} &= \left\langle L, \left[\cdot, \frac{\delta H}{\delta L}\right]\right\rangle + lc\left(\cdot, \frac{\delta H}{\delta L}\right) \end{aligned}$$

this implies that $\frac{\partial l}{\partial t} = 0$, we use the nondegeneracy and invariance of the bilinear form on ΨDO and we have that

$$\frac{\partial L}{\partial t} = \left[\frac{\partial H}{\partial L}, L\right] + lc\left(\cdot, \frac{\delta H}{\delta L}\right)$$

□

A special case of the equation 3.2 is the zero curvature equation obtained in the context of current algebras by Reyman and Semenov-Tian-Shansky [RSTS].

Conclusions

In this Thesis we have studied the Lie algebra of pseudodifferential symbols in one and several independent variables with an emphasis on the existence and classification of central extensions and their application to the contruction of integrable systems. Our algebraic point of view generalizes and unifies some previous constructions carried out for special algebras such as the algebra of difeomorphisms of the circle. Specially, highlight our general construction of twisted algebras and our study of pseudodifferential symbols in the case of several independent variables. In particular, our hierarchies of centrally extended algebras in the twisted and several variables cases are new examples of centrally extended algebras generalizing previous work by Khesin [K] and Kac-Peterson [KP]. We also believe of interest our new proof of the Dzhumadildaev classification theorem: while pseudodifferential symbols on the circle have been investigated in detail, see for example [KW, GR], it is notoriously difficult to investigate pseudodifferential symbols on higher dimensional manifolds, see [R, W], and therefore we think it is of importance to have an algebraic description of their properties, including a classification of their central extensions.

We also mention that several problems present themselves. For example, in the third chapter of this work we explain how to constuct hierarchies of integrable equations in our algebraic setting. We have left as an open problem the explicit constuction of some of these hierarchies for specific algebras.Of partcicular importance is the construction of hierarchies of equation in several independent variables using our work of Chapter 2 and the analogs of Theorem 79 and 75. We also left as open problems the study of twisted pseudodifferential symbols in the case of several independent variables, as a generalization of the work by Keshin, Lyubashenko and Roger [KLR] on extensions and contractions of the Lie algebra of q-pseudodifferential symbols on the circle, and the possible contruction of Lie groups whose Lie algebras are (extensions of) the algebras of pseudodifferential symbols considered here. It is known that this construction is possible in some particular cases for one independent variable, see [KZ, KLR], but it appears to be unknown whether we can carry out this construction for the case of several independent variables.

Chapter 4

Appendix

We present some calculations that are used in the proof of Theorem 48, and justify the remark 53. This appendix is the first version of a manuscript being prepared for publication. It has been written together with Marco Farinati (Universidad de Buenos Aires).

4.1 Hochschild homology and cohomology of pseudodifferential operators

4.1.1 The objects

Let k be a field of characteristic zero. The Weyl algebra, or the algebra of algebraic differential operators in the affine space can be described as the vector space $A_n = k[\{x_{\pm i} : i = 1, \ldots, n\}]$, with the multiplication law determined by the rules

$$[x_{+i}, x_{+j}] = 0 = [x_{-i}, x_{-j}], \; [x_{-i}, x_{+j}] = \delta_{ij}$$

This algebra acts faithfully on $k[x_1, \ldots, x_n]$ sending x_{+i} to the multiplication by x_i, and x_{-i} to $\frac{\partial}{\partial x_i}$. This algebra is filtered by the order of the differential operators, and also it has the Berstein filtration, where a monomial $x_{+1}^{a_1} \cdots x_{+n}^{a_n} x_{-1}^{b_1} \cdots x_{-n}^{b_n}$ has total degree $\sum_i a_i + \sum_i b_i$. Both filtrations induce commutative product on the associated graded algebra, and the canonical Poisson structure on k^{2n}.

One may localize on the x_{+i}'s, or on the x_{-i}'s, but not on both simultaneously, unless one consider formal series on $x_{\pm i}^{-1}$. We consider psuedodifferential algebra $\Psi_n DO$ and denoted by Ψ_n .

It is clear that $A_n \subset \Psi_n$ is a subalgebra. The algebra Ψ_n has also two filtrations: one by "order of differential operator", namely $|x_+^I x_-^J|_{diff} = |J| = \sum_i j_i$, and also by total degree: $|x_+^I x_-^J|_{tot} = |I| + |J|$. We call this total degree the Bernstein filtration. The associated graded algebra is commutative, one may identify it with Laurent polynomials $\mathrm{gr}\Psi_n = k[x_{+1}, x_{+1}^{-1}, \dots x_{+n}, x_{+n}^{-1}, x_{-1}, x_{-1}^{-1}, \dots x_{-n}, x_{-n}^{-1}]$. On the other hand, this algebra Ψ_n is *complete* with respect to the Bernstein filtration.

4.1.2 Main tools

On several arguments, we will use the following standard Lemma of filtered abelian groups. By a filtered abelian group M we mean a familiy of subgroups F_pM for each $p \in \mathbb{Z}$ such that $F_pM \subseteq F_{p+1}M$ for all p, $\cup_p F_pM = M$, and $0 = \cap_p F_pM$.

Lemma 80. *Let Z, A, B, C be filtered abelian groups and*

$$Z \xrightarrow{h} A \xrightarrow{f} B \xrightarrow{g} C$$

a complex of filtered groups (i.e. each morphism preserves the respective filtration). Assume that

$$\mathrm{gr}Z \xrightarrow{\mathrm{gr}h} \mathrm{gr}A \xrightarrow{\mathrm{gr}f} \mathrm{gr}B \xrightarrow{\mathrm{gr}g} \mathrm{gr}C$$

es exact in A and B.

1. *If the filtrations are bounded below, i.e. if $F_{p_0}M = 0$ for some $p_0 \in \mathbb{Z}$ ($M = A, B, C$), then the original complex*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact in B.

2. *In the general case, the completion*

$$\widehat{A} \xrightarrow{\widehat{f}} \widehat{B} \xrightarrow{\widehat{g}} \widehat{C}$$

is exact in B.

An interesting consequence is the completed version of Künneth formula:

Corollary 81. *Let A, B, C be completed filtered complexes such that $\mathrm{gr}(A\widehat{\otimes}B) = \mathrm{gr}A \otimes \mathrm{gr}B$ and let $f : C \to A\widehat{\otimes}B$ a map such that $\mathrm{gr}f : \mathrm{gr}C \to \mathrm{gr}A \otimes \mathrm{gr}B$ is a quasi-isomorphism. Then the original map is a quasi-isomorphism.*

Proof. We consider the complex $\mathrm{Cone}(f) = C \oplus \Sigma[A\widehat{\otimes}B]$ with differential $\partial_C + (-1)^{||} f + \partial_{A\otimes B}$. It has an obvious filtration induced by the filtration on A, B and C, and

$$\mathrm{grCone}(f) = \mathrm{gr}C \oplus \Sigma[\mathrm{gr}(A\widehat{\otimes}B)] = \mathrm{gr}C \oplus \Sigma[\mathrm{gr}A \otimes \mathrm{gr}B)] = \mathrm{Cone}(\mathrm{gr}f)$$

Since $\mathrm{gr}f$ is a quasi-isomorphism, this complex is acyclic. The lemma implies that its completion is acyclic, so $\mathrm{Cone}(f)$ is acyclic, namely, f is a quasi-isomorphism. $\square$

4.2 Case Ψ_1

Let us denote $\Psi = \Psi_1$, and $x_\pm = x_{\pm 1}$. Notice that $\widehat{\Psi}^e = \Psi_1 \widetilde{\otimes} \Psi_1^{op} \cong \Psi_1 \widetilde{\otimes} \Psi_1 = \Psi_2$. We also will use the letters $y_\pm = x_\pm \otimes 1$ and $z_\pm = 1 \otimes x_\pm$ in $\Psi_1 \otimes \Psi_1$.

Let W be the 2-dimensional vector space $W = ke_+ \oplus ke_-$ The main result of this section is the following:

Proposition 82. *The complex*

$$0 \longrightarrow \Psi_2 \otimes \Lambda^2 W \xrightarrow{d_1} \Psi_2 \otimes W \xrightarrow{d_0} \Psi_2 \xrightarrow{m} \Psi_1$$

is a resolution of Ψ_1 as $\Psi_1 \widehat{\otimes} \Psi_1^{op}$-module. The map m is the multiplication, and d_0 and d_1 are determined by

$$d_1(e_+ \wedge e_-) = (y_+ - z_+)e_- - (y_- - z_-)e_+$$

and

$$d_0(e_\pm) = y_\pm - z_\pm$$

and Ψ_1-linearity on the left and on the right. Explicitely,

$$d_1\left(\sum_{ijkl} a_{ijkl} y_+^i y_-^j z_+^k z_-^l e_+ \wedge e_-\right) = \sum_{ijkl} a_{ijkl} y_+^i y_-^j (y_+ - z_+) z_+^k z_-^l e_- - \sum_{ijkl} a_{ijkl} y_+^i y_-^j (y_- - z_-) z_+^k z_-^l e_+$$

Proof. If one declares $|e_\pm| = 1$ and $|e_+ \wedge e_-| = 2$ then this complex is canonicaly filtered using the Berstein filtration on Ψ_2 and Ψ_1, and the maps clearly preserve the filtration. The associated graded algebra may be identifyed with the Koszul complex

$$0 \to k[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}] \otimes \Lambda^2 W \to k[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}] \otimes W \to k[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}] \to k[x_\pm, x_\pm^{-1}] \to 0$$

associated to the regular sequence $\{y_+ - z_+, y_- - z_-\}$ in $k[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}]$, hence exact. We identify $k[y_\pm, z_\pm, y_\pm^{-1}, z_\pm^{-1}]/\langle y_+ - z_+, y_- - z_-\rangle \cong k[x_\pm, x_\pm^{-1}]$ via the map $z_\pm \mapsto x_\pm$, $y_\pm \mapsto x_\pm$.The proof of this proposition follows from the Lemma above. $\square$

Corollary 83. *The algebra $\Psi = \Psi_1$ satisfies a Van den Berg duality property with trivial dualizing module, also called the Calabi-Yau property for algebras.*

Proof. One way to prove this result is to compute $HH^\bullet(\Psi, \Psi^e)$ using this complex, getting Ψ^e in degree $2n$ and zero elsewhere, and then apply Van den Berg's theorem [VdB], but also one can use this complex to compute homology or cohomology for a general bimodule M, getting the following complexes:

- An homological complex, after applying $M \otimes_{\Psi^e} -$ and identifying $M \otimes_{\Psi^e} \Psi^e \otimes V \cong M \otimes V$:
$$0 \longrightarrow M \otimes \Lambda^2 W \longrightarrow M \otimes W \longrightarrow M \longrightarrow 0$$
The induced diferentials are
$$d_1(me_+ \wedge e_-) = (x_+m - mx_+)e_- - (x_-m - mx_-)e_+ = [x_+, m]e_- - [x_-, m]e_+$$
$$d_0(me_+ + m'e_-) = x_+m - mx_+ + x_-m' - m'x_- = [x_+, m] + [x_-, m']$$

- An cohomological complex, after applying $\mathrm{Hom}_{\Psi^e}(-, M)$ and identifying $\mathrm{Hom}_{\Psi^e}(\Psi^e \otimes V, M) \cong V^* \otimes M$:
$$0 \longrightarrow M \longrightarrow W^* \otimes M \longrightarrow \Lambda^2 W^* \otimes M \longrightarrow 0$$
and, in dual bases $e^+ \wedge e^-$, e^+, e^-, the differentials are
$$d^0(m) = [x_+, m]e^+ + [x_-, m]e^-$$
$$d^1(me^+ + m'e^-) = ([x_+, m] - [x_-, m'])e^+ \wedge e^-$$

So, after convenient change of signs, and reflecting degrees, one can identify the differentials in homology with cohomology; one concludes $H^\bullet(\Psi, M) \cong H_{2-\bullet}(\Psi, M)$ for all M. □

Theorem 84. *The Hochschild homology and cohomology of Ψ with coefficients in Ψ are given by*
$$HH^0(\Psi) = k,\ HH_2 \cong \Lambda^2 W$$
$$HH^1(\Psi) = kx_-^{-1}e^+ \oplus kx_+^{-1}e^- \cong HH_1(\Psi)$$
$$HH^2(\Psi) = k\frac{1}{x_+x_-}e^+ \wedge e^-,\ HH_0(\Psi) = k$$

Proof. We know that the center of Ψ is k, this computes HH^0 and using duality HH_0. Also it is well known (and easily computable) that $\Psi/[\Psi, \Psi] = kx_+^{-1}x_-{-1}$, so one knows HH_0 and by duality HH^2. It remains to compute HH^1. One can do it directly from this complex, but one can also compute using the isomorphism $HH^1(\Psi) = \mathrm{Der}(\Psi)/\mathrm{Innder}(\Psi)$ (in analogy with proposition 7). The second computation will be carried out later, it has the advantage that it gives standard representative, which are "$[\log x_\pm, -]$". □

Lemma 85. *The cup product in $H^\bullet(\Psi_1, \Psi_1)$ is non-zero.*

Proof. Let $L_\pm ='' [\log x_\pm, -]''$ be the derivation determined by

$$L_+(x_+) = 0,\ L_+(x_-) = -\frac{1}{x_+}$$

$$L_-(x_-) = 0,\ L_-(x_+) = \frac{1}{x_-}$$

The cup product is defined in the standard complex by the rule $(L_+ \smile L_-)(a \otimes b) = L_+(a)L_-(b)$, as a map $\Psi \otimes \Psi \to \Psi$. for instance

$$(L_+ \smile L_-)(x_- \otimes x_+) = -\frac{1}{x_+}\frac{1}{x_-}$$

$$(L_+ \smile L_-)(x_+ \otimes x_-) = 0$$

Since we work with a smaller complex, in order to compute the product of our cohomology classes, we need a comparison map between the small complex and the standard one. In one direction it is not hard, we look at the resolutions:

$$\begin{tikzcd}
\cdots \arrow[r] & \Psi \otimes \overline{\Psi}^{\otimes 3} \otimes \Psi \arrow[r] & \Psi \otimes \overline{\Psi}^{\otimes 2} \otimes \Psi \arrow[r] & \Psi \otimes \overline{\Psi} \otimes \Psi \arrow[r] & \Psi \otimes \Psi \arrow[r] & \Psi \\
 & 0 \arrow[u] \arrow[r] & \Psi \otimes \Lambda^2 W \otimes \Psi \arrow[u, hook] \arrow[r] & \Psi \otimes W \otimes \Psi \arrow[u, hook] \arrow[r] & \Psi \otimes \Psi \arrow[u, equal] \arrow[r] & \Psi \arrow[u, equal]
\end{tikzcd}$$

And can show directly that the inclusion $W \to \Psi$ and $\Lambda^2 W \to \Psi \otimes \Psi$ (the second maps $w_1 \wedge w_2 \mapsto w_1 \otimes w_2 - w_2 \otimes w_1$), extend linearly on the left and on the right, giving maps $\Psi \otimes W \otimes \Psi \to \Psi \otimes \Psi \otimes \Psi$ and $\Psi \otimes \Lambda^2 W \otimes \Psi \to \Psi \otimes \Psi^{\otimes 2} \otimes \Psi$, that actually give rise to a complex map. The point is that the difference between $w_1 w_2$ and $w_2 w_1$ is a scalar, so it is zero in $\overline{\Psi}$. As those maps the identity, they must be an homotopy equivalence. It is clear the comparison map gives the corresponding restrictions, namely, if $D : \Psi \to \Psi$ is a derivation (i.e. a cocycle in the standard complex) then it corresponds to the element $D|_W \in \mathrm{Hom}(W, \Psi)$, for example, L_+ corresponds to $-\frac{1}{x_-}e^-$ and L_- to $\frac{1}{x_+}e^+$. We need to show that the class of $L_+ \smile L_- \neq 0$. But

$$(L_+ \smile L_-)(x_+ \otimes x_- - x_- \otimes x_+) = -\frac{1}{x_+}\frac{1}{x_-}.$$

so $L_+ \smile L_-$ corresponds to $-\frac{1}{x_+ x_-}e^+ \wedge e^-$, wich is not zero since it is actually a generator of $H^2(\Psi_1, \Psi_1)$. $\square$

Corollary 86. *The Hochschild cohomology of Ψ_n is isomorphic to $\Lambda^\bullet D_n$ where D_n is the $2n$-dimensional vector subspace of $\mathrm{Der}(\Psi)$ generated by $[\log x_{i\pm}, -]$.*

Proof. First we remark that $\Psi_n = \Psi_{n-1}\widehat{\otimes}\Psi_1$, so the algebra structure is a consequence of the Künneth formula. Once we know that cohomology is generated in degree one, we only check that HH^1 is an abelian Lie algebra. This is a very easy computation, for instance, in one variable, wrinting $L_\pm := [\log x_\pm, -]$ one can easily check that

$$[L_+, L_-](a) = \left[\sum_{n=1}^{\infty} \frac{x_+^{-n} x_-^{-n}}{(n-1)!^2}, a\right]$$

Namely, this is not zero in $\mathrm{Der}(\Psi)$, but it is inner, so it is zero in $H^1(\Psi, \Psi)$. □

Remark 87. *Our computations imply that the Gerstenhaber bracket of the Lie algebra of pseudodifferential symbols is identically zero. This fact appears to be important for non-commutative geometry, see The homology of algebras of pseudodifferential symbols and the noncommutative residue (J.L. Bryliski and E. Getzler, K-Theory (1987), 385–403). We will develop this observation elsewhere.*

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