UNIVERSIDAD DE SANTIAGO DE CHILE

FACULTAD DE CIENCIA

DEPARTAMENTO DE MATEMATICA Y CIENCIA DE LA COMPUTACION

Categorification results for knots and graphs and some applications

Lida María Mendoza González

Profesor Guía: **Enrique Reyes García**

Tesis entregada a la Facultad de Ciencia de la Universidad de Santiago de Chile para optar al grado de Doctor en Ciencia con mención en Matemática.

Santiago de Chile

2015

Esta Tesis fue desarrollada bajo la supervisión del profesor guía Dr. Enrique Reyes García, acadmico del Departamento de Matemática y Ciencia de la Computación de la Universidad Santiago de Chile.

Profesor Guía
Dr. Enrique Reyes García

Profesor Informante
Dr. Jesús Juyumaya Rojas

Profesor Informante
Dra. María Ronco Vignau

Profesor Informante
Dr. Daniel Pons Rubio

Profesor Informante
Dr. Rodrigo Castro Marín

Director Doctorado en Ciencia
Mención Matemática
Dra. María Isabel Cortez Muñoz

Director Departamento de Matemática
y Ciencia de la Computación
Mg. Pedro Marín Álvarez

Abril, 2015

Resumen

En esta tesis presentamos algunos ejemplos de categorificación en topología y geometría,con el objetivo de hallar invariantes para nudos y grafos. Primeramente estudiamos la cohomología de Khovanov para nudos topológicos y como aplicación discutimos la torsión de Reidemeister para nudos como en [13] y definimos el operador de Laplace, el cual podría ser un invariante para nudos, al analizar su espectro, el cual es un problema abierto. En busca de nuevos invariantes, estudiamos la cohomología de Chekanov para nudos Legendrianos [4], la cual tiene estructura de álgebra y no sólo de complejo como en el caso de la teoría de de Khovanov; la estructura de álgebra nos ayuda a definir productos de Massey los cuales nos ayudan a la construcción de nuevos invariantes para nudos Legendrianos. Otra aplicación de esta álgebra es el estudio de las ecuaciones diferenciales, ya que podemos definir y solucionar ecuaciones diferenciales integrables non conmutativas sobre el álgebra de Chekanov. Finalmente consideramos grafos, usando Categorificación de Khovanov para grafos, definimos la torsión de Reidemeister para grafos y el operador de Laplace, como problema abierto o a futuro, se puede estudiar sobre los grafos Legendrianos y definir productos de Massey sobre su álgebra y obtener un posible invariante para grafos Legendrianos.

Abstract

We present some instances of categorification in geometry and topology, with the objective of finding invariants for knots and graphs. Specifically, after summarizing some of the classical work on knot invariants, we introduce the Khovanov cohomology for topological knots [13], the Chekanov cohomology for Legendrian knots [4], and a version of Khovanov theory valid for graphs [11]. Our main applications are the following: we obtain a Legendrian isotopy-invariant A_∞ algebra structure for Legendrian knots; we present new product invariants for Legendrian knots using (classical and generalized) Massey products; we construct (non-commutative) integrable systems based on the Chekanov algebra; we define a Reidemeister torsion and a Laplace operator on graphs arising from their Khovanov-type cohomology.

Agradecimientos

A Dios por la vida, salud y su ayuda en los momentos mas difíciles de mi vida.

A Vicenia González, mi madre por su apoyo, paciencia y compañia.

A Jarnishs Beltran, por su ayuda incondicional, por soportar cada momento esquizofrénico de mi vida en el periodo de la elaboración de esta tesis y por su cariño durante todos estos años.

A mi profesor guía: Dr. Enrique Reyes por su dedicación, paciencia y consejos. Por ser el generador de los conocimientos adquiridos durante el desarrollo de este trabajo.

A los profesores María Ronco, Rodrigo Castro, Daniel Pons y Jesús Juyumaya, por aceptar ser parte de la comisión calificadora.

A mi familia pues aunque lejos, siempre ha estado presente en los momentos difíciles, en especial a mi Padrino Juan Jimenez por sus innumerables consejos.

A mi hermana Yurlay Montalvan y sobrinos, por ser parte de mi vida y motivación para estudiar.

A los profesores del programa de doctorado en matemáticas, en especial Dr. Victor Guiñez por su apoyo y paciencia.

A mis compañeros de doctorado Patricio Cerda, Gonzalo Castro, María José Moreno y Solange Aransubia.

Al proyecto Basal USA1298, al proyecto FONDECYT 1111042 y a la USACH por el apoyo económico para la culminación de este doctorado.

Contents

Introduction

The objective of this work is to contribute to the understanding of the notion of categorification in geometry and topology, with the main goal of finding invariants for knots and graphs. We say that a mathematical object is a *knot invariant* if it remains unchanged under smooth deformations of the knot. (A formal definition of a deformation appears in Chapter 1). The first invariant of knots is the Alexander polynomial discovered around 1928, see [6]. This invariant was the only known invariant for more than 50 years. In 1980 Jones (see [13] and references therein) discovered a new invariant (now known as the Jones polynomial) which has proven to be much more powerful than the Alexander polynomial: there are knots which are not distinguishable via the later polynomial which can be told apart by the former polynomial. In the last decade of the XXth century M. Khovanov [13] realized that the Jones polynomial can be understood as the Euler-Poincare polynomial of a cohomology theory, and that this cohomology is itself an knot invariant. This is an example of *categorification*: invariants (numbers, polynomials, etcetera) are understood in terms of homological algebra, mainly cohomology and homology. In this thesis we study categorification in two contexts: First, we consider topological and contact knot theory; second, we consider graphs. As stated above, our main goal is to investigate the existence of invariants obtained with the help of homological tools. We are able to produce new invariants in the case of contact knot theory, and we also believe our work can be used to define an invariant for graphs which can be interpreted as a volume, in analogy with [20].

Let us explain in more detail what we do in this thesis.

1. **Introduction to topological knot theory.** We begin with an introduction to topological knots and to the study of classical invariants such as the fundamental group, and the Jones and Alexander polynomials. We have also included explicit examples of computations.

2. **Introduction to contact geometry and Legendrian knots.** We develop some aspects of the theory of contact manifolds and Legendrian knots. We also introduce the Bennequin

invariant and the rotation number, the so-called classical Legendrian invariants. As before, we include some non-trivial examples. In this part of the thesis we explain the limitations of these invariants for classification purposes, motivating the categorification carried out by Chekanov.

3. **Categorification.** In this chapter we explain two instances of categorification: the Khovanov homology for topological knots, and the Chekanov homology for Legendrian knots. We also present some non-trivial examples.

4. **Applications.** In this chapter we discuss several applications of the previous results. We study product structures (Massey products, some generalizations and A_∞-algebra structures) for Legendrian knots, and we construct integrable systems (commutative and non-commutative) using the Chekanov algebra; our construction (which contains in particular non-commutative version of the Korteweg-de Vries equation) is one of the few examples of non-commutative integrable systems (compare with [19]) obtained with the help of algebras naturally associated to geometric phenomena.

5. **Introduction to graphs and to the categorification of the chromatic polynomial.** We introduce graphs, mention their main properties, and present a construction of the chromatic polynomial which motivates its categorification via a construction which is analog to the Khovanov construction. We introduce the Reidemeister torsion for graphs and we define a volume form. Using the tools developed for this construction we also define a Laplace operator for graphs.

Part I

First Part: Knots

Chapter 1

Introduction to knot theory

1.1 Basic definitions

Following R. Fox (see for instance [6]), we understand the theory of knots as an instance of the location problem: Given spaces X, Y the aim is to classify and understand how X can be located within Y. To classify means that if X_1 and X_2 are located within Y, we say that they are equivalent if there exists a movement in Y which takes X_1 into X_2 (isotopy, for example) and we classify module this equivalence relation. If X is the circle S^1 and Y is the three-dimensional Euclidean space $\mathbb{R}^3$ then we have the classical theory of knots.

Definition 1. *The subset $K \subset \mathbb{R}^3$ is a knot if there is a homeomorphism of unit circle S^1 in $\mathbb{R}^3$ whose image is K.*

All knots are homomeorphic to S^1 and therefore they are homeomorphic to each other. We now present some examples of knots.

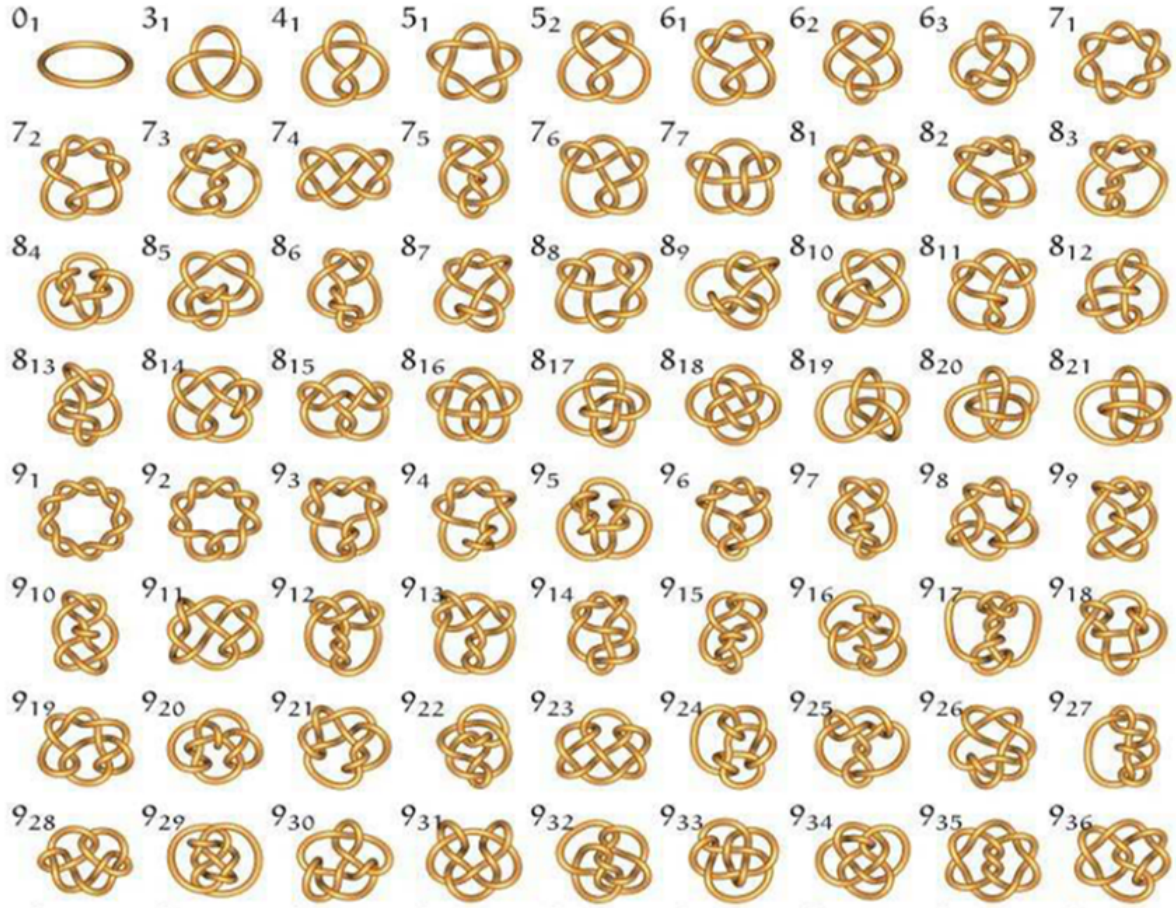


Figure 1.1: Examples of knots, see [6]

Regular Diagrams

A knot is usually specified by a projection; in fact all the examples we have presented in Figure 1 are projections of the corresponding knots. Consider the parallel projection given by $P : \mathbb{R}^3 \to \mathbb{R}^3$ where $P(x, y, z) = (x, y, 0)$. If K is a knot we say that $P(K) = K'$ is the projection of K. In addition, if K is assigned an orientation, K' inherits a natural orientation. However, K' Is not a simple closed curve in the plane, as it may have several points of intersection. A point p of K' is called a crossing if the inverse image $P^{-1}(p) \cap K$ contains more than one point of K. The order of $p \in K'$ is $\left|P^{-1} \cap K\right|$. Thus, a double point is a crossing point of order 2, a triple point is a crossing point of order 3, and so forth. The projection K'

We now present formal definitions after [6]:

Definition 2. *A polygonal knot is a knot formed by a finite union of line segments called edges, and whose endpoints are the vertices of the knot.*

Definition 3. *A polygonal knot K is in regular position if its projection P satisfies the following two conditions:*

(i) The only points of intersection of K are double points.

(ii) No double point of K is the image of a vertex, that is, the following situation cannot occur:

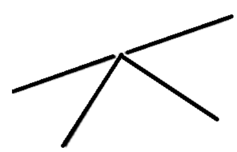


Figure 1.2: This situation cannot occur if K is in regular position.

The projection of a knot in regular position is said to be a regular projection. It is proven in [6] (Chapter I, Section 3) that any polygonal knot is equivalent (under a small rotation) to a polygonal knot in regular position. Now we note that at a double point of a projection it is not always clear whether the knot passes above or below itself; in order to remove this ambiguity we draw the projection near points of crossing using a continuous line for overcrossing and a discontinuous line for undercrossing. For example:

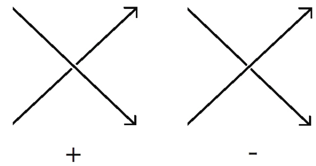
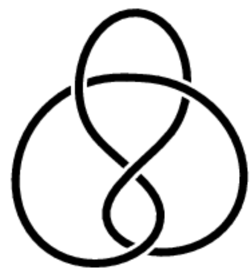
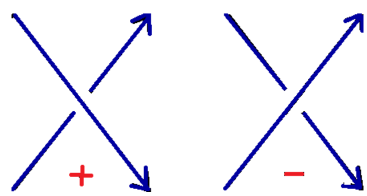


Figure 1.3: Regular position

Example 4. *In the first figure below we draw the figure eight knot. In the second figure we draw its projection. Note the discontinuous lines corresponding to undercrossings:*



Definition 5. *We say that a knot K is oriented when it is assigned a direction indicated by an arrow on its arcs. Oriented crossings are given signs of ± 1 as indicated below.*



We will define an equivalence relation among oriented knots. Before stating the formal definition, we recall that homeomorphisms h of $\mathbb{R}^3$ into itself preserve orientation or reverse orientation. The composition of homeomorphisms follow the following rules:

$$\begin{pmatrix} h_1 & h_2 & h_3 = h_1 \circ h_2 \\ preserves & preserves & preserves \\ inverts & preserves & inverts \\ preserves & inverts & inverts \\ inverts & inverts & preserves \end{pmatrix}$$

For example, the application identity preserves orientation and the application reflection, $R(x, y, z) = (x, y, -z)$, reverses orientation. More generally, if h is a linear transformation, h

preserves or reverses orientation depending whether its determinant is positive or negative; more generally, if h and its inverse are differentiable, then h preserves or inverts orientation depending on the sign of its Jacobian determinant.

Definition 6. *Two knots K_1 and K_2 are equivalent if there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ that preserves orientation such that $h(K_1) = K_2$. The equivalence of knots K is denoted by $K_1 \approxeq K_2$.*

We remark that our previous comments on homeomorphisms of $\mathbb{R}^3$ imply that indeed $\approxeq$ is an equivalence relation.

Tame and wild Knots

Definition 7. *A knot is tame if it is equivalent to a polygonal knot. Knots which are not tame are called wild knots.*

In [6] we can find a classical example of a wild knot (see Figure 1.5 below); in Figure 1.4 we present an example of a tame knot:

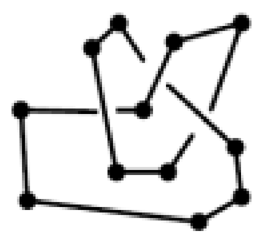


Figure 1.4: Tame knot

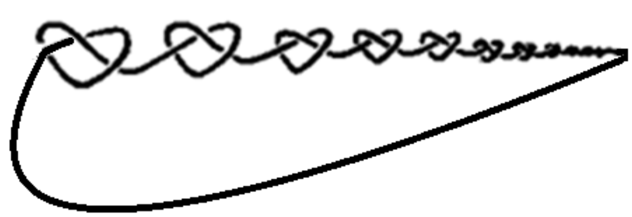


Figure 1.5: Wild knot

We mention a very important result due to Fox, see [6]: Every C^1 knot parameterized by arc length is tame. Hereafter we assume that all knots are tame.

Invertible Knots

Definition 8. *A knot K is invertible if there is an orientation-preserving homeomorphism of $\mathbb{R}^3$ into itself such that the restriction $h\,|_K$ is an orientation-reversing homeomorphism of K in itself.*

For example, the clover is invertible. We just have to turn it over as shown in the following figure:

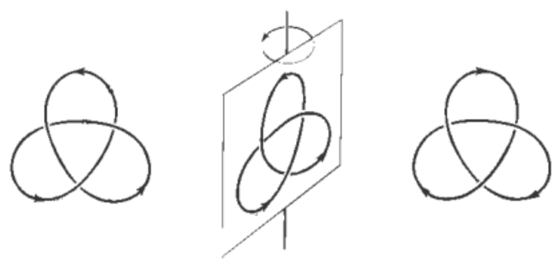


Figure 1.6: .

Definition 9. *A knot K is amphicheiral if there is an orientation-reversing homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) = K$.*

Definition 10. *The* mirror image *of a knot K is the image of K under the reflection R defined by $R(x, y, z) = (x, y, -z)$.*

Lemma 11. *A knot is amphicheiral if and only if there is an orientation preserving homeomorphism of $\mathbb{R}^3$ in $\mathbb{R}^3$ which sends K to its mirror image.*

Proof. If K is amphicheiral, there is an orientation-reversing homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) = K$. Let R be the reflexion defined above. Then, $h \circ R(K) = K = R(K)$, and therefore there is a homeomorphism $h \circ R$ that preserves orientation and sends K to its mirror image. The converse is similar. $\square$

Figure 1.7 shows that the clockwise and counterclockwise figure-eight knots are equivalent using orientation preserving movements. Hence, the figure-eight knot is amphicheiral.

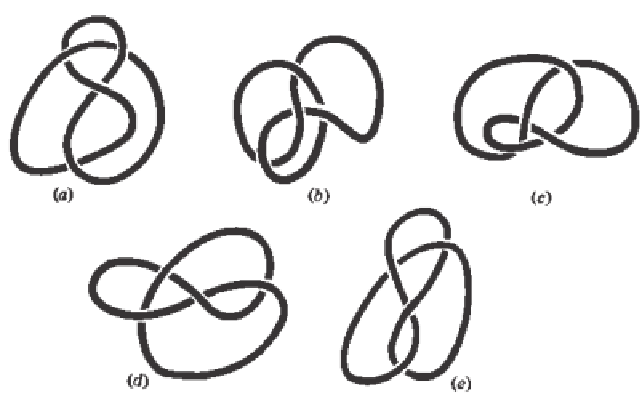


Figure 1.7: The figure-eight knot is amphicheiral.

1.2 Topological knot invariants

Topological knot invariants are invariant with respect to the more elaborate notion of the *isotopy type* of a knot, not simply equivalence. We recall that an isotopic deformation of a topological space X is a family of homeomorphisms h_t, $0 \leq t \leq 1$, of X onto itself such that h_0 is the identity, i.e., $h_0(p) = p$ for all $p \in X$, and the function H defined by $H(t,p) = h_t(p)$ is simultaneously continuous in t and p.

Definition 12. *The knots K_1 and K_2 are said to be of the same isotopy type if there exists an isotopic deformation $\{h_t\}$ of $\mathbb{R}^3$ such that $h_1(K_1) = K_2$.*

One of the most important questions in knot theory is to determine whether two knots are isotopic or not, and to classify knots under isotopy. K. Reidemeister, a German scientist, developed the first combinatorial approach to the theory of knots. Using this approach Reidemeister wrote the first book on knot theory in 1932, see [21]. Reidemeister's theorem says that to decide whether two knot diagrams represent isotopy-equivalent knots, it is sufficient to study their projections. More precisely, it is enough to consider the following *Reidemeister moves*:

Type I (This allows us to twist the section of the knot to produce a crossing, or remove one).

Type II This allows us to poke part of the knot under or over another piece of the knot (or unpoke a loop from under or over) to add (or remove) two crossings.

Type III (This allows us to move part of the knot from one side of a crossing to another).

The three types of movements are represented in Figure 1.8.

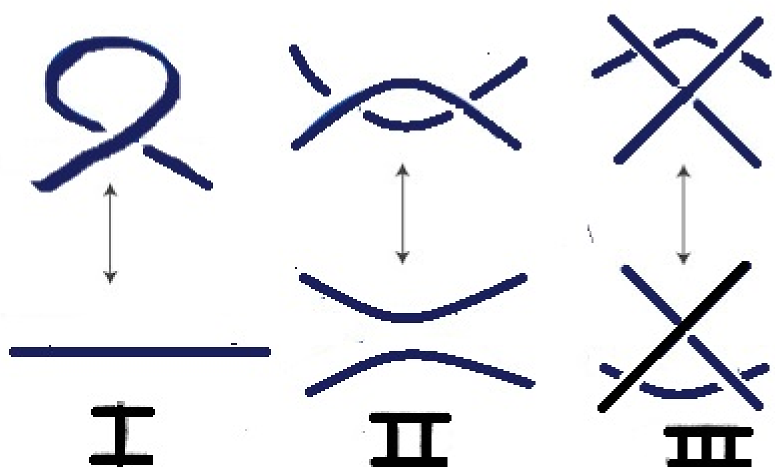


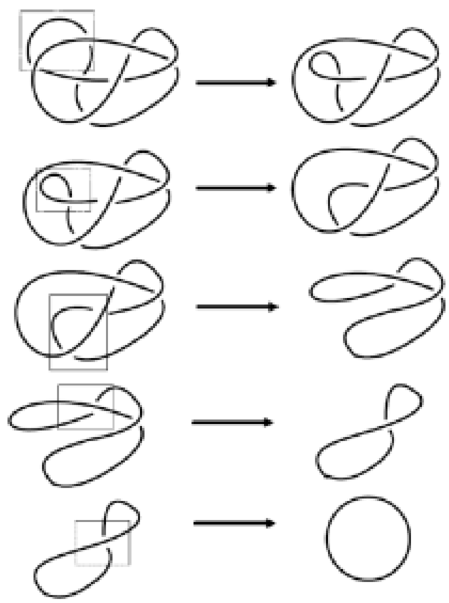
Figure 1.8: Reidemeister moves.

Theorem 13. *Two knots can be deformed into each other under ambient isotopy if and only if their diagrams can be transformed into one another by planar isotopy and the three Reidemeister moves.*

Proof. The first proof of this result is in K.Reidemeister's *Knotentheorie*, see [21]. A more standard reference is [6]. □

Applying these transformations to a knot we may reduce it to an isotopic knot:

Example 14. *We apply Reidemeister moves to a knot and we reduce it to the trivial knot;*



1.2.1 The fundamental group of a knot

One of the first invariants for general topological spaces is the fundamental group. The invariant in this case is a group. The construction of the fundamental group is as follows: Let X be a topological space, and consider the set Ω of all closed paths (or loops) that come from a fixed point $p \in X$ (called the base point). The set Ω can be divided into equivalence classes: two loops are equivalent if one can be deformed continuously into the other or, in other words, if there exists an isotopy sending one loop into the other. Two loops which are equivalent are said to be homotopic. For example, consider Figure 1.10:

The loops a_1 and a_2 are homotopic, and these in turn are homotopic to the loop that remains constant at p (the loop e). On the other hand, the loops a_3 and a_4 are also homotopic, but these

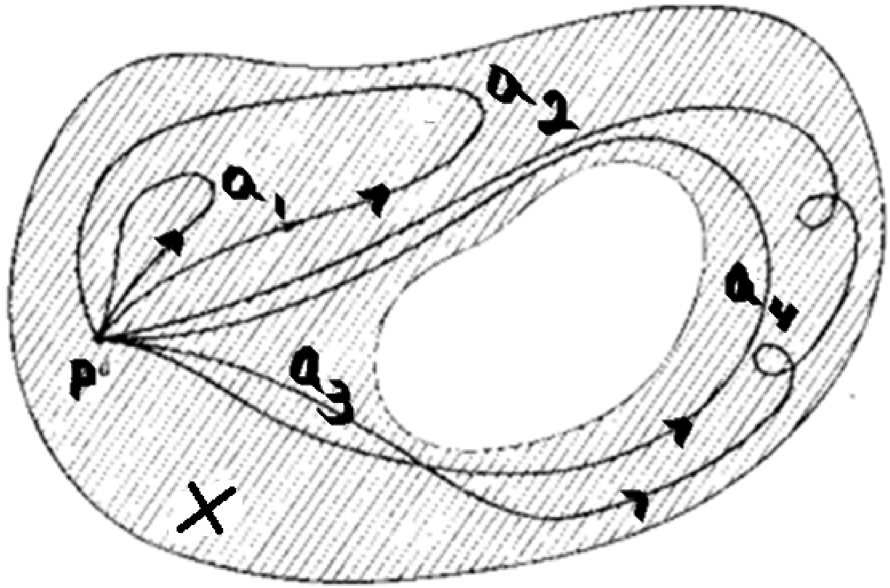


Figure 1.9: Construction of the fundamental group

two loops are not homotopic to e, because it is impossible to pass through the hole in a continuous fashion.

The class of loops which are homotopic to a particular loop α is represented by $[\alpha]$. If α is homotopic to β, the classes $[\alpha]$ and $[\beta]$ are identical. We define a multiplication between classes of loops as follows:

Take representatives of the classes $[\alpha]$ and $[\beta]$ which we want to multiply, say α and β respectively. Then, $[\alpha]$ $[\beta]$ is the class of the loop leaving p, running through α, returning to p, going to β and again travelling back to p. It can be shown that this operation is well defined, that is, it does not depend on the representatives chosen. Also, it can be shown that the set of equivalence classes with this multiplication has a group structure; the multiplication is associative, there is an identity (the class of the constant loop e), and for every element $[\alpha]$ there is an inverse $[\alpha]^{-1}$ such that $[\alpha][\alpha]^{-1} = [\alpha]^{-1}[\alpha] = [e]$.

We remark that the multiplication is not necessarily commutative: $[\alpha][\beta] \neq [\beta][\alpha]$ in general. The group of classes of homotopic loops of the space X with base point p is denoted by $\pi_1(X, p)$, and is called the fundamental group of X with base point p, see [6].

Definition 15. *If K is a knot and p is any point in $\mathbb{R}^3 \backslash K$, then the fundamental group*

$$\pi_1(\mathbb{R}^3 \backslash K, p),$$

for some $p \in \mathbb{R}^3 \backslash K$ is called the fundamental group (or knot group) of the knot K.

Sometimes reference to the base point p is omitted to ease notation. As all spaces considered are path-connected, different base points will give rise to isomorphic knot groups. Also, it is important to remark that two isotopic knots have isomorphic fundamental groups [6].

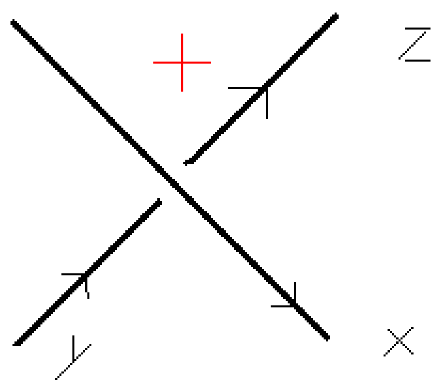
Theorem 16. *Let K_1 , K_2 be isotopic knots, then*

$$\pi_1(\mathbb{R}^3 \backslash K_1, p) \cong \pi_1(\mathbb{R}^3 \backslash K_2, p)$$

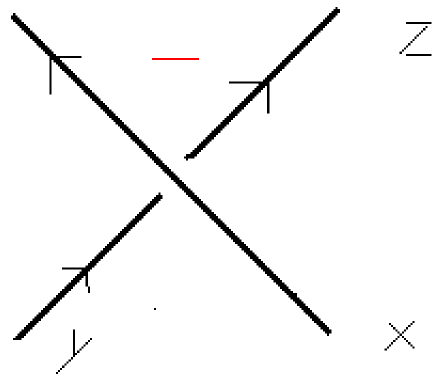
Proof. A proof appears in [6]. □

There exist some standard ways to calculate the fundamental group $\pi_1(\mathbb{R}^3 \backslash K, p)$, see for example [6] and [25]. We also cite a classical presentation due to Wirtinger:

Let D be an oriented diagram for K, and assume that it has n arcs. We label these arcs with symbols from a set X. For each of the n crossings, collect the rule $xy = zx$ if the crossing is positive and labelled as shown:



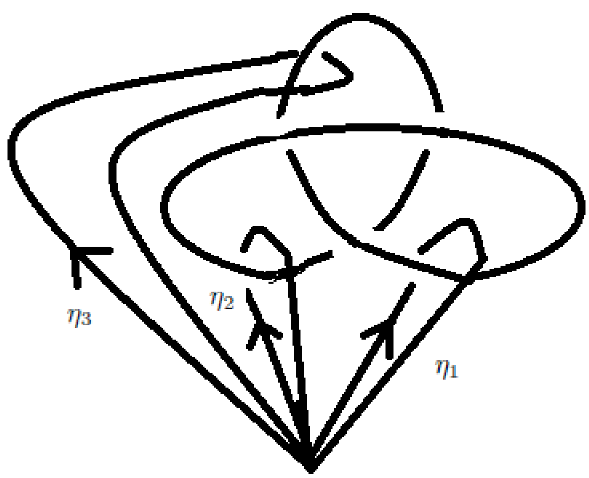
On the other hand, we collect the rule $xz = yx$ if the crossing is negative and labelled as shown:



The group on X defined by these rules is precisely $\pi_1(\mathbb{R}^3 \backslash K, p)$. A proof of this fact appears in [25].

We present two examples of calculations:

Example 17. *For the trefoil knot, we have:*



$$\eta_1 \eta_3 \eta_1^{-1} \eta_2 = 1$$
$$\eta_1^{-1} \eta_3 \eta_2 \eta_3^{-1} = 1$$
$$\eta_2^{-1} \eta_3^{-1} \eta_2 \eta_1 = 1$$

This is,

$\eta_1\eta_3\eta_1^{-1}\eta_2 = (\eta_1\eta_3)(\eta_2\eta_1)^{-1} = 1$......................................(1)
$\eta_1^{-1}\eta_3\eta_2\eta_3^{-1} = (\eta_3\eta_2)(\eta_1\eta_3)^{-1} = 1$.....................................(2)
$\eta_2^{-1}\eta_3^{-1}\eta_2\eta_1 = (\eta_2\eta_1)(\eta_3\eta_2)^{-1} = 1$.....................................(3)

From (1) and (2) we have :

$$(\eta_3\eta_2)(\eta_1\eta_3)^{-1}(\eta_1\eta_3)(\eta_2\eta_1)^{-1} = (\eta_3\eta_2)(\eta_2\eta_1)^{-1} = \left((\eta_2\eta_1)(\eta_3\eta_2)^{-1}\right)^{-1} = 1$$

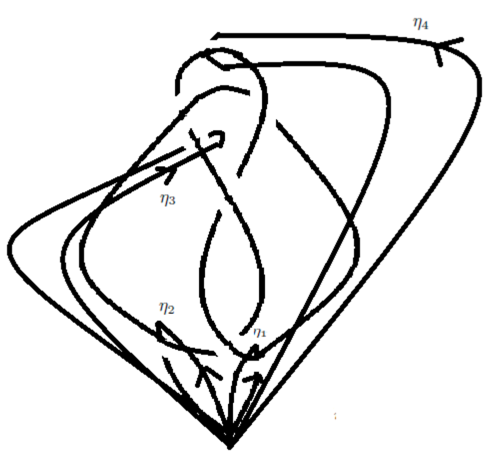
From (1) and (3) we have :

$$(\eta_1\eta_3)(\eta_2\eta_1)^{-1}(\eta_2\eta_1)(\eta_3\eta_2)^{-1} = (\eta_1\eta_3)(\eta_3\eta_2)^{-1} = \left((\eta_3\eta_2)(\eta_1\eta_3)^{-1}\right)^{-1} = 1,$$

Then, using (2) we have

$$\pi_1(\mathbb{R}^3 - K) = \langle \eta_1, \eta_2, \eta_3 / \eta_1\eta_3\eta_1^{-1}\eta_2 = \eta_2^{-1}\eta_3^{-1}\eta_2\eta_1 \rangle.$$

Example 18. *For the eight knot, we have:*



$$\eta_1\eta_3^{-1}\eta_1^{-1}\eta_2^{-1} = 1$$

$$\eta_3\eta_4\eta_3^{-1}\eta_1 = 1$$

$$\eta_1^{-1}\eta_4\eta_2\eta_4^{-1} = 1$$

$$\eta_2^{-1}\eta_4^{-1}\eta_2\eta_3 = 1$$

Then ;

$$\pi_1(\mathbb{R}^3 - K) = \langle \eta_1, \eta_2, \eta_3 / \eta_1\eta_3^{-1}\eta_1^{-1}\eta_2^{-1} = \eta_1^{-1}\eta_4\eta_2\eta_4^{-1} = \eta_2^{-1}\eta_4^{-1}\eta_2\eta_3 \rangle.$$

1.2.2 The Jones polynomial and the Kauffman Bracket

In 1984 V. Jones discovered an invariant which associates with each knot a Laurent polynomial (ie a polynomial that can have positive and negative powers). A simple way to define it is from another polynomial, the Kauffman bracket.

Definition 19. *The Kauffman bracket is a function of space-oriented diagrams to a Laurent polynomial ring with integer coefficients in a variable. This application assigns to each diagram D a polynomial $< D > \in \mathbb{Z}[A^{-1}, A]$, starting from the following rules:*

Let $\bigcirc$ be the trivial knot. Then,

1. $< D \cup \bigcirc >= (-A^{-2} - A^2) < D >$,

2. $< \bigcirc >= 1$,

3. $< D >= A < D_1 > +A^{-1} < D_2 >$. *This item is understood as follows. The sum of the polynomials $A < D_1 >$ and $A^{-1} < D_2 >$ replaces the polynomial $< D >$ after eliminating one crossing in the following way:*

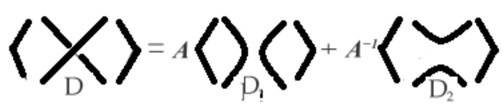
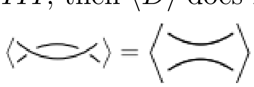
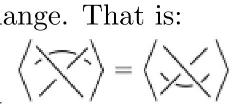


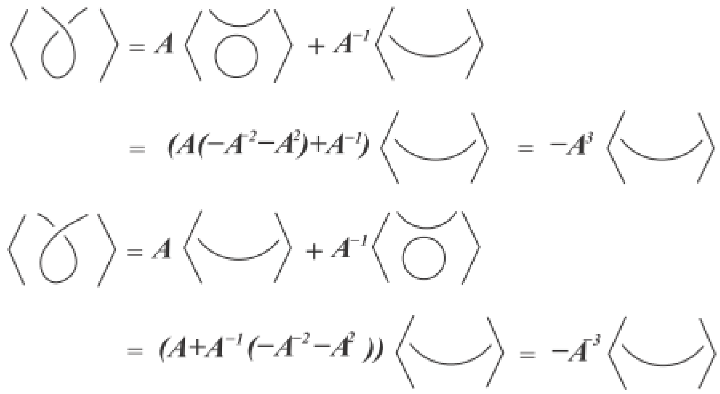
Figure 1.10: Eliminating one crossing

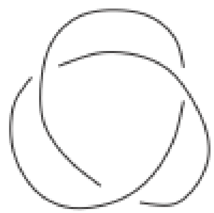
The Kauffman bracket is not a true invariant, as shown in the following lemma:

Lemma 20. *If a diagram is affected by a type I Reidemeister move, then its Kauffman bracket is also modified. In fact, we have:*

On the other hand, if a diagram D is changed by a movement of Reidemeister type II or type III, then $\langle D \rangle$ does not change. That is:

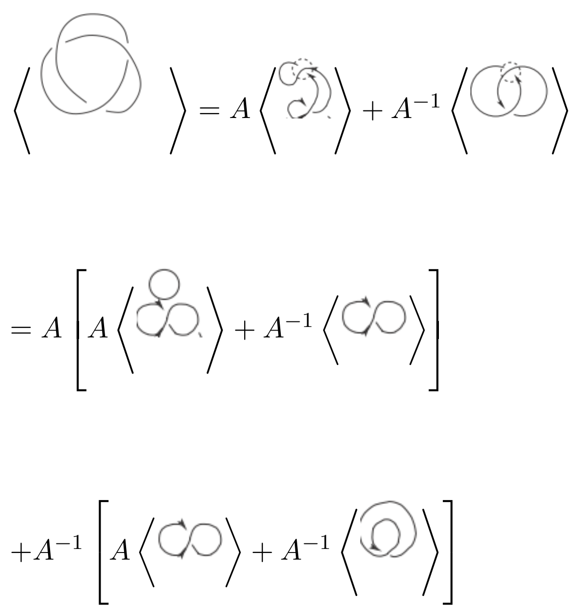
 and  .





Example 21. *Calculate the Kauffman bracket of the left-handed trefoil knot*

We apply the rules of Definition 19:



$$= A\left[A\left[A\left\langle \bigcirc\bigcirc\bigcirc \right\rangle + A^{-1}\left\langle \bigcirc\bigcirc \right\rangle\right] + A^{-1}\left[A\left\langle \bigcirc\bigcirc \right\rangle + A^{-1}\left\langle \bigcirc \right\rangle\right]\right]$$

$$+A^{-1}\left[A\left[A\left\langle \bigcirc\bigcirc \right\rangle + A^{-1}\left\langle \bigcirc \right\rangle\right] + A^{-1}\left[A\left\langle \bigcirc \right\rangle + A^{-1}\left\langle \circledcirc \right\rangle\right]\right]$$

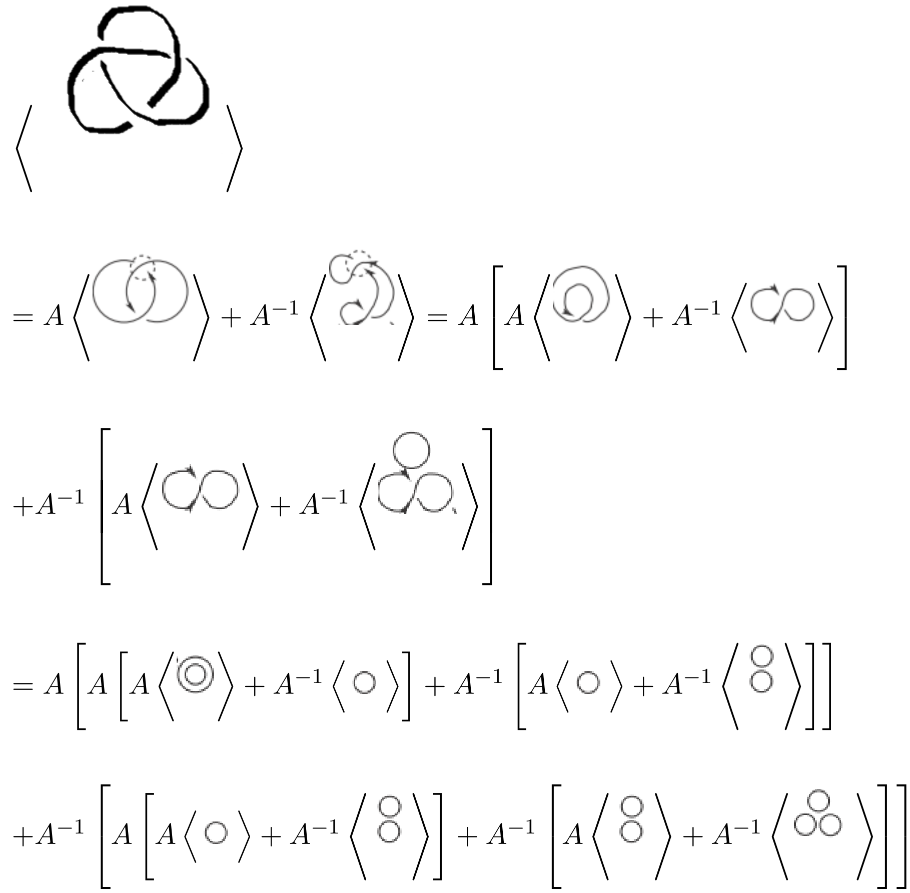
$$= A^3\left\langle \bigcirc\bigcirc\bigcirc \right\rangle + (3A + A^{-3})\left\langle \bigcirc\bigcirc \right\rangle + (3A^{-1})\left\langle \bigcirc \right\rangle$$

$$= (-A^{-2} - A^2)^2 A^3 + (-A^{-2} - A^2)(3A + A^{-3}) + 3A^{-1}$$

$$= A^7 - A^3 - A^{-5}\ .$$

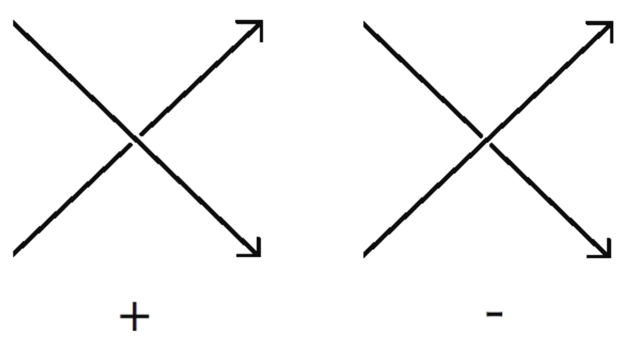
Example 22. *As a complement to the previous example, we compute the Kauffman bracket of the right-handed trefoil knot.*

We have:



$$= A^{-3}(-A^{-2} - A^{2})^{2} + (-A^{-2} - A^{2})(A^{3} + 3A^{-1}) + 3A = A^{-7} - A^{5} - A^{-3}\ .$$

Definition 23. *The knot writhe $W(D)$ of a diagram D is the sum of the signs of its crossings, bearing in mind that we represent the positive sign with $+1$ and the negative sign for -1, and that positive and negative crossings are defined as follows:*



Lemma 24. *The knot polynomial $f_D(A) = (-A)^{-3w(D)} < D >$ is an invariant for oriented knots.*

A proof of this result (and also the foregoing definitions) appears in Kauffman's paper [12].

For example, for the right-handed and left-handed trefoil knots we have the polynomials:

$$f(A) = (-A)^{-3w(\;)} \left\langle \quad \right\rangle = (-A^{-3(-3)})(A^7 - A^3 - A^{-5}) = -A^{16} + A^{12} + A^4$$

and

$$f(A) = (-A)^{-3w(\;)} \left\langle \quad \right\rangle = (-A^{-3(3)})(A^{-7} - A^5 - A^{-3}) = -A^{-16} + A^{-12} + A^{-4}$$

We can see that $f_D(A) \in Z\left[A^2, A^{-2}\right]$.

Definition 25. *The Jones polynomial of an oriented knot K is the Laurent polynomial $f_D(A)$ with $A = t^{\frac{-1}{4}}$ and with integer coefficients. [see L Kauffman [12]]. That is:*

$$V_K(t) = f_D(t^{\frac{-1}{4}}).$$

Example 26. :
$$\begin{aligned} V(t) = f(t^{\frac{-1}{4}}) &= -(t^{\frac{-1}{4}})^{16} + (t^{\frac{-1}{4}})^{12} + (t^{\frac{-1}{4}})^{4} \\ &= -t^{-4} + t^{-3} + t^{-1} \end{aligned}.$$

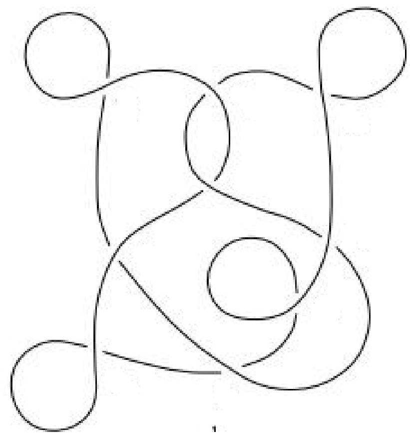
Example 27. *:*
$$\begin{aligned} V(t) = f(t^{\frac{-1}{4}}) &= -(t^{\frac{-1}{4}})^{-16} + (t^{\frac{-1}{4}})^{-12} + (t^{\frac{-1}{4}})^{-4} \\ &= -t^{4} + t^{3} + t \end{aligned}.$$

It follows from Lemma 24 and Definition 25 that the Jones polynomial is an invariant. We state this fact explicitly following [12]:

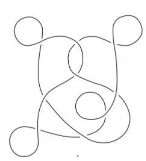
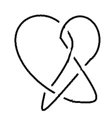
Theorem 28. *The polynomial $V_K(t)$ is an invariant of ambient isotopy.*

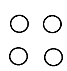
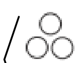
We also mention that there is another way of arriving at the Jones polynomial, discovered by Freyd, Yetter, Hoste, Lickorish, Millett and Ocneanu, see [9]. They constructed a polynomial in three variables (the *HOMFLY polynomial*) which reduces to the Jones polynomial for particular choices of these variables. Looking ahead to Chapter 3, we mention that the HOMFLY polynomial also admits a categorification, see for example the thesis [3] and [15], but we will not discuss it in this work.

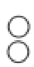
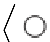
Example 29. *In this example we compute the Kauffman and Jones polynomials of the following knot:*



For the Kauffman polynomial we have,

$\Big\langle$  $\Big\rangle = (-A^{-3})^4 \Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle (A^{-7} + A^{-11}) + \Big\langle$  $\Big\rangle (5A^{-9} + 3A^{-13} + A^{-17})$

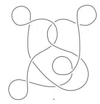
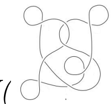
$+ \Big\langle$  $\Big\rangle (9A^{-11} + 5A^{-15}) + \Big\langle$  $\Big\rangle (7A^{-13})$

$$= (-A^{-2} - A^2)^3(A^{-7} + A^{-11}) + (-A^{-2} - A^2)^2(5A^{-9} + 3A^{-13} + A^{-17})$$

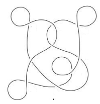
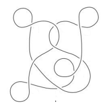
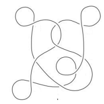
$$+(-A^{-2} - A^2)(9A^{-11} + 5A^{-15}) + (7A^{-13})$$

$$= -A^{-1} + A^{-5} - 2A^{-9} + A^{-13} - A^{-17} + A^{-21}$$

Now we compute the Jones polynomial using these calculations:

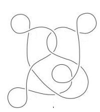
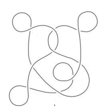
Writhe () = *W*() = *1*

and therefore we have:

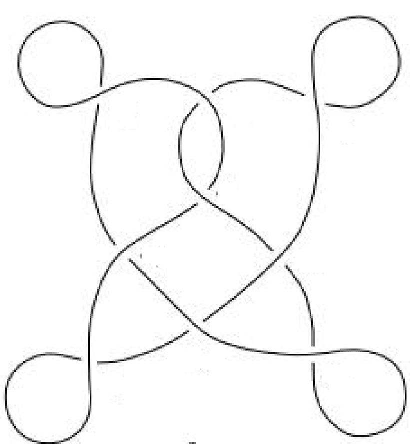
$f($  $)(A) = (-A)^{-3W(}$  $^{)}$ $\Big\langle$  $\Big\rangle$

$$= A^{-4} - A^{-8} + 2A^{-12} - A^{-16} + A^{-20} - A^{-24}\,,$$

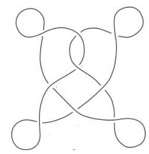
and so,

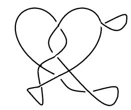
J $\Big\langle$  $\Big\rangle$ $(q) = f($  $)(q^{-1/2}) = q^2 - q^4 + 2q^6 - q^8 + q^{10} - q^{12}\,.$

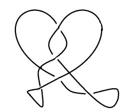
Example 30. *In this example we compute the Kauffman and Jones polynomials of the knot*

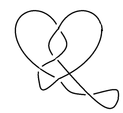


Our goal is to compare with the previous example. We have,

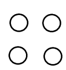
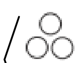
$\Big\langle$  $\Big\rangle$

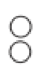
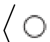
$= (-A^{-3}) \Big\langle$  $\Big\rangle$

$= (-A^{-3})^2 \Big\langle$  $\Big\rangle$

$= (-A^{-3})^3 \Big\langle$  $\Big\rangle$

$= (-A^{-3})^4 \Big\langle$  $\Big\rangle$

$= \Big\langle$  $\Big\rangle (A^{-7} + A^{-11}) + \Big\langle$  $\Big\rangle (5A^{-9} + 3A^{-13} + A^{-17})$

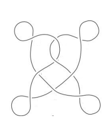
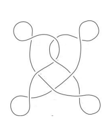
$+ \Big\langle$  $\Big\rangle (9A^{-11} + 5A^{-15}) + \Big\langle$  $\Big\rangle (7A^{-13})$

$$= (-A^{-2} - A^2)^3(A^{-7} + A^{-11}) + (-A^{-2} - A^2)^2(5A^{-9} + 3A^{-13} + A^{-17})$$

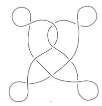
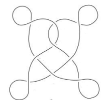
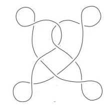
$$+(-A^{-2} - A^2)(9A^{-11} + 5A^{-15}) + (7A^{-13})$$

$$= -A^{-1} + A^{-5} - 2A^{-9} + A^{-13} - A^{-17} + A^{-21}$$

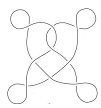
This is the Kauffman polynomial. Now we compute as before:

Writhe ( *)=W(*  *)=1*

and therefore we find, as in the previous example,

$f($  $) = (-A)^{-3W(}$  $^{)}\Big\langle$  $\Big\rangle$

$$= A^{-4} - A^{-8} + 2A^{-12} - A^{-16} + A^{-20} - A^{-24}$$

$J_{\Big\langle}$  $_{\Big\rangle}(q) = q^2 - q^4 + 2q^6 - q^8 + q^{10} - q^{12}$

These two examples show that two knots which look different, can have the same Jones polynomial. In fact, we can check using the Reidemeister moves that these two knots are equivalent, in the topological sense, and therefore they *must have* the same invariants. However, we will see in the next chapter that these two knots are not isotopic in the sense of contact geometry.

We note that there exist yet other invariants which we will not discuss. For example, the coloration number, the link number, etc. Information on them can be found in Kauffman's paper [12].

Chapter 2

Introduction to contact geometry and Legendrian knots

In this chapter we describe the basic concepts of contact manifolds and contact structure, and we present some examples. Then we introduce Legendrian knots, Legendrian Reidemeister moves, and their frontal and Lagrangian projections. Our goal is to prepare the way for the next chapter in which we study the Chekanov algebra for Legendrian knots.

2.1 Contact manifolds

Consider a $(2n+1)$-dimensional manifold M together with a differential 1-form α which satisfies the following condition:

$$\alpha \wedge (d\alpha)^n \neq 0 \, . \tag{2.1}$$

Such a form is called a contact form and the pair (M, α) is called a contact manifold.

We note that the condition (2.1) says that all contact manifolds are orientable. Now we recall that a *distribution* on M is an smooth assignment $q \to W_q$, in which for each $q \in M$, W_q is a vector subspace of T_qM. A special case of distribution is a sub-bundle $\varepsilon \subset TM$ of codimension 1 of TM, so that classically, ε is a "field of hyperplanes". This notion is connected with the theory of contact forms via the following lemma (see $[10, Lemma 1.1.1]$):

Lemma 31. *Let $\varepsilon \subset TM$ be a sub-bundle of codimension 1 of TM. We have,*

1. *There exists a differential 1-form α such that locally $\varepsilon = \ker \alpha$.*

2. *There exists a 1-form α such that $\varepsilon = \ker\alpha$ globally if and only if the quotient bundle TM/ε is trivial.*

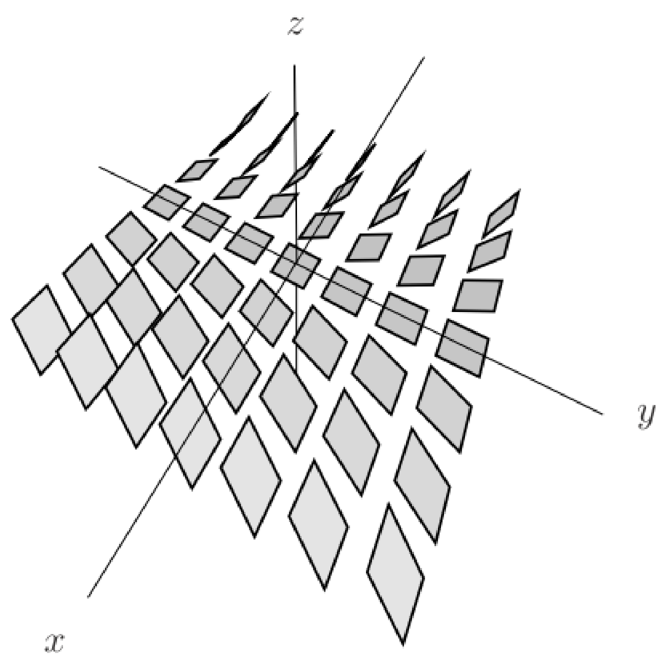
We recall that the quotient bundle TM/ε is defined as follows: for each $p \in M$, we consider the quotient vector space T_pM/ε_p, where the distribution $\varepsilon \subset TM$ is given by $p \mapsto \varepsilon_p \subset T_pM$. Then,

$$TM/\varepsilon = \bigcup_{p\in M} T_pM/\varepsilon_p\,.$$

We also recall that the bundle TM/ε is trivial if and only if it is isomorphic to a product bundle of the form $M \times W$ in which W is a one-dimensional vector space.

Motivated by this lemma, we also say that a distribution $\varepsilon = \ker\alpha \subset TM$ is called a contact structure if its dimension is $2n$ and its codimension 1 and it is maximally non-integrable[1]. The pair (M, ε) is called a contact manifold in a wide sense.

Example 32. *Let $M = \mathbb{R}^3$ with the 1-form $\alpha = dz + xdy$. Then, $d\alpha = dy \wedge dx$ and $\alpha \wedge (d\alpha) = dz \wedge dy \wedge dx$ which is the standard volume form of M. In this case we easily find that $\varepsilon = ker\alpha = \langle \partial_x, \partial_y - x\,\partial_z\rangle$. The contact structure looks as follows (Figure taken from [10]):*



More generally, in $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_1, y_1, ..., x_n, y_n, z)$ we have that $\alpha = dz + \sum_{j=i}^{n} x_j dy_j$ is a contact 1-form called the standard contact structure of $\mathbb{R}^{2n+1}$.

[1] We recall that a distribution $\varepsilon = \{\varepsilon_q\}_{q\in M}$ on M is (maximally) integrable if for each $q \in M$ there exists a submanifold N of M with $q \in N$ such that $T_qN = \varepsilon_q$. A distribution ε is *involutive* if for each pair of vector fields X, Y in ε we have $[X, Y] \in \varepsilon$. It is a classical result that ε is involutive if and only if it is integrable.

Example 33. *We consider* $\mathbb{R}^3$ *with cylindrical coordinates* (r, φ, z)*, and we define the 1-form* $\alpha_{ot} = \cos r\, dz + r \sin r\, d\varphi$*. In this case we find*

$$\alpha_{ot} \wedge d\,\alpha_{ot} = (r + \cos r\, \sin r) dz \wedge dr \wedge d\varphi = \left(1 + \frac{\sin r}{r} \cos r\right) r dr \wedge d\varphi \wedge dz\ .$$

Since $r dr \wedge d\varphi \wedge dz$ *is simply the standard volume form of* $\mathbb{R}^3$ *in cylindrical coordinates and the coefficient* $1 + \frac{\sin r}{r} \cos r$ *is a non-zero smooth function on* $\mathbb{R}^+$*, we conclude that* α_{ot} *is a contact structure of* $\mathbb{R}^3$*. This contact structure is called the "overtwisted contact strucure" of* $\mathbb{R}^3$*. It is*

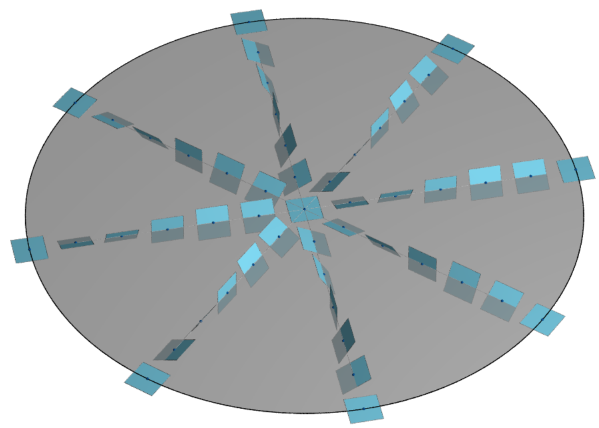


Figure 2.1: The overtwisted distribution $\xi_{ot} = \langle \partial_r, r \sin r \partial_z - \cos r \partial_\varphi \rangle$ (see Geiges, [10]).

known [see [10] p. 53] that this contact structure is not equivalent to the standard contact structure of $\mathbb{R}^3$ *considered in the previous example (i.e. there is no diffeomorphism* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *satisfying* $f^*\alpha_{ot} = \alpha$*).*

Definition 34. *Given a contact form* α*, there exists a unique vector field* R_α *defined uniquely by the following conditions:*

a) $d\alpha(R_\alpha, \cdot) = 0$

b) $\alpha(R_\alpha) = 1$. *This vector field is called the Reeb vector field of the contact structure.*

Since $d\alpha$ has rank $2n$, then $ker\, d\alpha$ has dimension 1, and so $ker\, d\alpha = \langle X \rangle$. Now, $d\alpha(R_\alpha, \cdot) = 0$, and it follows that $R_\alpha = fX$ but, since $\alpha(R_\alpha) = 1$, we have $f = 1$. Normalizing, we can write $R_\alpha = X = \partial_t$.

Lemma 35. *If* (M, α) *is a contact manifold, then so is* $(M, \lambda\alpha)$*, where* $\lambda : M \to \mathbb{R} - \{o\}$.

Proof. It is enough to prove that $\lambda\alpha \wedge (d\lambda\alpha)^n \neq 0$. Indeed:

$$\lambda\alpha \wedge (d\lambda\alpha)^n = \lambda\alpha \wedge [(d\lambda)\alpha + \lambda\,(d\alpha)]^n$$

$$= \lambda\alpha \wedge [(d\lambda) \wedge \alpha + \lambda \wedge (d\alpha)] \wedge [(d\lambda) \wedge \alpha + \lambda \wedge (d\alpha)]^{n-1}$$

$$= \lambda^2\alpha \wedge d\alpha \wedge [(d\lambda)\alpha + \lambda\,(d\alpha)] \wedge [(d\lambda) \wedge \alpha + \lambda \wedge (d\alpha)]^{n-2}$$

$$= \lambda^3\alpha \wedge (d\alpha)^2 \wedge [(d\lambda) \wedge \alpha + \lambda \wedge (d\alpha)]^{n-2}$$

$$= \lambda^{n+1}\alpha \wedge (d\alpha)^n$$

$$\neq 0.$$

Thus,$(M, \lambda\alpha)$ is a contact manifold. ☐

Lemma 36. *If M is a 3-dimensional manifold and α is a 1-form on M, the contact condition $\alpha \wedge (d\alpha)^n \neq 0$ is equivalent to $[x, y]_p \notin \varepsilon_p$ for each $p \in M$ and $x, y \in \varepsilon_p = \ker \alpha_p$.*

Proof. We compute:

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

$$= 0 - 0 - \alpha([X, Y])$$

$$= -\alpha([X, Y]) \neq 0.$$

Thus, the condition $\alpha \wedge (d\alpha)^n \neq 0$ is equivalent to $\alpha([X, Y]) \neq 0$ and the result follows. ☐

Example 37. *In the case of $\mathbb{R}^3$ equipped with the standard contact structure, we have*

$$\ker(dz + xdy) = \langle \partial_x, \partial_y - x\,\partial_z \rangle\ .$$

The above lemma implies that this form really determines a contact structure, for

$$[\partial_x, \partial_y - x\,\partial_z] = \partial_x(\partial_y - x\,\partial_z) - (\partial_y - x\,\partial_z)\partial_x = -\partial_z \notin \langle \partial_x, \partial_y - x\,\partial_z \rangle.$$

2.2 Legendrian knots

Definition 38. *A Legendrian knot in a three-dimensional contact manifold* (M, ε)*, is a embedded circle* $L \subset M$ *which is always tangent to the distribution* ε*. In other words, a Legendrian knot is a compact one-dimensional integral submanifold of the distribution* ε*.*

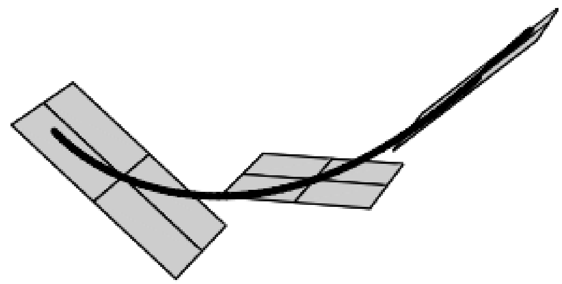


Figure 2.2: An example of a one-dimensional integral curve.

Legendrian knots always exist. In fact (see [10]) given an arbitrary knot $f : S^1 \to M$, there exists a Legendrian knot in M which is isotopic (in the topological sense) to f. Moreover, this Legendrian knot can be chosen so that it C^0-approximates the original knot.

Front and Lagrangian projections

Hereafter, ε represents the standard contact structure of $\mathbb{R}^3$.

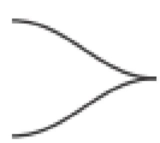
Definition 39. *Consider an embedding* γ *of* S^1 *into* $(\mathbb{R}^3, \varepsilon)$*, given by* $\gamma(s) = (x(s), y(s), z(s))$*.*

(*a*) *The front projection of a parametrized curve* $\gamma(s) = (x(s), y(s), z(s))$ *in* $(\mathbb{R}^3, \varepsilon)$*, is the curve* $\gamma_F(s) = (x(s), z(s))$*.*

(*b*) *The Lagrangian projection of a parametrized curve* $\gamma(s) = (x(s), y(s), z(s))$ *in* $(\mathbb{R}^3, \varepsilon)$ *is the curve* $\gamma_L(s) = (x(s), y(s))$*.*

Examples of front and Lagrangian projections appear in Figure 2.3 below.

Definition 40. *We will call a cusp of a Legendrian knot, a point of the form below appearing in the frontal diagram of the knot:*



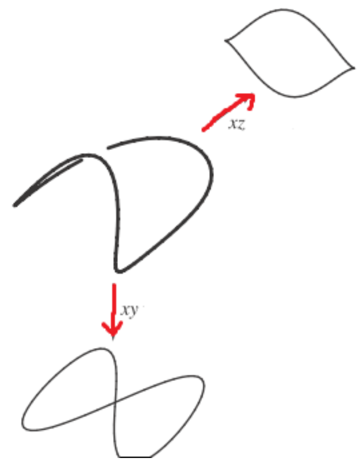


Figure 2.3: The xz projection is the *frontal* projection; the xy projection is the *Lagrangian* projection.

Example 41. *Consider the trefoil knot whose parametrization is given by:*

$$((2 + \sin 3t)\cos 2t, (2 + \sin 3t)\sin 2t, \cos 3t) \ .$$

This is a topological knot, which can be approximated by a Legendrian trefoil knot. Its Lagrangian and frontal projections are given "approximately" by

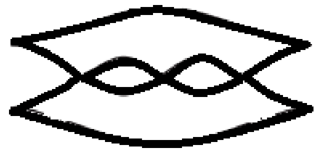
$$\gamma_L(t) = ((2 + \sin 3t)\cos 2t, (2 + \sin 3t)\sin 2t)$$

and

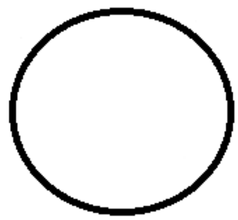
$$\gamma_F(t) = ((2 + \sin 3t)\cos 2t, \cos 3t) \ .$$

Graphically we have, for instance that

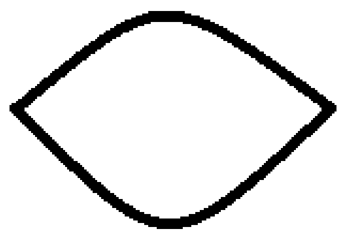
has frontal projection



Example 42. *As a further example, we consider the Legendrian unknot*



It has frontal projection



Two Legendrian knots K_0 and K_1 are equivalent if there is a *Legendrian isotopy* between them, that is, there exists a smooth family of Legendrian knots L_t, $t \,\epsilon[0,1]$, with $L_i = K_i$, para $i = 0, 1$.

The frontal diagram of a Legendrian knot can be distorted into the frontal diagram of an equivalent Legendrian knot using the following moves called *Legendrian Reidemeister moves*:

Theorem 43. *Two front diagrams represent Legendrian-isotopic Legendrian knots if and only if they are related by a regular homotopy and a sequence of moves shown in Figure 2.4.*

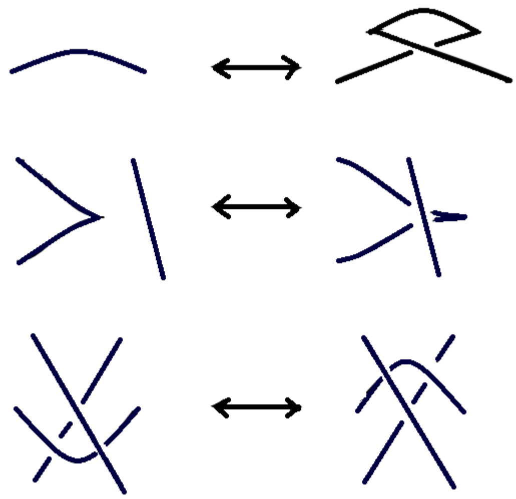


Figure 2.4: Legendrian Reidemeister moves for frontal projections.

Proof. The proof is in "Legendrian and transversal knots" by John Etnyre, see [8]. □

It is possible to pass from a Lagrangian projection to a frontal projection using moves called *resolutions*, as shown in Figure 2.5:

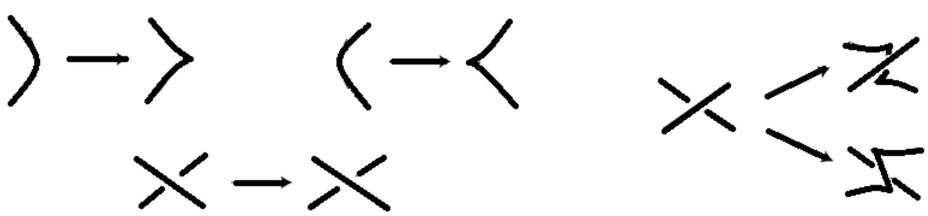


Figure 2.5: Realizing a knot type as a Legendrian knot in frontal projection.

Theorem 44. *Two Lagrangian diagrams represent Legendrian isotopic knots only if they are related by a sequence of moves shown in Figure 2.6*

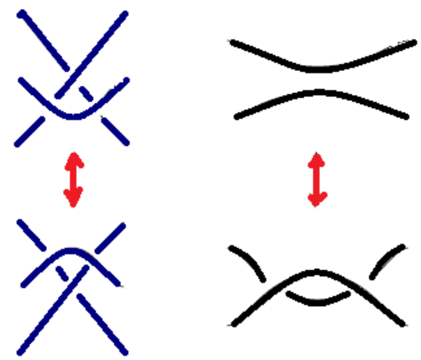


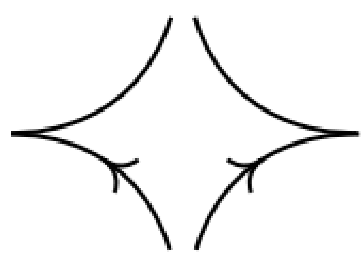
Figure 2.6: Legendrian Reidemeister moves in Lagrangian projection.

Proof. See [8]. Note that the theorem *is not* an equivalence as in the front projection case. □

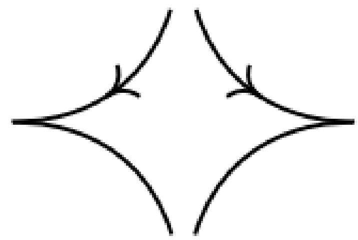
2.3 Classical invariants of Legendrian knots

We consider two invariants for Legendrian knots called *the classical invariants*: the Bennequin-Thurston and Maslov invariants. In order to define them we use frontal diagrams (i.e., frontal projections).

We will call *positive cusp* an oriented diagram of the form:



We will call *negative cusp* an oriented diagram of the form:



The Thurston-Bennequin invariant is characterized combinatorially by the following theorem (see [10]):

Theorem 45. *Let K be a Legendrian knot in $(\mathbb{R}^3, \varepsilon)$. Write K_F for the knot diagram of K obtained by the front projection. Then the Thurston–Bennequin invariant of K is given by:*

$$\begin{aligned} tb(K) &= \frac{1}{2}(\textit{number of cusps of } K_F + \textit{number of positive cusps} - \textit{number of negative cusps}) \\ &= W(K_F) - \frac{1}{2}\sharp(\,cusps(K_F)\,)\ . \end{aligned}$$

We can also characterize combinatorially the rotation number and the Maslov number of a Legendrian knot following [10]:

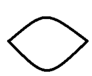
Theorem 46. *Let K be a Legendrian knot in $(\mathbb{R}^3, \varepsilon)$. Let K_F of the knot diagram of K obtained by the front projection. Then the rotation number of K is given by:*

$$Rot(K) = \frac{1}{2}(\textit{number of positive cusps} - \textit{number of negative cusps})\ ,$$

and the Maslov number of K is $Maslov(K) = 2\,Rot(K)$.

Let us consider two easy examples.

Example 47. *Trivial knot:*

$tb($  $) = 0 - \frac{1}{2}(1-1) = -1$

Example 48. *Trefoil Knot (see Example 41):*

$tb($  $) = -3 - \frac{1}{2}(2-4) = -2.$

It is natural to ask if the invariants we just reviewed are enough to characterize Legendrian knots. We now quote the first theorem on this topic, see [10] and [7].

Theorem 49. *(Eliashberg-Fraser) Let K_1 and K_2 be two topologically trivial Legendrian knots. Then they are Legendrian isotopic if and only if*

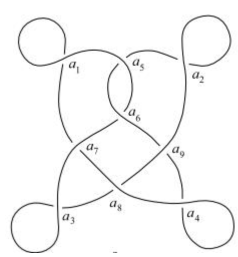
$$tb(K_1) = tb(K_2)$$

and

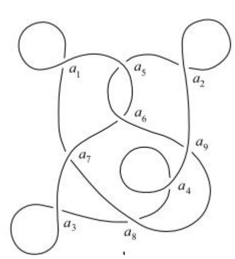
$$rot(K_1) = rot(K_2)\ .$$

Remark 50. *A topologically trivial Legendrian knot is a Legendrian knot bounding an embedded 2-disk. We also remark that the Eliashberg-Fraser theorem cannot be extended to topologically non-trivial Legendrian knots in full generality. In fact, we will see that there exist Legendrian knots for which $tb(K_1) = tb(K_2)$ and $rot(K_1) = rot(K_2)$, but they are not Legendrian isotopic.*

Example 51. *We consider the following pair of Chekanov knots, L_1 and L_2 respectively, as an specific example of the limitation of the classical invariants:*



and



A short computation yields:

$$tb(L_1) = 1 \ , \qquad rot(L_1) = 0$$

and

$$tb(L_2) = 1 \ , \qquad rot(L_2) = 0 \ ,$$

that is, $tb(L_1) = tb(L_2) = 1$ *and* $rot(L_1) = rot(L_2) = 0$*.*

As we remarked in the previous chapter, these two knots are isotopic as topological knots; in fact, it is not difficult to see this using (topological, not contact) Reidemeister moves. However, Chekanov [4] proved that these two knots are not Legendrian isotopic. This example is the original motivation for categorification via Chekanov homology which we will recall in the following chapter.

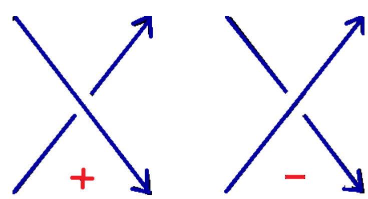
Chapter 3

Categorification

We divide this chapter into two parts, one corresponding to topological knots and the other corresponding to Legendrian knots.

3.1 The Khovanov homology

In this part we look at the Khovanov complex for a diagram of an oriented topological knot. Consider an oriented diagram D with n-crossings, denote by n_- the number of negative crossings, and by n_+ the number of positive crossings. We have the following figure:



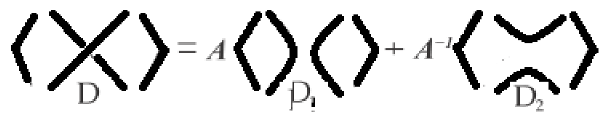
In order to explain the categorification of the Jones polynomial, we recall that the Kauffman bracket is not an invariant, as explained in Chapter 1, but we can define the polynomial

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle \ .$$

The *normalized Jones polynomial* is a topological isotopy invariant given by

$$J(D) = \frac{\hat{J}(D)}{q + q^{-1}} \ .$$

Consider the following property of the Kauffman bracket.



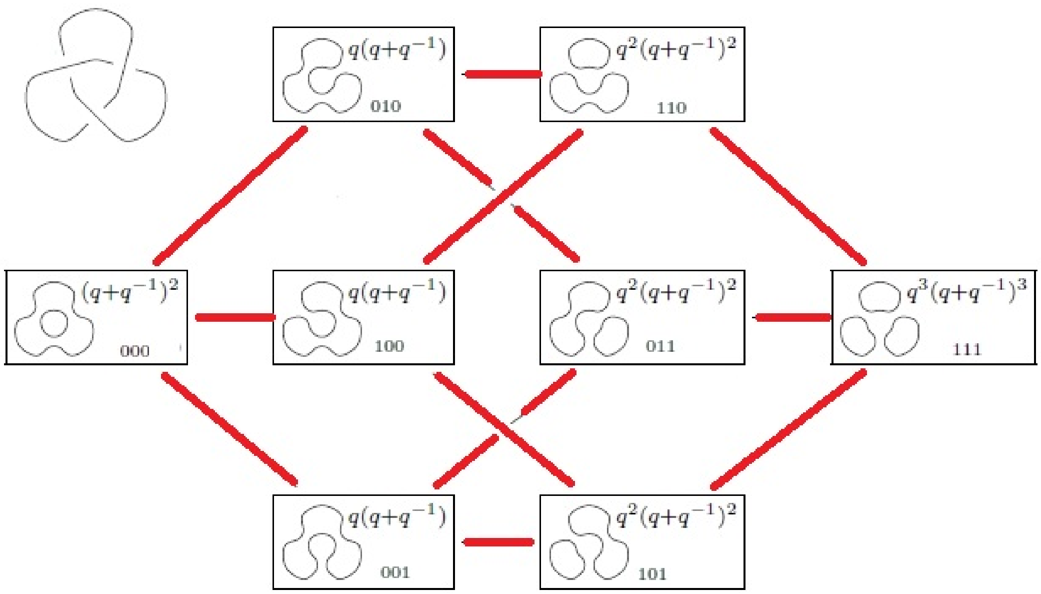
This property allows us to define 0- and 1-smoothings: The diagram D_1 is called a 1-smoothing, and the diagram D_2 is called a 0-smoothing. It follows that for a diagram with n crossings there are 2^n (0- or 1-) smoothings. Consider $\{0,1\}^n$ the set of words built using zeros and ones. We assume that each word represents the position of a vertex, thus forming a hypercube whose vertices υ belong to $\{0,1\}^n$. The way each vertex is labelled depends on the way each crossing of the given diagram is replaced by a 1- or 0-smoothing.

If we apply smoothings, we will "divide" the diagram into a collection of circles. Each smoothing determines a word in $\{0,1\}^n$ and therefore a vertex υ. Denote by Γ_υ this collection of circles in the plane. The number of ones in the vertex υ, we call r_υ, and the number of circles in Γ_υ will be denoted by k_υ. These data allows us to compute the Jones polynomial as follows:

$$\hat{J}(D) = \sum_{\upsilon \in \{0,1\}^n} \left\{(-1)^{n_-} q^{n_+ - 2n_-}\right\} \left\{(-1)^{r_\upsilon} q^{r_\upsilon} \left(q + q^{-1}\right)^{k_\upsilon}\right\} \tag{3.1}$$

We note that the first factor in each of the summands depends only on the original diagram, while the second factor depends on the vertex υ we are considering.

Example 52.



In this diagram we start by applying 0*-smoothings in all the crossings to the diagram of a trefoil knot. We obtain the vertex represented by the word* 000*. We write in each square the second factor of the formula above for* $\hat{J}(D)$ *corresponding to the vertex being considered. Now, starting from the vertex* 000 *we apply* 1*-smoothings obtaining all possible words in* $\{0,1\}^3$*.*

Now we add all possible polynomials forming the "state sum"

$$\begin{aligned}\Theta &= \left(q+q^{-1}\right)^2 - 3q\left(q+q^{-1}\right) + 3q^2\left(q+q^{-1}\right)^2 - q^3\left(q+q^{-1}\right)^3 \\ &= \left(q+q^{-1}\right)\left[q^{-1}+q^3-q^5\right]\end{aligned}$$

Now we compute $\hat{J}(D)$*:*

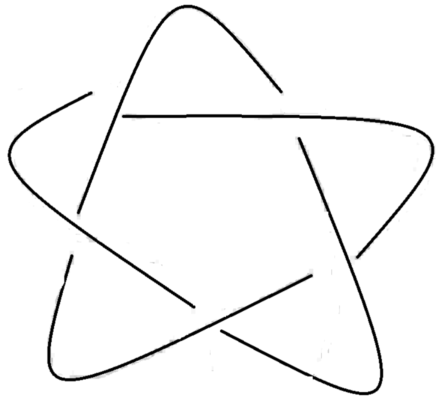
$\hat{J}($  $)$

$$\begin{aligned} &= (-1)^{n_-}q^{n_+-2n_-}\Theta \\ &= (-1)^{n_-}q^{n_+-2n_-}\left(q+q^{-1}\right)\left[q^{-1}+q^3-q^5\right] \\ &= (-1)^0 q^3\left(q+q^{-1}\right)\left[q^{-1}+q^3-q^5\right] \\ &= \left(q+q^{-1}\right)\left[q^2+q^6-q^8\right] \ .\end{aligned}$$

Therefore the Jones polynomial for the trefoil knot is given by

$J($  $) = \left[q^2+q^6-q^8\right] \ .$

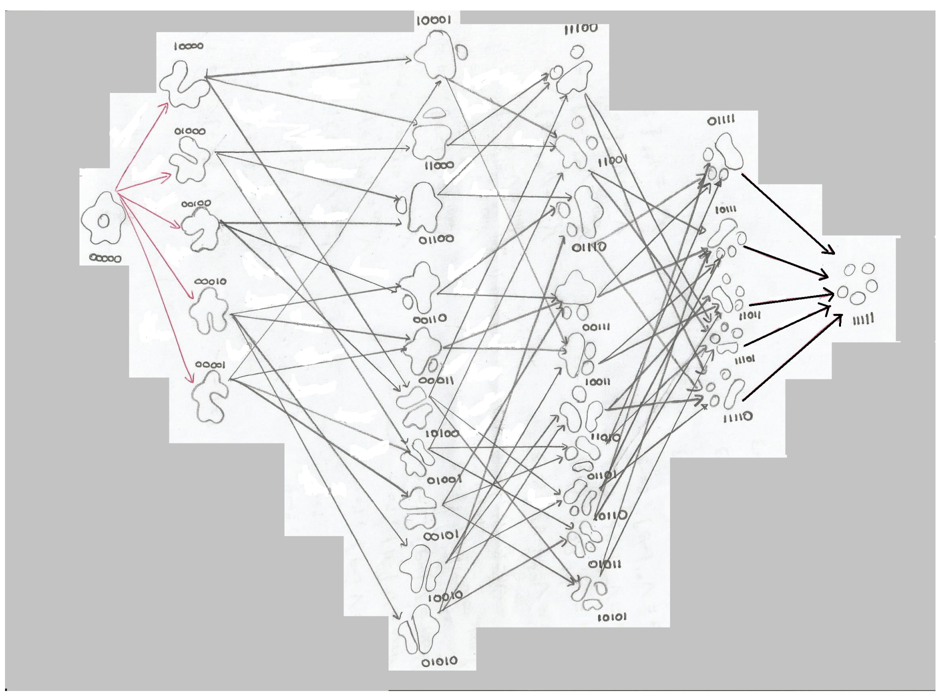
Example 53. *Consider*



We can check that:

$$\hat{J}(K) = -q^{-10} - q^{-12} - q^{-8} + 1 \ .$$

In fact, we apply smoothings and we obtain the figure:



Thus:

$$\begin{aligned}\hat{J}(K) = -q^{-2n}\,\big[&(q+q^{-1})^2 - 5q(q+q^{-1}) + 10q^2(q+q^{-1})^2 - 10q^3(q+q^{-1})^3 + \\ &5q^4(q+q^{-1})^4 - q^5(q+q^{-1})^5\big] \\ = -q^{-10} &- q^{-12} - q^{-8} + 1\ .\end{aligned}$$

Now we will see some algebraic concepts which we need to form the Khovanov complex:

Definition 54. *Let V be a vector space over $\mathbb{R}$. The tensor product of V with itself, $V \otimes V$, is the vector space over $\mathbb{R}$ generated by all elements $a \otimes b$ $(a, b \in V)$ satisfying the following conditions:*

1. $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
2. $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$
3. $(na) \otimes b = n(a \otimes b) = a \otimes (nb)$

for $n \in \mathbb{R}$ y a, a_1, a_2, b, b_1, $b_2 \in V$.

In our application to the Khovanov complex we use the vector space V generated by two different elements v_+, v_-. The definition of tensor product shows that in this case a basis for $V \otimes V$ is $\{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$. It can be shown that $(V \otimes V) \otimes V \cong V \otimes (V \otimes V)$ and that $\mathbb{R} \otimes V \cong V$. So, without ambiguity, we can write $V^{\otimes 3} = V \otimes V \otimes V$, or more generally ,$V^i = V \otimes \cdots \otimes V$, the tensor product of i factors. A basis for V^i is $\{v_1 \otimes \cdots \otimes v_i |\ v_k \in \{v_+, v_-\}$.

For convenience we eliminate "$\otimes$ " and write $v_+ \otimes v_-$ as $v_+ v_-$.

Our aim is to assign an $\mathbb{Z}$-graded vector space to each vertex of the hypercube and to define a notion of graded dimension, in such a way that the factors $q^{r_v}\,(q+q^{-1})^{k_v}$ that appear in each rectangle (corresponding to the vertex v in the example above) is the graded dimension of this $\mathbb{Z}$-graded vector space.

Definition 55. *A graduation on a vector space W is a choice of decomposition of W into a direct sum of subspaces W_m, that is,*

$$W = \bigoplus_m W_m$$

Definition 56. *Let $W = \bigoplus_m W_m$ a $\mathbb{Z}$-graded vector space where W_m denotes the subspace of homogeneous elements with degree m. The graded dimension of W is the power series*

$$q\,dimW := \sum_m q^m\,dim(W_j)\ .$$

Now we go back to the vector space V generated by two basis elements v_+, v_-. We graduate V by setting $degree(v_\pm) = \pm 1$. Also, we equip V with co-multiplication $\Delta : V \to V \otimes V$ and multiplication $m : V \otimes V \to V$ defined by

$$\Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+\ ; \qquad \Delta(v_-) = v_- \otimes v_-\ ;$$

and

$$m(v_+ \otimes v_+) = v_+\ ; \qquad m(v_\pm \otimes v_\mp) = v_-\ ; \qquad m(v_- \otimes v_-) = 0\ .$$

Moreover, we can define the co-unit $\epsilon \in V^*$ by means of $\epsilon(v_+) = 0$ and $\epsilon(v_-) = 1$, and we assign degrees to ϵ, m y Δ as follows:

$$\deg(\varepsilon) = 1 \ \text{ y } \ \deg(m) = \deg(\Delta) = -1\ .$$

These definitions transform V into a commutative Frobenius algebra, as observed by Khovanov in [14].

Now we consider graduations of tensor products. Set $W_1 := Span\{v_+\}$, and let $W_{-1} := Span\{v_-\}$, then

$$V = W_1 \oplus W_{-1}$$

On the other hand, $V \otimes V$ has base elements $\{v_+v_+, v_+v_-, v_-v_+, v_-v_-\}$. Their degrees are $\{2, 0, 0, -2\}$ respectively. We chose a graduation as follows:

$$W_2 := Span\{v_+v_+\}\ , \qquad W_0 := Span\{v_+v_-, v_-v_+\}\ , \qquad W_{-2} := Span\{v_-v_-\}\ .$$

Then,

$$V \otimes V = W_2 \oplus W_0 \oplus W_{-2}\ .$$

Example 57. *The graded dimensions of V and $V \otimes V$ are:*

$$q\dim V = q^1 dimW_1 + q^{-1} dimW_{-1} = (q + q^{-1})$$

and

$$q\dim(V\otimes V) = q^2 dimW_2 + q^0 dimW_0 + q^{-2} dimW_{-2} = q^2 + 2 + q^{-2} = (q+q^{-1})^2 \ .$$

Now we can construct the Khovanov complex $C^{*,*}(D)$ associated to an oriented diagram D. We begin with the following definition:

Definition 58. *For a graded vector space W and an integer l, we define a new graduate vector space: $W\{l\}^m = W^{m-l}$. We can see that $q\dim(W\{l\}) = q^l\, q\dim(W)$.*

Here and below we use the notation introduced in our discussion of the Jones polynomial as a state sum (see Equation 3.1).

We recall that a diagram with n crossings has 2^n vertices. For each word $\upsilon \in \{0,1\}^n$ we define the associated graduate vector space V_υ as

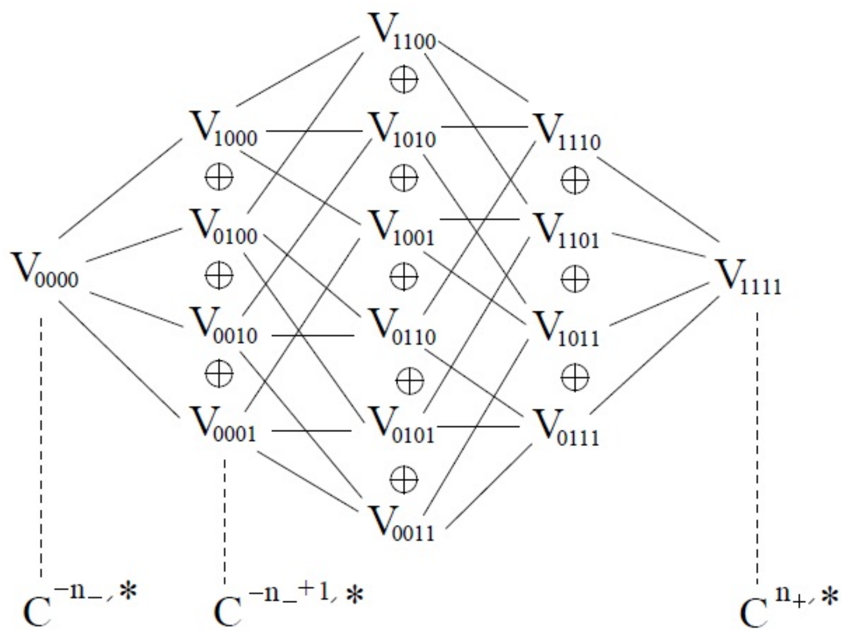
$$V_\upsilon = V^{\otimes k_\upsilon}\{r_\upsilon + n_+ - 2n_-\} \ .$$

Note that this space depends *only* on the number of circles produced by the smoothings being used and the number of "1"'s in the word υ. Thus, for instance, in Example 52, the spaces V_{100}, V_{010} and V_{001} are all the same.

For each $i \in \mathbb{Z}$ we also define the graduate vector space

$$C^{i,*}(D) = \bigoplus_{\substack{\upsilon\in\{0,1\}^n \\ r_\upsilon = i+n_-}} V_\upsilon \ .$$

Thus, if $C^{i,*}(D)$ is not trivial, it is the direct sum of the column vector space $r_\upsilon - n_-$ of the hypercube given by all the words $\upsilon \in \{0,1\}^n$, see figure below. The internal grading (indicated by $*$) comes from the fact that each V_υ is a graded vector space.



An element ϑ of $C^{i,j}(D)$ is said to have homological degree i y q-graduation j. If $\vartheta \in V_\upsilon \subset C^{*,*}(D)$ with homological degree i y q-graduation j, then $i = r_\upsilon - n_-$. We compute j as follows:

Since $\vartheta \in V_\upsilon$, then $\vartheta \in (V_\upsilon)^j = (V^{\otimes k_\upsilon}\{r_\upsilon + n_+ - 2n_-\})^j$. Now, let $deg(\vartheta)$ be the degree of ϑ as an element of the unshifted tensor product $V^{\otimes k_\upsilon}$. Then, by Definition 58 we have

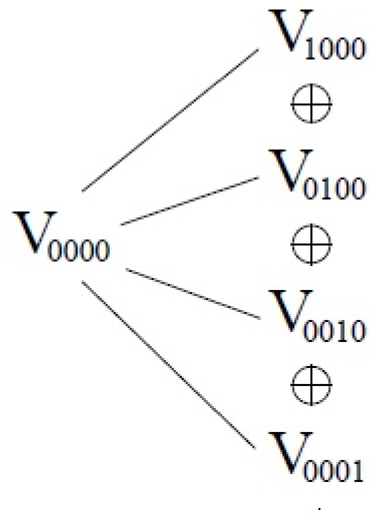
$$deg(\vartheta) = j - \{r_\upsilon + n_+ - 2n_-\} = j - i + n_- - n_+ \ ,$$

since $i = r_\upsilon - n_-$. Thus, the index j, the q-graduation of θ, is

$$j = \deg(\vartheta) + i + n_+ - n_- \ .$$

Now we can define a differential "d" on the complex $C^{*,*}(D)$. Since each $C^{i,*}(D)$ is a direct sum, we must define the morphism $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$ taking this fact into account. Suppose that $V_\upsilon \subseteq C^{i,*}(D)$ and that $V_{\upsilon'} \subseteq C^{i+1,*}(D)$. We make three suppositions: first, the word υ' has "1"'s at least in the same positions as the word υ; second, υ' has exactly one more "1" than υ; third, k_υ and $k_{\upsilon'}$ differ exactly by 1; then there is a morphism of degree 1 from V_υ to

$V_{\upsilon'}$ as shown below



We say that the pair (υ, υ') is *admissible*.

In order to find admissible pairs, we proceed as follows: if the word 1000, for example, changes to the word 1100 this change is represented by $1*00$, and we say that there is an arrow from 1000 to 1100. Then each υ' such that $(1000, \upsilon')$ is admissible is obtained by "moving" the symbol $*$ without moving the "1"'s appearing in 1000. Pictorially, we have

$$* = \begin{cases} 0 & \text{, corresponds to the start of the arrow} \\ 1 & \text{, corresponds to the end of the arrow.} \end{cases}$$

We can indicate the morphism between V_{1000} and V_{1*00} by $d_{1*00} : V_{1000} \to V_{1*00}$. More generally, if the pair (υ, υ') is admissible, we indicate the morphism between them as $d_{(\upsilon,\upsilon')} : V_\upsilon \to V_{\upsilon'}$. Note that if (υ, υ') is admissible, then Γ_υ (a set of circles arising from smoothing, see page 44) and $\Gamma_{\upsilon'}$ differ exactly by one circle, and this difference is obtained by splitting one circle in Γ_υ or by joining two circles of Γ_υ into one. If two circles in Γ_υ are joined, we apply the multiplication m to their corresponding vector spaces, and if one circle is split into two we apply Δ to its corresponding vector space. To all the other circles in Γ_υ we apply the identity map. This recipe defines the map $d_{(\upsilon,\upsilon')}$.

Now we combine the maps $d_{(\upsilon,\upsilon')}$ in order to construct the differential d^i. If (υ, υ') is admissible, then $\upsilon' = \upsilon$, except for the fact that one "0" in υ has been changed for a $*$. We define

$$sign(\upsilon) = (-1)^{\text{number of } 1's \text{ to the left of } *},$$

and then we set

$$d_v = \sum_{(v,v')\ \text{admissible}} sign(v)\, d_{(v,v')}\ .$$

Finally, we define $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$ as

$$d^i = \sum_{\substack{v\in\{0,1\}^n \\ r_v = i+n_-}} d_v\ .$$

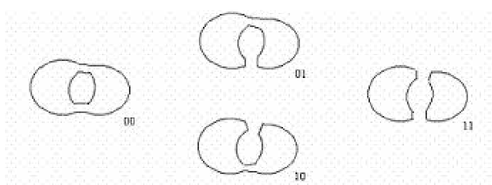
Proposition 59. $d^{i+1} \circ d^i = 0$

Proof. This fact is proven in the original paper by Khovanov, [13], and it also appears in [2] and the Lectures [27]. □

Proposition 60. *The homotopy type of $(C^{*,*}(D), d)$ is invariant under the transformations of Reidemeister. In particular, the (co)homology of $(C^{*,*}(D), d)$ is invariant under isotopy.*

Proof. See Lecture 2 of [27]. □

Example 61. *Consider the Hopf link*  *. Applying smoothings we obtain*



In this example we note that:

$n_- = 2$

$n_+ = 0$

$n = 4$

Where vector spaces V_v are given by;

$$\left.\begin{aligned} V_{00} &= V^{\otimes 2}\,\{-4\} \\ V_{01} &= V^{\otimes 1}\,\{-3\} \\ V_{10} &= V^{\otimes 1}\,\{-3\} \\ V_{11} &= V^{\otimes 2}\,\{-2\} \end{aligned}\right. \quad \text{and so}$$

$C^{*,*}(D) = \bigoplus_{\substack{\upsilon\in\{0,1\}^n \\ r_\upsilon = i+n_-}} V_\upsilon$. Then:

1. if $\upsilon = 00$, then $r_\upsilon = o$, so $i = -n_-$ and

   $C^{-n_-,*}(D) = V^{\otimes 2}\{-4\}$.

2. if $\upsilon = 01$, then $r_\upsilon = 1$, so $i = 1 - n_-$ and

   $C^{1-n_-,*}(D) = V^{\otimes 1}\{-3\} \oplus V^{\otimes 1}\{-3\}$.

3. if $\upsilon = 11$, then $r_\upsilon = 2$, so $i = 2 - n_-$ and

   $C^{2-n_-,*}(D) = V^{\otimes 2}\{-2\}$.

Thus,

$$0 \to C^{-n_-,*}(D) \to C^{1-n_-,*}(D) \to C^{2-n_-,*}(D) \to 0$$

$$0 \to C^{-2,*}(D) \to C^{-1,*}(D) \to C^{2,*}(D) \to 0, \text{ since } n_- = 2$$

$$0 \to V^{\otimes 2}\{-4\} \to V^{\otimes 1}\{-3\} \oplus V^{\otimes 1}\{-3\} \to V^{\otimes 2}\{-2\} \to 0$$

Forming sequence:

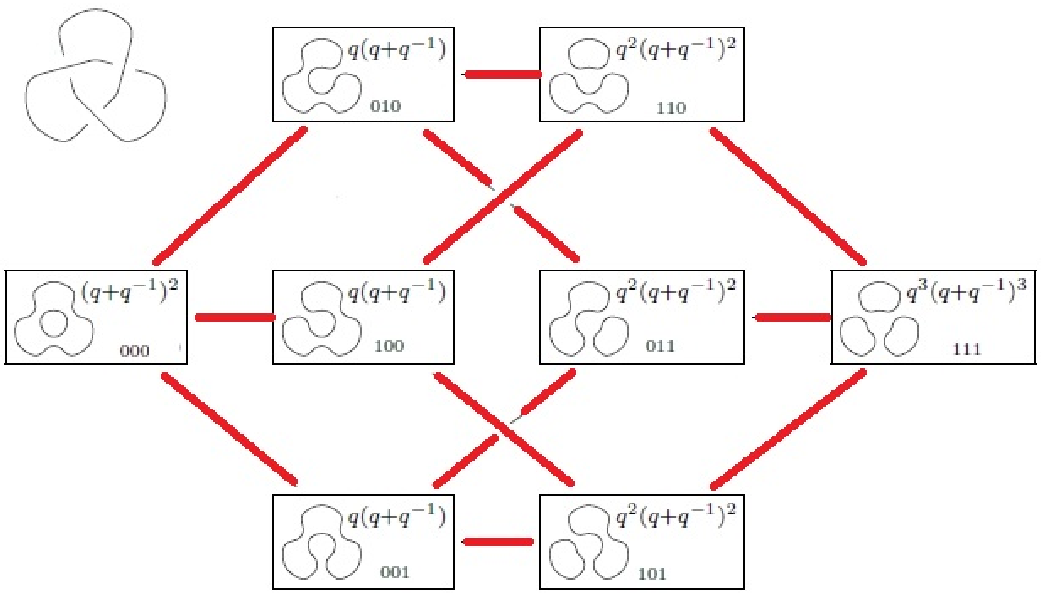
$0 \xrightarrow{d_0} V^{\otimes 2}\{-4\} \xrightarrow{d_1} V^{\otimes 1}\{-3\} \oplus V^{\otimes 1}\{-3\} \xrightarrow{d_2} V^{\otimes 2}\{-2\} \xrightarrow{d_3} 0$ where

$$d_0 = 0$$
$$d_1(v_1 \otimes v_2) = (m(v_1 \otimes v_2), m(v_1 \otimes v_2))$$
$$d_2 = \Delta(v_1) - \Delta(v_2) \ .$$

We can write these applications in matrix form, fixing an ordered basis, this is:

$$d_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ y } d_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Example 62. *We consider the trefoil knot. Applying smoothings we decompose it in several states: first we apply 0-smoothings to all three crossings. Then we apply exactly one 1-smoothing to each of the 0-smoothings, leaving in a column the states that have the same number of circles. See the figure below.*



The graph above tells us that in the initial state we have the vector space $V^{\otimes 3}$. Using this space we apply smoothings and we obtain vector spaces associated with each column. For example: $(V^{\otimes 2}) \oplus (V^{\otimes 2}) \oplus (V^{\otimes 2})$, $V \oplus V \oplus V$ y $V^{\otimes 2}$. Now we can construct a complex

$$0 \to V^{\otimes 3} \to (V^{\otimes 2}) \oplus (V^{\otimes 2}) \oplus (V^{\otimes 2}) \to V \oplus V \oplus V \to V^{\otimes 2} \to 0\ .$$

As in the previous example we have:

$d^1(v_1 \otimes v_2 \otimes v_3) = (m(v_1v_2)v_3, v_1m(v_2v_3), v_2m(v_1v_3))$

$d^2(v_1v_2, v_3v_4, v_5v_6) = (m(v_3v_4) - m(v_1v_2), m(v_5v_6) - m(v_1v_2), m(v_5v_6) - m(v_3v_4))$

$d^3(v_1, v_2, v_3) = \triangle\ (v_1) - \triangle\ (v_2) + \triangle\ (v_3)$

Each application d^i can be written as a matrix using a basis of the corresponding vector space. For example, the set

$$(v_+v_+v_+, v_-v_-v_-, v_+v_-v_-, v_-v_-v_+, v_-v_+v_-, v_+v_+v_-, v_+v_-v_+, v_-v_+v_+)$$

is a basis for the vector space $V^{\otimes 3}$*, and*

$$(v_+v_+,0,0),(0,v_+v_+,0),(0,0,v_+v_+),(v_-v_-,0,0),(v_+v_-,0,0),$$
$$(v_-v_+,0,0),(0,v_-v_-,0),(0,v_+v_-,0),(0,v_-v_+,0),(0,0,v_-v_-),$$
$$(0,0,v_+v_-),(0,0,v_-v_+)$$

is an ordered basis for the vector space $(V^{\otimes 2})\oplus(V^{\otimes 2})\oplus(V^{\otimes 2})$*. Now we compute:*

$d^1(v_+v_+v_+)=(m(v_+v_+)v_+,v_+m(v_+v_+),v_+m(v_+v_+))$

$=(v_+v_+,v_+v_+,v_+v_+)$

$=(v_+v_+,0,0)+(0,v_+v_+,0),(0,0,v_+v_+)$

We obtain the following matrix representing d^1*:*

$$d^1=\begin{pmatrix}1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&1&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&0&0&0&1&1&0\\0&0&0&0&1&1&0&0\\0&0&0&0&1&0&1&0\end{pmatrix}$$

Proceeding in the same way we find d^2 *and* d^3*:*

$$d^2 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$d^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

Definition 63. *The Euler characteristic $\chi_q(C)$ of a graded chain complex is the alternated sum*

$$\chi_q(C) = \sum_{1 \leq i \leq n} (-1)^i q \dim(H^i)$$

An interesting observation due to Bar-Natan, [2], is that if the chain groups are finite-dimensional, then it is not necessary to compute cohomology for calculating the Euler characteristic. We provide a full proof, roughly following [11]:

Theorem 64. *If the chain groups are finite dimensional, then the Euler characteristic can be computed using only the dimensions of the chain groups, that is,*

$$\chi_q(C) = \sum_{1 \leq i \leq n} (-1)^i q \dim(H^i) = \sum_{1 \leq i \leq n} (-1)^i q \dim(C^i)$$

Proof. We consider a finite chain complex

$$C: \ 0 \longrightarrow c^0 \longrightarrow c^1 \longrightarrow . \ . \ . \longrightarrow c^n \longrightarrow 0$$

with cohomology groups $H^0, H^1, H^2, ..., H^n$. We have:

$$\begin{aligned}
\chi_q(C) &= \sum_i (-1)^i q \dim(H^i) \\
&= \sum_i (-1)^i \sum_j q^j \dim(H^{i,j}) \\
&= \sum_i (-1)^i \sum_j q^j \left[\dim\ker d^{i,j} - \dim img\, d^{i+1,j}\right] \\
&= \sum_i \sum_j (-1)^i q^j \dim\ker d^{i,j} - \sum_i \sum_j (-1)^i q^j \dim img\, d^{i+1,j} \\
&= \sum_i \sum_j (-1)^i q^j \dim\ker d^{i,j} - \sum_i \sum_j (-1)^i q^j \left[\dim\left(c^{i+1,j}\right) - \dim\ker d^{i+1,j}\right] \\
&= \sum_i \sum_j (-1)^i q^j \dim\ker d^{i,j} - \sum_i \sum_j (-1)^i q^j \dim\left(c^{i+1,j}\right) \\
&+ \sum_i \sum_j (-1)^i q^j \dim\ker d^{i+1,j} \\
&= \sum_i \sum_j (-1)^i q^j \dim\ker d^{i,j} - \sum_r \sum_j (-1)^{r-1} q^j \dim\left(c^{r,j}\right) \\
&\quad + \sum_r \sum_j (-1)^{r-1} q^j \dim\ker d^{r,j} \\
&= \sum_i \sum_j (-1)^i q^j \dim\ker d^{i,j} + \sum_r \sum_j (-1)^r q^j \dim c^{r,j} \\
&- \sum_r \sum_j (-1)^r q^j \dim\ker d^{r,j} \\
&= \sum_i \sum_j (-1)^i q^j \dim\ker d^{i,j} + \sum_i \sum_j (-1)^i q^j \dim\left(c^{i,j}\right) \\
&- \sum_i \sum_j (-1)^i q^j \dim\ker d^{i,j} \\
&= \sum_i \sum_j (-1)^i q^j \dim\left(c^{i,j}\right)
\end{aligned}$$

□

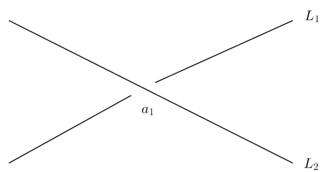
Using this result, we can state one of the main observations due to Mikhail Khovanov [13], see also [2, 27]:

Theorem 65. *The graded Euler characteristic of the Khovanov complex* $(C^{*,*}(D), d)$ *is the un-normalised Jones polynomial* $\hat{J}(D)$ *defined in Section* 3.1.

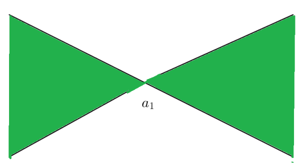
3.2 The Chekanov homology for Legendrian knots

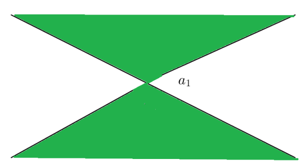
The Chekanov homology is the homology of a particular differential algebra $(\mathcal{A}, d)$ constructed using the crossings of a given Legendrian knot. The algebra $\mathcal{A}$ is simply a free algebra in a finite number of generators and therefore its construction is standard. We explain how to construct the differential d using contact geometry.

Let K be an oriented Legendrian knot in $\mathbb{R}^3$ equipped with its standard contact structure, and assume that we know its Lagrangian diagram $\gamma_L(K)$.
We denote by $a_1, ..., a_n$ the double points in the Lagrangian projection and we define $\mathcal{A}$ as the unitary graduated tensor algebra over $\mathbb{Z}_2$ generated by the set $\{a_1, ..., a_n\}$.
Let us fix a double point a_1 in $\gamma_L(K)$. Then, there exist two lines L_1 and L_2 such that

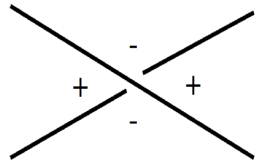


Therefore, these two lines divide the plane in four quadrants (two for above and two for below)

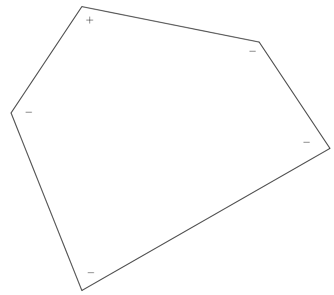




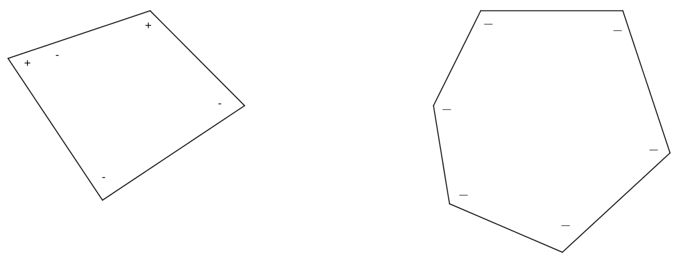
We will define the differential over $\mathcal{A}$ counting piecewise immersions $f : D^2 \longrightarrow \mathbb{R}^2$ from the unit disk D^2 into $\mathbb{R}^2$, such that the boundaries of these immersions are polygons which vertices are some of the double points of $\gamma_L(K)$. Graphically, for each double point we equip each quadrant with a sign, in the following way:



Of all the polygons we have obtained from immersions as above, we consider only those immersions which in their interior have exactly one $(+)$ sign and the others signs are $(-)$:



Thus, for instance, we do not consider the cases

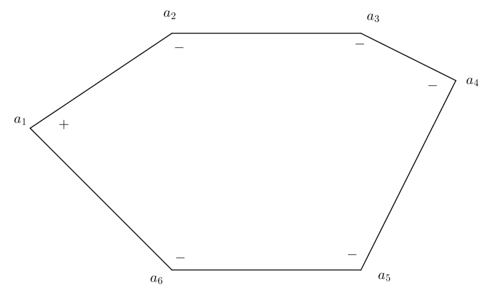


It follows that for each generator a_i there exists a finite number of polygons with a $(+)$ sign in the vertex a_i and $(-)$ sign in the other vertexes; we will call this set of immersions $\triangle(a_i : a)$. We note that each polygon inherits an orientation from the orientation of the Lagrangian projection of the knot.

Definition 66. *Given a generator a_i and a set of generators $b = \{b_1, ..., b_n\}$, let $f \in \triangle(a_i : b)$ be an immersion*

$$f : D^2 \longrightarrow \mathbb{R}^2$$

which preserves orientation and that applies the boundary ∂D^2 to the Lagrangian projection $\gamma_L(K)$, with the property that $f|_{\partial D^2}$ is an immersion except in b. We form a word taking the points of b consecutively in a counterclockwise sense, that is,



Remark 67. *We note that in the figure above we are counting only the vertices with a $(-)$ sign, as explained in the definition.*

Definition 68. *The differential $\partial : \mathcal{A} \to \mathcal{A}$ defined on a generator a_i is constructed using the words defined above and adding over all the elements of $\Delta(a_i : b)$, that is:*

$$\partial a_i = \sum_{\Delta(a_i:b)} \#\Delta(a_i : b) b_1 \cdots b_n \ ,$$

where $\# \triangle (a_i : b)$ is the number of elements in the set $\triangle(a_i : b)$ counted module 2. We extend ∂ to all $\mathcal{A}$ by linearity and the Leibnitz rule.

Definition 69. *(The algebra graduation) In order to define a graduation of $\mathcal{A}$ we use the frontal projection $\gamma_F(K)$ of Legendrian knots. Given a crossing or a double point a_i in $\gamma_F(K)$, we define the degree of a_i ($deg\, a_i$) as follows:*

$$deg\, a_i = \begin{cases} -m(p_i) + 1, & \text{if } a_i \text{ is a right cusp} \\ -m(p_i), & \text{if } a_i \text{ is a crossing} \end{cases} \ ,$$

where p_i is a path contained in the projection $\gamma_F(K)$ which begins and ends in a_i (we move following the segment with the steepest slope).

In case that a_i is a right cusp, it can be shown that p_i coincides with all the projection $\gamma_L(K)$ and therefore:

$$deg\, a_i = \begin{cases} 1, & \text{if } a_i \text{ is a cusp oriented upwards} \\ 1 - m(k), & \text{if } a_i \text{ is a cusp oriented downwards} \end{cases}$$

Theorem 70. *$(\mathcal{A}, \partial)$ is a differential graded algebra, that is,*

- $\partial^2 = 0$
- ∂ *is of degree* -1
- $\partial(ab) = (\partial a)b + (-1)^{|a|}a(\partial b)$, *for* $a, b \in \mathcal{A}$.

Moreover, the homology of $(\mathcal{A}, \partial)$ *is unchanged under Legendrian isotopy: the homology rings of isotopic Legendrian knots are isomorphic as graded rings.*

Proof. Chekanov [4]. See also [8]. □

Construction of linearized Legendrian homology

We let $(\mathcal{A}, \partial)$ be the Chekanov algebra. We define a descending chain of subalgebras of $\mathcal{A}$ as follows: $\mathcal{A}^n$ is the subalgebra generated (as a vector space) by all words of length greater or equal than n. Certainly, $\mathcal{A}^n$ is not only a subalgebra but also a bilateral ideal. We have the sequence

$$\mathcal{A} = \mathcal{A}^0 \supseteq \mathcal{A}^1 \supseteq \mathcal{A}^2 \supseteq \cdots \supseteq \mathcal{A}^n \supseteq \cdots$$

Following Chekanov [4], we modify the differential ∂ so that the new differential respects this filtration. This is done using *augmentations*:

Definition 71. *An augmentation of* $(\mathcal{A}, \partial)$ *is a multiplicative homomorphism* $\varepsilon : \mathcal{A} \to \mathbb{Z}_2$ *such that:*

- $\varepsilon(1) = 1$
- $\varepsilon \circ \partial = 0$
- *If* $|a| \neq 0$, *then* $\varepsilon(a) = 0$

Definition 72. *Given an augmentation* ε *of* $(\mathcal{A}, \partial)$, *we can define an automorphism* g *of* $\mathcal{A}$ *as*

$$g(a) = a + \varepsilon(a)$$

for all $a \in \mathcal{A}$. *We also define a new differential* $\partial^g : \mathcal{A} \to \mathcal{A}$ *as*

$$\partial^g = g \circ \partial \circ g^{-1} .$$

It can be checked that this new differential respects the word filtration defined above, that is,

$$\partial^g(\mathcal{A}^n) \subseteq \mathcal{A}^n \ .$$

Definition 73. *Given an augmentation ε in $\mathcal{A}$, we define the nth-order Legendrian homology as*

$$L_n^{\varepsilon}CH_*(L) = H_*(\mathcal{A}^n/\mathcal{A}^{n+1}, \partial^g) \ .$$

If $n = 1$, we write $L_1^{\varepsilon}CH_*(L) = L^{\varepsilon}CH_*(L)$ and we call it the linearized Legendrian homology.

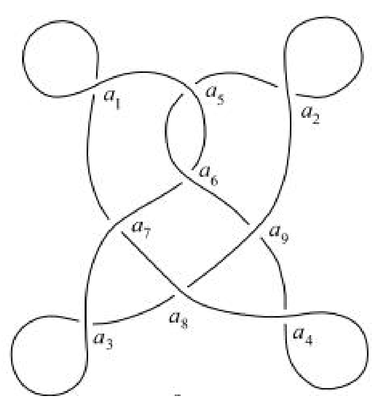
Theorem 74. *(Chekanov) The set of isomorphism classes of linearized Legendrian homology for all possible augmentations, is invariant under Legendrian isotopy.*

Proof. This important theorem is in [4], see also [8]. We remark that it also holds for nth-order Legendrian homology. $\square$

Definition 75. *The Poincare-Chekanov polynomial is given by:*

$$P_{(A_1,\partial^g)}(t) = \sum_{i=|a|} dim \ L_1CH_i(A_1, \partial_i^g)t^i$$

Example 76. *Consider the following Legendrian knot L_1:*



We calculate its linearized Legendrian homology. The differential algebra on the generators a_1, $\ldots, a_9$ is constructed as follows:

$$\text{grad}\, a_i = 1, \ \textit{for } i \leq 4$$

$$\text{grad}\, a_5 = 2$$

$$\text{grad}\, a_6 = -2$$

$$\text{grad}\, a_i = 0, \ \textit{for } i \geq 7$$

and the differential is computed to be

$$\begin{aligned}
\partial a_1 &= 1 + a_7 + a_7 a_6 a_5 \\
\partial a_2 &= 1 + a_9 + a_5 a_6 a_9 \\
\partial a_3 &= 1 + a_8 a_7 \\
\partial a_4 &= 1 + a_8 a_9 \\
\partial a_5 &= 0 \\
\partial a_6 &= 0 \\
\partial a_7 &= 0 \\
\partial a_8 &= 0 \\
\partial a_9 &= 0 \ .
\end{aligned}$$

Now suppose that we have an augmentation ε; it follows that there is a morphism g such that $g(a_i) = a_i + c_i$, where $c_i = \varepsilon(a_i)$; we define the differential $\partial^g =: g \circ \partial \circ g^{-1}$, and we observe that the differential on $\mathcal{A}^n/\mathcal{A}^{n+1}$ appearing in Definition 73 is exactly $\partial^g_n = \pi_n \partial^g$, in which π_n is the canonical projection. In particular, it must happen that $\partial^g_0 = 0$, and therefore using the definition of g,

$$\begin{aligned}
0 = \partial^g_0(a_1) &= 1 + c_7 + c_7 c_6 c_5 \\
0 = \partial^g_0(a_2) &= 1 + c_9 + c_5 c_6 c_9 \\
0 = \partial^g_0(a_3) &= 1 + c_8 c_7 \\
0 = \partial^g_0(a_4) &= 1 + c_8 c_9 \ .
\end{aligned}$$

It follows that $c_7 = c_8 = c_9 = 1$, and that the other c_i are zero. Then we get

$$\begin{aligned}
g(a_1) &= a_1 \\
g(a_2) &= a_2 \\
g(a_3) &= a_3 \\
g(a_4) &= a_4 \\
g(a_5) &= a_5 \\
g(a_6) &= a_6 \\
g(a_7) &= a_7 + 1 \\
g(a_8) &= a_8 + 1 \\
g(a_9) &= a_9 + 1 \ .
\end{aligned}$$

Now we can calculate the differential ∂^g_1:

$$
\begin{aligned}
\partial_1^g(a_1) = \pi_1\partial^g(a_1) = \pi_1 \circ g \circ \partial(a_1) = \pi_1 \circ g(1 + a_7 + a_7a_6a_5)\\
= \pi_1(1 + a_7 + 1 + (a_7 + 1)a_6a_5)\\
= a_7\\
\partial_1^g(a_2) = \pi_1\partial^g(a_2) = \pi_1 \circ g \circ \partial(a_2) = \pi_1 \circ g(1 + a_9 + a_5a_6a_9)\\
= \pi_1(1 + a_9 + 1 + a_5a_6(a_9 + 1))\\
= a_9\\
\partial_1^g(a_3) = \pi_1\partial^g(a_3) = \pi_1 \circ g \circ \partial(a_3) = \pi_1 \circ g(1 + a_8a_7)\\
= \pi_1(1 + (a_8 + 1)(a_7 + 1))\\
= a_7 + a_8\\
\partial_1^g(a_4) = \pi_1\partial^g(a_4) = \pi_1 \circ g \circ \partial(a_4) = \pi_1 \circ g(1 + a_8a_9)\\
= \pi_1(1 + (a_8 + 1)(a_9 + 1))\\
= a_9 + a_8\\
\partial_1^g(a_2) = \pi_1\partial^g(a_2) = \pi_1 \circ g \circ \partial(a_2) = \pi_1 \circ g(0) = 0\\
\partial_1^g(a_2) = \pi_1\partial^g(a_2) = \pi_1 \circ g \circ \partial(a_2) = \pi_1 \circ g(0) = 0\\
\partial_1^g(a_2) = \pi_1\partial^g(a_2) = \pi_1 \circ g \circ \partial(a_2) = \pi_1 \circ g(0) = 0\\
\partial_1^g(a_2) = \pi_1\partial^g(a_2) = \pi_1 \circ g \circ \partial(a_2) = \pi_1 \circ g(0) = 0\\
\partial_1^g(a_2) = \pi_1\partial^g(a_2) = \pi_1 \circ g \circ \partial(a_2) = \pi_1 \circ g(0) = 0
\end{aligned}
$$

Thus, we obtain the graded differential

$$(\partial_1^g)_i \quad : \quad (A_1)_i \longrightarrow (A_1)_{i-1}$$

and we can find the homology groups

$$
\begin{aligned}
L_1CH_0 &= \ker(\partial_1^g)_0 \big/ img\,(\partial_1^g)_1\\
&= \ker(\partial_1^g)_0 : (A_1)_0 \longrightarrow (A_1)_{-1} \big/ img\,(\partial_1^g)_1 : (A_1)_1 \longrightarrow (A_1)_0\\
&= \{a_7, a_8, a_9\} / \{a_7, a_8, a_9\}\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
L_1CH_1 &= \ker\left(\partial_1^g\right)_1 \big/ img\left(\partial_1^g\right)_2 \\
&= \ker\left(\partial_1^g\right)_1 : (A_1)_1 \longrightarrow (A_1)_0 \big/ img\left(\partial_1^g\right)_2 : (A_1)_2 \longrightarrow (A_1)_1 \\
&= \{a_1 + a2 + a3 + a4\} / \{0\} \\
&= \langle a_1 + a2 + a3 + a4 \rangle
\end{aligned}
$$

$$
\begin{aligned}
L_1CH_2 &= \ker\left(\partial_1^g\right)_2 \big/ img\left(\partial_1^g\right)_1 \\
&= \ker\left(\partial_1^g\right)_2 : (A_1)_2 \longrightarrow (A_1)_1 \big/ img\left(\partial_1^g\right)_3 : (A_1)_3 \longrightarrow (A_1)_2 \\
&= \{a_5\} / \{0\} \\
&= \langle a_5 \rangle
\end{aligned}
$$

$$
\begin{aligned}
L_1CH_{-2} &= \ker\left(\partial_1^g\right)_{-2} \big/ img\left(\partial_1^g\right)_{-3} \\
&= \ker\left(\partial_1^g\right)_0 : (A_1)_0 \longrightarrow (A_1)_{-1} \big/ img\left(\partial_1^g\right)_{-1} : (A_1)_{-1} \longrightarrow (A_1)_{-2} \\
&= \{a_6\} / \{0\} \\
&= \langle a_6 \rangle
\end{aligned}
$$

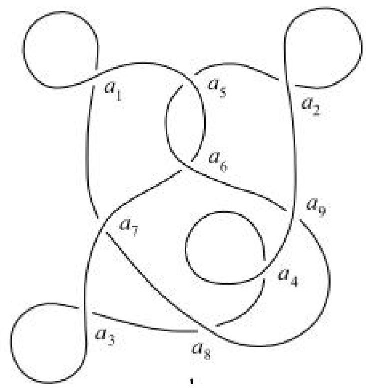
Thus, in particular,

$$
\begin{aligned}
\dim L_1CH_0 = 0 \\
\dim L_1CH_1 = 1 \\
\dim L_1CH_2 = 1 \\
\dim L_1CH_{-2} = 1\ ,
\end{aligned}
$$

and we can compute the Poincare polynomial:

$$
\begin{aligned}
P_{\left(A_1, \partial_1^g\right)}(t) &= \sum_{i=|a|} t^i \dim L_1CH_i\left(A_1, \partial_1^g\right) \\
&= t + t^2 + t^{-2}\ .
\end{aligned}
$$

Example 77. *In an analogous fashion, we can study the knot* L_2



In this case we find two polynomials: $t+1$ *and* $2+t$. *It follows that these two knots are not Legendrian isotopic.*

Chapter 4

Applications

In this chapter we present some applications of the foregoing theory. We have three applications in mind: the calculation of Reidemeister torsion and definition of a Laplace operator for topological knots; the existence of "Massey products" in the theory of Legendrian knots; and the generation of (noncommutative) integrable systems from the Chekanov algebra.

The Reidemeister torsion (see [26]) is important because it yields a volume form for knots. This observation was made by Ortiz-Navarro [20] in the case of the Khovanov cohomology. We recall his work, we present a non-trivial example, and we extend slightly his work by proposing a Laplace operator for knots. Massey products are important because they yield new invariants for Legendrian knots. This construction is new, although there is already a related paper on the subject, see [5] and Section 4.2 below. Finally, we have decided to show how to define integrable equations within the algebraic framework of Chekanov homology. We believe this is important because it yields a natural construction of *noncommutative* integrable equations based on a noncommutative differential algebra which is geometrically motivated and which is completely different to the algebras considered by researchers in geometric aspects of differential equations [24].

4.1 The Reidemeister torsion

4.1.1 The general theory of Reidemeister torsion

The exposition below follows mainly the book [26].

Definition 78. *Let F be a field and let D be a finite dimensional vector space over F. Suppose that $dimD = k$, and let us fix two ordered bases $b = (b_1, ..., b_k)$ and $c = (c_1, ..., c_k)$ of D. For each*

$j = 1, ..., k$, we set

$$b_j = \sum_{i=1}^{k} a_{ij} c_i \ .$$

The matrix $(a_{ij})_{i,j=1,...,k}$ is called the transition matrix between the bases b and c. We write

$$det(a_{ij}) = [b/c] \ ,$$

so that $[b/c] \in F^* = F - \{0\}$.

We set $b \sim c$ if and only if $[b/c] = 1$. It is easy to see that $\sim$ is an equivalence relation.

Now, let C, D and E be vector spaces over F and assume that we have a short exact sequence of vector spaces

$$0 \longrightarrow C \xrightarrow{i} D \xrightarrow{\pi} E \longrightarrow 0 \ .$$

Then, it is easy to see that $dimD = dimC + dimE$. We let $c = (c_1, ..., c_p)$ be a basis for C, $d = (d_1, ..., d_k)$ be a basis for D and $e = (e_1, ..., e_q)$ be a basis for E (so that $p + q = k$). Since π is onto, we can lift each e_i to some $\widetilde{e_i} \in D$, such that $\pi(\widetilde{e_i}) = e_i$. We call the set $\widetilde{e_i} = (\widetilde{e_1}, ..., \widetilde{e_q})$ a pullback of e. It follows that the set

$$c\,\widetilde{e} = (c_1, ..., c_p, \widetilde{e_1}, ..., \widetilde{e_q})$$

is a basis for D.

Now we consider the case of an acyclic cochain complex on F,

$$C = (0 \longrightarrow C^0 \xrightarrow{\partial^0} C^1 \longrightarrow \cdots \xrightarrow{\partial^{m-2}} C^{m-1} \xrightarrow{\partial^{m-1}} C^m \longrightarrow 0)$$

We will construct a short exact sequence as above. Let

$$B^{r-1} = Im(\partial^{r-1} : C^{r-1} \to C^r) \subset C^r.$$

Since C is acyclic, the first isomorphism theorem tells us that

$$C^r/B^{r-1} = C^r/ker(\partial^r : C^r \to C^{r+1}) = Im\partial^r = B^r \ .$$

This means that $0 \to B^{r-1} \xrightarrow{i} C^r \xrightarrow{\partial^r} B^r \to 0$ is a short exact sequence. We choose a base b^r of B^r, $r = 1, ..., m$. By the construction above, the ordered set $b^{r-1}\,\widetilde{b^r}$ is a basis of C^r. We compare it with the basis c^r of C^r as in Definition 78:

Definition 79. *The Reidemeister torsion of C is*

$$\tau(C) = \prod_{r=0}^{m} \left| [b^{r-1} \widetilde{b^r} / c^r]^{(-1)^{r+1}} \right| \in F^* \quad .$$

The following proposition appears in [26]:

Proposition 80.

1. *$\tau(C)$ does not depend either on the choice of b^r or on the choice of pullback $\widetilde{b^r}$.*

2. *$\tau(C)$ depends on the basis c^r, which is called a distinguished basis for the cochain space C^r.*

3. *If another basis is equivalent to the distinguished basis, then $\tau(C)$ is the same for both bases. Indeed, if C' is the same acyclic chain complex C but we consider different bases for the spaces C^r, say ${c^r}'$, then*
$$\tau(C') = \tau(C) \prod_{i=0}^{m} \left[c^i / {c^i}' \right]^{(-1)^{i+1}} \quad .$$

Now we consider a cochain complex (D, Δ) which is not necessarily acyclic. By the first isomorphism theorem we have that the sequence

$$0 \to ker\Delta^i \xrightarrow{\iota} D^i \xrightarrow{\Delta^i} Im\Delta^i \to 0 \tag{4.1}$$

is exact, where ι is the inclusion map. Also,

$$0 \to B^{i-1} \xrightarrow{\iota} ker\Delta^i \xrightarrow{\pi} H^i \to 0 \tag{4.2}$$

is also exact, where π is the projection map. Thus, if we have a basis b^{i-1} for $B^{i-1} = Im\Delta^{i-1}$, and a basis $[h^i]$ for H^i for each i, we use the exact sequence (4.2) and we obtain a basis $[b^{i-1}h^i]$ for $ker\Delta^i$, where $\pi(h^i) = [h^i]$. We now use (4.1) and we conclude that $[b^{i-1}\, h^i\, \widetilde{b^i}]$ is a basis for D^i.

Definition 81. *The torsion de Reidemeister of D is*

$$\tau(D) = \left| \prod_{r=0}^{m} [b^{r-1}\, h^r\, \widetilde{b^r} / c^r]^{(-1)^{r+1}} \right| \in F^* \quad .$$

As before, it is proven in [26] that $\tau(D)$ is well-defined:

Proposition 82. *The torsion $\tau(C)$ does not depend on the choice of b^r or its pull back $\widetilde{b^r}$. On the other hand, if the basis c^r is changed, the torsion changes by a factor, as in Proposition 80.*

The application of the above theory to the Khovanov complex has been carried out by Ortiz-Navarro [20]. One interesting possibility, motivated by [23, 24], is to define a Laplace operator on this complex. We can do it in great generality.

4.1.2 The Laplacian of a chain complex

Let $A = \bigoplus_i C^i$ and suppose that $F = \mathbb{R}$. There exists a unique Euclidean metric $\langle .,. \rangle_i$ over C^i such the distinguished basis c^i of C^i is an orthonormal basis. Define $\partial^* : C^i \to C^{i+1}$ via

$$\langle \partial_i^* a, b \rangle_{i+1} = \langle a, \partial_i b \rangle_i \qquad a \in C^i, b \in C^{i+1}$$

and

$$\Delta_i = \partial^* \partial + \partial \partial^* : C^i \to C^i \ .$$

This is our Laplace operator. We leave as an open problem to investigate it in detail in the context of Khovanov theory. For example, we do not know how (or whether) the operator Δ_i changes under isotopy. On the other hand, due to its intrinsic importance, we have decided to prove the following general result, essentially an abstract version of classical Ray-Singer theory [23, 24].

Theorem 83. *If the homology of the complex $C(K, \partial)$ is trivial, then*

$$\log \tau(K, \partial) = \frac{1}{2} \sum_q^{\infty} q(-1)^{q+1} \log \det(-\Delta_q)$$

Proof. Recall that we have defined Δ_i using an Euclidean metric. We consider for each q the set $B_q = \partial_{q+1} C_{q+1} \subseteq C_q$, $\bar{B}_{q-1} = \partial_q^* C_{q-1} \subseteq C_q$; now, since ∂_q^* is the formal adjoint of ∂_q and :

$$\begin{aligned} \partial_q^* &: C_{q-1} \longrightarrow C_q \\ \partial_q &: C_q \longrightarrow C_{q-1} \\ \Delta_q &= -\partial_q^* \partial_q - \partial_{q+1} \partial_{q+1}^* \end{aligned}$$

we obtain $B_q \perp \bar{B}_{q-1}$. In fact, let $v_q \in B_q$, $\bar{v}_{q-1} \in \bar{B}_{q-1}$, then there exist u_{q+1} y v_{q-1}, such that

$v_q = \partial_{q+1} u_{q+1}$ and $\bar{v_{q-1}} = \partial_q^* v_{q-1}$; from here we obtain

$$\begin{aligned}
\left\langle v_q, \bar{v_{q-1}} \right\rangle &= \left\langle \partial_{q+1} u_{q+1}, \partial_q^* v_{q-1} \right\rangle \\
&= \left\langle \partial_q \partial_{q+1} u_{q+1}, v_{q-1} \right\rangle \\
&= 0
\end{aligned}$$

since $\partial_q \partial_{q+1} = 0$. On the other hand, since C_q admits a Hodge decomposition, then

$$C_q = Hq \oplus \operatorname{Im} g\ \partial_q^* C_{q-1} \oplus \operatorname{Im} g\ \partial_{q+1} C_{q+1}$$

in which Hq is the kernel of Δ_q , this is,

$$C_q = Hq \oplus \bar{B}_{q-1} \oplus B_q$$

We have that $H_q \cong \mathbf{H}_q$ which is trivial, and so

$$C_q = \bar{B}_{q-1} \oplus B_q$$

Let us check that the Laplacian on C_q can be decomposed thus:

$$\begin{aligned}
\Delta_q &= -\partial_q^* \partial_q - \partial_{q+1} \partial_{q+1}^* : C_q \longrightarrow C_q \\
\partial_q^* \partial_q &: C_q = \bar{B}_{q-1} \oplus B_q \longrightarrow \partial_q^* \partial_q (\bar{B}_{q-1} \oplus B_q) \subseteq \bar{B}_{q-1} \\
\partial_q^* \partial_q &: \bar{B}_{q-1} \oplus B_q \longrightarrow \bar{B}_{q-1} \\
\partial_{q+1} \partial_{q+1}^* &: \bar{B}_{q-1} \oplus B_q \longrightarrow B_q
\end{aligned}$$

Thus, we can separate Δ_q in two operators:

$$\begin{aligned}
\partial_q^* \partial_q &: \bar{B}_{q-1} \oplus B_q \longrightarrow \bar{B}_{q-1} \text{ acts on } \bar{B}_{q-1} \\
\partial_{q+1} \partial_{q+1}^* &: \bar{B}_{q-1} \oplus B_q \longrightarrow B_q \text{ acts on } B_q
\end{aligned}$$

Let $\{b_1, b_2, b_3, ..., b_{r_q}\}$ be a basis of B_q and $\left\{\bar{b}_1, \bar{b}_2, \bar{b}_3, ..., \bar{b}_n\right\}$ a basis of $\bar{B}_{q-1}$, then:

$$
\begin{aligned}
\Delta_q b_1 &= -\partial_{q+1}\partial^*_{q+1} b_1 = \sum_{i=1}^{r_q} a_{i1} b_i \\
&. \\
&. \\
&. \\
\Delta_q b_{r_q} &= -\partial_{q+1}\partial^*_{q+1} b_{r_q} = \sum_{i=1}^{r_q} a_{ir_q} b_i \\
\Delta_q \bar{b}_{r_q+1} &= -\partial^*_q \partial_q \bar{b}_{r_q+1} = \sum_{j=r_q+1}^{n} a_{j,r_q+1} \bar{b}_j \\
&. \\
&. \\
&. \\
\Delta_q \bar{b}_n &= -\partial^*_q \partial_q \bar{b}_n = \sum_{j=r_q+1}^{n} a_{j,n} \bar{b}_j
\end{aligned}
$$

Thus, the matrix of Δ_q in the basis $\left\{b_1, b_2, b_3, ..., b_{r_q}, \bar{b}_{r_q+1}, ..., \bar{b}_n\right\}$ is :

$$
\begin{pmatrix}
a_{11} & . & . & . & a_{1r_q} & 0 & . & . & . & 0 \\
a_{21} & . & . & . & a_{2r_q} & 0 & . & . & . & 0 \\
. & . & . & . & . & 0 & . & . & . & 0 \\
. & . & . & . & . & 0 & . & . & . & 0 \\
. & . & . & . & . & 0 & . & . & . & 0 \\
a_{r_q1} & . & . & . & a_{r_qr_q} & 0 & . & . & . & 0 \\
0 & . & . & . & . & 0 & l_{r_q+1,r_q+1} & . & . & l_{r_q+1,n} \\
0 & . & . & . & . & 0 & . & . & . & . \\
0 & . & . & . & . & 0 & . & . & . & . \\
0 & . & . & . & . & 0 & . & . & . & . \\
0 & . & . & . & . & 0 & . & . & . & . \\
0 & . & . & . & . & 0 & l_{n,r_q+1} & . & . & l_{n,n}
\end{pmatrix}
$$

The operator $\partial_q : \bar{B}_{q-1} \longrightarrow B_q$, is a bijection since : $ker\partial_q = B_q$, in fact: we know that $B_q \subseteq ker\partial_q$, and then we can consider $c_q \in ker\partial_q$, we write $c_q = \tilde{b}_{q-1} + b_q$, so that $\partial_q c =$

$\partial_q(\tilde{b}_{q-1}+b_q)=\partial_q\tilde{b}_{q-1}=0$. We wish to prove that $\tilde{b}_{q-1}=0$.

We can write $\tilde{b}_{q-1}=\partial_q^* c_{q-1}$. We compute

$$\begin{aligned}\left\langle \tilde{b}_{q-1},\tilde{b}_{q-1}\right\rangle &= \langle \partial_q^* c_{q-1},\partial_q^* c_{q-1}\rangle\\ &= \langle \partial_q\partial_q^* c_{q-1},c_{q-1}\rangle\\ &= \left\langle \partial_q\tilde{b}_{q-1},c_{q-1}\right\rangle\\ &= 0\end{aligned}$$

Thus, $\tilde{b}_{q-1}=0$, and therefore $ker\partial_q=B_q$.

Now for the image:

$$\partial\Big|_{B_q}=0,\ \partial_q(\tilde{B}_{q-1})\subseteq\partial_q c_q\subseteq B_{q-1}$$

We prove that $B_{q-1}=Im\left(\partial\Big|_{\tilde{B}_{q-1}}\right)$. For this, no element of $\tilde{B}_{q-2}$ should be the image of ∂_q. We take $\tilde{b}_{q-2}\in\tilde{B}_{q-2}$ such that $\partial_q c_q=\tilde{b}_{q-2}=\partial_q^* c_{q-2}$. We compute:

$$\begin{aligned}\left\langle \tilde{b}_{q-2},\tilde{b}_{q-2}\right\rangle &= \langle \partial_q c_q,\partial_q^* c_{q-2}\rangle\\ &= \langle \partial_q^2 c_q,c_{q-2}\rangle\\ &= 0\end{aligned}$$

Thus, $\tilde{b}_{q-2}=0$. This means that $\partial_q(\tilde{B}_{q-1})=B_{q-1}$. We have the diagram

$$\begin{array}{ccc} \tilde{B}_{q-1}\oplus B_q & & \xrightarrow{\partial_q} B_{q-1}\\ \downarrow & & \nearrow\\ \tilde{B}_{q-1}\oplus B_q\big/ker\partial_q\cong\tilde{B}_{q-1} & & \end{array}$$

$$\begin{aligned}\partial &: C_q\longrightarrow C_{q-1}\\ \partial &: \bar{B}_{q-1}\oplus B_q\longrightarrow\bar{B}_{q-2}\oplus B_{q-1}\end{aligned}$$

Therefore, $\partial_q\ :\bar{B}_{q-1}\longrightarrow B_{q-1}$ is invertible, and so there exists ∂_q^{-1} such that $\partial_q\partial_q^{-1}=\partial_q^{-1}\partial_q=1$

and moreover:

$$\begin{aligned}
\partial_q \partial_q^{-1} &= \partial_q(-\Delta_q)(-\Delta_q)^{-1}\partial_q^{-1} \\
&= \partial_q(\partial_{q+1}\partial_{q+1}^* + \partial_q^*\partial_q)(-\Delta_q)^{-1}\partial_q^{-1} \\
&= \partial_q\partial_{q+1}^*\partial_{q+1}(-\Delta_q)^{-1}\partial_q^{-1} \\
&= \partial_q\partial_{q+1}^*(-\Delta_q)^{-1}
\end{aligned}$$

This is,

$$\partial_q^{-1} = \partial_{q+1}^*(-\Delta_q)^{-1}$$

Moreover, we can see that the Laplace operators over B_q and $\bar{B}_{q-1}$ are equivalent:

$$\begin{array}{ccc}
\bar{B}_{q-1} \oplus B_q & \xrightarrow{(-\Delta_q)} & \bar{B}_{q-1} \oplus B_q \\
\partial_{q+1}\partial_{q+1}^* \downarrow & & \downarrow \partial_q^*\partial_q \\
B_{q-1} & \xrightarrow[\partial_q^{-1}]{} & \bar{B}_{q-1}
\end{array}$$

We consider d_q and $\bar{d}_{q-1}$ the determinants of $\partial_{q+1}\partial_{q+1}^*$ and $\partial_q^*\partial_q$ over B_q , $\bar{B}_{q-1}$ respectively, since they are $\partial_{q+1}\partial_{q+1}^* \cong \partial_q^*\partial_q$, then their matrices have equal determinants, this is, $\left|\partial_{q+1}\partial_{q+1}^*\right| = \left|\partial_q^*\partial_q\right|$, from where $d_{q-1} = \bar{d}_{q-1}$, thus

$$\begin{aligned}
|(-\Delta_q)| &= \left|(\partial_{q+1}\partial_{q+1}^* + \partial_q^*\partial_q)\right| \\
&= \left|\partial_{q+1}\partial_{q+1}^*\right| \left|\partial_q^*\partial_q\right| \\
&= d_q\bar{d}_{q-1} \\
&= d_q d_{q-1}
\end{aligned}$$

Let us choose a basis $\mathbf{b}_q$ of $B_q = \partial_{q+1}C_{q+1}$ for each q, and then we write each element of the basis in terms of the q-cell. We obtain a matrix $\dot{\mathbf{B}}_q$, for each element of the basis b_{q-1} of B_{q-1}, since ∂_q

is invertible, there exists $\overset{-}{b_{q-1}}$, such that

$$
\begin{aligned}
\partial_q \overset{-}{b_{q-1}} &= b_{q-1} \\
\overset{-}{b_{q-1}} &= \partial_q^{-1} b_{q-1} \\
\overset{-}{b_{q-1}} &= \partial_q^*(-\Delta)^{-1} b_{q-1} \\
\overset{-}{\dot{\mathbf{B}}_{q-1}} &= \partial_q^*(-\Delta)^{-1} \dot{\mathbf{B}}_{q-1}
\end{aligned}
$$

Now we consider α_q, the transition matrix (or change of basis) of $\mathbf{b}_q$ and $\overset{-}{\mathbf{b}_{q-1}}$:

$$
\begin{aligned}
\mathbf{b}_q &= \alpha_q \ \overset{-}{\mathbf{b}_{q-1}} \\
\alpha_q{}^*(\mathbf{b}_q) &= \overset{-}{\mathbf{b}_{q-1}}
\end{aligned}
$$

then;

$$
\begin{aligned}
\alpha_q &= \dot{\mathbf{B}}_q \oplus \overset{-}{\dot{\mathbf{B}}_{q-1}} \\
\alpha_q{}^* &= \dot{\mathbf{B}}_q{}^* \oplus \overset{-}{\dot{\mathbf{B}}_{q-1}}{}^*
\end{aligned}
$$

$$
\begin{aligned}
\alpha_q \alpha_q^* &= \dot{B}_q \dot{B}_q^* \oplus \tilde{B}_{q-1} \tilde{B}_{q-1}^* \\
&= \dot{B}_q \dot{B}_q^* \oplus \dot{B}_{q-1} \left(-\triangle_{q-1}\right)^{-1} \partial^* \partial \left(-\triangle_{q-1}\right)^{*-1} \dot{B}_{q-1}^* \\
&= \dot{B}_q \dot{B}_q^* \oplus \dot{B}_{q-1} \left(-\triangle_{q-1}\right)^{*-1} \dot{B}_{q-1}^*
\end{aligned}
$$

On the other hand,

$$\det \alpha_q \alpha_q^* = \left| \left[b_q, \tilde{b}_{q-1}/e_q \right] \right|^2$$

Then,

$$
\begin{aligned}
\left| \left[b_q, \tilde{b}_{q-1}/e_q \right] \right|^2 &= \det \left(\dot{B}_q \dot{B}_q^* \right) \det \left(\dot{B}_{q-1} \dot{B}_{q-1}^* \right) \det \left(-\triangle_{q-1}\right)^{-1} \\
&= \det \left(\dot{B}_q \dot{B}_q^* \right) \det \left(\dot{B}_{q-1} \dot{B}_{q-1}^* \right) \left(d_{q-1}\right)^{-1}
\end{aligned}
$$

Thus;

$$\left| \left[b_q, \tilde{b}_{q-1}/e_q \right] \right|^2 = \det \left(\dot{B}_q \dot{B}_q^* \right) \det \left(\dot{B}_{q-1} \dot{B}_{q-1}^* \right) \left(d_q d_{q-1}\right)$$

$$2\log\left|\left[b_q,\tilde{b}_{q-1}/e_q\right]\right| = \log\left(\det\left(\dot{B}_q\dot{B}_q^*\right)\right) + \log\left(\dot{B}_{q-1}\dot{B}_{q-1}^*\right) + \log\left(d_{q-1}\right)^{-1}$$

$$\begin{aligned}
2\log\left|\left[b_q,\tilde{b}_{q-1}/e_q\right]\right| &= \log\left(\det\left(\dot{B}_q\dot{B}_q^*\right)\right) + \log\left(\dot{B}_{q-1}\dot{B}_{q-1}^*\right) + (-1)\left[\log d_q + \log d_{q-1}\right] \\
\log\left|\left[b_q,\tilde{b}_{q-1}/e_q\right]\right| &= \frac{1}{2}\left(\log(1) + \log(1) + (-1)\log d_{q-1}\right) \\
\log\left|\left[b_q,\tilde{b}_{q-1}/e_q\right]\right| &= \frac{1}{2}(-1)\log d_{q-1}
\end{aligned}$$

Using the equality from the beginning:

$$\begin{aligned}
\log\tau(K,.) &= \sum_{q=1}^{N}(-1)^q\log\left|\left[b_q,\tilde{b}_{q-1}/e_q\right]\right| \\
&= \sum_{q=1}^{N}(-1)^q\left[\frac{1}{2}(-1)\log d_{q-1}\right] \\
&= \frac{-1}{2}\sum_{q=1}^{N}(-1)^q\log d_{q-1} \\
&= \frac{-1}{2}\sum_{q=1}^{N}(-1)^q\log\det\Delta_q\left|_{B_{q-1}}\right. \\
&= \frac{-1}{2}\left\{\sum_{q=1}^{N}(-1)^q\log\det\Delta_q\left|_{B_{q-1}}\right. - N(-1)^N\log\det\Delta_N\left|_{B_{N-1}}\right.\right\} \\
&= \frac{-1}{2}\sum_{q=1}^{N}(-1)^q\log\det\Delta_q\,|B_{q-1} + \sum_{q=0}^{N-1}q(-1)^q\log\det\Delta_q\left|_{B_{q-1}}\right. \\
&- \sum_{q=1}^{N}q(-1)^q\log\det\Delta_q\left|_{B_{q-1}}\right.
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{2}\left\{\sum_{q=1}^{N}(-1)^q \log\det \Delta_q \big|_{B_{q-1}} - \sum_{q=1}^{N} q(-1)^q \log\det \Delta_q \big|_{B_{q-1}} \right. \\
&\quad \left. + \sum_{q=0}^{N-1} q(-1)^q \log\det \Delta_q \big|_{B_{q-1}} \right\} \\
&= \frac{-1}{2}\left\{ -\sum_{q=1}^{N}(q-1)(-1)^q \log\det \Delta_q \big|_{B_{q-1}} + \sum_{q=0}^{N-1} q(-1)^q \log\det \Delta_q \,|B_{q-1} \right\} \\
&= \frac{-1}{2}\left\{ \sum_{q=1}^{N}(q-1)(-1)^{q-1} \log\det \Delta_q \big|_{B_{q-1}} + \sum_{q=0}^{N-1} q(-1)^q \log\det \Delta_q \big|_{B_{q-1}} \right\},\ j = q-1 \\
&= \frac{-1}{2}\left\{ \sum_{j=0}^{N-1} j(-1)^j \log\det \Delta_{j+1} \big|_{B_j} + \sum_{q=0}^{N-1} q(-1)^q \log\det \Delta_q \big|_{B_{q-1}} \right\} \\
&\quad \frac{-1}{2}\left\{ \sum_{j=0}^{N-1} j(-1)^j \log\det(\partial_{j+1}\partial_{j+1}^*) \big|_{B_j} + \sum_{q=0}^{N-1} q(-1)^q \log\det(\partial_q\partial_q^*) \big|_{B_{q-1}} \right\} \\
&= \frac{-1}{2}\left\{ \sum_{q=0}^{N-1} q(-1)^q \log\det(\partial_{q+1}\partial_{q+1}^*) \big|_{B_q} + \sum_{q=0}^{N-1} q(-1)^q \log\det(\partial_q\partial_q^*) \big|_{B_{q-1}} \right\} \\
&= \frac{-1}{2}\left\{ \sum_{q=0}^{N-1} q(-1)^q \log\det(\partial_{q+1}\partial_{q+1}^*) \big|_{B_q} + \sum_{q=0}^{N-1} q(-1)^q \log\det(\partial_q^*\partial_q) \Big|_{\tilde{B}_{q-1}} \right\} \\
&= \frac{-1}{2}\left\{ \sum_{q=0}^{N-1} q(-1)^q \left[\log\det(\partial_{q+1}\partial_{q+1}^*) \big|_{B_q} + \log\det(\partial_q^*\partial_q) \Big|_{\tilde{B}_{q-1}}\right] \right\} \\
&= \frac{-1}{2}\left\{ \sum_{q=0}^{N-1} q(-1)^q \log\left[\left\{\det(\partial_{q+1}\partial_{q+1}^*) \big|_{B_q} * \det(\partial_q^*\partial_q) \Big|_{\tilde{B}_{q-1}}\right\}\right] \right\} \\
&= \frac{-1}{2}\left\{ \sum_{q=0}^{N-1} q(-1)^q \log\det(\Delta_q) \big|_{C_q} \right\} \\
&= \frac{-1}{2}\left\{ \sum_{q=0}^{N} q(-1)^q \log\det(\Delta_q) \right\}
\end{aligned}
$$

□

Applications to knots

Now we consider some computational examples. In accordance with the conventions of Khovanov theory, we will use cochains instead of chains. We set $C^i = D_{-i}$ for a given complex $(D_q)_{q\in\mathbb{Z}}$.

Then, the Laplace operator becomes

$$\Delta_q = -\partial_q \partial_q^* - \partial_{q+1}^* \partial_{q+1} \ .$$

Example 84. *Consider the Hopf link (Example 61). We have*

$$\partial_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$\partial_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

then

$$\begin{aligned}
\Delta_0 &= -\partial_0 \partial_0^* - \partial_1^* \partial_1 \\
&= 0 - \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\Delta_1 &= -\partial_1 \partial_1^* - \partial_2^* \partial_2 \\
&= -\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} -4 & 2 & 0 & 0 \\ 2 & -4 & -2 & 0 \\ 0 & -2 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\Delta_2 = -\partial_2\partial_2^* - \partial_3^*\partial_3$$

$$= - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} - 0$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Example 85. *Consider the left trefoil knot as in Example 62:*

$$\partial_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\partial_2 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\partial_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

The laplacian of the trefoil knot, $\Delta_q = -\partial_q\partial_q^* - \partial_{q+1}^*\partial_{q+1}$*, becomes*

$\Delta_0 = -\partial_0\partial_0^* - \partial_1^*\partial_1$

$$= \mathit{0}\text{-} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Delta_1 = -\partial_1\partial_1^* - \partial_2^*\partial_2$. *A long calculation yields:*

$$\Delta_1 = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 \end{pmatrix}$$

$\Delta_2 = -\partial_2\partial_2^* - \partial_3^*\partial_3$

$\Delta_2 =$

$$
-\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-2 & -1 & 1 & 0 & 0 & 0 \\
-1 & -2 & -1 & 0 & 0 & 0 \\
1 & -1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & -2 & 2 \\
0 & 0 & 0 & -2 & -4 & -2 \\
0 & 0 & 0 & 2 & -2 & -4
\end{pmatrix}
$$

4.2 Product structures for Legendrian knots and PDEs: a manuscript

In this section we show how to construct invariants for Legendrian knots using cohomology operations know as (generalized) Massey products, and we present a construction of differential equations using the Chekanov algebra. we wish to construct these equations as examples of (non-conmutative) integrable equations arising in a geometric setting, as a complement to the abstract analysis of non commutative integrable equations appearing in Olver-Sokolov [24]. We leave as an open problem the question of whether invariants for these equations (or explicit solutions!) give us geometric information on Legendrian knots. This section is written in the form of a manuscript to be submitted for publication. The manuscript is divided in three parts. first, we introduce the algebraic tools needed to define or product invariant. Second, we consider classical (as in May [17] and Kraines [16]) and generalized (after Babenko-Taimanov [1]) Massey products and we prove (very easily) that they provide Legendrian invariants. Third, we consider differential equations of two kinds: We present a Maurer-Cartan equation satisfied by representatives of classical Massey products, thereby giving some dynamical interpretation to the product invariants, and we construct hierarchies of (non-commutative) integrable equations naturally induced by the Chekanov construction reviewed in section 3.2.

With respect to this last part of the manuscript, we present here a motivating example. Let us construct a non-commutative Korteweg-de Vries equation:

Definition 86. *Let (A, ∂) be a differential graded algebra. We consider ∂ as a linear map from A to A and we extend it linearly to a derivation $\widetilde{\partial}$ on $\widetilde{A} = \sum_{n\geq 0} A^{\otimes n}$ simply by stating that $\widetilde{\partial}|_A = \partial$, and that $\widetilde{\partial}$ satisfies the Leibnitz rule with respect to the product $\otimes$.*

Now let $D(A, \widetilde{\partial})$ be the algebra of formal differential symbols of $\widetilde{A}$, this is, the algebra generated by $\widetilde{A}$ and the symbol ε with the relation

$$\xi \circ a = a\xi + \widetilde{\partial}(a) \tag{4.3}$$

for all $a \in \widetilde{A}$. For example, using this definition we have:

$$\xi^n \circ a = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \widetilde{\partial}^j(a)\xi^{n-j} \tag{4.4}$$

for all $a \in A$ y $n \geq 0$. In the algebra $D(A, \widetilde{\partial})$ we can set up *Lax equations*

$$\frac{dL}{dt} = [B, L] \ ,$$

in which $[L, B] = LB - BL$. For instance, set $\xi(u) = u'$ for $u \in \widetilde{A}$, and define

$$B = 4\xi^3 + 6u\xi + 3u'$$

and

$$L = \xi^2 + u \ .$$

We have:

$$[B, L] = (\xi^3 + 6u\xi + 3u')(\xi^2 + u) - (\xi^2 + u)(4\xi^3 + 6u\xi + 3u') \ ,$$

which yields, after a long but straightforward computation,

$$[B, L] = u''' + 3uu' + 3u'u \ .$$

Therefore the Lax equation above becomes

$$u_t = u''' + 3uu' + 3u'u$$

precisely the non-commutative Korteweg-de Vries equation, see [19]. In the manuscript that follow we carry out a similar (but more general) analysis for the case of the Chekanov algebra. We can be much more explicit, since for this algebra the differential ∂ is given by geometric data.

Massey products, A_∞-algebras, differential equations and Chekanov homology[1]

Lida Mendoza and Enrique G. Reyes

Departamento de Matemática y Ciencia de la Computación,

Universidad de Santiago de Chile

Casilla 307 Correo 2, Santiago, Chile.

E-mail: lida.mendoza@usach.cl ; ereyes@fermat.usach.cl

Abstract

We consider (classical and generalized) Massey products on the Chekanov homology of a Legendrian knot, and we prove that they are invariant under Legendrian isotopies. We also construct a minimal A_∞-algebra structure on the Chekanov algebra of a Legendrian knot, we prove that this structure is invariant under Legendrian isotopy, and we observe that its higher multiplications allow us to find representatives for classical Massey products. Finally, we consider differential equations: we remark that these Legendrian invariants admit a "dynamical interpretation", in the sense that they provide solutions for a Maurer-Cartan equation posed on an infinite-dimensional bigraded Lie algebra, and we show how to set up and solve a (twisted) Kadomtsev-Petviashvili hierarchy of equations starting from the Chekanov algebra of a Legendrian knot.

4.2.1 Introduction

We say that a manifold M of dimension $2n+1$ is a contact manifold if it admits a maximally non-integrable distribution η; if we write η as the kernel of a one-form α, [11], then the non-integrability condition translates into the fact that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing. *Legendrian submanifolds* are n-dimensional submanifolds of M which are everywhere tangent to the "contact distribution" η. If $\eta = \ker(\alpha)$, then Legendrian submanifolds correspond to n-dimensional integral submanifolds of the exterior differential system determined by α.

We restrict to the case $n = 1$. In this case, *compact* Legendrian submanifolds are knots. A classical problem is to *classify* Legendrian knots in a given three-dimensional contact manifold M. Because of their definition as integral submanifolds, the classification of Legendrian knots is different from the classification of topological knots: if we define a Legendrian isotopy as a

[1] A slightly reviewed version of this manuscript has been accepted for publication in *Journal of Nonlinear Mathematical Physics*, 2015. The main change between this version and the accepted version is briefly indicated below as a footnote.

deformation of a Legendrian knot through Legendrian knots, it is known that a single isotopy class of topological knots admits infinitely many Legendrian isotopy classes of Legendrian knots, see [11]. Thus, in order to obtain Legendrian classification results one has to go well beyond standard topological invariants such as the Alexander or Jones polynomials.

A powerful Legendrian invariant for Legendrian knots in $M = \mathbb{R}^3$ with the contact structure determined by $\eta = \ker(\alpha)$, in which $\alpha = dz + xdy$, was introduced by Chekanov in [4]. This invariant comes "categorified", in contradistinction with, for instance, the classical Jones polynomial from topological knot theory whose categorification was achieved in [17]: the Chekanov invariant is defined via the homology of a non-commutative differential algebra determined by the contact geometry of the ambient contact manifold (M, η). Certainly, once we have invariants determined by homology, it is natural to investigate whether other similar invariants exist.

We show in this paper that it is indeed possible to construct further Legendrian invariants simply by using classical Massey products [18, 20] and their generalizations [2]. We also apply a classical construction of A_∞-algebras, [13, 21, 30], to the Chekanov algebra of a Legendrian knot, and we observe that this A_∞-algebra is also a Legendrian invariant. Following [19], we observe that this construction allows us to understand the (classical) Massey product invariants in terms of higher multiplications of A_∞-algebras. There are two reasons why this observation may be of importance: first, these higher multiplications can be computed in a relatively straightforward fashion, see [19, 21]; second, it yields a "dynamical interpretation" for Massey product Legendrian invariants as we now explain.

Motivated by Witten's profound work on the Jones polynomial, [29], it is natural to ask about the possible physical interpretation of the invariants considered here. We present a first attempt to such an interpretation: we observe that a result by He, [12], implies that the higher multiplications of the A_∞-algebra constructed from the Chekanov algebra provide a solution to a Maurer-Cartan equation posed on an infinite-dimensional bigraded Lie algebra.

Finally, we consider a twisted Kadomtsev-Petviashvili (KP) hierarchy of equations defined with the help of the Chekanov algebra. One reason for believing that this construction may be of interest is that it provides us with instances of *noncommutative* integrable equations, such as the ones investigated in [24], arising quite naturally from a non-trivial geometric context.

Our work is organized as follows. Section 2 is an introduction to A_∞-algebras, and Section 3 is a rather detailed review of (generalized) Massey products after [18, 20] and [2]. In Section 4 we introduce contact manifolds and Legendrian knots, we summarize the construction of the Chekanov algebra, and we explain in what sense classical and generalized Massey products determine Legendrian invariants. Finally, in Section 5 we show that the Legendrian invariants arising from classical Massey products solve a Maurer-Cartan equation and we introduce our twisted KP hierarchy.

Remark 87. *While writing up our results we found out that previous work on Legendrian knot invariants and classical Massey products (on linearized Chekanov (co)homology) had been carried out in [5]. The existence of this interesting paper prompted us to include generalized Massey products after [20] and [2], which do not appear in [5]. We also note that differential equations are not considered in this reference.*

4.2.2 A_∞-algebras

Let $\mathbb{K}$ be a field, and A a $\mathbb{Z}$-graded $\mathbb{K}$-vector space, $A = \oplus_{i\in\mathbb{Z}} A^i$. An A_∞-algebra structure on A is a family of graded linear maps $m_n : A^{\otimes n} \to A$, $n \geq 1$, such that the degree of m_n is $2-n$ and the identities

$$\sum_{\substack{r+s+t=n \\ r,t\geq 0\,;\, s\geq 1}} (-1)^{rs+t} m_{r+1+t} \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0 \tag{4.5}$$

hold for all $n \geq 1$. For example, if $n = 1$ then $r, t = 0$ and $s = 1$, so that m_1 is a degree 1 map and the identity (4.5) is simply $m_1 \circ m_1 = 0$, that is, (A, m_1) is a cochain differential complex. Also, if $n = 2$, then (4.5) can be written as

$$m_1 \circ m_2 = m_2 \circ (id \otimes m_1) + m_2 \circ (m_1 \otimes id)\ ,$$

and therefore m_2 is a bilinear map which behaves as a multiplication and the differential m_1 satisfies the graded Leibnitz rule with respect to m_2. Note that m_2 is not necessarily associative. Indeed, the third identity arising from (4.5) is

$$m_2 \circ (m_2 \otimes id) - m_2 \circ (id \otimes m_2) = m_1 \circ m_3 + m_3 \circ (id^{\otimes 2} \otimes m_1 + id \otimes m_1 \otimes id + m_1 \otimes id^{\otimes 2})\ , \tag{4.6}$$

so that m_2 is associative if the right hand side of this equation is identically zero. Thus, if $m_3 = 0$, we conclude that (A, m_1, m_2) is a differential graded algebra with a differential of degree 1. Conversely, every differential graded algebra is an A_∞-algebra with $m_3 = m_4 = \cdots = 0$.

Remark 88. *Let us write $d = m_1$ and $d^3 = m_1 \otimes id^{\otimes 2} + id \otimes m_1 \otimes id + id^{\otimes 2} \otimes m_1$. It is trivial to see that $d^3 \circ d^3 = 0$, so that (A, d) and $(A^{\otimes 3}, d^3)$ are cochain differential complexes. In this* *notation, identity (4.6) becomes, simply,*

$$m_2 \circ (m_2 \otimes id) - m_2 \circ (id \otimes m_2) = d \circ m_3 + m_3 \circ d^3 \ .$$

Now, the functions $f = m_2 \circ (m_2 \otimes id)$ and $g = m_2 \circ (id \otimes m_2)$ are cochain maps of degree zero from $A^{\otimes 3}$ to A, and m_3 is a map of degree -1 satisfying the above equation. This says precisely that m_3 is an homotopy between the maps f and g. In other words, m_2 is associative up to *homotopy, and this homotopy is also part of the A_∞ algebra structure.*

A_∞-algebras first appeared in topology, more precisely in the theory of loop spaces, see [25, 26]. A short review of their properties —and a guide to earlier literature— is in [19].

Definition 89. *Let (A, m_n) and (B, m'_n) be two A_∞-algebras. An A_∞-morphism $f : A \to B$ is a family of linear maps, $f_n : A^{\otimes n} \to B$, $n \geq 1$, of degree $1-n$ such that the following Stasheff morphism identities hold:*

$$\sum_{\substack{r+s+t=n \\ r,t\geq 0\,;\, s\geq 1}} (-1)^{rs+t} f_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = \sum_{j=1}^{n} \sum_{i_1+\cdots+i_j=n} (-1)^u m'_j(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_j}) \ , \quad (4.7)$$

where $i_k \geq 1$ for all k and $u = (i_{j-1}-1) + 2(i_{j-2}-1) + \cdots + (j-2)(i_2-1) + (j-1)(i_1-1)$.

Note that the first Stasheff morphism identity is simply $f_1\, m_1 = m'_1\, f_1$ that is, it says that f_1 is a cochain map. We say that a morphism f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism of complexes, *i.e.* the induced map $H(f_1) : H(A) \to H(B)$ is an isomorphism.

Now we review Merkulov's construction [21] of an A_∞-algebra starting from a differential graded algebra. As we already explained, every differential graded algebra (A, d) *is* an A_∞ algebra, but the importance of [21] is that it allows us to construct an explicit A_∞ structure on A with non-zero higher multiplications. We recall that if ϕ, ψ are two graded linear maps on the differential graded algebra (A, d), the *supercommutator* of ϕ and ψ is $[\phi, \psi] = \phi\,\psi - (-1)^{|\phi|\,|\psi|}\psi\,\phi$. Merkulov's construction relies on the following assumption:

(M) Let (A, d) be a differential graded algebra. We assume that there exist a subcomplex W of A, and a vector space homomorphism $Q : A \to A$ of degree -1, such that the image of the map $Id - [d, Q] : A \to A$ is in fact in W.

Note that it is not required that W is a subalgebra of A. We define a sequence of linear maps $\lambda_n : A^{\otimes n} \to A$, $n \geq 1$, as follows: λ_1 is determined only by the condition $Q\,\lambda_1 = -Id$, and for

$n \geq 2$ we set

$$\lambda_2(v \otimes w) = v \cdot w, \tag{4.8}$$

$$\lambda_n = \sum_{s+t=n\,;\, s,t\geq 1} (-1)^{s+1} \lambda_2[Q\lambda_s \otimes Q\lambda_t] \,. \tag{4.9}$$

The following theorem holds (Merkulov, [21]):

Theorem 90. *Let (A, d) be a differential graded algebra and assume that condition (M) holds. Define linear maps $m_n : W^{\otimes n} \to W$, $n \geq 1$ via*

$$m_1 = d \,, \tag{4.10}$$

$$m_n = (Id - [d, Q]) \circ \lambda_n \,, \quad n \geq 2 \,, \tag{4.11}$$

in which λ_n are the maps constructed above. The maps m_n satisfy the identities (4.5)*, and therefore they determine an A_∞-algebra structure on the complex W.*

Theorem 90 is also discussed in [30] and [19]. As pointed out for example in [19], it is an explicit realization of a very general result due to Kadeishvili [13]. Now we follow [19] in identifying an appropriate subcomplex W and a linear map Q satisfying assumption (M):

Let $A = \bigoplus_{p\in\mathbb{Z}} A^p$ be a differential graded algebra with differential d of degree 1. We denote by B^p and Z^p the coboundaries and cocycles of A^p. Then, there are subspaces H^p of Z^p and L^p of A^p such that

$$Z^p = B^p \oplus H^p \quad \text{and} \quad A^p = Z^p \oplus L^p = B^p \oplus H^p \oplus L^p \,. \tag{4.12}$$

We set $W = \bigoplus_{p\in\mathbb{Z}} H^p$ and we define the map Q as follows: $Q^p : A^p \to A^{p-1}$ is given by

$$Q^p|_{L^p} = Q^p|_{H^p} = 0 \,, \quad Q^p|_{B^p} = \left(d^{p-1}|_{L^{p-1}}\right)^{-1} \,.$$

It is easy to see that Q determines an homotopy between Id and pr, where $pr : A \to A$ is the projection from A onto W, that is, we have $Id - pr = d\,Q + Q\,d$ and therefore Assumption (M) holds with W and Q as above. We also note that $d|_{H^p} = 0$, so that in fact, the operation m_1 of Theorem 90 is identically zero and therefore (see Remark 1) the operation m_2 is an *associative* multiplication on W. Using the first isomorphism theorem, we identify the complex W with the cohomology of A, that is, $W = ker(d)/Im(d)$. Hereafter we write HA instead of W, to remind us of this identification. Following [19] we rewrite Merkulov's result thus:

Proposition 91. *Consider the functions λ_n defined above, and set $m_n = pr \circ \lambda_n : HA^{\otimes n} \to HA$ for $n \geq 2$. Then, $(HA, 0, m_2, m_3, \dots)$ is an A_∞-algebra and $f = \{-Q\, \lambda_n\}$ is a quasi-isomorphism of A_∞-algebras between HA and A.*

An A_∞-algebra constructed as above is called a *Merkulov model* or a *minimal model* of the differential graded algebra A, in analogy with D. Sullivan's minimal models for differential graded commutative algebras introduced in the context of rational homotopy theory [27]. We also note that in the context of A_∞-algebras, being quasi-isomorphic is a *transitive* property, as stressed for example in [28], and therefore all Merkulov models of A (which obviously depend on the choice of the subspaces H^p and L^p introduced above) are quasi-isomorphic.

4.2.3 Massey products

Once a minimal model $(HA, 0, m_2, m_3, \dots)$ of a differential graded algebra (A, d) is available, it is very natural to investigate the *associative algebra* (HA, m_2) and to ask about the meaning of the higher multiplications m_n, $n \geq 3$. As observed in [19], these higher multiplications are connected to classical Massey products. Since we use (classical, generalized) Massey products to define Legendrian isotopy invariants of Legendrian knots, we review them in some detail. We follow the sign conventions of [20]. In particular, we write $\bar{a} = (-1)^{1+\deg(a)}a$, so that $d\bar{f} = -\overline{df}$ and $\overline{ab} = -\bar{a}\,\bar{b}$.

Classical Massey products

Let (A, d) be a differential graded algebra with $\deg(d) = 1$. If $\alpha_1, \alpha_2 \in HA$, their *length two* Massey product $\langle \alpha_1, \alpha_2 \rangle$ is the singleton $\{\alpha_1\, \alpha_2\}$; we define the length 3 Massey product as follows:

Suppose that $\alpha_1, \alpha_2, \alpha_3 \in HA$ and assume that $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$. We pick representatives $a_{i-1,i} \in A$ of the cohomology classes α_i. Because we are assuming that $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$, there exist cochains a_{02} and a_{13} such that

$$da_{02} = \bar{a}_{01}a_{12}\ , \qquad \text{and} \qquad da_{13} = \bar{a}_{12}a_{23}\ . \tag{4.13}$$

With these choices we can check that $a_{03} = \bar{a}_{02}\, a_{23} + \bar{a}_{01}\, a_{13}$ satisfies $d\, a_{03} = 0$. The length 3 Massey product *of the cocycles* a_{01}, a_{12} and a_{23} is the set $MP_3(a_{01}, a_{12}, a_{23})$ of all cohomology classes of the cocycles $a_{03} = \bar{a}_{02}\, a_{23} + \bar{a}_{01}\, a_{13}$ arising from different choices of cochains a_{02} and a_{13}.

Proposition 92. *The length 3 Massey product $MP_3(a_{01}, a_{12}, a_{23})$ depends only on the cohomology classes of the cocycles a_{01}, a_{12}, a_{23}.*

Indeed, we can check that $MP_3(a_{01}, a_{12}, a_{23}) = MP_3(a_{01} + db, a_{12}, a_{23}) = MP_3(a_{01}, a_{12} + db, a_{23}) = MP_3(a_{01}, a_{12}, a_{23} + db)$ for any cochain b. Proposition 92 follows easily from this observation as, for example, it implies that $MP_3(a_{01} + db_1, a_{12} + db_2, a_{23}) = MP_3(a_{01} + db_1, a_{12}, a_{23}) = MP_3(a_{01}, a_{12}, a_{23})$. We will provide some further details of the proof in the general case of length n Massey products, to be discussed below. This result allows us to make the following definition:

Definition 93. *Let $\alpha_1, \alpha_2, \alpha_3$ be three cohomology classes in HA such that $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$. Their length 3 Massey product is $< \alpha_1, \alpha_2, \alpha_3 >= MP_3(a_{01}, a_{12}, a_{23})$, in which a_{01}, a_{12}, a_{23} are arbitrary cocycle representatives of $\alpha_1, \alpha_2, \alpha_3$.*

Now we consider the general case. Let $(a_1, \ldots, a_n)$ be an n-tuple of cocycles. We say that a collection of cochains (a_{ij}), $0 \leq i < j \leq n$, $(i, j) \neq (0, n)$, is an *MP_n-defining system* for $(a_1, \ldots, a_n)$ if it satisfies the following conditions:

1. $a_{i-1,i} = a_i \quad$ for $1 \leq i \leq n$.
2. $d\, a_{ij} = \sum_{i<r<j} \bar{a}_{ir}\, a_{rj} \quad$ for $0 \leq i < j \leq n$ and $1 < j - i < n$.
3. $\deg(a_{ir}\, a_{rj}) = 1 + \deg(a_{ij}) \quad$ for all $i < r < j$.

Lemma 94. *Property 2 of an MP_n-defining system is consistent with $d^2 = 0$. Moreover, the cochain*

$$a_{0n} = \sum_{0<r<n} \bar{a}_{0r}\, a_{rn} \tag{4.14}$$

is a cocycle.

Proof. We check the first claim by induction. It is straightforward to see that

$$d\left(\sum_{i<r<j} \bar{a}_{ir}\, a_{rj}\right) = \\ \sum_{i<r<j}\ \sum_{i<s<r} (-1)^{\deg(a_{ir})+\deg(a_{is})} a_{is}a_{sr}a_{rj} - \sum_{i<r<j}\ \sum_{r<s<j} (-1)^{1+\deg(a_{rs})} a_{ir}a_{rs}a_{sj}\ ,$$

and consideration of the signs $(-1)^{\deg(a_{ir})+\deg(a_{is})}$ and $(-1)^{1+\deg(a_{rs})}$ using Property 3 above allows us to conclude that the right hand side of this equation is zero. The computations needed to check that the cochain a_{0n} given by (4.14) is a cocycle are similar. ☐

The length n Massey product *of the cocycles* a_i, $1 \leq i \leq n$, is the set $MP_n(a_1, \ldots, a_n)$ of cohomology classes of the cocycles a_{0n} associated to all possible MP_n-defining systems for $(a_1, \ldots, a_n)$. As in the length 3 case we have the following crucial observation:

Proposition 95. *The length n Massey product of the cocycles a_i, $1 \leq i \leq n$, depends only on the cohomology classes of these cocycles.*

Proof. Let us fix t with $1 \leq t \leq n$ and let b be a cochain with $\deg(b) + 1 = \deg(a_t)$. As explained after Proposition 92, it is enough to prove that

$$MP_n(a_1, \ldots, a_t, \ldots, a_n) = MP_n(a_1, \ldots, a_t + db, \ldots, a_n)$$

and, by symmetry, it is enough to check that

$$MP_n(a_1, \ldots, a_t, \ldots, a_n) \subseteq MP_n(a_1, \ldots, a_t + db, \ldots, a_n).$$

Matters being so, let x be a cohomology class in $MP_n(a_1, \ldots, a_t, \ldots, a_n)$. Then, there exists an MP_n-defining system (a_{ij}) for $(a_1, \ldots, a_t, \ldots, a_n)$, $0 \leq i < j \leq n$ and $(i,j) \neq (0,n)$, such that x is the cohomology class of the cocycle $a_{0n} = \sum_{0<r<n} \bar{a}_{0r}\, a_{rn}$. We can exhibit an MP_n-defining system (a'_{ij}) for $(a_1, \ldots, a_t + db, \ldots, a_n)$ such that the corresponding cocycle a'_{0n} is cohomologous to the original cocycle a_{0n} . Indeed, we set:

$$\begin{aligned}
a'_{ij} &= a_{ij} \quad \text{if } i \neq t-1 \quad \text{and } j \neq t\ , \\
a'_{t-1,t} &= a_{t-1,t} + db = a_t + db\ , \\
a'_{it} &= a_{it} - a_{i,t-1}\, b \quad \text{for } i < t-1\ , \\
a'_{t-1,j} &= a_{t-1,j} - \bar{b}\, a_{tj} \quad \text{for } j > t\ .
\end{aligned}$$

It is long, but straightforward, to check that (a'_{ij}) is in fact an MP_n-defining system for $(a_1, \ldots, a_t + db, \ldots, a_n)$. Now we consider the cocycle a'_{0n}. Again, a rather simple calculation yields

$$\begin{aligned}
a'_{0n} &= a_{0n} + (-1)^{\deg(b)} d(b a_{1n}) \quad \text{if } t = 1\ , \\
a'_{0n} &= a_{0n} \quad \text{if } 1 < t < n\ , \\
a'_{0n} &= a_{0n} - d(a_{0,n-1} b) \quad \text{if } t = n\ .
\end{aligned}$$

Thus, the cohomology class of a'_{0n} is also the class x we started with, and we conclude that $x \in MP_n(a_1, \ldots, a_t + db, \ldots, a_n)$. ☐

The proof of Proposition 95 above is modelled after Kraines' work [18]. It allows us to define the length n Massey product on cohomology classes:

Definition 96. *Let $\alpha_1, \ldots, \alpha_n$ be n cohomology classes in HA, let a_i be a cocycle representative of α_i, $1 \leq i \leq n$, and assume that there exists an MP_n-defining system (a_{ij}) for $(a_1, \ldots, a_n)$. Then, the length n Massey product of $\alpha_1, \ldots, \alpha_n$ is $< \alpha_1, \ldots, \alpha_n > = MP_n(a_1, \ldots, a_n)$.*

Remark 97.

1. *Let us assume that the cohomology class α_i belongs to H^{s_i}, $1 \leq i \leq n$, and that the Massey product $< \alpha_1, \ldots, \alpha_n >$ exists, so that there is an MP_n-defining system (a_{ij}) such that the cohomology class of the cocycle $a_{0n} = \sum_{0<r<n} \bar{a}_{0r}\, a_{rn}$ belongs to $< \alpha_1, \ldots, \alpha_n >$. Conditions 1–3 satisfied by (a_{ij}) imply that for each $0 < r < n-1$,*

$$\deg(a_{0r}\, a_{rn}) = s_1 + \cdots + s_n - n + 2 \,,$$

and therefore we conclude that $< \alpha_1, \ldots, \alpha_n > \subseteq H^{s_1 + \cdots + s_n - n + 2}$.

2. *We remark that, as defined, the length n Massey product is a* partial *operation, not defined on arbitrary n-tuples of cohomology classes. A necessary and sufficient condition for the product $< \alpha_1, \ldots, \alpha_n >$ to exist is that the length $(n-1)$ Massey products $< \alpha_1, \ldots, \alpha_{n-1} >$ and $< \alpha_2, \ldots, \alpha_n >$ vanish simultaneously, see [18] and [23] for further information.*

The behavior of Massey products under differential algebra morphisms, see [18] and [20], is crucial for us:

Proposition 98. *Let (R, d) and (S, d') be differential graded algebras. Massey products are natural with respect to differential algebra morphisms, that is, if $f : R \to S$ is a differential algebra morphism and if the Massey product $< \alpha_1, \ldots, \alpha_n >$ exists, so does $< f_*\alpha_1, \ldots, f_*\alpha_n >$, and*

$$f_* < \alpha_1, \ldots, \alpha_n > \subseteq < f_*\alpha_1, \ldots, f_*\alpha_n > \ . \tag{4.15}$$

In particular, if f is a quasi-isomorphism, then (4.15) is an equality.

Finally, it remains the issue of *computing* Massey products. The following result connecting Massey products with A_∞-structures ([19], Theorem 3.1) tells us how to proceed:

Theorem 99. *Let (A, d) be a differential graded algebra. Up to a sign, the higher multiplications on the minimal model HA of A give Massey products: for any $n \geq 3$ if $\alpha_1, \ldots, \alpha_n \in HA$ are such that $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined, then*

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \ldots, \alpha_n \rangle \ ,$$

where $b = 1 + |\alpha_{n-1}| + |\alpha_{n-3}| + |\alpha_{n-5}| + \cdots$.

Generalized Massey products

We generalize the constructions of the previous subsection following Babenko and Taimanov [2]. If $(\mathcal{A}, d)$ is a graded differential algebra over a field k, we let $M(\mathcal{A})$ be the set of all upper triangular infinite matrices with entries in $\mathcal{A}$ such that only finitely many entries are different from zero. Addition and multiplication on $M(\mathcal{A})$ are defined in a natural way. In particular, if $A = (a_{ij})_{i,j\geq 1}$ and $B = (b_{ij})_{i,j\geq 1}$ belong to $M(\mathcal{A})$, then $AB = \left(\sum_{k\geq 1} a_{ik}\, b_{kj}\right)_{i,j\geq 1}$. We also extend the "bar" notation from the previous section, $\overline{a} = (-1)^{1+deg(a)}\, a$ for *homogeneous* elements of $\mathcal{A}$, to a linear map on $\mathcal{A}$ by setting $\overline{\lambda a + b} = \lambda \overline{a} + \overline{b}$ for $\lambda \in k$ and a, b homogeneous. This linear map extends in an obvious way to a linear map on $M(\mathcal{A})$.

We also extend the differential on $\mathcal{A}$ to a differential on $M(\mathcal{A})$ by setting $dA = (da_{ij})_{i,j\geq 1}$ for $A \in M(\mathcal{A})$. Obviously the extended map d is linear and it satisfies $d^2 = 0$. Also, we readily check that

$$d(AB) = (dA)B + \widetilde{A} dB \ ,$$

in which $\widetilde{A} = \left((-1)^{\deg(a_{ij})} a_{ij}\right)$, so that $\widetilde{A} = -(\overline{a}_{ij}) = -\overline{A}$.

Now, let $(a)_{ij}$ be the matrix in which the (i,j)-entry is equal to a and all the other entries are zero. Given $A \in M(\mathcal{A})$ we define $Ker(A)$ as the $\mathcal{A}$-module spanned by the matrices $(1)_{ij}$ such that $A \cdot (1)_{ij} = (1)_{ij} \cdot A = 0$. We note that if $B \in Ker(A)$, then $\overline{B} \in Ker(A)$ as well.

Definition 100. *A matrix $A \in M(\mathcal{A})$ is called a formal connection on $\mathcal{A}$. We say that A is flat if it satisfies the Maurer-Cartan equation*

$$dA - \overline{A}\, A \equiv 0 \quad mod\ Ker(A) \ . \tag{4.16}$$

The matrix $\mu(A) = dA - \overline{A}\, A$ is called the curvature of A.

The existence of generalized Massey products is a consequence of the following result:

Proposition 101. *Let A be a flat formal connection on $\mathcal{A}$. Then, the matrix $\mu(A)$ is closed.*

Proof. We compute:

$$d(dA - \overline{A}A) = -\{(d\overline{A})A + \widetilde{\overline{A}} dA\} = -(\mu(\overline{A}) + \overline{\overline{A}}\,\overline{A})A + A(\mu(A) + \overline{A}A) = -\mu(\overline{A})A + A\mu(A) \ .$$

On the other hand,

$$-\mu(\overline{A}) = \overline{dA} - \overline{\overline{A}A} = \overline{dA - \overline{A}A} = \overline{\mu(A)} \ ,$$

and so

$$d\mu(A) = \overline{\mu(A)}\, A + A\, \mu(A) \ .$$

Now, since A is flat, $\mu(A)$ and $\overline{\mu(A)}$ belong to $Ker(A)$, and therefore $d\mu(A) = 0$. □

It follows from Proposition 101 that the entries of the curvature matrix $\mu(A) = (\mu_{ij})_{i,j\geq 1}$ of a flat formal connection A determine a matrix of cohomology classes $([\mu_{ij}])_{i,j\geq 1}$. After Babenko and Taimanov [2] we make the following definition:

Definition 102. *Let A be a flat formal connection on $\mathcal{A}$ and let $\mu(A) = (\mu_{ij})_{i,j\geq 1}$ be the corresponding curvature matrix. The generalized Massey product corresponding to A is the matrix of cohomology classes $[\mu(A)] = ([\mu_{ij}])_{i,j\geq 1}$.*

We note that as they stand, generalized Massey products are defined on $\mathcal{A}$, and not on the cohomology of $\mathcal{A}$. However, Babenko and Taimanov generalize Proposition 95 on classical Massey products in a very interesting fashion. First of all, we make the following definition.

Definition 103. *Let $A = (a_{ij})$ be a flat formal connection on $\mathcal{A}$. The initial data of the Maurer-Cartan equation* (4.16) *is the set of all cohomology classes of entries a_{ij} of A which are cocycles of A.*

It can be checked, see [2, Prop. 1], that the matrix of cohomology classes $([\mu_{ij}])_{i,j\geq 1}$ of $\mu(A)$, in which A is a flat connection, depends only on the initial data of the Maurer-Cartan equation $\mu(A) \equiv 0 \mod Ker(A)$, induced by A. Thus, this product can be considered as defined in the cohomology of $\mathcal{A}$. In [2] is shown that it is a true generalization of the classical Massey product considered in the previous section, and also of the matric Massey products introduced by May in [20].

The following proposition, [2, Prop. 2], generalizes Proposition 98:

Proposition 104. *Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of differential graded algebras. We induce a map $\widehat{f} : M(\mathcal{A}) \to M(\mathcal{B})$ via $\widehat{f}((a_{ij})_{i,j\geq 1}) = (f(a_{ij}))_{i,j\geq 1}$. This map takes flat connections in $\mathcal{A}$ to flat connections in $\mathcal{B}$ and therefore we obtain a map on generalized Massey products*

$$f^*([\mu(A)]) = [\mu(\widehat{f}(A))] \ .$$

Moreover, if f is a quasi-isomorphism, then f^* is one-to-one.

4.2.4 Legendrian knots and the Chekanov algebra

Contact structures

Consider a $(2n+1)$-dimensional manifold M together with a differential 1-form α which satisfies the condition

$$\alpha \wedge (d\alpha)^n \neq 0 \,. \tag{4.17}$$

Such a form is called a contact form, and the pair (M, α) is called a contact manifold. If we set $\eta = \ker(\alpha)$, then η is a maximally non-integrable distribution on M and we recover the definition of contact manifolds appearing in [11].

Our main example of a contact manifold is $M = \mathbb{R}^3$ with $\alpha = dz + xdy$. It is easy to see that in this case $\eta = \ker(\alpha) = \langle \partial_x, \partial_y - x\,\partial_z \rangle$, see Figure 1.

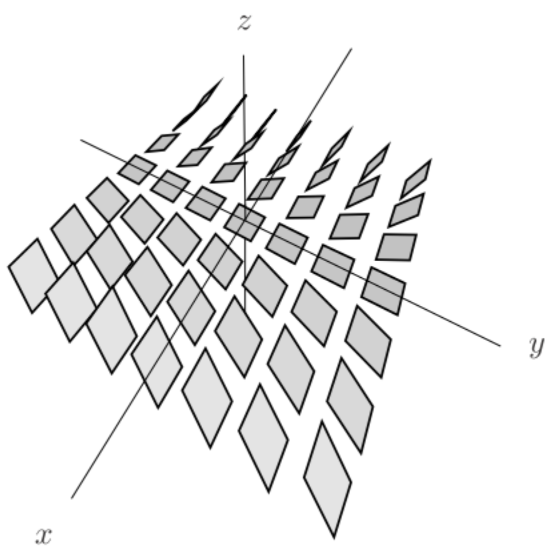


Figure 4.1: The contact distribution η on $\mathbb{R}^3$.

Remark 105. *It has been observed several times (see for instance [8] or the more recent [3]) that there exists a relationship between Lorentzian and contact geometry. Indeed, it is not difficult to define a Lorentzian metric on a contact manifold (M, α) so that the contact distribution is spacelike and the timelike direction is determined by the* Reeb vector field *of α (*i.e.*, the vector field R determined by the conditions $R \in ker(d\alpha)$ and $\alpha(R) = 1$, see [11]). In the case $(M, \alpha) = (\mathbb{R}^3, dz + xdy)$, we simply set*

$$ds^2 = dx^2 + dy^2 - (dz + xdy)^2 \,.$$

Elementary calculations yield that indeed ∂_x and $\partial_y - x\,\partial_z$ are spacelike and that the Reeb vector field $R = \partial_z$ is timelike. Now, an obvious question is whether this relation between contact and Lorentzian geometry may be useful for physics. We are not overly optimistic: we notice that the above metric satisfies the Einstein equations $(R_{ab} - (1/2)Rg_{ab}) + \Lambda g_{ab} = 8\pi T_{ab}$ with $\Lambda = -3/4$ and non-zero components of T_{ab} given by $T_{11} = T_{22} = -1/8\pi$. However*, the very definition of contact structures implies that this spacetime model does not admit Cauchy surfaces, even locally. Thus, in this naturally constructed model is not possible to set up sensible evolution problems.*

Legendrian knots

Definition 106. *A Legendrian knot in a three-dimensional contact manifold (M, α), is an embedded circle $L \subset M$ which is always tangent to the distribution $\eta = ker(\alpha)$. In other words, a Legendrian knot is a compact one-dimensional integral submanifold of η.*

Legendrian knots always exist, see [11, Theorem 3.3.1]: given an arbitrary knot $f : S^1 \to M$, there exists a Legendrian knot in M which is isotopic (in the topological sense) to f.

We also specify when two Legendrian knots K_0 and K_1 are equivalent: we say that they are *Legendrian isotopic* if there is a Legendrian isotopy between them, this is, there exists a smooth family of Legendrian knots L_t, $t \in [0, 1]$, with $L_i = K_i$, para $i = 0, 1$.

Hereafter we consider only the contact manifold $(\mathbb{R}^3, \alpha)$, in which $\alpha = dz + xdy$, and η will always represent the maximally non-integrable distribution $\ker(\alpha)$ on $\mathbb{R}^3$.

Definition 107. *Consider a Legendrian knot K in $(\mathbb{R}^3, \alpha)$ given by $\gamma(s) = (x(s), y(s), z(s))$, $s \in S^1$.*

1. *The front projection of $\gamma(s) = (x(s), y(s), z(s))$ in $(\mathbb{R}^3, \eta)$, is the curve $\gamma_F(s) = (x(s), z(s))$ in the xz-plane. We denote this projection by $\Pi_F(K)$.*

2. *The Lagrangian projection of $\gamma(s) = (x(s), y(s), z(s))$ in $(\mathbb{R}^3, \alpha)$ is the curve $\gamma_L(s) = (x(s), y(s))$ in the xy-plane. We denote this projection by $\Pi_L(K)$.*

Generically, the projection $\Pi_L(K)$ is an immersed curve with only double points.

The Chekanov algebra

Chekanov homology is the homology of a particular differential algebra $(\mathcal{A}, \partial)$ constructed using the crossings of the Lagrangian projection $\Pi_L(K)$ of a given Legendrian knot K. We denote by

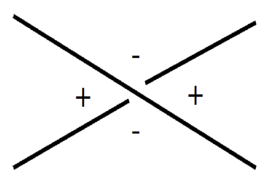
$e = \{a_1, ..., a_n\}$ the double points in the Lagrangian projection $\Pi_L(K)$, and we define $\mathcal{A}$ as the unitary tensor algebra over $\mathbb{Z}_2$ generated by the set e. The unit corresponds to the empty word.

First of all we describe the grading of $\mathcal{A}$. Suppose that we are given an immersion $\widetilde{\gamma} : S^1 \to \mathbb{R}^2$; we define the winding number of $\widetilde{\gamma}$ as $wind(\widetilde{\gamma}) = deg(d\widetilde{\gamma}/ds)$. Then we can check, [11], that $rot(K) = wind(\Pi_L(K))$ is a Legendrian invariant of K. We define a function from the set e to $\mathbb{Z}/2rot(K)$: if $a \in e$, we take a regular path $\gamma : [0, \pi] \to \Pi_L(K)$ from a to itself, starting from the upper strand (the one with bigger z-coordinate) to the lower strand (the one with smaller z-coordinate), and we define a curve $\Gamma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{RP}^1$ by taking the projection of $d\gamma/ds(s)$, $s \in [0, \pi]$ and then clockwise rotating from $[d\gamma/ds(\pi)]$ to $[d\gamma/ds(0)]$ for $s \in [\pi, 2\pi]$. We define

$$deg(a) = deg(\Gamma) \qquad mod\ 2\, rot(K)\ ,$$

and we extend to a full grading of $\mathcal{A}$ via $deg(a \otimes b) = deg(a) + \deg(b)$. Chekanov explains in [4] why the grading takes place in $\mathbb{Z}/2rot(K)$ and not in $\mathbb{Z}$.

Now we define the differential. Let us fix a double point a in $\Pi_L(K)$. Then, there exist two lines L_1 and L_2 which locally divide the plane into four quadrants. We equip each quadrant with a sign in the following way:



Now, given a and $b_1, ..., b_n \in e$, we define $\Delta(a; b_1, \cdots, b_n)$ as the set of immersed polygons in $\mathbb{R}^2$ with edges in $\Pi_L(K)$ and vertices in $a, b_1, ..., b_n$, and which cover an "up" quadrant near b and "down"quadrants near $a_1, \ldots, a_n$. We denote by $\#\Delta(a; b_1, \cdots, b_n)$ the cardinality of $\Delta(a; b_1, \cdots, b_n)$ mod 2. Then we define the differential ∂ at a by

$$\partial(a) = \sum_{n\in\mathbb{N}} \sum_{(b_1,\cdots,b_n)\in e^n} \#\Delta(a; b_1, \cdots, b_n) b_1 \cdots b_n\ .$$

We extend ∂ to all $\mathcal{A}$ by linearity and the graded Leibnitz rule, $\partial(ab) = (\partial a)b + (-1)^{|a|}a(\partial b)$. The fundamental theorem proven by Chekanov in [4] is:

Theorem 108. *The map ∂ is a differential on $\mathcal{A}$ of degree -1. Moreover, the homology of $(\mathcal{A}, \partial)$ is unchanged under Legendrian isotopy: the homology rings of isotopic Legendrian knots are isomorphic as graded rings.*

The A_∞-algebra of a Lengendrian knot and Legendrian invariants

We[2] consider the full Chekanov algebra $\tilde{A} = \oplus_{i=0}^{\infty} A_i$ equipped with a differential $\partial : \tilde{A} \to \tilde{A}$ of degree -1. We set $\partial_i = \partial|_{A_i} : A_i \to A_{i-1}$ and $C^{-i} = A_i$. Then, as is standard, $C = \oplus_{i=0}^{\infty} C^i$ is a $\mathbb{Z}$-graded differential algebra with a degree 1 differential which we continue denoting by ∂. We apply Merkulov's result recalled in Section 3 and we obtain:

Theorem 109. *Let K be a Legendrian knot. There exists an A_∞-algebra structure on the Chekanov cohomology $CH(K)$ of K: there exist higher multiplications m_n, $n \geq 2$, on $CH(K)$ such that $(CH(K), 0, m_2, m_3, \cdots)$ satisfy the higher order associative identities* (4.5). *Moreover, there exists a quasi-isomorphism of A_∞-algebras between $CH(K)$ and the Chekanov algebra $C(K)$.*

Let us consider two Legendrian knots K_1 and K_2 connected by a Legendrian isotopy. Then, the homology rings of their corresponding Chekanov algebras, $C(K_1)$ and $C(K_2)$ respectively, are isomorphic. The transitivity of quasi-isomorphisms in the context of A_∞-algebras imply that the minimal models of $C(K_1)$ and $C(K_2)$, $CH(K_1)$ and $CH(K_2)$ respectively, are quasi-isomorphic. Now, $CH(K_2)$ *is also* a minimal model for $CH(K_1)$, again because of the transitivity of quasi-isomorphisms for A_∞-algebras. It follows from the analysis carried out by Kajiura (see [13], Corollaries 5.8 and 5.10) that $CH(K_1)$ and $CH(K_2)$ are actually A_∞-isomorphic. We have:

Corollary 110. *The minimal model $(CH(K), 0, m_2, m_3, \cdots)$ of the Chekanov algebra $C(K)$ of a Legendrian knot K is invariant under Legendrian isotopy.*

Now, since the A_∞-algebras $CH(K_1)$ and $CH(K_2)$ are minimal, they are not only isomorphic as A_∞-algebras but also they are isomorphic as associative rings. The naturality of classical and generalized Massey products (Propositions 98 and 104) yield the following result:

Corollary 111. *The classical and generalized Massey products of the Chekanov algebra $C(K)$ of a Legendrian knot K are invariant under Legendrian isotopy. Moreover, in the classical case, we can effectively compute these invariants using the A_∞-algebra structure of $CH(K)$, as explained in Theorem* 99.

The invariants defined here *are* useful. Civan and his coworkers prove in [5] the existence of an A_∞-algebra structure on a linearized complex $LC(K)$ built from the Chekanov algebra, see [4], and they show that there exists an infinite family of knots that are distinguishable from their Legendrian mirrors by using classical Massey products on the cohomology of $LC(K)$.

[2] In this part of the paper we need to use a full $\mathbb{Z}$-graduation in order to connect the Chekanov theory with Massey products and A_∞-algebras. As explained in the accepted version of this manuscript and 4.2.5 below, this is possible to do thanks to the work carried out in [10].

4.2.5 Chekanov algebra and differential equations

In this section we prove that the invariants constructed in the previous section can be provided with a dynamical interpretation, in the sense that they can be considered as solutions of a differential equation of Maurer-Cartan type, and we also construct nonlinear evolution equations on the Chekanov algebra. As we stated in Section 1, we believe these examples of differential equations are interesting because they are instances of *noncommutative* equations, such as the ones investigated in [24], which arise quite naturally from a non-trivial geometric context.

Maurer-Cartan equations

We follow [12]. Let A be a $\mathbb{Z}$-graded associative algebra over a field $\mathbb{K}$. We use $|\cdot|$ to denote the degree of homogeneous elements of A. We consider the tensor coalgebra

$$T(A) = \bigoplus_{n\geq 1} A^{\otimes n}$$

equipped with the coassociative coproduct Δ uniquely determined by

$$\Delta(x) = \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)$$

in which $x = a_1 \otimes \cdots \otimes a_n \in T(A)$. This coalgebra admits a natural $\mathbb{Z} \times \mathbb{N}$ bi-graduation,

$$\mathrm{bideg}(a_1 \otimes \cdots \otimes a_n) = \left(\sum |a_i| \, , \, n \right) ,$$

if $a_i \in A$ are homogeneous elements of A. This bi-graduation induces a bi-graduation on $Hom(T(A), A)$: $\mathrm{bideg}(\varphi) = (i, j)$ if and only if φ is a graded $\mathbb{K}$-linear map of degree i and $\varphi : A^{\otimes j+1} \to A$. We write $C^{i,j}(A) = Hom^i(A^{\otimes j+1}, A)$.

Now we define

$$L = \bigoplus_{\substack{i\in\mathbb{Z} \\ j\geq 0}} C^{i,j}(A) \, . \tag{4.18}$$

A crucial observation, [12], is that L is a bi-graded differential Lie algebra. Its Lie bracket and differential are defined as follows. For $\varphi \in C^{i,j}(A)$ and $\phi \in C^{s,t}(A)$, $\varphi \diamond \phi \in C^{i+s\,,\,j+t}(A)$ is the map

$$\varphi \diamond \phi(a_1, a_2, \ldots, a_{j+t+1}) = \sum_{k\leq j} (-1)^{\epsilon} \varphi(a_1, \ldots, a_k \phi(a_{k+1}, \ldots, a_{k+t+1}), a_{(k+1)+t+1}, \ldots, a_{j+t+1}) \, ,$$

where $\epsilon = s\sum_{p=1}^{k}|a_p| + k\,t$, and we are identifying homogeneous elements of the form $a_1 \otimes \cdots \otimes a_r$ with $(a_1, \ldots, a_r)$. We then define the bigraded Lie bracket $[\ ,\ ] : L \otimes L \to L$ as

$$[\varphi, \phi] = \varphi \diamond \phi - (-1)^{is+jt}\phi \diamond \varphi\ ,$$

in which $\varphi \in C^{i,j}(A)$ and $\phi \in C^{s,t}(A)$, and the bigraded differential δ —of bidegree $(0,1)$— as $\delta = [m, \cdot]$, in which $m \in C^{0,1}(A) = Hom^0(A \otimes A, A)$ denotes the multiplication operator of A.

Now we consider an A_∞-algebra $(A, m_1, m_2, \ldots)$ with differential $m_1 = 0$, so that (A, m_2) is an associative graded algebra. As in Section 3, we have $A = \oplus_{i\in\mathbb{Z}} A^i$, $m_n : A^{\otimes n} \to A$, $n \geq 1$, and $\deg(m_n) = 2 - n$. We set $\widetilde{m}_n = m_{n+2}$ and $\widetilde{A}^n = A^{-n}$. Then, $\widetilde{m}_n : \widetilde{A}^{\otimes n+2} \to \widetilde{A}$ and $\deg(\widetilde{m}_n) = n$, so that $\widetilde{m}_n \in Hom^n(\widetilde{A}^{\otimes(n+1)+1}, \widetilde{A})$ and $\operatorname{bideg}(\widetilde{m}_n) = (n, n+1)$. In terms of $\widetilde{A}$ and operations $\widetilde{m}_n$, identities (4.5) now read

$$\sum_{\substack{l+j=n\\ 0\leq i\leq l+1\\ l,j\geq 0}} (-1)^k \widetilde{m}_l(a_1, \ldots, a_i, \widetilde{m}_j(a_{i+1}, \ldots, a_{i+j+2}), a_{i+j+3}, \ldots, a_{n+3}) = 0\ , \tag{4.19}$$

for $n \geq 0$, in which $0 \leq i \leq n+3$, and $k = j(|a_1| + \cdots + |a_i|) + i + j(l - i - 1)$.

We consider the minimal model $(CH(K), 0, m_2, m_3, \ldots)$ of a Legendrian knot K constructed in Section 5. With notation as above, Theorem 2.2 of [12] yields:

Theorem 112. *Let F be the bi-graded cocommutative coalgebra determined by the conditions*

$$F = span\{f_1, f_2, \ldots\}\ ; \quad bideg(f_i) = (i,i)\ ; \quad \Delta : F \to F \otimes F\ ; \quad \Delta(f_n) = \sum_{\substack{i+j=n\\ i,j\geq 1}} (-1)^{ij} f_i \otimes f_j\ .$$

Consider the $\mathbb{Z}$-graded vector space $\widetilde{CH(K)}$ equipped with multiplication operators $\{\widetilde{m}_n\}_{n\geq 0}$ and construct the bi-graded differential Lie algebra L as in (4.18). Define also a linear map $\alpha : F \to L$ via $f_i \mapsto \widetilde{m}_i$, $bideg(\alpha) = (0,1)$. This map satisfies the Maurer-Cartan equation

$$\delta \circ \alpha + \frac{1}{2}\mu \circ (\alpha \oplus \alpha) \circ \Delta = 0\ , \tag{4.20}$$

in which μ indicates the Lie product of L.

Since we observed in Theorem 99 that the functions $\tilde{m}_n$ belong to the higher Massey products of the Chekanov algebra $C(K)$ of the Legendrian knot K, we interpret Theorem 112 as providing a dynamic interpretation for our higher order Legendrian invariants.

Remark 113. *Maurer-Cartan equations, this time posed on A_∞-algebras, have been considered by Kajiura in [14, 15]. He has made two important observations. First, he has pointed out that the field equations of motion of string field theory are the Maurer-Cartan equations on an A_∞-algebra determined by the string theory in question. Second, he has remarked that the construction of a minimal model for an A_∞-algebra $\mathcal{A}$ (a generalization of Theorem 90, see [13, 21] and [14, 15]) amounts to constructing a solution to a Maurer-Cartan equation on $\mathcal{A}$.*

Integrable equations

We begin with the Chekanov algebra $(C(K), \partial)$ of a Legendrian knot K, and we assume that this algebra is generated by $\{a_1, \ldots, a_n\}$. In order to work with a differential graded algebra over a field, we abelianize $(C(K), \partial)$ following [10]. We consider the free associative algebra $C(K)_{\mathbb{Q}}$ generated by $\{a_1, \ldots, a_n\}$ over $\mathbb{Q}[t, t^{-1}]$, in which t is a formal parameter. It is shown in [10] that this algebra can be equipped with a $\mathbb{Z}$-grading (which reduces to Chekanov's grading if we set formally $t = 1$) and a differential ∂ satisfying $\partial(\mathbb{Q}[t, t^{-1}]) = 0$ and

$$\partial(v\, w) = (\partial v)\, w + (-1)^{deg(v)}\, v\, (\partial w)\ . \tag{4.21}$$

The algebra $C(K)_{\mathbb{Q}}$ is made into a (graded) commutative algebra by setting

$$w\, v = (-1)^{deg(v)\, deg(w)} v\, w\ .$$

It is proven in [10] that the homology of $C(K)_{\mathbb{Q}}$ is an invariant of the Legendrian isotopy class of K.

Now we linearize. We introduce a word-length filtration on $C(K)_{\mathbb{Q}}$ as follows: $C(K)^n_{\mathbb{Q}}$ is the subalgebra of $C(K)_{\mathbb{Q}}$ generated, as a vector space over $\mathbb{Q}[t, t^{-1}]$, by all words in $C(K)_{\mathbb{Q}}$ of length at least n. The *linearization* $LC(K)_{\mathbb{Q}}$ of $C(K)_{\mathbb{Q}}$ is the quotient space $C(K)_{\mathbb{Q}}/C(K)^2_{\mathbb{Q}}$, which we can consider as being embedded into $C(K)_{\mathbb{Q}}$. For each generator a_i we set $\partial_1(a_i) = \pi \circ \partial(a_i)$, in which $\pi : C(K)_{\mathbb{Q}} \to LC(K)_{\mathbb{Q}}$ is the standard projection, and we obtain a $\mathbb{Q}[t, t^{-1}]$-linear map $\partial_1 : LC(K)_{\mathbb{Q}} \to LC(K)_{\mathbb{Q}}$. Extending this map to $C(K)_{\mathbb{Q}}$ so that the extension satisfies the graded Leibniz rule (4.21), we obtain a *graded derivation* δ on $C(K)_{\mathbb{Q}}$. We consider $C(K)_{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra equipped with the derivation δ. The (graded) commutative associative algebra $(C(K)_{\mathbb{Q}}\,, \delta)$ is our basic arena for setting differential equations.

We define a $\mathbb{Q}[t, t^{-1}]$-algebra automorphism $S : C(K)_{\mathbb{Q}} \to C(K)_{\mathbb{Q}}$ by setting
$S(a_i) = (-1)^{deg(a_i)}\, a_i$ and extending linearly. Then, the graded Leibniz rule for δ becomes

$\delta(v\,w) = (\delta\,v)\,w + S(v)\,\delta w$, and the identity $\delta \circ S = S \circ \delta$ also holds. Following Demidov [6, 7] we consider the algebra of twisted pseudo-differential operators ΨDO_S given by

$$\Psi DO_S = \left\{ \sum_{\infty < i \leq n} f_i\, D^i : n \in \mathbb{Z} \text{ and } f_i \in C(K)_{\mathbb{Q}} \right\}$$

with multiplication determined by the rule

$$D^n \cdot f = \sum_{k=0}^{\infty} \begin{pmatrix} n \\ k \end{pmatrix} S^{n-k}(\delta^k f)\, D^{n-k}$$

for any $f \in C(K)_{\mathbb{Q}}$ and $n \in \mathbb{Z}$. The vector space ΨDO_S equipped with this multiplication becomes an associative (but not commutative) algebra called the algebra of *twisted pseudo-differential operators* of $C(K)_{\mathbb{Q}}$. Another instance of twisted algebras of this kind appears in [16].

Now we are ready to introduce differential equations. For $L \in \Psi DO_S$, we consider

$$\frac{\partial L}{\partial t} = \left[(L^{k-1})_+, L \right] \,, \tag{4.22}$$

in which $(\cdot)_+$ indicates projection into the subalgebra of ΨDO_S consisting of *differential* operators. This is our example of a non-commutative differential equation. Indeed, (4.22) gives rise to a (twisted) Kadomtsev-Petviashvili (KP) hierarchy of partial differential differential equations for the coefficients of the pseudo-differential operator L, and it is well-known that it encodes several (hierarchies of) integrable equations such as the Korteweg-de Vries and Boussinesq hierarchies.

We finish this paper with the observation that Equation (4.22) can be solved explicitly in a formal setting. We follow the classical work [22] as retold by [6, 7] and [9].

Let us equip the algebra $C(K)_{\mathbb{Q}}$ with a valuation v (valuations on rings are considered for example in [1]). We let I be the valuation ideal and $\pi : C(K)_{\mathbb{Q}} \to C(K)_{\mathbb{Q}}/I$ the canonical projection. We assume that $v \circ S = v$ and that $\delta I \subset I$, so that in particular $S(I) \subset I$ and the derivation δ and morphism S descend to the quotient ring $C(K)_{\mathbb{Q}}/I$.

Definition 114. *The space of formal pseudo-differential and differential operators of infinite order are, respectively, $\widehat{\Psi DO_S}$ and $\widehat{\mathcal{D}_S}$, in which*

$$\widehat{\Psi DO_S} = \left\{ \sum_{\alpha \in \mathbb{Z}} a_\alpha\, D^\alpha : a_\alpha \in C(K)_{\mathbb{Q}} \text{ and } \exists C \in \mathbb{R}^+, N \in \mathbb{Z}^+ \text{ so that } \pi(a_\alpha) > C\,\alpha - N \;\forall\; \alpha \gg 0 \right\}$$

and

$$\widehat{\mathcal{D}_{\mathcal{S}}} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \, D^\alpha : P \in \widehat{\Psi DO_S} \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\} .$$

We also define the Volterra group (notation as in [6, 7])

$$V_{C(K)_{\mathbb{Q}}} = 1 + \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \, D^\alpha \in \Psi DO_S : a_\alpha = 0 \text{ for } \alpha \geq 0 \right\} .$$

We have the fundamental result

Theorem 115. *The sets*

$$\widehat{\Psi DO_S}^{\times} = \{ P \in \widehat{\Psi DO_S} : \pi(P) \in V_{C(K)_{\mathbb{Q}}/I} \}$$

and

$$\widehat{\mathcal{D}_{\mathcal{S}}}^{\times} = \{ X \in \widehat{\mathcal{D}_{\mathcal{S}}} : \pi(P) = 1 \}$$

are groups: for each P in $\widehat{\Psi DO_S}^{\times}$ and each X in $\widehat{\mathcal{D}_{\mathcal{S}}}^{\times}$ there exist unique inverses given by $P^{-1} = \sum_{n \geq 0} (1-P)^n$ and $X^{-1} = \sum_{n \geq 0} (1-X)^n$. Moreover, for any $P \in \widehat{\Psi DO_S}^{\times}$ there exist unique operators $W \in V_{C(K)_{\mathbb{Q}}}$ and $Y \in \widehat{\mathcal{D}_{\mathcal{S}}}^{\times}$ such that $P = W^{-1} Y$. In other words, the group $\widehat{\Psi DO_S}^{\times}$ admits the global factorization

$$\widehat{\Psi DO_S}^{\times} = V_{C(K)_{\mathbb{Q}}} \, \widehat{\mathcal{D}_{\mathcal{S}}}^{\times} .$$

This theorem is essentially due to Mulase, see [22]; a recent detailed account of Mulase's result is in [9]. The twisted version of Mulase's factorization theorem (the version which we need here) is proven in [7]. The importance of Theorem 115 for us is that it allows us to solve (4.22). Indeed, reasoning as in [9] we have:

Theorem 116. *Consider the system of equations*

$$\frac{dL}{dt} = \left[(L^k)_+ , L \right] \tag{4.23}$$

with initial condition $L(0) = L_0 \in \Psi DO_S$, and let $Y(t) \in \widehat{\mathcal{D}_{\mathcal{S}}}^{\times}$ and $S(t) \in V_{C(K)_{\mathbb{Q}}}$ be the unique solution to the factorization problem

$$\exp(t \, L_0^k) = S^{-1}(t) \, Y(t) .$$

The unique solution to Equation (4.23) *with* $L(0) = L_0$ *is*

$$L(t) = Y\, L_0\, Y^{-1} \; . \tag{4.24}$$

4.2.6 Conclusions

In this paper we have presented Legendrian invariants for Legendrian knots using classical ([18, 20]) and generalized ([2]) Massey products. Classical (in the sense of [18]) Massey products on *linearized* Chekanov homology have been used earlier in [5] to distinguish between some Legendrian knots and their mirror images. The observation that matric and generalized Massey products as in [20, 2] also yield Legendrian invariants seem to be new. We leave explicit applications for another publication. We have also consider the issue of a possible "physical interpretation" of the Massey product invariants. To examine this issue, we have used the following: equip the (co)homology $CH(K)$ of the Chekanov algebra $C(K)$ of a Legendrian knot K with a minimal A_∞-algebra structure following [13, 14, 15, 21, 19]; we have: *(a)* if $\alpha_1, \ldots, \alpha_n$ are (co)homology classes in $CH(K)$ and m_n is a higher multiplication in this A_∞-algebra, then ([19]) $m_n(\alpha_1, \ldots, \alpha_n)$ belongs to the classical Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$, if this product is defined; and *(b)* the higher multiplication operations of a minimal A_∞-algebra solve a Maurer-Cartan equation posed on a bi-graded differential Lie algebra ([12]). Thus, classical Massey product invariants admit a "dynamical" interpretation, in the sense that representatives of them solve a "nonlinear field equation". We leave open the question of whether this interpretation extends to the case of generalized Massey products, and also whether Legendrian isotopic Legendrian knots determine gauge transformations of our Maurer-Cartan equation. Finally, in this work we have presented a natural class of differential equations arising from the Chekanov algebra of a Legendrian knot and we have showed how to solve it in an algebraic setting following [9]. We note that in contradistinction with the classical KP hierarchy case, see [9] and references therein, we do not have, as yet, a hamiltonian interpretation for (4.22). Also, as A. Eslami-Rad has pointed out to us, it is natural to believe that (4.22) may encode geometric information on Legendrian knots, since the Chekanov algebra *is* a combinatorial translation of contact homology, see [10]. We hope that the explicit solution (4.24) will allow us to extract (at least some of) it.

Acknowledgements Both authors have been partially supported by FONDECYT. L.M. also acknowledges financial support from MECESUP. Important remarks and observations by Anahita Eslami-Rad are also grateful acknowledged. In particular, it was her who stressed the importance

of solving Equation (4.22).

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Part II

Second Part: Graphs

Chapter 5

Introduction to graphs and the categorification of the chromatic polynomial

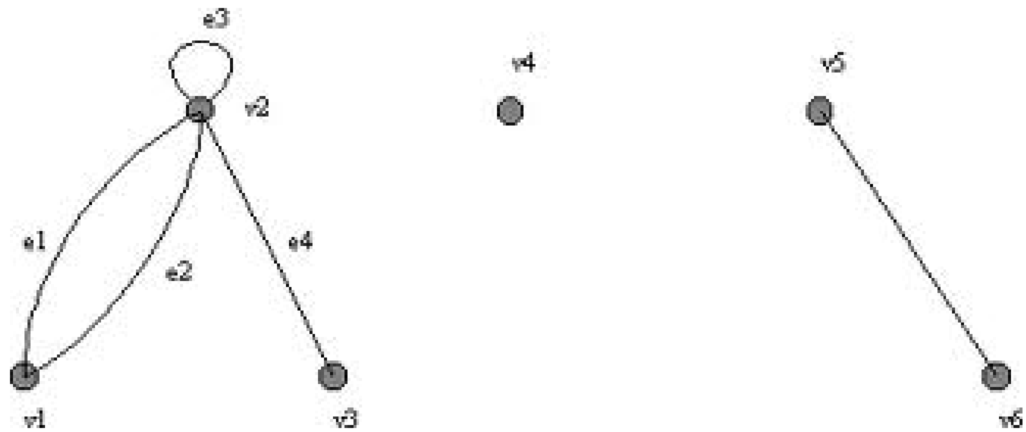
5.1 Introduction

There exist combinatorial invariants not only for knots, but also for graphs. An example is the chromatic polynomial. It is therefore natural to try to categorify graph invariants and to study some of the consequences of this exercise. In this chapter, we review the categorification of the chromatic polynomial carried out by Laura Helme-Guizon in her Ph.D thesis, see [11], and we use it to define a volume form for graphs. Using our previous work in section 4.1 we also define a Laplace operator for graphs. We leave as an open problem to study its analytic properties such as its spectrum, and the precise relationship between this operator and the standard discrete graph Laplace operator.

We begin with some basic definitions of graph theory, mainly to fix our notation. Our main reference is [11].

Definition 117. *A graph G is a duple, $G = (V; E)$, in which E is a finite non-empty set of elements called vertices, and E is a collection of subsets of V of the form $\{a, b\}$, in which $a, b \in E$. These subsets are called edges of G and are pictured simply by lines joining pairs of vertices.*

The vertices of G are usually denoted by $v_1, v_2, v_3, ..., v_n$, and are represented by points. The edges are usually denoted by $e_1, e_2, e_3, ..., e_m$. For example, the following three disjoint figures are graphs:



Definition 118. *Let us fix a graph $G = (V; E)$. A chain in G is a sequence $(e_{i_1}, \dots, e_{i_k})$, $i_1 < i_2 < \cdots < i_k$, of edges of G, such that each side e_{i_j} in the sequence has a vertex in common with the edge $e_{i_{j-1}}$, with the edge $e_{i_{j+1}}$, or with both.*

1. *The length of the chain $(e_{i_1}, \dots, e_{i_k})$, $i_1 < i_2 < \cdots < i_k$, is k.*

2. *An elementary chain is a chain which does not repeat vertices.*

3. *A simple chain is a chain which does not repeat sides.*

Definition 119. *We classify graphs as follows:*

1. *Multigraph: a graph with several edges between pairs of vertices.*

2. *Simple graph: a graph without loops, that is, without sides of the form $\{a, a\}$, $a \in V$.*

3. *Complete graph: a graph G such that for every pair of vertices in G there exists at least one edge joining them.*

4. *Subgraph of $G = (V, E)$: a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$ such that if $\{a, b\} \in E'$ then $a, b \in V'$*

5. *Partial graph of $G = (V, E)$: a graph of the form $G' = (V, E')$ for a subset $E' \subset E$.*

6. *Connected graph: a graph $G = (V, E)$ such that for each pair of vertices $a, b \in V$, there exists a chain in G joining a and b.*

We can also consider oriented graphs if instead of defining edges using subsets $\{a, b\}$ of vertices we consider ordered pairs (a, b) of vertices. Definitions analogous to the ones above can be easily adapted to this case.

5.2 The chromatic polynomial

We introduce some definitions and notations from graph theory which we need for the categorification of the chromatic polynomial.

Give a graph G, we can define a polynomial $C_G(n)$ in the variable n called the chromatic polynomial of G. This polynomial, which has integer coefficients, counts the number of ways that G can be colored using n distinct colors.

5.2.1 The Definition of $C_G(n)$

We define the chromatic polynomial using recursion. First of all we have:

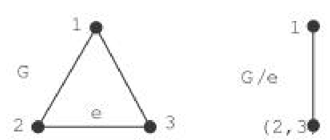
Definition 120. *The chromatic polynomial of a graph consisting of k disconnected vertices is*

$$C_{G_k}(n) = n^k \ .$$

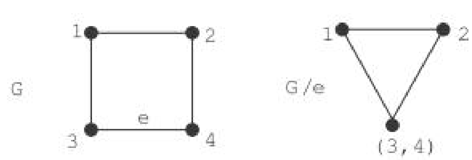
Now we let $G\setminus e$ denote G with the edge e removed, and we let $G\#e$ denote G with the edge e contracted or shrunk along itself so that its two endpoints of e become one. We define the chromatic polynomial $C_G(n)$ recursively as the difference of the chromatic polynomials of the graphs $G\setminus e$ and $G\#e$,

$$C_{G_k}(n) = C_{G\setminus e}(n) - C_{G\#e}(n) \ .$$

Example 121. *We show a graph G and the graph $G\setminus e$:*



Another elementary example is:

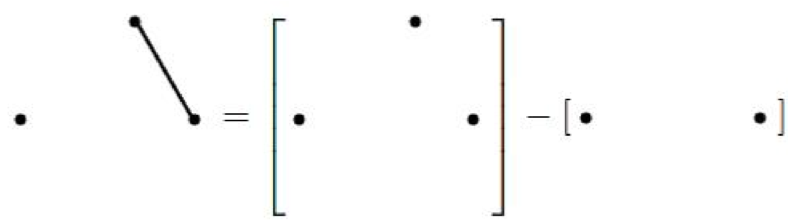


Examples of chromatic polynomials follow:

Example 122. *If G is the trivial graph formed by only one vertex,*

we have $C_{G_k}(n) = n$

Example 123. *Consider the graph G below; we decompose it into $G \diagdown e$ and $G\#e$, and we compute the chromatic polynomial $C_{G_k}(n)$ of $G \diagdown e$ and $G\#e$ as follows:*

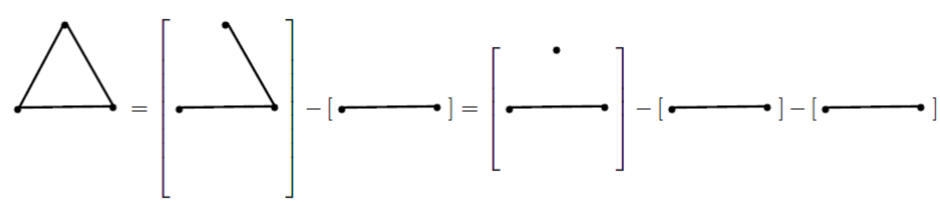


Thus, $C_{G_k}(n) = (\cdot \quad \cdot \quad \cdot) - (\cdot \quad \cdot) = n^3 - n^2$.

In an analogous way we compute the chromatic polynomial of the following simple graphs:

Example 124.

$C_{G_k}(n) = (\cdot \qquad \cdot) - (\cdot) = n^2 - n$;



$$\begin{aligned} C_{G_k}(n) &= n^3 - n^2 - (n^2 - n) - (n^2 - n) \\ &= n^3 - 3n^2 + 2n \end{aligned}$$

Remark 125. *There are other formulas for $C_{G_k}(n)$ which are useful for us, since they show that C_G is indeed an invariant of the graph:*

For each $s \subseteq E(G)$, let $[G : s]$ be the graph whose vertex set is $V(G)$ and whose edge set is s, let $k(s)$ be the number of connected components of $[G : s]$. We have

$$C_G(n) = \sum_{s \subseteq E(G)} (-1)^{|s|} n^{k(s)} .$$

Equivalently, grouping the terms s with the same number of edges yields a "state sum" formula:

$$C_G(n) = \sum_{s \subseteq E(G)} (-1)^{|s|} n^{k(s)} .$$

5.3 Categorification of the chromatic polynomial

5.3.1 Cochain complexes

First of all, we order the edges of our graph G. Our cochain complex depends on this order, so let us assume that $G = (V, E)$ is a graph with n edges, and that these edges have a fixed order. We consider all subgraphs of G which contain all vertices of G. These subgraphs are called *spanning subgraphs* of G and can be described uniquely by words $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ in $\{0, 1\}^n$, in which $\varepsilon_k = 1$ if the kth edge e_k is in the subgraph and $\varepsilon_k = 0$ otherwise.

Conversely, for each $\varepsilon \in \{0, 1\}^n$ we can associate uniquely a set of edges s_ε of G, and hence a subgraph G_ε corresponding to ε.

Now we let M be the graded vector space generated by v_- and v_+ with $degree(v_-) = -1$ and $degree(v_+) = 1$. For each vertex $\epsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ of the cube $\{0, 1\}^n$ we let $k(\varepsilon)$ be the number of connected components of G_ε. We attach a copy of M to each connected component and then we consider the product $M_\varepsilon(G) = M^{\otimes k(\varepsilon)}$.

Definition 126. *The $i-th$ cochain group $C^i(G)$ of the cochain complex $C(G)$ is the direct sum of all vector spaces $M_\varepsilon(G)$ of length $|\varepsilon| = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = i$, that is,*

$$C^i(G) = \bigoplus_{|\varepsilon|=i} M_\varepsilon(G) .$$

Each group $C^i(G)$ is a graded vector space. Its graduation is determined by the degree of its elements. We write

$$C^i(G) = \bigoplus_{j \geq 0} C^{i,j}(G)$$

where $C^{i,j}(G)$ denotes the subspace of elements of degree j of $C^i(G)$.

5.3.2 The differential

Let us repeat how we associate vector spaces to subgraphs of G. Given a graph G, the first thing we do is to order the edges of the graph. Then we apply a procedure identical to the one used for the construction of the Khovanov complex for knots, that is, we consider "$0-$smoothings" and "$1-$smoothings". In the present case, these smoothings refer to the elimination of edges of the graph. In this way we obtain a set of "connected components": each connected component is either a subgraph or a single vertex. To each connected component we associate a copy of a vector space M. For instance, if the graph G has n vertices and we eliminate all edges joining these vertices (this is, we apply all possible $0-$smoothings), then we end up with exactly n connected components, and we associate to these components the vector space $M^{\otimes n}$.

Now we define the differential. Assume that we begin with a maximal set of connected components (that is, we have eliminated all edges of G). Then we proceed to join vertices using $1-$smoothings (in other words, we join vertices through an edge in such a way to obtain an union of subgraphs of G). This process defines a degree-preserving "multiplication". We set $m : M \otimes M \longrightarrow M$ is defined as follows:

$$
\begin{aligned}
m(v_- \otimes v_+) &= m(v_+ \otimes v_-) = v_- \\
m(v_- \otimes v_-) &= 0 \\
m(v_+ \otimes v_+) &= v_+ \, .
\end{aligned}
$$

We use this multiplication to define the differential. Since each space $C^i(G)$ is a direct sum, we define the differential by pieces, as in the Khovanov theory for knots. We follow the explanation in [11].

1. We decide which vertices ε of the hypercube $\{0,1\}^n$ are to be related by maps d_ϵ: set $\varepsilon_i = (0, \cdots, 1, \cdots, 0)$ (that is, ε_i is the vertex with 0 everywhere except at i). Then, there exists a map from ε to $\widehat{\varepsilon}$ if and only if $\widehat{\varepsilon} - \varepsilon = \varepsilon_i$ for some $1 \leq i \leq n$. If there exists a map from ε to $\widehat{\varepsilon}$, this map will be denoted by $d_{\varepsilon'}$, in which ε' is equal to ε except at the i^{th}-position at which $\varepsilon'_i = *$.

2. We note that if there is a map from ε to $\widehat{\varepsilon}$, then $G_{\widehat{\varepsilon}}$ has exactly one more edge than G_ε. *If* adding this edge yields a subgraph with the same number of connected components than before adding it, then $d_{\varepsilon'}$ is the identity and it does not contribute to the differential.

   On the other hand, *if* adding the edge decreases the number of components by one, then we set

$$d_{\varepsilon'} : M^{\otimes k(\varepsilon)} \to M^{\otimes (k(\varepsilon)-1)} = \begin{cases} Id & \text{On tensor factors corresponding to components} \\ & \text{not affected by the adding of the edge} \\ m & \text{Otherwise} \end{cases}$$

Note that $d_{\varepsilon'}$ is degree preserving.

3. Now, let us recall that we have the graded vector spaces

$$C^i(G) = \sum_{|\varepsilon|=i} M^{\otimes k(\epsilon)} .$$

The differential $d^i : C^i(G) \longrightarrow C^{i+1}(G)$ is given by

$$d^i = \sum_{|\varepsilon'|=i} (-1)^{\varepsilon} d_{\varepsilon'}$$

where $|\varepsilon|$ is the number of $1's$ in the word ε' and the sign $(-1)^{\varepsilon'}$ is given by:

$$(-1)^{\varepsilon} = \begin{cases} 1 & \text{If the number of } 1's \text{ to the left of } * \text{ is even} \\ -1 & \text{If the number of } 1's \text{ to the left of } * \text{ is odd .} \end{cases}$$

Remark 127. *What we have done is precisely to define the differential using "admissible pairs" as we did when we discussed Khovanov homology for knots. In fact, we must define a morphism*

$$d^i : C^{i,*}(G) \to C^{i+1,*}(G) .$$

Suppose that $M_\upsilon \subseteq C^{i,*}(G)$ *and that* $M_{\upsilon'} \subseteq C^{i+1,*}(G)$ *; we make three suppositions: first, the word* υ' *has "1's" at least in the same positions as the word* υ*; second,* υ' *has exactly one more* 1 *than* υ*. Then there is a morphism from* M_υ *to* $M_{\upsilon'}$*. We say that the pair* (υ, υ') *is* admissible.

In order to find admissible pairs, we proceed as follows: if the word 1000*, for example, changes to the word* 1100 *this change is represented by* $1*00$*, and we say that there is an arrow from* 1000 *to* 1100*. All* υ' *such that the pair* $(1000, \upsilon')$ *is admissible are obtained by "moving" the symbol* $*$ *without moving the* 1*'s appearing in* 1000*. We can indicate the morphism between* M_{1000} *and*

M_{1*00} *by* $d_{1*00} : M_{1000} \to M_{1*00}$ *. Now we define as before:*

$$d_\varepsilon = \begin{cases} I : M_\varepsilon(G) \longrightarrow M_{\tilde{\varepsilon}}(G), \text{ if } G_\varepsilon = G_{\tilde{\varepsilon}} \\ m : M_\varepsilon(G) \otimes M_\varepsilon(G) \longrightarrow M_{\tilde{\varepsilon}}(G) \text{ , if } G_\varepsilon \neq G_{\tilde{\varepsilon}} \end{cases}$$

and

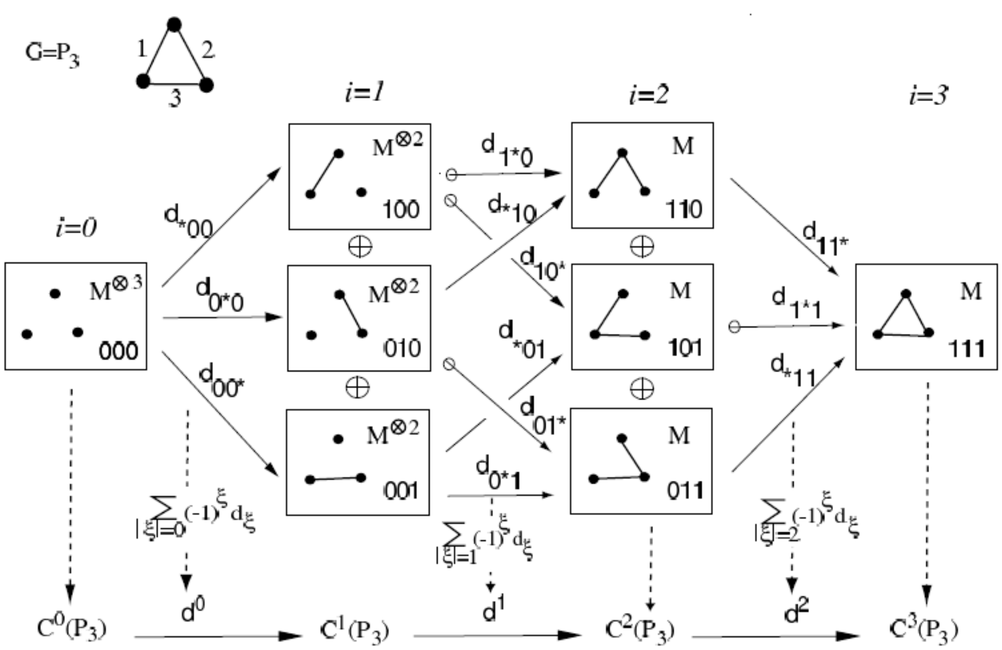
$$\begin{aligned} d^i &: C^{i,*}(G) \to C^{i+1,*}(G) \\ d^i &: = \sum_{|\varepsilon|=i} (-1)^\varepsilon d_\varepsilon \end{aligned}$$

The following theorem is proven in [11]. We note that in this reference the authors use a different graduation for the vector space M.

Theorem 128. $d^{i+1} \circ d^i = 0$ *. Moreover, the Euler characteristic of the cochain complex $C^i(G)$ is equal to the chromatic polynomial of G evaluated at $q + q^{-1}$.*

It is also proven in [11] (see their Theorem 2.12) that the above construction does not depend on the ordering of the edges.

Example 129. *We complete an example considered in [11], using our matrix computations of the differential, as in the case of knots. In this example we denote the vector space generated by $\{v_-, v_+\}$ by V.*



We have the differential:

$$\begin{aligned} d^0(v_1, v_2, v_3) &= (m(v_1 \otimes v_2) \otimes v_3, v_1 \otimes m(v_2 \otimes v_3), m(v_1 \otimes v_3) \otimes v_2) \\ d^1(v_1 \otimes v_2, v_3 \otimes v_4, v_5 \otimes v_6) &= (m(v_3 \otimes v_4) - m(v_1 \otimes v_2), m(v_5 \otimes v_6) - m(v_1 \otimes v_2), m(v_5 \otimes v_6) \\ &- m(v_3 \otimes v_4)) \\ d^2(v_1, v_2, v_3) &= v_1 - v_2 + v_3 \ . \end{aligned}$$

We find the matrix representation of each d^i as follows. Let us consider the basis

$$\{(v_+v_+v_+), (v_-v_-v_-), (v_+v_-v_-), (v_-v_-v_+), (v_-v_+v_-), (v_+v_+v_-), (v_+v_-v_+), (v_-v_+v_+)\}$$

for $V^{\otimes 3}$, and the basis

$$\{(v_+v_+, 0, 0), (0, v_+v_+, 0), (0, 0, v_+v_+), (v_-v_-, 0, 0)$$

$$(0, v_-v_-, 0), (0, 0, v_-v_-), (v_-v_+, 0, 0), (0, v_-v_+, 0), (0, 0, v_-v_+),$$

$$(v_+v_-, 0, 0), (0, v_+v_-, 0), (0, 0, v_+v_-)\}$$

for $V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2}$. Then, in these bases, the matrix of d^0 is:

$$d^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

In a similar fashion, considering also the basis

$$\{(v+,0,0),(0,v+,0),(0,0,v+),(v-,0,0),(0,v-,0),(0,0,v-)\}$$

for $V \oplus V \oplus V$*, and the basis*

$$\{v_+, v_-\}$$

for V*, we obtain the matrices*

$$d^1 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

and

$$d^2 = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \; .$$

We use these differential operators to compute cohomology. We have that $H^0(C^0)$ *and* $H^1(C^1)$ *are not trivial:*

$$H^0(C^0) = \ker d^0 \diagup \{0\} = \langle (0,1,0,0,0,0,0,0) \rangle \; .$$

$$H^1(C^1) = \ker d^1 \diagup imgd^0$$

$$=$$

$$\left\langle \begin{array}{c} [1,1,1,0,0,0,0,0,0,0,0,0],[0,0,0,1,0,0,0,0,0,0,0,0],[0,0,0,0,1,0,0,0,0,0,0,0] \\ [0,0,0,0,0,1,0,0,0,0,0,0],[0,0,0,0,0,0,1,1,1,0,0,0],[0,0,0,0,0,0,-1,0,0,1,0,0] \\ [0,0,0,0,0,0,0,-1,0,0,1,0],[0,0,0,0,0,0,1,1,0,0,0,1] \end{array} \right\rangle$$

$$\diagup$$

$$\left\langle \begin{array}{c} [1,1,1,0,0,0,0,0,0,0,0,0,0],[0,0,0,1,0,1,0,0,0,0,0,0],[0,0,0,0,1,1,0,0,0,0,0,0] \\ [0,0,0,1,1,0,0,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,1,1,1,0],[0,0,0,0,0,0,1,0,0,0,1,1] \\ [0,0,0,0,0,0,1,1,1,0,0,0] \end{array} \right\rangle$$

$$=$$

$$\left\langle \begin{array}{c} [0,0,0,1,0,0,0,0,0,0,0,0],[0,0,0,0,1,0,0,0,0,0,0,0],[0,0,0,0,0,1,0,0,0,0,0,0], \\ [0,0,0,0,0,0,-1,0,0,1,0,0],[0,0,0,0,0,0,0,-1,0,0,1,0] \\ [0,0,0,0,0,0,1,1,0,0,0,1],[0,0,0,0,0,0,0,0,0,1,1,1,0] \end{array} \right\rangle$$

On the other hand, $H^2(C^2)$ and $H^3(C^3)$ are trivial. We compute explicitly only $H^2(C^2)$:

$$H^2(C^2) = \ker d^2 \diagup imgd^1$$

$$=$$

$$\left\langle \begin{array}{c} [1,1,0,0,0,0],[-1,0,1,0,0,0], \\ [0,0,0,-1,0,1],[0,0,0,1,1,0], \end{array} \right\rangle \diagup \left\langle \begin{array}{c} [-1,-1,0,0,0,0],[1,0,-1,0,0,0] \\ [0,0,0,-1,-1,0],[0,0,0,1,0,-1] \end{array} \right\rangle = \{0\} \ .$$

$$H^3(C^3) = \{0\} \ .$$

We use these computations to calculate the Euler characteristic:

$$
\begin{aligned}
\chi_q(C) \quad &= \textstyle\sum_{i=0}^{3}(-1)^i q\dim C^i \\
&= q\dim C^0 - q\dim C^1 + q\dim C^2 - q\dim C^3 \\
&= q\dim V^{\otimes 3} - 3q\dim V^{\otimes 2} + 3q\dim V - q\dim V \\
&= (q+q^{-1})^3 - 3(q+q^{-1})^2 + 3(q+q^{-1}) - (q+q^{-1}) \\
&= (q+q^{-1})^3 - 3(q+q^{-1})^2 + 2(q+q^{-1}) \\
&= C_G(q+q^{-1})\ .
\end{aligned}
$$

We can check this result by using explicitly the generators of each complex:

- $C^0 = V^{\otimes 3} = \langle v_+v_+v_+, v_-v_-v_-, v_+v_-v_-, v_-v_-v_+, v_-v_+v_-, v_+v_+v_-, v_+v_-v_+, v_-v_+v_+\rangle$

  $q\dim C^0 = q^3 + q^{-3} + q^{-1} + q^{-1} + q^{-1} + q + q + q = 3q + 3q^{-1} + q^{-3} + q^3$

- $C^1 = V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2} = \left\langle \begin{array}{c} (v_+v_+,0,0),(0,v_+v_+.0),(0,0,v_+v_+),(v_-v_-,0,0) \\ (0,v_-v_-,0),(0,0,v_-v_-),(v_-v_+,0,0),(0,v_-v_+,0) \\ (0,0,v_-v_+),(v_+v_-,0,0),(0,v_+v_-,0),(0,0,v_+v_-) \end{array} \right\rangle$

  $q\dim C^1 = 3q^2 + 3q^{-2} + 6$

- $C^2 = V \oplus V \oplus V = \left\langle \begin{array}{c} (v_+,0,0),(0,v_+,0),(0,0,v_+) \\ (v_-,0,0),(0,v_-,0),(0,0,v_-) \end{array} \right\rangle$

  $q\dim C^2 = 3q + 3q^{-1}$

- $C^3 = V = \langle v_+, v_-\rangle$

  $q\dim C^3 = q + q^{-1}\ .$

We have, as before,

$$
\begin{aligned}
\chi_q(C) &= \textstyle\sum_{i=0}^{3}(-1)^i q \dim C^i \\
&= 3q + 3q^{-1} + q^{-3} + q^3 - (3q^2 + 3q^{-2} + 6) \\
&+3q + 3q^{-1} - (q + q^{-1}) \\
&= 5q + 5q^{-1} - 3q^2 - 3q^{-2} + q^{-3} + q^3 - 6 \\
&= (q + q^{-1})^3 - 3(q + q^{-1})^2 + 2(q + q^{-1}) \\
&= C_G(q + q^{-1})\ .
\end{aligned}
$$

5.4 The Reidemeister torsion for graphs

As an application of this categorification, we define the Reidemeister torsion of a graph as the torsion of the complex $C^i(G)$. Since a change in the ordering of edges yields isomorphic complexes (Theorem 2.12 of [11]) this torsion is well-defined, in the sense of Proposition 82. We refer to the notation of Section 4.1.

Example 130. *We consider the graph G of the previous example. We compute:*

$$
\begin{aligned}
d^0(v_1, v_2, v_3) &= (m(v_1 \otimes v_2) \otimes v_3, v_1 \otimes m(v_2 \otimes v_3), m(v_1 \otimes v_3) \otimes v_2) \\
d^1(v_1 \otimes v_2, v_3 \otimes v_4, v_5 \otimes v_6) &= (m(v_3 \otimes v_4) - m(v_1 \otimes v_2), m(v_5 \otimes v_6) - m(v_1 \otimes v_2), m(v_5 \otimes v_6) \\
&- m(v_3 \otimes v_4)) \\
d^2(v_1, v_2, v_3) &= v_1 - v_2 + v_3\ .
\end{aligned}
$$

These operators have matrix representations

$$d^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$d^1 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

$$d^2 = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

It follows that:

1. $0 \longrightarrow C^0 \xrightarrow{d^0} C^1$

$$H^0(C^0) = \ker d^0 \diagup \{0\}$$
$$= \langle (0,1,0,0,0,0,0,0) \rangle$$

$$b_0 = imgd^0 =$$
$$\left\langle \begin{array}{l} [1,1,1,0,0,0,0,0,0,0,0,0], [0,0,0,1,0,1,0,0,0,0,0,0], [0,0,0,0,0,1,1,0,0,0,0,0,0] \\ [0,0,0,0,0,0,0,0,1,1,1,0], [0,0,0,0,0,0,0,1,0,0,0,1,1], [0,0,0,0,0,0,1,1,1,0,0,0] \end{array} \right\rangle$$

$$\tilde{b}_0 = \left\langle \begin{array}{c} [1,0,0,0,0,0,0,0],[0,0,1,0,0,0,0,0],[0,0,0,1,0,0,0,0] \\ [0,0,0,0,1,0,0,0],[0,0,0,0,0,1,0,0],[0,0,0,0,0,0,0,1,0] \\ [0,0,0,0,0,0,0,1] \end{array} \right\rangle$$

$$B_{c^0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \textit{ then:}$$

$$detB_{c^0} = 1$$

2. $C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2$

$$H^1(C^1)$$
$$= \left\langle \begin{array}{l} [0,0,0,1,0,0,0,0,0,0,0,0],[0,0,0,0,1,0,0,0,0,0,0,0],[0,0,0,0,0,1,0,0,0,0,0,0], \\ [0,0,0,0,0,0,-1,0,0,1,0,0],[0,0,0,0,0,0,0,-1,0,0,1,0] \\ [0,0,0,0,0,0,1,1,0,0,0,1],[0,0,0,0,0,0,0,0,1,1,1,0] \end{array} \right\rangle$$

$$\tilde{b}^1 = \left\langle \begin{array}{c} (1,0,0,0,0,0,0,0,0,0,0,0,0),(0,1,0,0,0,0,0,0,0,0,0,0,0) \\ (0,0,0,0,0,1,0,0,0,0,0,0),(0,0,0,0,0,0,0,0,0,0,0,1,0) \end{array} \right\rangle$$

$$B_{c^1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \textit{ then:}$$

$$detB_{c^1} = 1$$

3. $C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3$

$H^2(C^2) = \{0\}$

$$\tilde{b}_2 = \langle (1,0,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,0,1) \rangle$$

$$imgd^1 = \langle (-1,-1,0,0,0,0), (1,0,-1,0,0,0), (0,0,0,-1,-1,0), (0,0,0,1,0,-1) \rangle$$

$$B_{c^2} = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \textit{ then:}$$

$$detB_{c^2} = 1$$

4. $C^2 \xrightarrow{d^2} C^3 \xrightarrow{d^3} 0$

$H^3(C^3) == \{0\}$

$\tilde{b}^3 = \{0\}$

$imgd^2 = \langle (1,0), (0,1) \rangle$

$B_{c^3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *then:*

$detB_{c^3} = 1$. *We conclude that* $\tau(G) = 1$.

5.5 The Laplace operator for graphs

Let $A = \bigoplus_i C^i(G)$ and suppose that $F = \mathbb{R}$. There exists a unique Euclidean metric $\langle ., . \rangle_i$ over $C^i(G)$ such the distinguished basis c_i of C^i is an orthonormal basis. We define $d^* : C^i \to C^{i+1}$ via $\langle d_i^* a, b \rangle_{i+1} = \langle a, d_i b \rangle_i \quad a \in C^i, b \in C^{i+1}$, and we also consider the corresponding Laplace operator $\Delta_i = -d_i d_i^* - d_{i+1}^* d_{i+1} : C^i \to C^i$.

As already mentioned, this Laplace operator seems to be quite different from the standard discrete Laplacian associated to graphs. We leave a study of their possible relations as an open problem. We finish this thesis with a explicit computation of a Laplacian for a complex associated to a graph:

Example 131. *We consider again the graph G of Example 129 and the chain complex*

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \longrightarrow 0\ .$$

We recall that

$$\Delta_i = -d_i d_i^* - d_{i+1}^* d_{i+1}$$

and therefore the Laplace operator of G is given as follows:

$\Delta_0 = -d_0 d_0^* - d_1^* d_1$

$$\Delta_0 = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$-\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

$$
= \begin{pmatrix}
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & -1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1
\end{pmatrix}
$$

$$
- \begin{pmatrix}
2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 2 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & -1 & 2
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & -3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & -3
\end{pmatrix}
$$

$\Delta_1 = -d_1 d_1^* - d_2^* d_2$

Δ_1

$$
- \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

$$
- \begin{pmatrix}
1 & 0 \\
-1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & -1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1
\end{pmatrix}
$$

$$= -\begin{pmatrix} 2 & 1 & -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2 & -2 \\ 0 & 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & -2 & 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -1 & 1 \\ 0 & 0 & 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 1 & -1 & -5 \end{pmatrix}$$

$$\Delta_2 = -d_2 d_2^* - d_3^* d_3$$

$$\Delta_2 = -\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} - 0 = -\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} .$$

Chapter 6

Conclusions

In this thesis we have presented some instances of categorification in geometry and topology. Our main objective has been to explore the possibility of finding invariants for knots and graphs. We first introduced the Khovanov cohomology for topological knots [13]. As an application, we discussed Reidemester torsion for knots following [20], and we defined a Laplace operator. This approach seems promisory, but we leave open the question whether analysis of this Laplace operator yields invariants for knots. We believe it would be very interesting if, for instance, we could find invariants using its spectral properties.

We then moved to the study of Chekanov cohomology for Legendrian knots, see [4]. The fact that now we had an algebra structure to our dispossal (and not only a differential complex as in the Khovanov case) allowed us to use further homological tools in our search for invariants: we were able to present product invariants for Legendrian knots using (classical and generalized) Massey products. This work has been already published ("Massey products, A_∞-algebras, differential equations, and Chekanov homology", *Journal of Nonlinear Mathematical Physics*, Vol. 22, No. 3 (2015), 342–360). We believe that the generalized Massey product invariants (perhaps considered in their linearized version as in [5]) may be useful for distinguishing Legendrian *links*, but we have left their explicit application for the near future. As a further application of the algebra structure of the Chekanov theory we have considered differential equations. These equations are interesting because they are instances of *integrable* nonlinear equations arising from geometric data. We leave open the natural and interesting question whether solutions to these equations would yield further information on Legendrian knots.

Finally, we have considered graphs. There is a version of Khovanov theory valid for graphs, see [11], and we have exploited it for defining a Reidemeister torsion and a Laplace operator on graphs. In this part of our work there are further open problems we could consider. For example, we could investigate *Legendrian* graphs, and we could also consider in more detail the analytic properties of our graph Laplace operator.

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