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On qualitative properties of fractional-difference equations on abstract spaces and their applications to lattices models.

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Abstract

On qualitative properties of fractional-difference equations on abstract spaces and their applications to lattices models.

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In this thesis we study the existence, uniqueness, qualitative properties and regularity of solutions for different classes of discrete-time fractional difference equations in abstract spaces using an effective technique based in an operator-theoretical method. In addition we present explicit examples of equations that can be considered in our abstract results.

Keywords: difference equations; maximal ℓ_p-regularity; weighted bounded vector-valued spaces; fractional difference operator.

No hay rama de la matemática, por lo abstracta que sea, que no pueda aplicarse algún día a los fenómenos del mundo real.

Nikolái Ivánovich Lobachevski
(1792–1856)
Matemático Ruso

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Introduction

The objective of this thesis is to present a study of the existence, uniqueness, regularity and qualitative properties of bounded solutions for some classes of abstract discrete-time fractional difference equations by using operator theoretical methods in spaces of vector-valued functions. In the last time, recent technological innovations have caused a considerable interest in the study of dynamical processes that present a mixed continuous and discrete nature, see [3, 4, 12]. For instance, discrete-time linear models appear in the study of the solution to optimal control problems in dynamic programming [21]. Moreover, they are also used for modeling coal liquefaction mechanisms [67] and robust energy filtering in signal processing [49], among others fields of interest. In the biological context, qualitative behavior of discrete models with delays has been examined in [36] and [75]. See also [15, 18, 59]. The same happens with the analysis of mixed partial differential equations and integral equations [28, 66]. A classical textbook is the monograph by W. J. Rugh [64].

On the other hand, starting with the works of S. Blünck [20] and P. Portal [61, 62], the existence and uniqueness of solutions for discrete systems that belong to the Lebesgue space of vector-valued sequences began to be considered by many authors [42, 50, 51]. A recent textbook on this topic is the monograph of Agarwal, Cuevas and Lizama [6], where several applications in different contexts are given. After the works of L. Weis [71], and H. Amann [9], characterizations of Lebesgue regularity using multiplier theorems for operator valued symbols have appeared in several papers in the last decade. See for instance the ones of Bu [23, 24], Chill and Srivastava [26], the special volume [22] and references therein. For instance in [44] Kovács, Li and Lubich studied maximal regularity using the results of Blünck for numerical schemes. Kemmochi [43], in the same line, introduced the notion of maximal regularity for the finite difference method. Other contributions can be found in the references [7,47,48].

On the other hand, modeling with fractional difference equations is a recent and promising area of research that has been developed from different sides of interest. For instance, Atici and Şengül [14] develop some basics results of discrete fractional calculus. These authors introduce and solve the Gompertz fractional difference equation for tumor growth models. See also the works of Atici and Eloe [13] for related results in this direction. The methodology used in such discrete fractional calculus was extended by Lizama in [50] to the context of abstract models, including in this way the handling of difference differential equations by methods of functional analysis and operator theory. Studies on qualitative properties, as for example the existence of positive solutions for discrete fractional systems, have been provided by Goodrich [29,38,39]. Other interesting contributions are due to Ferreira [35], Holm [41], Kovács, Li and Lubich [44], Dassios [30,31], Wu, Baleanu et. al. [32,72–74], Čermák et. al. [25] and Tarasov et. al. [68–70].

In this thesis, we characterize well-posedness of some linear discrete-time fractional difference equations in Lebesgue spaces of sequences. Our results are based on UMD spaces, the concept of R-boundedness, and the notion of α and α^τ resolvent families of operators. Note that R-boundedness has been an useful tool in the functional analytic approach to partial differential equations. In [71] L. Weis shows that R-boundedness provides a proper setting for R-boundedness theorems for operator-valued Fourier multipliers.

We will consider the following three problems, the first and second in the setting of Lebesgue spaces whose domains are $\mathbb{Z}$ and $\mathbb{N}_0$ respectively, the third one is in the context of weighted vector-valued spaces with domain $\mathbb{N}_0$.

In the first problem, our concern is the following fractional difference equation

$$\Delta^\alpha u(n) + \lambda\Delta^\beta u(n) = Au(n) + f(n), \quad n \in \mathbb{Z}, \quad \alpha,\beta > 0, \quad \lambda \geq 0, \tag{0.0.1}$$

where $f \in \ell_p(\mathbb{Z}, X)$, A is a closed linear operator with domain $D(A)$ defined on a Banach space X and Δ^γ denotes the fractional difference operator of order $\gamma > 0$ as defined recently by Abadías and Lizama [1]. Roughly speaking, it corresponds to a slight variant of the Grünwald-Letnikov derivative. Equation (0.0.1) is an example of a wide class of mixed evolution equations that can be considered either as models for partial differential equations that are continuous in space but discrete in time [17], or systems of difference equations [5, Chapter 3; 34]. Typical models that are included in this problem correspond to the discrete time Klein-Gordon equation

$$\Delta^2 u(n,x) = u_{xx}(n,x) - bu(n,x) + G(u)(n,x), \quad n \in \mathbb{Z}, \quad x \in \Omega \subset \mathbb{R}^N \tag{0.0.2}$$

where $\Delta^2 u(n,x) := u(n+2,x) - 2u(n+1,x) + u(n,x)$, and the discrete time telegraph equation

$$\tau \Delta^2 u(n,x) + \Delta u(n,x) = \rho u_{xx}(n,x), \quad n \in \mathbb{Z}, \quad \tau \geq 0, \; \rho > 0, \quad x \in J \subset \mathbb{R}, \tag{0.0.3}$$

as well as fractional versions of them [11, 37]. The discrete version of the Basset equation [16, 58]

$$\Delta^2 u(n) + \lambda \Delta^{3/2} u(n) + bu(n) = f(n), \quad n \in \mathbb{Z}, \quad \lambda, b > 0, \tag{0.0.4}$$

will be also included in our framework. The study of the uniqueness and causality of p-summable solutions [19] suggests to consider the above equations on $\mathbb{Z}$. More precisely, given a Banach space X, we ask the following problem: Is it possible to characterize solely in terms of the data of a given mixed evolution equation, the existence and uniqueness of solutions that belong to the vector-valued space of sequences $\ell_p(\mathbb{Z}, X)$?.

We success solving this open problem for the equation (0.0.1). It is worthwhile to observe that, for instance, the model (0.0.1) includes the Basset equation (0.0.4) taking $X = \mathbb{C}, A = bI, \alpha = 2$ and $\beta = 3/2$ whereas it also includes the linearized Klein-Gordon equation (0.0.1) choosing $X = L^2(\Omega), A = \partial_{xx} - bI, \alpha = 2$ and $\lambda = 0$.

This problem will be solved by the following way. First, after giving the definition on maximal ℓ_p-regularity, we prove our main result, namely, if

$$\{(1-e^{-it})^{\alpha}+\lambda(1-e^{-it})^{\beta}\}_{t\in\mathbb{T}_0}\subset\rho(A),\quad \lambda\geq 0,\quad \alpha,\beta>0,\quad \mathbb{T}_0:=(-\pi,\pi)\setminus\{0\},$$

where $\rho(A)$ denotes the resolvent set of A, then the following assertions are equivalent:

(i) For all $f\in\ell_p(\mathbb{Z},X)$ the problem $\Delta^{\alpha}u(n)+\lambda\Delta^{\beta}u(n)=Au(n)+f(n)$, $n\in\mathbb{Z}$, has a unique solution in $\ell_p(\mathbb{Z},[D(A)])$;

(ii) $M(t):=((1-e^{-it})^{\alpha}+\lambda(1-e^{-it})^{\beta}-A)^{-1}$ is an ℓ_p-multiplier from X to $[D(A)]$;

(iii) The set $\{M(t)\}_{t\in\mathbb{T}_0}$ is R-bounded.

Furthermore in the context of Hilbert spaces a simpler criterion is also provided, replacing the condition (iii) above by

$$\sup_{t\in\mathbb{T}_0}\|M(t)\|<\infty.$$

As a consequence, we analyze the nonlinear equation

$$\Delta^{\alpha}u(n)+\lambda\Delta^{\beta}u(n)=Au(n)+G(u)(n)+g(n),\quad n\in\mathbb{Z},$$

where $g\in\ell_p(\mathbb{Z},X)$ and $G:\ell_p(\mathbb{Z},X)\to\ell_p(\mathbb{Z},X)$ are given. We show that if $G(0)=G'(0)=0$ and g is small enough, then the nonlinear equation has at least one solution in $\ell_p(\mathbb{Z},X)$. Finally we prove, as an application of our characterization, that for all $0<\alpha,\beta<2$ and $b>2^{\alpha}+\lambda 2^{\beta}$ we can find $\varepsilon^*>0$ such that for all $\varepsilon\in(0,\varepsilon^*)$, there exists $u^{\epsilon}\in\ell_p(\mathbb{Z},L^2(\mathbb{R}))$ that solves the problem

$$\Delta^{\alpha}u^{\epsilon}(n,x)+\lambda\Delta^{\beta}u^{\epsilon}(n,x)=u^{\epsilon}_{xx}(n,x)-bu^{\epsilon}(n,x)+G(u^{\epsilon})(n,x)+\varepsilon f(n,x),$$

for all $n\in\mathbb{Z}$, $x\in\Omega\subset\mathbb{R}^N$.

It is important to observe that the results of this problem are included in the recently published joint paper [56].

In the second problem, we deal with the fractional difference equation with delay of the form

$$\begin{cases} \Delta^{\alpha} u(n) = Tu(n) + \beta u(n-\tau) + f(n), \; n \in \mathbb{N}_0 \\ \quad u(j) = x_j, \; j = -\tau, \ldots, 0, 1 \end{cases} \tag{0.0.5}$$

where $\tau \in \mathbb{N}_0$, β is a real number, T is a linear bounded operator defined on a Banach space X, $1 < \alpha \leq 2$, f is a vector-valued function and Δ^{γ} denotes the fractional difference operator of order $\gamma > 0$ in sense of Riemann-Liouville.

An interesting feature that involves (0.0.5) is that the fractional difference operator Δ^{α} can be realized as sampling, by means of the Poisson distribution, of the classical fractional Riemann-Liouville operator. See the work of Lizama [51, Theorem 3.5] where this remarkable connection has been discovered. This nonlocal operator has recently appeared in several research of increasing interest to different but related fields. For instance, in relation to the notion of Césaro operators of order $\alpha > 0$ [2], chaos for fractional delayed logistic maps [72] and almost automorphic solutions of fractional difference equations [1].

The analysis of ℓ_p-maximal regularity for difference equations of fractional order $\alpha > 0$ in the form

$$\begin{cases} \Delta^{\alpha} u(n) = Tu(n) + f(n), \quad n \in \mathbb{N}_0, \\ \quad u(0) = 0, \end{cases}$$

where T is a bounded operator defined on a Banach space X was studied in [50] for the range $0 < \alpha \leq 1$ and in [52] for $1 < \alpha \leq 2$. In [53] ℓ_p-maximal regularity for the equation (0.0.6) with infinite delay was studied in $\mathbb{Z}$ for all $\alpha > 0$ when T is an unbounded operator. Recently, in [54] the authors characterized the ℓ_p-maximal regularity for the finite delayed equation

$$\begin{cases} \Delta^{\alpha} u(n) = Tu(n) + \beta u(n-\tau) + f(n), \quad n \in \mathbb{N}_0, \quad n \geq 1, \quad \beta \in \mathbb{R}, \\ \quad u(j) = 0, \quad j = -\tau, ..., 0, \quad \tau \in \mathbb{N}_0, \end{cases} \tag{0.0.6}$$

whenever $0 < \alpha \leq 1$. However, the validity of such characterization for the case $1 < \alpha \leq 2$ was left as an open problem.

The main purpose of this part of the work is to give a positive answer to this open problem. Note that the maximal ℓ_p-regularity for the equation (0.0.6) with infinite delay was studied in $\mathbb{Z}$ for all $\alpha > 0$ when T is an unbounded operator in the work of Lizama and Murillo [53].

This problem is studied as follows. Firstly, we introduce the new concept of α^τ resolvent operators in the range $1 < \alpha \leq 2$, which is an important tool for the construction of the solution of (0.0.6). This family, denoted by $\{M_\alpha(n)\}_{n\geq -\tau}$, incorporates directly the finite delay in its definition. Then, we will prove that a general solution for our model, with initial conditions $u(j) = x_j$, $j = -\tau, \ldots, 0, 1$, can be written as

$$\begin{aligned} u(n) &= M_\alpha(n)u(0) + F_\alpha(n-1)[u(1) - u(0)] \\ &\quad + \beta \sum_{j=1}^{\tau} F_\alpha(n-2+j-\tau)u(-j) + (F_\alpha * f)(n-2),\ n \geq 2. \end{aligned} \tag{0.0.7}$$

Here, $h_\alpha(n) = (\alpha-1)^n$ and $F_\alpha(n) = (M_\alpha * h_\alpha)(n)$. Note that in the case $\alpha = 2$ and $\beta = 0$, the resolvent family $M_2(n)$ perfectly coincides with the notion of discrete cosine operator which was introduced and studied by Chojnacki [27] in the context of UMD Banach spaces.

We remark that the representation (0.0.7) is not straightforward but it is one of the main tasks that we have overcome in order to achieve the solution of our problem.

Finally we prove the main result of this part of the thesis. We will show that if X is a UMD space and the condition $\sup_{n\in\mathbb{N}_0} \|M_\alpha(n)\| < \infty$ is satisfied, then the maximal ℓ_p-regularity of equation (0.0.6) and the R-boundedness of the sets

$$\left\{ z^{2-\alpha}(z-1)^\alpha (z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} : |z| = 1, z \neq 1 \right\},$$

$$\left\{ z^{-\tau} (z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} : |z| = 1, z \neq 1 \right\},$$

are equivalent. This characterization coincides perfectly as the counterpart of the result achieved in the paper [53] by Lizama and Murillo where also an R-boundedness condition on two sets is needed. We note that in practice, tools to check this condition are generally not easy to find. However, the monograph of Agarwal, Cuevas and Lizama [6] shows a way in the general case. For the case of Hilbert spaces, we observe that R-boundedness can be replaced merely by uniform boundedness. For such a case, we are able to provide a very simple criterion on T that ensures maximal ℓ_p-regularity of equation (0.0.6), namely:

$$||T|| < \omega_{\alpha,\beta,\tau} := \min_{|z|=1} |f_{\alpha,\beta,\tau}(z)| < 1 \quad \text{where} \quad f_{\alpha,\beta,\tau}(z) := z^{2-\alpha}(z-1)^{\alpha} - \beta z^{-\tau}.$$

We finish the study of our problem with the following examples:

$$x(n+3) - 2x(n+2) + qx(n+1) + rx(n) = f(n),$$

with initial conditions: $x(0) = x(1) = x(2) = 0$. We show that maximal ℓ_p-regularity of this equation for $f \in \ell_p(\mathbb{N}_0)$ is guaranteed whenever $1 < q < 2$ and $1 - q < r < -1 + \sqrt{2-q}$.

In the fractional case $1 < \alpha \leq 2$, we consider

$$\Delta^{\alpha} x(n) = (1-q)x(n) - rx(n-1) + f(n)$$

and a sufficient condition for maximal ℓ_p- regularity on the parameters $r, q \in \mathbb{R}$ is provided: $\omega(r) := \min_{|z|=1} |z^{3-\alpha}(z-1)^{\alpha} + r| < 1$ and $1 - \omega(r) < q < 1 + \omega(r)$. We observe that the results of this chapter can be found in the joint paper [55] that has been submitted for publication.

Finally in the third problem, we consider the following nonlinear fractional difference equation

$$\begin{cases} \Delta^{\alpha} u(n) = Tu(n) + f(n, u(n)),\ n \in \mathbb{N}_0 \\ \quad u(0) = x \\ \quad u(1) = y \end{cases} \tag{0.0.8}$$

where T is a linear bounded operator defined on a Banach space X, $1 < \alpha \leq 2$ and f is a nonlinear function defined on $\mathbb{N}_0 \times X$ with values in X.

A classical example of equations that can be modeled as (0.0.8) is the time-discrete nonconvolution equation which is a class of fractional integro-differential equations and PDE discretized only in time, written as follows

$$\Delta^\alpha u(t,x) = \int k(x,s)u(n,s)ds + f(n,u(n,x)),\ n \in \mathbb{N}_0,\ x \in \Omega \subset \mathbb{R}^N,$$

where f is a suitable forcing term and k is a complex-valued measurable function [52]. This discrete fractional equations admits the form (0.0.8) with

$$Tf(x) := \int k(x,s)u(n,s)ds,$$

which defines a bounded operator on suitable spaces of functions.

Recently Lizama and Velasco, in [57], studied the existence of weighted bounded solutions for a time-discrete nonlinear fractional equation of the form

$$\Delta^\alpha u(n) = Tu(n) + f(n,u(n)),\ n \in \mathbb{N}_0, \tag{0.0.9}$$

with initial condition $u(0) = u_0$ whenever $0 < \alpha \leq 1$. However, for another values of α, it was left as an open problem. The main objective of this third part of the thesis is to provide an answer for this open problem.

We are able to show existence of weighted bounded solutions of (0.0.9) for the range $1 < \alpha < 2$.

In order to prove our results, we need a special sequence of bounded operators introduced by Lizama and Murillo in [52] called α-resolvent families, which have an important role in the representation of the solution by means of a kind of discrete variation of parameters formula. A second main ingredient is the use of a special vector-valued Banach space of weighted sequences $l_f^\infty(\mathbb{N}_2; X)$, whose properties allow to prove the existence of solutions of (0.0.9) under certain conditions on the nonlinear term.

On the other hand, we note that in the continuous case, the compactness of the operator T is neccesary, but in the discrete case, there are examples of operator T that do not need this condition [10, Section 7]. In spite of this situation, we will show in this work that when T is a non-compact operator, we can still have existence of solutions for the fractional model (0.0.9). See Corollary 4.3.6 and Example 4.4.1.

This problem is solved in the following way. We first recall the concept of α-resolvent sequences of bounded operators, denoted by $S_\alpha(n)$, that was introduced by Lizama and Murillo [52] for $1 < \alpha \leq 2$. These α-resolvent families allow us to obtain an explicit representation of the solution for the fractional difference equation (0.0.9) with initial values $u(0) = u_0$ and $u(1) = u_1$ namely

$$u(n) = S_\alpha(n)u_0 + (S_\alpha * h_\alpha)(n-1)[u_1 - u_0] + (S_\alpha * h_\alpha * f)(n-2),\ n \geq 2$$

see [52, Theorem 3.8]. Here h_α is defined by the sequence $h_\alpha(n) = (\alpha - 1)^n$. We note that in the border case $\alpha = 2$, the resolvent sequence $S_2(n)$ coincides with the notion of discrete time cosine function introduced by Chojnacki [27] who studied it in the context of UMD-spaces.

Next we study the nonlinear problem (0.0.8). For this purpose, we firstly give a formulation of the solution motivated by the representation of the solution in the linear case (see Theorem 4.2.5 below). Next, we recall from the work of Lizama and Velasco [57] the following vector-valued Banach spaces of weighted sequences

$$l_f^\infty(\mathbb{N}_2; X) := \left\{ u : \mathbb{N}_2 \to X : \sup_{n \geq 2} \frac{\|u(n)\|}{nn!} < \infty \right\}.$$

This space is called the factorial number system space [57]. Observe that the sequence $nn!$ provides a suitable weight in order to find the existence of solutions in $l_f^\infty(\mathbb{N}_2; X)$ for the equation (0.0.8). See Lemma 4.3.4 below for properties about this sequence.

Information about the growth rate of the sequence $u(n)$ as $n \to \infty$ is obtained. See Theorem 4.3.5 and Theorem 4.3.8, where the Banach fixed point theorem and Leray-Schauder alternative theorem, respectively, are used. Moreover, we show the following practical result:

Suppose that $\|T\| < \dfrac{\alpha^\alpha(2-\alpha)^{2-\alpha}}{4}$ and $f : \mathbb{N}_0 \times X \to X$ satisfies the following hypothesis:

(F) $f(0,0) \neq 0$, $f(1,0) \neq 0$, and there exist a sequence $a \in \ell^1(\mathbb{N}_0)$ and constants $c \geq 0$ and $b > 0$ such that

$$\|f(k,x)\| \leq a(k)(c\,\|x\| + b)$$

for all $k \in \mathbb{N}_0$, $x \in X$.

(L) The function f satisfies a Lipschitz type condition in $x \in X$ uniformly in $k \in \mathbb{N}_0$, with Lipschitz constant

$$L_f < \frac{64}{3}\left(\frac{\alpha^\alpha(2-\alpha)^{2-\alpha}}{4} - \|T\|\right).$$

Then, the problem (0.0.8) with initial conditions $u(0) = u(1) = 0$ has an unique solution in $l^\infty_f(\mathbb{N}_2; X)$.

Finally we give concrete examples and applications of the results obtained in this work. We note that the results of this chapter can be found in the recent paper [46], submitted for publication.

This thesis is organized in four chapters. In the first chapter, we give basic definitions and fix some notation. Moreover, we present a result about Fourier multipliers that we use in chapters 1 and 2, see [20]. Chapters 2, 3 and 4 are devoted to the detailed study of the three problems described above.

Chapter 1

Preliminaries

In this chapter we give some preliminary concepts related to fractional differences, discrete Fourier and zeta transforms, UMD spaces, R-boundedness and operator-valued Fourier multipliers theorems defined on UMD spaces and important properties that we will use in the forthcoming chapters. For more details see [6, 33] and the references therein.

1.1 The Riemann-Liouville fractional derivative

In this section we introduce the notion of the fractional difference operator. Denote by $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and by $s(\mathbb{N}_0, X)$ the vectorial space of all vector-valued sequences $f : \mathbb{N}_0 \to X$.

Recall that given $f, g \in s(\mathbb{N}_0, X)$, we define the finite convolution product as follows

$$(f * g)(n) = \sum_{j=0}^{n} f(n-j)g(j), \quad n \in \mathbb{N}_0. \tag{1.1.1}$$

We also recall the following definition.

Definition 1.1.1. For a sequence $f \in s(\mathbb{N}_0, X)$. We define the forward Euler operator as follows

$$\Delta f(n) = f(n+1) - f(n), \quad n \in \mathbb{N}_0,$$

and $\Delta^0 \equiv I$, where I is the identity operator. Also we define for a fixed integer positive m,

$$\Delta^m = \Delta \circ \Delta^{m-1}$$

Now we recall the following definition of Cesáro numbers that was introduced by Zygmund in [76, p. 77] and rediscovered in several instances. See also [60, formula (27) with $h = 1$]

Definition 1.1.2. For a fixed real number α, define the Cesáro numbers of order α as follows

$$k^\alpha(j) = \begin{cases} \dfrac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!} & n \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\alpha \in \mathbb{R} \setminus \{-1, -2, \dots\}$, we have $k^\alpha(n) = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}$, where Γ denotes the Euler gamma function..

Note also that the sequence $(k^\alpha(n))_{n\in\mathbb{N}_0}$ satisfies the semigroup property, that is, $k^\alpha * k^\beta = k^{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{C}$. As a function of n, k^α is increasing for $\alpha > 1$, decreasing for $0 < \alpha < 1$. Furthermore, $k^\alpha(n) \le k^\beta(n)$ for $\beta \ge \alpha > 0$ and $n \in \mathbb{N}_0$. See for instance [76, Theorem III.1.17, p. 42 formula (2)] and [2, Section 2].

The following definition of fractional sum was introduced by Lizama in [50, Formula 2.2] (see also [51]). This definition corresponds to a particular case of fractional sum proposed by Eloe and Atici, Eloe and Abdeljawad (see [4, 12, 13]).

Definition 1.1.3. Let $\alpha > 0$ and $f \in s(\mathbb{N}_0, X)$ be given. We define the fractional sum of order α as follows

$$\Delta^{-\alpha} f(n) = \sum_{k=0}^{n} k^\alpha(n-k) f(k), \quad n \in \mathbb{N}_0. \tag{1.1.2}$$

The following definition corresponds to an analogous version of fractional derivative in the sense of Riemann-Liouville, see [12].

Definition 1.1.4. Let $f \in s(\mathbb{N}_0, X)$ be given, we define the fractional difference operator of order $\alpha > 0$ (in sense of Riemann-Liouville) as follows

$$\Delta^\alpha f(n) := \Delta^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

Observe that recently in [51], Lizama gives a relationship between the discrete fractional difference operator and the discrete fractional derivative operator in the sense of Riemann-Liouville through the so called Poisson transform amd given by

$$\Delta^\alpha u(n) = \int_0^\infty p_{n+[\alpha]}(t) D^\alpha u(t) dt, \tag{1.1.3}$$

where $p_n(t) = \dfrac{e^{-t}t^n}{n!}$. Using this formula we obtain, for instance, the representation of Césaro numbers by means of the Poisson transform of the function $g_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t \geq 0$.

1.2 Zeta and Fourier transforms

The transform method is most suitable for linear difference equations and discrete systems. It is used in the analysis of signal processing, digital control and commnications. The Z-transform of a sequence $f \in s(\mathbb{N}_0, X)$ is defined by

$$\tilde{f}(z) \equiv Z[f(n)] := \sum_{j=0}^{\infty} z^{-j} f(j), \tag{1.2.1}$$

where z is a complex number. Note that this series is convergent for $|z| > R$, for a sufficiently large R. The number R is called the radius of convergence of the series (1.2.1).

Some useful properties of the Z-transform are stated in the following result. See for instance [6, Proposition 1.2.2].

Proposition 1.2.1. *The following properties hold:*

a) (Linearity) Let $\tilde{x}(z)$ be the Z-transform of $x(n)$ with radius of convergence R_1 and $\tilde{y}(z)$ be the Z-transform of $y(n)$ with radius of convergence R_2. Then for any complex numbers a, b, we have

$$Z[ax(n) + by(n)] = a\tilde{x}(z) + b\tilde{y}(z), \quad \text{for } |z| > \max\{R_1, R_2\}.$$

b) (Right shifting) Let R be the radius of convergence of $\tilde{x}(z)$. If $x(-i) = 0$ for $i = 1, 2, \dots, k$, then

$$Z[x(n-k)] = z^{-k}\tilde{x}(z), \quad \text{for } |z| > R.$$

c) (Left shifting) Let R be the radius of convergence of $\tilde{x}(z)$. Then

$$Z[x(n+k)] = z^k\tilde{x}(z) - \sum_{r=0}^{n-1} z^{k-r}x(r).$$

In particular

$$Z[x(n+1)] = z\tilde{x}(z) - zx(0), \quad \text{for } |z| > R.$$

and

$$Z[x(n+2)] = z^2\tilde{x}(z) - z^2x(0) - zx(1), \quad \text{for } |z| > R.$$

d) (Convolution) For the finite convolution defined in equation (1.1.1)

$$Z[(x * y)(n)] = \tilde{x}(z)\tilde{y}(z).$$

The same formula holds if the convolution is defined by

$$(x * y)(n) = \sum_{j=0}^{\infty} x(n-j)y(j).$$

e) (Uniqueness) Suppose that there are two vector-valued sequences $x(n)$ and $y(n)$ such that $\tilde{x}(z) = \tilde{y}(z)$ for $|z| > R$. Then $x(n) = y(n)$.

The formula of the inverse Z-transform is given by

$$x(n) = \frac{1}{2\pi i}\int_C \tilde{x}(z)z^{n-1}dz = \text{ sum of residues of } \tilde{x}(z)z^{n-1}, \tag{1.2.2}$$

where C is a circle centered at the origin of the complex plane, that encloses all poles of $\tilde{x}(z)z^{n-1}$.

On the other hand, the discrete time Fourier transform of a sequence $f \in s(\mathbb{Z}, X)$ is defined by

$$\widehat{f}(t) := \sum_{j=-\infty}^{\infty} e^{-ijt}f(j), \quad t \in (-\pi, \pi), \tag{1.2.3}$$

whenever the right side of the above identity exists.

The notation $\widehat{f}(t)$ helps to highlight the periodicity property of this transform and emphasizes the relationship of the discrete-time Fourier transform to the Z-transform. Also the discrete-time Fourier transform plays an important role in representing and analyzing discrete-time signal and systems.

Note that the convolution theorem for discrete-time Fourier transform holds, i.e., $\widehat{f * g}(t) = \widehat{f}(t)\widehat{g}(t)$. Here the convolution is defined in analogous way to the case of the Z-transform:

$$(f * g)(n) = \sum_{j=-\infty}^{\infty} f(j)g(n-j).$$

Further properties of the discrete-time Fourier Transform are analogous to those of the Z-transform, since it is the evaluation of the Z-transform around the unit circle in the complex plane.

The following inverse transform recovers the discrete-time sequence

$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(t)e^{int}dt.$$

1.3 The discrete-time Fourier transform in $\ell_p(\mathbb{Z}, X)$

In what follows, we detail the definition and properties of the discrete-time Fourier transform in the vector-valued Lebesgue space of sequences $\ell_p(\mathbb{Z}, X)$. We denote by $\mathcal{S}(\mathbb{Z}, X)$ the space of all vector-valued sequences $f : \mathbb{Z} \to X$ such that for each $k \in \mathbb{N}_0$ there exists C_k with $p_k(f) := \sup_{n\in\mathbb{Z}} |n|^k \|f(n)\| < C_k$.

Recall that $\mathcal{S}(\mathbb{Z}, X)$ is norm dense in $\ell_p(\mathbb{Z}, X)$ for $1 < p < \infty$. We also denote by $C^n_{per}(\mathbb{R}; X), n \in \mathbb{N}_0$, the space of all 2π-periodic X-valued and n-times continuously differentiable functions defined in $\mathbb{R}$.

Let $\mathbb{T} := (-\pi, \pi)$ and $\mathbb{T}_0 := (-\pi, \pi) \setminus \{0\}$. We introduce the space of test functions as $C^\infty_{per}(\mathbb{T}; X) := \bigcap_{n\in\mathbb{N}_0} C^n_{per}(\mathbb{R}; X)$ endowed with the topology induced by the countable family of seminorms:

$$q_k(\varphi) = \max_{k\in\mathbb{N}_0} \sup_{t\in[-\pi,\pi]} \|\varphi^{(k)}(t)\|.$$

If $X = \mathbb{C}$ we simply denote $C^\infty_{per}(\mathbb{T}; X) = C^\infty_{per}(\mathbb{T})$ and $\mathcal{S}(\mathbb{Z}; X) = \mathcal{S}(\mathbb{Z})$. We also consider the following spaces of vector-valued distributions

$$\mathcal{S}'(\mathbb{Z}; X) := \{T : \mathcal{S}(\mathbb{Z}) \to X \ : \ T \text{ is linear and continuous}\}$$

and

$$\mathcal{D}'(\mathbb{T}; X) := \{T : C^\infty_{per}(\mathbb{T}) \to X \ : \ T \text{ is linear and continuous}\}.$$

Observe that we can identify $\ell_p(\mathbb{Z}; X)$ with a subspace of $\mathcal{S}'(\mathbb{Z}; X)$ via the mapping

$$T_f(\psi) := \langle T_f, \psi\rangle := \sum_{n\in\mathbb{Z}} f(n)\psi(n), \quad \psi \in \mathcal{S}(\mathbb{Z}), \tag{1.3.1}$$

and we have $T_f \in \mathcal{S}'(\mathbb{Z}, X)$. The space $C^\infty_{per}(\mathbb{T}; X)$ can be also identified with a subspace of $\mathcal{D}'(\mathbb{T}; X)$ via the linear map

$$L_S(\varphi) := \langle L_S, \varphi\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)S(t)dt, \quad \varphi \in C^\infty_{per}(\mathbb{T}),$$

and we get $L_S \in \mathcal{D}'(\mathbb{T}; X)$.

The discrete time Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{Z}, X) \to C^\infty_{per}(\mathbb{T}, X)$ is defined by

$$\mathcal{F}\varphi(t) \equiv \widehat{\varphi}(t) := \sum_{j=-\infty}^{\infty} e^{-ijt}\varphi(j), \qquad t \in (-\pi, \pi].$$

It is an isomorphism whose inverse is defined by

$$\mathcal{F}^{-1}\varphi(n) \equiv \check{\varphi}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)e^{int}dt, \quad n \in \mathbb{Z}, \tag{1.3.2}$$

where $\varphi \in C^\infty_{per}(\mathbb{T}; X)$. This isomorphism, allows to define the discrete time Fourier transform (DTFT) between the spaces of distributions $\mathcal{S}'(\mathbb{Z}; X)$ and $\mathcal{D}'(\mathbb{T}; X)$ as follows

$$\langle \mathcal{F}T, \psi\rangle \equiv \mathcal{F}(T)(\psi) := \widehat{T}(\psi) \equiv \langle T, \check{\psi}\rangle, \quad T \in \mathcal{S}'(\mathbb{Z}; X), \quad \psi \in C^\infty_{per}(\mathbb{T}), \tag{1.3.3}$$

whose inverse $\mathcal{F}^{-1} : \mathcal{D}'(\mathbb{T}; X) \to \mathcal{S}'(\mathbb{Z}; X)$ is given by

$$\langle \mathcal{F}^{-1}L, \psi\rangle \equiv \mathcal{F}^{-1}(L)(\psi) := \check{L}(\psi) \equiv \langle L, \widehat{\psi}\rangle, \quad L \in \mathcal{D}'(\mathbb{T}; X), \quad \psi \in \mathcal{S}(\mathbb{Z}).$$

In particular, we get

$$\langle \mathcal{F}T_f, \varphi\rangle = \langle T_f, \check{\varphi}\rangle = \sum_{n\in\mathbb{Z}} f(n)\check{\varphi}(n), \quad \varphi \in C^\infty_{per}(\mathbb{T}), \quad f \in \ell_p(\mathbb{Z}, X). \tag{1.3.4}$$

The convolution of a distribution $T \in \mathcal{S}'(\mathbb{Z}, X)$ with a function $a \in \ell_1(\mathbb{Z})$ is defined by

$$\langle T * a, \varphi\rangle := \langle T, a \circ \varphi\rangle, \quad \varphi \in \mathcal{S}(\mathbb{Z}),$$

where

$$(a \circ \varphi)(n) := \sum_{j=0}^{\infty} a(j)\varphi(j+n).$$

Note that $(a \circ \varphi) \in \mathcal{S}(\mathbb{Z})$.

1.4 The discrete fractional derivative on $\mathbb{Z}$

In this section we introduce an extended notion of fractional difference operator, analogous to Definition 1.1.4. We denote by $\ell_p(\mathbb{Z}, X)$ the vector space of all vector-valued sequences $f : \mathbb{Z} \to X$ such that

$$\sum_{j=-\infty}^{\infty} \|f(n)\|^p < \infty$$

for $1 \leq p < \infty$. In the case when $X = \mathbb{C}$ or $X = \mathbb{R}$, we denote it by $\ell_p(\mathbb{Z})$. Now we recall that given $f \in \ell_p(\mathbb{Z}, X)$ and $g \in \ell_1(\mathbb{Z})$, we can define the convolution product as

$$(f * g)(n) = \sum_{j=-\infty}^{n} f(n-j)g(j) = \sum_{j=0}^{\infty} f(j)g(n-j), \quad n \in \mathbb{Z}.$$

The following definition of fractional sum was introduced in [1].

Definition 1.4.1. Let $\alpha > 0$ and $f : \mathbb{Z} \to X$ be given. We define the fractional sum of order α as follows

$$\Delta^{-\alpha} f(n) := (k^\alpha * f)(n) = \sum_{j=-\infty}^{n} k^\alpha(n-k)f(k), \quad n \in \mathbb{Z}, \tag{1.4.1}$$

whenever it exists.

The following definition corresponds to an analogous version of discrete fractional derivative in the sense of Grünwald-Letnikov, see [60, formula (27) with h=1].

Definition 1.4.2. Let $f : \mathbb{Z} \to X$ be given, we define the the fractional difference operator of order $\alpha > 0$ (in sense of Grünwald-Letnikov) by

$$\Delta^\alpha f(n) := (k^{-\alpha} * f)(n) = \sum_{j=-\infty}^{n} k^{-\alpha}(n-j)f(j) = \sum_{j=0}^{\infty} k^{-\alpha}(j)f(n-j). \tag{1.4.2}$$

The above definition of fractional difference operator of order α was first introduced by Abadías and Lizama [1], after previous work of Lizama [50], as follows:

$$W^{\alpha} f(n) := (-1)^m \Delta^m W^{-(m-\alpha)} f(n), \quad n \in \mathbb{Z},$$

where $m := [\alpha] + 1$. In the above cited references it is also called Weyl difference operator of order α and is denoted by W instead of Δ. Their equivalence with (1.4.2) was recently proved.

From [50], the following generation formula holds

$$\sum_{j=0}^{\infty} k^{\beta}(j) z^j = \frac{1}{(1-z)^{\beta}}, \; \beta \in \mathbb{R}, \; |z| < 1,$$

see also [76, p.42 formulae (1) and (8)]. In particular, for all $\alpha \in \mathbb{R}_+$ we have that the radial limit exists and

$$\widehat{k^{-\alpha}}(t) = \sum_{j=0}^{\infty} k^{-\alpha}(j) e^{-itj} = \frac{1}{(1-e^{-it})^{-\alpha}} = \Big(1 - e^{-it}\Big)^{\alpha}, \quad t \in \mathbb{T}_0. \tag{1.4.3}$$

Observe that $k^{-\alpha} \in \ell_1(\mathbb{Z})$ (see also [76, p.42 formula (2)]). We also recall the following lemma stated in [53] which will be used in the proof of our main result.

Lemma 1.4.3. *Let $u, v \in \ell_p(\mathbb{Z}; X)$ and $a \in \ell_1(\mathbb{Z})$. The following assertions are equivalent:*

i) $(a * v)(n) = u(n)$ *for all* $n \in \mathbb{Z}$.

ii) $< u, \check{\phi} > = < v, (\phi \cdot \widehat{a}_-)\check{} >$ *for all* $\phi \in C^{\infty}_{per}((-\pi, \pi), \mathbb{R})$, *where*

$$(\phi \cdot \widehat{a}_-)\check{}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}(-t) \phi(t) e^{int} dt, \quad n \in \mathbb{Z}.$$

1.5 UMD spaces

This section is devoted to the study of a class of Banach spaces, the so-called UMD-spaces, which share many of the good properties of Hilbert spaces and include the L^p spaces for $1 < p < \infty$.

Definition 1.5.1. Let X be a Banach space. We say that X has the *Unconditional Martingale Difference property, that is X is a UMD space, if for each $p > 1$, there exists a constant $C_p > 0$ such that for any $(f_n)_{n\in\mathbb{N}_0} \subset L^p(\Omega, \Sigma, \mu; X)$ and any choice of signs $(\xi_n)_{n\in\mathbb{N}_0} \subset (-1, 1)$ and any $N \in \mathbb{Z}_+$ we have the following estimate*

$$\left\| f_0 + \sum_{n=1}^{N} \xi_n (f_n - f_{n-1}) \right\|_{L^p(\Omega,\Sigma,\mu;X)} \leq C_p \left\| f_N \right\|_{L^p(\Omega,\Sigma,\mu;X)} .$$

We recall that those Banach spaces X for which the Hilbert transform defined by

$$(Hf)(t) = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \frac{1}{\pi} \int_{\varepsilon\leq|s|\leq R} \frac{f(t-s)}{s} ds$$

is bounded on $L_p(\mathbb{R}, X)$ for some $p \in (1, \infty)$ are called $\mathcal{HT}$ spaces. The limit in the above formula has to be understood in the L_p sense.

More information and details on the Hilbert transform and the UMD Banach spaces can be found in [9, Section III.4.3-III.4.5]. Note that the notions of UMD space and $\mathcal{HT}$ space are equivalent (see [6]).

Some examples of UMD spaces include the Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, $1 < p < \infty$, Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD space. Moreover, a UMD space is reflexive and therefore, $L^1(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ (if Ω is a infinite set) and $C^s([0, 2\pi]; X)$ are not UMD.

1.6 R-Boundedness

The notion of R-boundedness is a significant tool in the study of abstract multiplier operators. Preliminary concepts for the definition and properties of R-boundedness that we will use may be found in [33]. In what follows, we denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y endowed with the uniform operator topology; when $X = Y$, we denote it by $\mathcal{B}(X)$. We recall the following definition.

Definition 1.6.1. Let X and Y be Banach spaces. A family $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called *R-bounded if there exists a constant $c > 0$ such that*

$$\sum_{\varepsilon_j \in \{-1,1\}^n} \left\| \sum_{j=1}^{n} \varepsilon_j T_j x_j \right\| \leq c \sum_{\varepsilon_j \in \{-1,1\}^n} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \tag{1.6.1}$$

for all $T_1, \dots, T_n \in \mathcal{T}$, $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$. The least c such that (1.6.1) *is satisfied is called the R-bound of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$*

Now, we recall some properties of R-bounded sets.

Proposition 1.6.2. *i) Every R-bounded set is a uniformly bounded set.*

ii) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

iii) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X, Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded as well, and $R(\mathcal{T} + \mathcal{S}) \leq R(\mathcal{T}) + R(\mathcal{S})$.

iv) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded, and $R(\mathcal{S}\mathcal{T}) \leq R(\mathcal{S})R(\mathcal{T})$.

v) A subset $\mathcal{T} \subset \mathcal{B}(X,Y)$ of the form $\mathcal{T} = \{\lambda I : \lambda \in \mathcal{U}\}$ is R-bounded, whenever $\mathcal{U} \subset \mathbb{C}$ is bounded.

1.7 Operator-valued multipliers

We finish this section recalling the following Fourier multiplier theorem for operator-valued symbols established by S. Blünck [20]. This theorem corresponds to a discrete version of a result proved by Weis [71] and Amann [8, Section III.4.3-III.4.5] which establishes sufficient conditions to ensure when a operator-valued symbol is a multiplier in the context of UMD Banach spaces.

Recall that the Banach space $\mathcal{B}(X,Y)$ is equipped with the uniform operator topology. First we introduce the following notion of ℓ_p-multiplier.

Definition 1.7.1. Let X, Y be Banach spaces, $1 < p < \infty$. A function $M \in C^{\infty}_{per}(\mathbb{T}, \mathcal{B}(X,Y))$ is an ℓ_p-multiplier (from X to Y) if there exists a bounded operator $T : \ell_p(\mathbb{Z}; X) \to \ell_p(\mathbb{Z}; Y)$ such that

$$\sum_{n\in\mathbb{Z}} (Tf)(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} (\varphi \cdot M_-\check{)}(n) f(n) \tag{1.7.1}$$

for all $f \in \ell_p(\mathbb{Z}; X)$ and all $\varphi \in C^{\infty}_{per}(\mathbb{T})$. Here

$$(\varphi \cdot M_-\check{)}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)dt, \quad n \in \mathbb{Z}.$$

We finally recall the following Fourier multiplier theorem for operator valued symbols due to S. Blünck, see [6, 20] for more details. This theorem will be crucial for the characterization of maximal regularity. Blünck's theorem and its converse establish an equivalence between R-bounded sets and l_p-multipliers.

Theorem 1.7.2. [20, Theorem 1.3] *Let $p \in (1, \infty)$ and let X be a UMD space. Let*

$M : \mathbb{T}_0 \longrightarrow \mathcal{B}(X)$ be differentiable function such that the set

$$\big\{M(t), (z-1)(z+1)M'(t) : z = e^{it},\, t \in \mathbb{T}_0\big\}$$

is R-bounded. Then there exists an operator $T_M \in \mathcal{B}(\ell_p(\mathbb{Z}, X))$ such that

$$(\widehat{T_M f})(z) = M(t)\widehat{f}(z), \quad \textit{for all} \quad z = e^{it},\, t \in \mathbb{T}_0. \tag{1.7.2}$$

The converse of Blünck's Theorem also holds without any restriction on the Banach space X.

Theorem 1.7.3. [20, Proposition 1.3] *Let $p \in (1,\infty)$ and let X be a UMD space. Let $M : \mathbb{T}_0 \longrightarrow \mathcal{B}(X)$ be an operator-valued function. Suppose that there exists an operator $T_M \in \mathcal{B}(\ell_p(\mathbb{Z}, X))$ such that the identity* (1.7.2) *holds. Then the set*

$$\{M(t) : t \in \mathbb{T}_0\}$$

is R-bounded.

Note that in the case of considering two Banach spaces, we obtain a slight modification of Blünck's Theorem.

Theorem 1.7.4. [20, Theorem 1.3] *Let $p \in (1,\infty)$ and let X, Y be UMD spaces. Let $M \in C^\infty_{per}(\mathbb{T}_0, \mathbb{B}(X,Y))$ such that the sets*

$$\big\{M(t), (e^{it}-1)(1+e^{it})M'(t) :\ t \in \mathbb{T}_0\big\}$$

are both R-bounded. Then M is an ℓ_p-multiplier (from X to Y) for $1 < p < \infty$.

The converse of the last theorem is fulfilled without any restriction on the Banach spaces X, Y in the following sense.

Theorem 1.7.5. [20, Proposition 1.4] *Let $p \in (1,\infty)$ and let X, Y be Banach spaces. Let $M : \mathbb{T}_0 \to \mathcal{B}(X,Y)$ be an operator valued function. Suppose that there* *is a bounded operator* $T_M : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, Y)$ *such that* (1.7.1) *holds. Then the set* $\{M(t) : t \in \mathbb{T}_0\}$ *is R-bounded.*

Chapter 2

Lebesgue regularity for fractional differential-difference equations with fractional damping

In this chapter we provide necessary and sufficient conditions for the existence and uniqueness of solutions belonging to the vector-valued space of sequences $\ell_p(\mathbb{Z}, X)$ for equations that can be modeled in the form

$$\Delta^\alpha u(n) + \lambda \Delta^\beta u(n) = Au(n) + G(u)(n) + f(n),\ n \in \mathbb{Z},\ \alpha, \beta > 0,\ \lambda \geq 0,$$

where X is a Banach space, $f \in \ell_p(\mathbb{Z}, X)$, A is a closed linear operator with domain $D(A)$ defined on X and G is a nonlinear function. The operator Δ^γ denotes the fractional difference operator of order $\gamma > 0$ in the sense of Grünwald-Letnikov. Our class of models includes the discrete time Klein-Gordon, telegraph and Basset equations, among other differential difference equations of interest. We prove a simple criterion that shows the existence of solutions assuming that f is small and G is a nonlinear term.

2.1 A characterization of maximal regularity

In this section, we first provide a characterization of the existence and uniqueness of solutions in $\ell_p(\mathbb{Z}, [D(A)])$ for the general model

$$\Delta^\alpha u(n) + \lambda\Delta^\beta u(n) = Au(n) + f(n), \tag{2.1.1}$$

where $\alpha, \beta > 0,\ \lambda \geq 0,\ A$ is a closed linear operator defined on a Banach space X and $f : \mathbb{Z} \to X$ is a vector-valued sequence. Recall that the above model is an abbreviated form to write a partial differential equation which is continuous in space but discrete in time. For example, the equation

$$u(n+2,x) - 2u(n+1,x) + u(n,x) + \lambda[u(n+1,x) - u(n,x)] = \partial_{xx}u(n,x) + f(n,x),$$

where $n \in \mathbb{Z}$, $x \in \Omega \subset \mathbb{R}^N$, fits in the abstract setting of the model (2.1.1) with $\alpha = 2,\ \beta = 1$ and $A = \partial_{xx}$.

We introduce the following definition, also called ℓ_p-well-posedness in the literature.

Definition 2.1.1. Let $1 < p < \infty$. We say that (2.1.1) has maximal ℓ_p-regularity if for each $f \in \ell_p(\mathbb{Z}, X)$ there exists a unique $u \in \ell_p(\mathbb{Z}, [D(A)])$ that satisfies (2.1.1).

We are ready to prove our main result in this chapter. Recall that $\mathbb{T} = (-\pi, \pi)$ and $\mathbb{T}_0 = \mathbb{T} \setminus \{0\}$.

Theorem 2.1.2. *Let A be a closed linear operator defined on an UMD space X. Set $\alpha, \beta > 0$ and $\lambda \geq 0$. Suppose that*

$$\{(1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta\}_{t\in\mathbb{T}_0} \subset \rho(A)$$

and define $M(t) := ((1 - e^{-it})^\alpha + \lambda(1 - e^{-it})^\beta - A)^{-1}$. Then the following assertions are equivalent

(i) (2.1.1) *has maximal ℓ_p-regularity.*

(ii) $M(t)$ *is an* ℓ_p*-multiplier from* X *to* $[D(A)]$.

(iii) *The set* $\{M(t)\}_{t\in\mathbb{T}_0}$ *is R-bounded.*

Proof. We first show $(iii) \Rightarrow (ii)$. Let $\{M(t) : t \in \mathbb{T}_0\}$ be R-bounded. We will show that the set $\{(e^{it}-1)(1+e^{it})M(t) : t \in \mathbb{T}_0\}$ is also R-bounded. Defining for each $t \in \mathbb{T}_0$, $f_\alpha(t) := (1-e^{-it})^\alpha$ and $f_\beta(t) := (1-e^{-it})^\beta$ it can be shown that

$$M'(t) = -M(t)^2(f'_\alpha(t) + \lambda f'_\beta(t)).$$

Since $f'_\alpha(t) = i\alpha f_\alpha(t)\dfrac{1}{e^{it}-1}$, it follows that $(e^{it}-1)(1+e^{it})M'(t) = i(\alpha f_\alpha(t) + \lambda\beta f_\beta(t))(1+e^{it})M(t)^2$. From [6, Proposition 2.2.5] we deduce that the set $\{(e^{it}-1)(1+e^{it})M'(t) : t \in \mathbb{T}_0\}$ is R-bounded and the claim is proved. Consequently, by Theorem 1.7.4, we obtain (i). The implication $(ii) \implies (iii)$ follows immediatly from Theorem 1.7.5. Let now show that $(i) \implies (ii)$. Let $f \in \ell_p(\mathbb{Z}, X)$ be given. Then there exists a unique $u_f \in \ell_p(\mathbb{Z}, [D(A)])$ solution of (2.1.1). We define $T_{\alpha,\beta} : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, [D(A)])$ the linear operator given by $T_{\alpha,\beta}(f) = u_f$. By the Closed Graph Theorem, we get that $T_{\alpha,\beta}$ is bounded. Let $\varphi \in C^\infty_{per}(\mathbb{T})$, $f \in \ell_p(\mathbb{Z}, X)$ and $u = T_{\alpha,\beta}f$. Since $k^{-\alpha} \in \ell_1(\mathbb{Z})$ we obtain the following identity,

$$\begin{aligned}
(k^{-\alpha} \circ \check{S})(n) :&= \sum_{j=0}^{\infty} k^{-\alpha}(j)\check{S}(j+n) = \sum_{j=0}^{\infty} k^{-\alpha}(j)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n+j)t}S(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\Big(\sum_{j=0}^{\infty} e^{ijt}k^{-\alpha}(j)\Big)S(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\widehat{k}^{-\alpha}(-t)S(t)dt = (\widehat{k}^{-\alpha}_{-} \cdot S)\check{}(n),
\end{aligned} \tag{2.1.2}$$

valid for any $S \in C^\infty_{per}(\mathbb{T}, \mathcal{B}(X,Y))$. Therefore, using the hypothesis and the observation that we have $M \in C^{\infty}_{per}(\mathbb{T}, \mathcal{B}(X, [D(A)]))$ we get

$$\begin{aligned}
\langle T_{\alpha,\beta} f, \check{\varphi}\rangle &= \langle u, \check{\varphi}\rangle = \sum_{n\in\mathbb{Z}} \check{\varphi}(n)u(n) = \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)u(n)dt \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} (1-e^{it})^{\alpha} e^{int}\varphi(t)((1-e^{it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - A)^{-1}u(n)dt \\
&+ \lambda \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} (1-e^{it})^{\beta} e^{int}\varphi(t)((1-e^{it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - A)^{-1}u(n)dt \\
&- \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} ((1-e^{it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - A)^{-1}Au(n)e^{int}\varphi(t)dt \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\widehat{k}^{-\alpha}(-t)\varphi(t)M(-t)u(n)dt \\
&+ \lambda \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\widehat{k}^{-\beta}(-t)\varphi(t)M(-t)u(n)dt \\
&- \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)Au(n)dt \\
&= \langle u, ((\widehat{k_-^{-\alpha} + \lambda k_-^{-\beta}}) \cdot \varphi \cdot M_-\check{)}\rangle - \langle Au, (\varphi \cdot M_-\check{)}\rangle \\
&= \langle u, (k_-^{-\alpha} + \lambda k_-^{-\beta}) \circ (\varphi \cdot M_-\check{)}\rangle - \langle Au, (\varphi \cdot M_-\check{)}\rangle,
\end{aligned}$$

where in the last equality we have used (2.1.2) with $S = \varphi \cdot M_-$. Therefore

$$\begin{aligned}
\langle u, \check{\varphi}\rangle &= \langle (k_-^{-\alpha} + \lambda k_-^{-\beta}) * u, (\varphi \cdot M_-\check{)}\rangle - \langle Au, (\varphi \cdot M_-\check{)}\rangle \\
&= \langle \Delta^{\alpha} u + \lambda \Delta^{\beta} u - Au, (\varphi \cdot M_-\check{)}\rangle. \qquad (2.1.3)
\end{aligned}$$

We conclude that $\langle T_{\alpha,\beta} f, \check{\varphi}\rangle = \langle f, (\varphi \cdot M_-\check{)}\rangle$ and then $M(t)$ is an ℓ_p-multiplier.

It only remains to prove that (ii) implies (i). We first claim that $N(t) := (1 - e^{-it})^{\alpha}((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - A)^{-1}$ and $S(t) = (1-e^{-it})^{\beta}((1-e^{-it})^{\alpha} + \lambda(1-e^{-it})^{\beta} - A)^{-1}$ are ℓ_p-multipliers. Indeed, since $N(t) = f_{\alpha}(t)M(t)$ and $S(t) = f_{\beta}(t)M(t)$ where $f_{\alpha}(t) = (1-e^{-it})^{\alpha}$ and $f_{\beta}(t) = (1-e^{-it})^{\beta}$, the R-boundedness of $N(t)$ and

$S(t)$ follows. On the other hand, the identities:

$$\begin{aligned}(e^{it}-1)(1+e^{it})N'(t) &= -i\alpha N(t)(1+e^{it}) \\ &\quad + i\alpha N(t)^2(1+e^{it}) + i\lambda\beta N(t)S(t)(1+e^{it}) \\ (e^{it}-1)(1+e^{it})S'(t) &= -i\beta S(t)(1+e^{it}) \\ &\quad + i\alpha S(t)N(t)(1+e^{it}) + i\lambda\beta S(t)^2(1+e^{it})\end{aligned}$$

show that the sets $\{(e^{it}-1)(1+e^{it})N'(t)\ :\ t\in\mathbb{T}_0\}$ and $\{(e^{it}-1)(1+e^{it})S'(t)\ :\ t\in\mathbb{T}_0\}$ are R-bounded and then the claim holds by Theorem 1.7.4. Let $f\in\ell_p(\mathbb{Z},X)$ be given. By hypothesis, there exists $u\in\ell_p(\mathbb{Z},[D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} u(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} (\varphi\cdot M_-\check{)}(n)f(n) \tag{2.1.4}$$

for all $\varphi\in C^{\infty}_{per}(\mathbb{T})$. On the other hand, there exist $v,w\in\ell_p(\mathbb{Z};[D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} v(n)\check{\psi}(n) = \sum_{n\in\mathbb{Z}} (\psi\cdot N_-\check{)}(n)f(n) \tag{2.1.5}$$

$$\sum_{n\in\mathbb{Z}} w(n)\check{\eta}(n) = \sum_{n\in\mathbb{Z}} (\eta\cdot S_-\check{)}(n)f(n) \tag{2.1.6}$$

for all $\psi,\eta\in C^{\infty}_{per}(\mathbb{T})$. Since $N(t)=\widehat{k}^{-\alpha}(t)M(t)$ we have

$$(\psi\cdot N_-\check{)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\psi(t)\widehat{k}^{-\alpha}(-t)M(-t)dt.$$

Choosing $\varphi(t)=\psi(t)\widehat{k^{-\alpha}}(-t)$ in (2.1.4) we obtain $\langle v,\check{\psi}\rangle = \langle u,(\psi\cdot\widehat{k_-^{-\alpha}}\check{)}\rangle$ and hence by Lemma 1.4.3 we get that $k^{-\alpha}*u\in\ell_p(\mathbb{Z},X)$ and

$$\Delta^{\alpha}u(n) = k^{-\alpha}*u(n) = v(n), \quad n\in\mathbb{Z}. \tag{2.1.7}$$

Analogously, since $S(t)=\widehat{k}^{-\beta}(t)M(t)$ we can choose $\varphi(t)=\eta(t)\widehat{k^{-\beta}}(-t)$ in (2.1.4) and then, by Lemma 1.4.3 we get that $k^{-\beta}*u\in\ell_p(\mathbb{Z},X)$ and

$$\Delta^{\beta}u(n) = k^{-\beta}*u(n) = v(n), \quad n\in\mathbb{Z}. \tag{2.1.8}$$

Now, from the identity $N(t) + \lambda S(t) = AM(t) + I$ we obtain after multiplication by $e^{int}\varphi(t)$ and then integration on the interval $(-\pi, \pi)$, the identity $(\varphi \cdot N_-\check{)}(n) + \lambda(\varphi \cdot S_-\check{)}(n) = A(\varphi \cdot M_-\check{)}(n) + \check{\varphi}(n)I$, for all $\varphi \in C^{\infty}_{per}(\mathbb{T})$. Then we get

$$\langle f, (\varphi \cdot N_-\check{)}\rangle + \langle f, \lambda(\varphi \cdot S_-\check{)}\rangle = \langle f, A(\varphi \cdot M_-\check{)}\rangle + \langle f, \check{\varphi}\rangle.$$

Replacing (2.1.4), (2.1.5) and (2.1.6) in the above identity we obtain

$$\sum_{n\in\mathbb{Z}} v(n)\check{\varphi}(n) + \lambda \sum_{n\in\mathbb{Z}} w(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} Au(n)\check{\varphi}(n) + \sum_{n\in\mathbb{Z}} \check{\varphi}(n) f(n),$$

for all $\varphi \in C^{\infty}_{per}(\mathbb{T})$. Considering (2.1.7), (2.1.8) and replacing $\varphi_k(t) := e^{-ikt}$, $k \in \mathbb{Z}$ in the above identity, we conclude that u satisfies the equation (2.1.1).

In order to show uniqueness, we consider $u : \mathbb{Z} \to [D(A)]$ one solution of (2.1.1) with $f \equiv 0$. For all $\varphi \in C^{\infty}_{per}(\mathbb{T})$ and using (2.1.3) we obtain $\langle u, \check{\varphi}\rangle = \langle \Delta^{\alpha} u + \lambda\Delta^{\beta} u - Au, (\varphi \cdot M_-\check{)}\rangle = 0$. Choosing $\varphi_k(t) := e^{-ikt}$, $k \in \mathbb{Z}$ we obtain $u \equiv 0$. This proves (i) and the theorem. □

The following statement follows from the closed graph theorem and Theorem 2.1.2.

Corollary 2.1.3. *In the context of Theorem 2.1.2, if condition (iii) is valid, we have $u, \Delta^{\alpha} u, \Delta^{\beta} u, Au \in l_p(\mathbb{Z}, X)$. Moreover, there exists a constant $C > 0$ such that*

$$\|\Delta^{\alpha} u\|_p + \lambda \left\|\Delta^{\beta} u\right\|_p + \|Au\|_p \le C \|f\|_p . \tag{2.1.9}$$

As a consequence of Theorem 2.1.2, we easily have a corresponding one in the case of Hilbert spaces, where R-boundedness is equivalent to norm boundedness [33, Chapter 3, Remark 3.2].

Corollary 2.1.4. *Let H be a Hilbert space and $\alpha, \beta > 0$, $\lambda \ge 0$. Suppose that $\{(1 - e^{-it})^{\alpha} + \lambda(1 - e^{-it})^{\beta}\}_{t\in\mathbb{T}_0} \subset \rho(A)$. The following assertions are equivalent:*

(i) For all $f \in \ell_p(\mathbb{Z}, H)$ there exists a unique $u \in \ell_p(\mathbb{Z}, H)$ such that $u(n) \in D(A)$ for all $n \in \mathbb{Z}$ and u satisfies (2.1.1)*;*

(ii) $\sup_{t\in\mathbb{T}_0} \left\|((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}\right\| < \infty.$

Now, we can consider the nonlinear perturbed version of (2.1.1) given by

$$\Delta^\alpha u(n) + \lambda\Delta^\beta u(n) = Au(n) + G(u)(n) + \rho f(n) \tag{2.1.10}$$

where $\rho > 0$, $f \in \ell_p(\mathbb{Z}, X)$ and $G : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, X)$. We can show a result concerning the existence of $\ell_p(\mathbb{Z}, X)$-solutions of (2.1.10) in terms of the symbol of the equation and the regularity of G.

Theorem 2.1.5. *Let X be a UMD space, $1 < p < \infty$, $\alpha, \beta > 0$ and $\lambda \geq 0$. Suppose that $\{(1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta\}_{t\in\mathbb{T}_0} \subset \rho(A)$. If the following conditions hold*

(i) the set $\{((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}\}_{t\in\mathbb{T}_0}$ is R-bounded;

(ii) G is continuously Fréchet differentiable at $u = 0$ and $G(0) = G'(0) = 0$.

Then there exists ρ^* such the equation (2.1.10) has a solution $u = u_\rho \in \ell_p(\mathbb{Z}, X)$ for each $\rho \in [0, \rho^*)$.

Proof. We note that $|||u||| := \|\Delta^\alpha u\| + \lambda\left\|\Delta^\beta u\right\| + \|Au\| + \|u\|$ defines a norm in $\ell_p(\mathbb{Z}, [D(A)])$ and hence $(\ell_p(\mathbb{Z}, [D(A)]), |||\cdot|||)$ becomes a Banach space. Let $L : \ell_p(\mathbb{Z}, [D(A)]) \to \ell_p(\mathbb{Z}, [D(A)])$ be defined as $(Lu)(n) := \Delta^\alpha u(n) + \lambda\Delta^\beta u(n) - Au(n)$. By Corollary 2.1.4, and since hypothesis (i) holds, we have that the inequality (2.1.9) holds, and then we obtain $|||u||| \leq C\|Lu\|$ for some constant $C > 0$. Also, by definition of L, we have $\|Lu\| \leq |||u|||$. Then L defines an isomorphism. Given $\rho \in [0, 1)$, we define:

$$H[u, \rho] = -Lu + G(u) + \rho f$$

By hypothesis (ii), we have $H[0, 0] = 0$ and H is continuously differentiable at $(0, 0)$. In addition, $H^1_{(0,0)} = -L$ which is invertible. Therefore, using the implicit function theorem, we deduce the existence of ρ^* such that for all $\rho \in [0, \rho^*)$, there exists $u_\rho \in \ell_p(\mathbb{Z}, X)$ such that $H[u_\rho, \rho] = 0$. This proves the theorem. □

2.2 Examples

We verify the conditions provided in Theorem 2.1.5 in order to show the existence and uniqueness of $\ell_p(\mathbb{Z}, X)$ solutions for the following equation

$$\Delta^\alpha u(n,x) + \lambda\Delta^\beta u(n,x) + bu(n,x) - \frac{\partial^2}{\partial x^2}u(n,x) = u^2(n,x) + \varepsilon f(n,x) \qquad (2.2.1)$$

where $\lambda \geq 0$ is fixed, b is a real number, $\varepsilon > 0$ and $f \in \ell_p(\mathbb{Z}, X)$ is an external force whose size is controlled by ε. Note that the linear part of the equation (2.2.1) corresponds to the discrete time Telegraph equation when $\alpha = 2, \beta = 1, b = 0, \lambda = \frac{1}{\tau}$ and $A = \frac{\rho}{\tau}\partial_{xx}$. Also, it coincides with the discrete time Klein-Gordon equation for $\alpha = 2, \lambda = 0$ and $A = \partial_{xx} - bI$.

Equation (2.2.1) can be modeled as (2.1.10) for $Au = u'' - bu$ defined on $L^2(\mathbb{R})$ and $G(u)(n,x) = u^2(n,x)$ for instance. It is well known that the operator $Bu = u''$ with domain $D(B) = \{u \in H_0^1(\mathbb{R}) : u'' \in L^2(\mathbb{R})\}$ generates a contraction C_0-semigroup on $L^2(\mathbb{R})$, therefore the following estimate for their resolvent operator holds

$$\left\|(\mu - B)^{-1}\right\| \leq \frac{1}{\Re(\mu)}, \text{ for all } \quad \Re(\mu) > 0. \qquad (2.2.2)$$

We observe that, for all $0 < \alpha, \beta \leq 2$ we have

$$\begin{aligned}
\Re((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta) &= (2-2\cos(t))^{\frac{\alpha}{2}}\cos\left(\alpha\arctan\left(\frac{\sin(t)}{1-\cos(t)}\right)\right) \\
&\quad + \lambda(2-2\cos(t))^{\frac{\beta}{2}}\cos\left(\beta\arctan\left(\frac{\sin(t)}{1-\cos(t)}\right)\right) \\
&> (2-2\cos(t))^{\frac{\alpha}{2}}\cos\left(\frac{\alpha\pi}{2}\right) + \lambda(2-2\cos(t))^{\frac{\beta}{2}}\cos\left(\frac{\beta\pi}{2}\right) \\
&> -(2^\alpha + \lambda 2^\beta).
\end{aligned}$$

As a consequence, for all $b \geq 2^\alpha + \lambda 2^\beta$ we get that $(1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta \in \rho(A)$ and (2.2.2) shows that

$$\sup_{t\in\mathbb{T}_0}\left\|((1-e^{-it})^\alpha + \lambda(1-e^{-it})^\beta - A)^{-1}\right\| \leq \frac{1}{b-(2^\alpha+\lambda 2^\beta)} < \infty.$$

Furthermore, G is a continuous Fréchet differentiable function at $u = 0$ and clearly satisfies that $G(0) = 0$ and $G'(0) = 0$. Then, by Theorem 2.1.5, we conclude that whenever $b > 2^\alpha + \lambda 2^\beta$ there exists a number $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ there exists a solution $u^\varepsilon \in \ell_p(\mathbb{Z}, X)$ of the perturbed fractional damping difference equation (2.2.1).

Remark 2.2.1. In particular, this example shows that the discrete Klein-Gordon equation admits non-trivial square-summable solutions defined on $\mathbb{Z}$, for small and square-summable external forcing terms whenever $b > 4$. In the case of the generalized discrete Basset equation

$$\Delta^2 u(n) + \lambda \Delta^\beta u(n) + bu(n) = f(n), \quad n \in \mathbb{Z}, \quad \lambda, b > 0, \quad \beta > 0,$$

we obtain that for any $f \in \ell_p(\mathbb{Z})$, there exist p-summable solutions whenever $b > 4$ and $\lambda < \dfrac{b-4}{2^\beta}$.

Chapter 3

Lebesgue regularity for fractional difference equations with delays

In this chapter, we provide a new and effective characterization for the existence and uniqueness of solutions for nonlocal time-discrete equations with delays of the form

$$\begin{cases} \Delta^{\alpha} u(n) = Tu(n) + \beta u(n-\tau) + f(n), \, n \in \mathbb{N}_0 \\ \quad u(j) = x_j, \, j = -\tau, \dots, 0, 1 \end{cases}$$

where $\tau \in \mathbb{N}$, β is a real number, T is a linear bounded operator defined on a Banach space X, $1 < \alpha \leq 2$, f is a vector-valued function and Δ^{γ} denotes the fractional difference operator of order $\gamma > 0$ in sense of Riemann-Liouville. This characterization is given solely in terms of the R-boundedness of the data of the problem, and in the context of the class of UMD Banach spaces.

3.1 Resolvent families with delay: $1 < \alpha \leq 2$

In this section we introduce an operator theoretical method to study the existence and uniqueness of solutions for the following problem.

$$\begin{cases} \Delta^{\alpha} u(n) = Tu(n) + \beta u(n-\tau) + f(n), \quad n \in \mathbb{N}_0, \, \tau \in \mathbb{N}_0, \, \beta \in \mathbb{R} \\ \quad u(j) = x_j, \, j = -\tau, \dots, 0, 1, \end{cases} \tag{3.1.1}$$

where $1 < \alpha \leq 2$ and $T \in \mathcal{B}(X)$. Note that the case $0 < \alpha \leq 1$ was studied by [54].

Definition 3.1.1. Let T be a bounded linear operator defined in a Banach space X, and let $1 < \alpha \leq 2$ and $\tau \in \mathbb{N}$ be given. We say that T is a generator of an α^τ-resolvent sequence if there exists a sequence of bounded and linear operators $\{M_\alpha(n)\}_{n \geq -\tau} \subset \mathcal{B}(X)$ that satisfies the following properties

(i) $M_\alpha(0) = M_\alpha(1) = I$,

(ii) $M_\alpha(-j) = 0$, $j = 1, \ldots, \tau$,

(iii) $M_\alpha(n+2) - M_\alpha(n+1) = T(M_\alpha * k^{\alpha-1})(n) + k^{\alpha-1}(n+2)I - (\alpha-1)k^{\alpha-1}(n+1)I + \beta(M_\alpha^\tau * k^{\alpha-1})(n)$ for each $n \in \mathbb{N}_0$,

where $\{M_\alpha^\tau(n)\}_{n \in \mathbb{N}_0}$ is defined by $M_\alpha^\tau(n) := M_\alpha(n - \tau)$.

Remark 3.1.2. Note that in the case when $\beta = 0$, Definition 3.1.1 coincides with the definition of resolvent sequence defined in [52].

Example 3.1.3. Suppose that $\{z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T\}_{z \in C} \subset \rho(T)$, where $\rho(T)$ denotes the resolvent set of T and C is a circle centered at the origin that encloses all singularities of $z^n(z-(\alpha-1))(z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1}$ in its interior. Then the operator defined by

$$M_\alpha(n) = \frac{1}{2\pi i}\int_C z^n(z-(\alpha-1))(z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} dz,\ n \in \mathbb{N},\ n \geq 2$$

$M_\alpha(0) = M_\alpha(1) = I$ and $M_\alpha(j) = 0$, $j = -\tau, \ldots, -1$ defines an α^τ-resolvent sequence of operators with generator T. This fact can be formally checked using the time discrete Fourier transform method to equation (3.1.1) and comparing it with the formula given in Theorem 3.1.5 below.

Now, we recall the following Lemma proved in [53].

Lemma 3.1.4. *Let* $1 < \alpha \leq 2$, $a : \mathbb{N}_0 \longrightarrow \mathbb{C}$ *and* $S : \mathbb{N}_0 \longrightarrow X$ *be given. Then*

$$\Delta^\alpha(a * S)(n) = \sum_{j=0}^{n} \Delta^\alpha S(n-j)a(j) + S(0)a(n+2) - \alpha S(0)a(n+1) + S(1)a(n+1).$$

Before establishing the main result of this section, we define for $1 < \alpha \le 2$

$$h_\alpha(n) := \begin{cases} (\alpha-1)^n & n \in \mathbb{N}_0 \\ 0 & otherwise \end{cases}$$

and

$$F_\alpha(n) := \begin{cases} (M_\alpha * h_\alpha)(n) & n \in \mathbb{N}_0 \\ 0 & otherwise. \end{cases}$$

Theorem 3.1.5. *Let $1 < \alpha \le 2$ and $f \in s(\mathbb{N}_0; X)$ be given. Assume that T is a generator of an α^τ-resolvent sequence $M_\alpha(n)$. Then the unique solution of* (3.1.1) *is given by*

$$\begin{aligned} u(n+2) &= M_\alpha(n+2)u(0) + (M_\alpha * h_\alpha)(n+1)[u(1) - u(0)] \\ &\quad + \beta \sum_{j=1}^{\tau} F_\alpha(n-\tau+j)u(-j) + (M_\alpha * h_\alpha * f)(n), \quad n \in \mathbb{N}_0. \end{aligned}$$

Proof. We define a vector-valued sequence v as follows. For $n = -\tau, \ldots, 0, 1$, $v(n) = x_n$ and for $n \ge 2$,

$$v(n) = M_\alpha(n)u(0) + (M_\alpha * h_\alpha)(n-1)[u(1) - u(0)] + \beta \sum_{j=1}^{\tau} F_\alpha(n-2-\tau+j)u(-j). \tag{3.1.2}$$

First, we will show that v is a solution of (3.1.1) with $f = 0$. Indeed, applying Δ^α to v, we get

$$\begin{aligned} \Delta^\alpha v(n+2) &= \Delta^\alpha M_\alpha(n+2)u(0) + \Delta^\alpha (M_\alpha * h_\alpha)(n+1)[u(1) - u(0)] \\ &\quad + \beta \sum_{j=1}^{\tau} \Delta^\alpha F_\alpha(n-\tau+j)u(-j),\ n \in \mathbb{N}_0. \end{aligned}$$

From Definition 3.1.1, we have that

$$\begin{aligned} \Delta^\alpha M_\alpha(n+2) &= \Delta^\alpha M_\alpha(n+1) + T\Delta^\alpha F_\alpha(n) + \Delta^\alpha k^{\alpha-1}(n+2)I \\ &\quad - (\alpha-1)\Delta^\alpha k^{\alpha-1}(n+1)I + \beta \Delta^\alpha (M_\alpha^\tau * k^{\alpha-1})(n), \end{aligned}$$

for all $n \in \mathbb{N}_0$. Since $\Delta^\alpha k^{\alpha-1} \equiv 0$, then

$$\Delta^\alpha M_\alpha(n+2) = \Delta^\alpha M_\alpha(n+1) + T\Delta^\alpha(M_\alpha * k^{\alpha-1})(n) + \beta\Delta^\alpha(M_\alpha^\tau * k^{\alpha-1})(n),\ n \in \mathbb{N}_0. \tag{3.1.3}$$

From Lemma 3.1.4, we obtain that

$$\begin{aligned}\Delta^\alpha(M_\alpha * k^{\alpha-1})(n) &= (\Delta^\alpha k^{\alpha-1} * M_\alpha)(n) + M_\alpha(n+2) - \alpha M_\alpha(n+1)\\ &\quad + (\alpha-1)M_\alpha(n+1)\\ &= M_\alpha(n+2) - M_\alpha(n+1)\\ &= \Delta M_\alpha(n+1).\end{aligned} \tag{3.1.4}$$

Thus, replacing (3.1.4) in (3.1.3), we obtain

$$\Delta^\alpha M_\alpha(n+2) = \Delta^\alpha M_\alpha(n+1) + T\Delta M_\alpha(n+1) + \beta\Delta M_\alpha^\tau(n+1),\ n \in \mathbb{N}_0,$$

or equivalently

$$\Delta\Delta^\alpha M_\alpha(n+1) = T\Delta M_\alpha(n+1) + \beta\Delta M_\alpha^\tau(n+1). \tag{3.1.5}$$

From (3.1.5), if $\Delta^\alpha M_\alpha(0) = TM_\alpha(0) + \beta M_\alpha(-\tau) = T$ we get $\Delta^\alpha M_\alpha(n+1) = TM_\alpha(n+1) + \beta M_\alpha^\tau(n+1)$ for all $n \in \mathbb{N}_0$. Indeed, from Definition 1.1.2 and definition of k^α, we have

$$\begin{aligned}\Delta^\alpha M_\alpha(n) &= \Delta^2(k^{2-\alpha} * M_\alpha)(n)\\ &= (k^{2-\alpha} * M_\alpha)(n+2) - 2(k^{2-\alpha} * M_\alpha)(n+1) + (k^{2-\alpha} * M_\alpha)(n).\end{aligned}$$

In particular, for $n = 0$

$$\Delta^\alpha M_\alpha(0) = (k^{2-\alpha} * M_\alpha)(2) - 2(k^{2-\alpha} * M_\alpha)(1) + (k^{2-\alpha} * M_\alpha)(0). \tag{3.1.6}$$

Since

$$\begin{aligned}(k^{2-\alpha} * M_\alpha)(2) &= k^{2-\alpha}(0)M_\alpha(2) + k^{2-\alpha}(1)M_\alpha(1) + k^{2-\alpha}(2)M_\alpha(0) \\ &= T + (5-2\alpha)I\end{aligned} \tag{3.1.7}$$

and

$$(k^{2-\alpha} * M_\alpha)(1) = k^{2-\alpha}(1)M_\alpha(0) + k^{2-\alpha}(0)M_\alpha(1) = (3-\alpha)I \tag{3.1.8}$$

as well as

$$(k^{2-\alpha} * M_\alpha)(0) = k^{2-\alpha}(0)M_\alpha(0) = I. \tag{3.1.9}$$

Replacing (3.1.9), (3.1.8) and (3.1.7) in (3.1.6), we get that

$$\Delta^\alpha M_\alpha(0) = T + (5-2\alpha)I - 2(3-\alpha)I + I = T. \tag{3.1.10}$$

On the other hand $TM_\alpha(0) + \beta M_\alpha^\tau(0) = T$. By Lemma 3.1.4,

$$\begin{aligned}\Delta^\alpha(M_\alpha * h_\alpha)(n) &= (\Delta^\alpha M_\alpha * h_\alpha)(n) + h_\alpha(n+2) - (\alpha-1)h_\alpha(n+1) \\ &= (\Delta^\alpha M_\alpha * h_\alpha)(n) \\ &= T(M_\alpha * h_\alpha)(n) + \beta(M_\alpha^\tau * h_\alpha)(n).\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}\Delta^\alpha v(n) = {} & TM_\alpha(n)u(0) + \beta M_\alpha^\tau(n)u(0) + T(M_\alpha * h_\alpha)(n-1)[u(1)-u(0)] \\ & + \beta(M_\alpha^\tau * h_\alpha)(n-1)[u(1)-u(0)] + \beta\sum_{j=1}^{\tau} TF_\alpha(n-2-\tau+j)u(-j) \\ & + \beta^2\sum_{j=1}^{\tau}(F_\alpha^\tau * h_\alpha)(n-2-\tau+j)u(-j) = Tv(n) + \beta v(n-\tau).\end{aligned}$$

Then, (3.1.2) is the solution of the equation (3.1.1) with $f = 0$. Now, we define a vector-valued sequence w as follows

$$w(n) = \begin{cases} (M_\alpha * h_\alpha * f)(n-2), & n \geq 2, \\ 0 & n = -\tau, \ldots, 1. \end{cases}$$

Since $M_\alpha(n) = 0$, for all $n = -\tau, \ldots, -1$, from Lemma 3.1.4 and the last claim, we obtain

$$\begin{aligned} \Delta^\alpha w(n) &= \Delta^\alpha (M_\alpha * h_\alpha * f)(n-2) \\ &= (\Delta^\alpha (M_\alpha * h_\alpha) * f)(n-2) + (M_\alpha * h_\alpha)(0) f(n) - \alpha (M_\alpha * h_\alpha)(0) f(n-1) \\ &+ (M_\alpha * h_\alpha)(1) f(n-1) \\ &= T(M_\alpha * h_\alpha * f)(n-2) + \beta (M_\alpha^\tau * h_\alpha * f)(n-2) + f(n) \\ &= Tw(n) + \beta w(n-\tau) + f(n), \end{aligned}$$

for all $n \geq 2$. Then, w solves (3.1.1) with initial conditions $w(j) = 0$, $j = -\tau, \ldots, 0, 1$. Finally we claim that $u = v + w$ is solution of (3.1.1). Indeed,

$$\begin{aligned} \Delta^\alpha u(n) &= Tv(n) + \beta v(n-\tau) + Tw(n) + \beta w(n-\tau) + f(n) \\ &= Tu(n) + \beta u(n-\tau) + f(n),\ n \in \mathbb{N}_0 \end{aligned}$$

and $u(j) = x_j$, for $j = -\tau, \ldots, 1$ and the theorem is proved. □

3.2 Maximal ℓ_p-regularity.

Let $T \in \mathcal{B}(X)$ and $f \in s(\mathbb{N}_0; X)$ be given. In this section, we consider the following nonlocal time-discrete equation with delay $\tau \in \mathbb{N}$:

$$\begin{cases} \Delta^\alpha u(n) = Tu(n) + \beta u(n-\tau) + f(n),\ n \in \mathbb{N}_0, \\ \quad u(j) = 0,\ j = -\tau, \ldots, 1, \end{cases} \tag{3.2.1}$$

where $1 < \alpha \leq 2$ and $\beta \in \mathbb{R}$. Assume that T is a generator of an α^τ-resolvent sequence $M_\alpha(n)$. Since $u(j) = 0$ for all $j = -\tau, ..., 1$ we obtain by Theorem 3.1.5, that the solution of (3.2.1) can be represented by

$$u(n) = (M_\alpha * h_\alpha * f)(n-2), \quad n \in \mathbb{N}, \quad n \geq 2.$$

Furthermore, from Lemma 3.1.4, we have the representation

$$\Delta^\alpha u(n) = T(M_\alpha * h_\alpha * f)(n-2) + \beta(M_\alpha^\tau * h_\alpha * f)(n-2) + f(n). \tag{3.2.2}$$

This motivates the following definition.

Definition 3.2.1. Let $1 < p < \infty$, $1 < \alpha \leq 2$ and $T \in \mathcal{B}(X)$ be given and suppose that T is a generator of an α^τ-resolvent sequence $M_\alpha(n)$. We say that the equation (3.2.1) has maximal ℓ_p-regularity if the operators $\mathcal{K}_\alpha$ and $\mathcal{P}_\alpha$, defined by

$$(\mathcal{K}_\alpha f)(n) := T\sum_{j=0}^{n} M_\alpha(n-j)(h_\alpha * f)(j) \quad \text{and}$$
$$(\mathcal{P}_\alpha f)(n) := \sum_{j=0}^{n} M_\alpha^\tau(n-j)(h_\alpha * f)(j),$$

are linear bounded operators in $\ell_p(\mathbb{N}_0; X)$ for some $p > 1$.

Remark 3.2.2. Observe that, in contrast with the continuous context, the discrete maximal ℓ_p-regularity ensures the stability of the solution and its fractional difference in the sense that $|u(n)| \to 0$ and $|\Delta^\alpha u(n)| \to 0$ as $n \to \infty$.

In what follows we need the following hypothesis:

$(H)_\alpha$ $\sup_{n\in\mathbb{N}_0} \|M_\alpha(n)\| < \infty$ and $(z^{2-\alpha}(z-1)^\alpha I - \beta z^{-\tau} - T)$ is invertible, $|z| = 1$, $z \neq 1$.

Now, we prove the main result of this chapter.

Theorem 3.2.3. *Let $1 < p < \infty$, $1 < \alpha \leq 2$ and let X be a UMD space. Let $T \in \mathcal{B}(X)$ be given such that T is a generator of an α^τ-resolvent sequence $M_\alpha(n)$ and the hypothesis $(H)_\alpha$ is satisfied. Then the following assertions are equivalent.*

(i) Equation (3.2.1) *has maximal ℓ_p-regularity.*

(ii) The sets

$$\left\{z^{2-\alpha}(z-1)^{\alpha}(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1} : |z|=1, z\neq 1\right\},$$

$$\left\{z^{-\tau}(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1} : |z|=1, z\neq 1\right\}$$

are R-bounded.

Proof. Suppose that *(ii)* holds. Then we define $N(t) = z^{2-\alpha}(z-1)^{\alpha}(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1}$ y $S(t) = z^{-\tau}(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1}$ for all $z=e^{it}$, $t\in(-\pi,\pi)$. Moreover, if we denote $f_\alpha(t)=e^{2it}(1-e^{-it})^{\alpha}$, then we can rewrite $N(t)=f_\alpha(t)(f_\alpha(t)-\beta e^{-it\tau}-T)^{-1}$ y $S(t)=e^{-it\tau}(f_\alpha(t)-\beta e^{-it\tau}-T)^{-1}$. Since $f'_\alpha(t)=\left(2i+\dfrac{i\alpha}{e^{it}-1}\right)f_\alpha(t)$, a simple computation gives us

$$\begin{aligned}
N'(t) &= \left(2i+\frac{i\alpha}{e^{it}-1}\right)(N(t)-N(t)^2)-\beta i\tau N(t)S(t)\\
S'(t) &= -i\tau S(t)-\beta i\tau S(t)^2-\left(2i+\frac{i\alpha}{e^{it}-1}\right)N(t)S(t).
\end{aligned}$$

Then,

$$\begin{aligned}
(z-1)(z+1)N'(t) &= a_\alpha(t)N(t)-a_\alpha(t)N(t)^2-\beta b_\tau(t)N(t)S(t)\\
(z-1)(z+1)S'(t) &= -b_\tau(t)S(t)-\beta b_\tau(t)S(t)^2-a_\alpha(t)N(t)S(t),
\end{aligned}$$

where $a_\alpha(t)=2i(z-1)(z+1)+i\alpha(z+1)$ and $b_\tau(t)=-i\tau(z+1)(z-1)$ are clearly bounded for $z=e^{it}$, $t\in(-\pi,\pi)$. We conclude from [6, Proposition 2.2.5] that the sets

$$\left\{(z-1)(z+1)N'(t) : z=e^{it},\, t\in\mathbb{T}_0\right\} \text{ and}$$

$$\left\{(z-1)(z+1)S'(t) : z=e^{it},\, t\in\mathbb{T}_0\right\}$$

are R-bounded. Then, by Blünck's Theorem 1.7.2, we conclude that there exist operators $T_\alpha, U_\alpha\in\mathcal{B}(\ell_p(\mathbb{Z};X))$ such that

$$\begin{aligned}
\widehat{(T_\alpha f)}(z) &= N(t)\hat{f}(z),\ z=e^{it},\ t\in\mathbb{T}_0,\\
\widehat{(U_\alpha f)}(z) &= S(t)\hat{f}(z),\ z=e^{it},\ t\in\mathbb{T}_0,
\end{aligned}\tag{3.2.3}$$

for all $f \in \ell_p(\mathbb{Z};X)$. From the identity

$$T(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1} = (z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau})(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1}-I,$$

and (3.2.3), we obtain that the left hand side in the identity

$$\begin{aligned} &T(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1}\hat{f}(z) = \\ &(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau})(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)^{-1}\hat{f}(z)-\hat{f}(z), \end{aligned} \tag{3.2.4}$$

defines an operator $R_\alpha \in \mathcal{B}(\ell_p(\mathbb{Z};X))$ given by $R_\alpha f(n) = T_\alpha f(n) - \beta U_\alpha f(n) - f(n)$. Now, for each $f \in \ell_p(\mathbb{Z};X)$, we define the operator

$$K_\alpha f(n) = \begin{cases} T(M_\alpha * h_\alpha * f)(n), & n \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the Z-transform of $M_\alpha * h_\alpha$ exists by hypothesis $(H)_\alpha$ and definition of h_α, and

$$(z^{2-\alpha}(z-1)^{\alpha}-\beta z^{-\tau}-T)\widehat{M_\alpha * h_\alpha}(z) = z^2 I.$$

Then, from the identity (3.2.3), we have that the discrete Fourier transform of $K_\alpha f(n-2)$ coincides with the discrete Fourier transform of $R_\alpha f(n)$ for $n \geq 2$. So, $K_\alpha f(n-2) = R_\alpha f(n)$ for each $n \geq 2$ by uniqueness. On the other hand, we define

$$P_\alpha f(n) = \begin{cases} (M_\alpha^\tau * h_\alpha * f)(n) & n \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

Using again the identity (3.2.3), we obtain that the discrete Fourier transform of $P_\alpha f(n-2)$ coincides with the discrete Fourier transform of $U_\alpha f(n)$. So, $P_\alpha f(n-2) = U_\alpha f(n)$ for each $n \geq 2$ by uniqueness. This proves (i). Now, we suppose that (i) is satisfied. We define the following operators

$$C_\alpha f(n) = \begin{cases} \mathcal{K}_\alpha f(n) & n \in \mathbb{N}_0 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$D_\alpha f(n) = \begin{cases} \mathcal{P}_\alpha f(n) & n \in \mathbb{N}_0 \\ 0 & otherwise. \end{cases}$$

Then, C_α and D_α are bounded linear operators on $\ell_p(\mathbb{Z}; X)$. Let $T_\alpha f(n) := C_\alpha f(n-2) + f(n)$, y $U_\alpha f(n) := D_\alpha f(n-2)$, $n \in \mathbb{Z}$. Given $z = e^{it}$, $t \in (-\pi, \pi)$, we have that

$$\begin{aligned} \widehat{T_\alpha f}(z) &= \sum_{j\in\mathbb{Z}} z^{-j} T_\alpha f(j) = \sum_{j=2}^{\infty} z^{-j} C_\alpha f(j-2) + \sum_{j\in\mathbb{Z}} z^{-j} f(j) \\ &= z^{-2} \sum_{j=0}^{\infty} z^{-j} C_\alpha f(j) + \sum_{j\in\mathbb{Z}} z^{-j} f(j) = z^{-2} \sum_{j=0}^{\infty} z^{-j} C_\alpha f(j) + \widehat{f}(z). \end{aligned}$$

By hypothesis $(H)_\alpha$, the Z-transform of $M_\alpha * h_\alpha$ exists for $|z| = 1$, $z \neq 1$. Finally, using the identity (3.2.3), we obtain

$$\begin{aligned} \widehat{T_\alpha f}(z) &= z^{-2} T(\widehat{\mathcal{M}_\alpha * h_\alpha})(z) \widehat{f}(z) + \widehat{f}(z) \\ &= z^{-2} T z (z - (\alpha - 1))(z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} \frac{z}{z - (\alpha - 1)} \widehat{f}(z) \\ &= T(z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} \widehat{f}(z) + \widehat{f}(z) \\ &= (z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau})(z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} \widehat{f}(z) - \widehat{f}(z) + \widehat{f}(z) \\ &= (N(t) - \beta S(t)) \widehat{f}(z), \end{aligned}$$

where $\mathcal{M}_\alpha$ is defined by

$$\mathcal{M}_\alpha(n) = \begin{cases} M_\alpha(n) & n \in \mathbb{N}_0 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} \widehat{U_\alpha f}(z) &= z^{-2} \sum_{j=0}^{\infty} z^{-j} \mathcal{P}_\alpha(j) f(j) = z^{-2} (\widehat{\mathcal{M}_\alpha^\tau * h_\alpha})(z) \widehat{f}(z) = z^{-2} z^{-\tau} (\widehat{\mathcal{M}_\alpha * h_\alpha}) \widehat{f}(z) \\ &= z^{-2-\tau} z (z - (\alpha - 1))(z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} \frac{z}{z - (\alpha - 1)} \widehat{f}(z) \\ &= z^{-\tau} (z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} \widehat{f}(z) = S(t) \widehat{f}(z), \end{aligned}$$

where $\mathcal{M}_\alpha^\tau$ is defined by $\mathcal{M}_\alpha^\tau(n) = \mathcal{M}_\alpha(n-\tau)$.

Then, from Theorem 1.7.3, we conclude that (ii) holds. $\square$

Remark 3.2.4. In the case of Hilbert spaces, R-boundedness coincides with boundedness. See e.g. [6]. As a consequence, condition (ii) of Theorem 3.2.3 can be replaced by the following equivalent assertion:

$$\sup_{|z|=1,\, z\neq 1} \left\| z^{2-\alpha}(z-1)^\alpha (z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} \right\| < \infty \quad \text{and}$$

$$\sup_{|z|=1,\, z\neq 1} \left\| z^{-\tau} (z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1} \right\| < \infty.$$

Remark 3.2.5. With the same proof and obvious modifications, the theorem is also true when we consider a finite number of delays in the equation (3.2.1).

We immediately obtain the following corollary (compare with [53, Corollary 4.5]).

Corollary 3.2.6. *If the hypothesis of Theorem 3.2.3 hold, then* $u, \Delta^\alpha u, Tu \in \ell_p(\mathbb{N}_0; X)$ *and there exists a constant* $C > 0$ *(independent of* $f \in \ell_p(\mathbb{N}_0; X)$*) such that the following inequality holds*

$$\|\Delta^\alpha u\|_{\ell_p(\mathbb{N}_0;X)} + \|u\|_{\ell_p(\mathbb{N}_0;X)} + \|Tu\|_{\ell_p(\mathbb{N}_0;X)} \leq C \|f\|_{\ell_p(\mathbb{N}_0;X)}.$$

3.3 Applications

Let us consider the following difference equation

$$x(n+3) - 2x(n+2) + qx(n+1) + rx(n) = f(n), \tag{3.3.1}$$

where $q, r \in \mathbb{R}$. This equation was studied in the homogeneous case by Györi and Ladas in [40] and in [34, Section 5.1]. We study a particular case of this equation with initial conditions $x(0) = x(1) = x(2) = 0$. Note that this equation can be reformulated as follows

$$\Delta^2 x(n) = (1-q)x(n) - rx(n-1) + f(n-1), \tag{3.3.2}$$

with initial conditions $x(-1) = x(0) = x(1) = 0$. Consequently, equation (3.3.2) can be posed into the scheme of (3.2.1) with $\alpha = 2$, $T = (1-q)I$, $\beta = -r$ and $\tau = 1$. We first compute the family $\{M_\alpha(n)\}_{n\geq -1}$ in order to obtain a solution x of (3.3.2). Indeed, using the inverse formula of Z-transform, we get that

$$\begin{aligned}
M_2(n) &= \frac{1}{2\pi i}\int_C z^{n-1} z(z-1)((z-1)^2 + rz^{-1} - (1-q))^{-1} dz \\
&= \frac{1}{2\pi i}\int_C \frac{z^{n+1}(z-1)}{(z^3 - 2z^2 + qz + r)} dz \\
&= \frac{1}{2\pi i}\int_C \frac{z^{n+1}(z-1)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} dz \\
&= \frac{(\lambda_1^{n+2} - \lambda_1^{n+1})}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{(\lambda_2^{n+2} - \lambda_2^{n+1})}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} + \frac{(\lambda_3^{n+2} - \lambda_3^{n+1})}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)},
\end{aligned}$$

where C is a circle centered at the origin that encloses the roots $\lambda_1, \lambda_2, \lambda_3$ of the equation $z^3 - 2z^2 + qz + r = 0$ in its interior.

It follows from Schur-Cohn criterion (see [34, Theorem 5.1]) or Samuelson criterion (see for example [65]) that all these roots lie inside of the unit disk $\mathbb{D}$ if and only if $|r-2| < 1+q$ and $|q+2r| < 1-r^2$ which, in turn, is equivalent to $1 < q < 2$ and $1-q < r < -1+\sqrt{2-q}$. See Figure 1 below. Then, under this restriction on the parameters of equation (3.3.2), we obtain that $\sup_{n\in\mathbb{N}_0}\|M_2(n)\| < \infty$. It means that the first part of the condition $(H)_2$ holds. In particular, we also have that $z^3 - 2z^2 + qz + r \neq 0$ for $|z| = 1$ and consequently

$$\sup_{|z|=1, z\neq 1} |(z-1)^2((z-1)^2 + rz^{-1} - (1-q))^{-1}| < \infty,$$

and

$$\sup_{|z|=1, z\neq 1} |z^{-1}((z-1)^2 + rz^{-1} - (1-q))^{-1}| < \infty.$$

Therefore all the conditions given in Theorem 3.2.3 hold and we conclude that whenever $1 < q < 2$ and $1-q < r < -1+\sqrt{2-q}$ and $f \in \ell_p(\mathbb{N}_0)$, there exists a unique $u \in \ell_p(\mathbb{N}_0)$ solving (3.3.2).

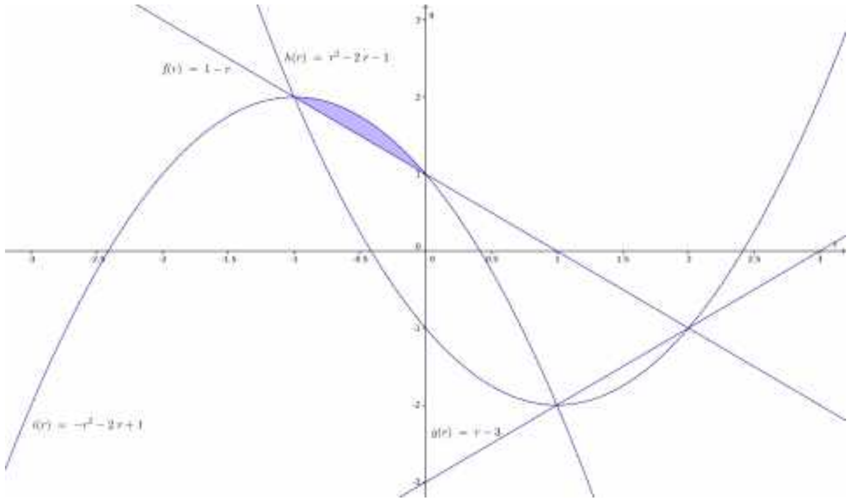

Figure 3.1: The sector $|r-2| < 1+q$ and $|q+2r| < 1-r^2$

In order to handle fractional models, the following result will be useful.

Corollary 3.3.1. *Let* $1 < \alpha \leq 2$, $\beta \in \mathbb{R}, \tau \in \mathbb{N}_0$. *Let* X *be a Hilbert space and* $T \in \mathcal{B}(X)$ *satisfying the following condition*

$$||T|| < \omega_{\alpha,\beta,\tau} := \min_{|z|=1} |f_{\alpha,\beta,\tau}(z)| < 1 \quad \textit{where} \quad f_{\alpha,\beta,\tau}(z) := z^{2-\alpha}(z-1)^{\alpha} - \beta z^{-\tau}.$$

Then equation (3.2.1) *has maximal* ℓ_p*-regularity.*

Proof. We first prove that T is the generator of an α^{τ}-resolvent sequence $M_\alpha(n)$ and the hypothesis $(H)_\alpha$ is satisfied. Indeed, by hypothesis and an application of the minimum principle, we obtain that $f_{\alpha,\beta,\tau}(z) \in \rho(T)$ and

$$(f_{\alpha,\beta,\tau}(z) - T)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{(f_{\alpha,\beta,\tau}(z))^{n+1}},$$

whenever $|z| \leq 1$. Hence there exists a circle Γ centered at the origin of radius $R < 1$ such that

$$M_\alpha(n) = \frac{1}{2\pi i}\int_C z^n(z-(\alpha-1))(z^{2-\alpha}(z-1)^\alpha - \beta z^{-\tau} - T)^{-1}dz,\, n \in \mathbb{N},\, n \geq 2$$

$M_\alpha(0) = M_\alpha(1) = I$ and $M_\alpha(j) = 0$, $j = -\tau, \ldots, -1$ defines an α^τ-resolvent family. Observe that we also have

$$||(f_{\alpha,\beta,\tau}(z) - T)^{-1}|| \leq \frac{1}{|f_{\alpha,\beta,\tau}(z)| - ||T||} < \frac{1}{\omega_{\alpha,\beta,\tau} - ||T||}. \tag{3.3.3}$$

As a consequence, for all $n \in \mathbb{N}$, we have

$$\|M_\alpha(n)\| < \frac{R^{n+1}(R + |\alpha - 1|)}{\omega_{\alpha,\beta,\tau} - ||T||},$$

and then $\sup\limits_{n\in\mathbb{N}_0} ||M_\alpha(n)|| < \infty$. This proves the claim. Moreover,

$$\sup_{|z|=1, z\neq 1} \|z^{1-\alpha}(z-1)^\alpha(f_{\alpha,\beta,\tau}(z) - T)^{-1}\| < \infty \quad \text{and}$$

$$\sup_{|z|=1, z\neq 1} \|z^{-\tau}(f_{\alpha,\beta,\tau}(z) - T)^{-1}\| < \infty.$$

Then part (ii) of Theorem 3.2.3 holds and we conclude that equation (3.2.1) has maximal ℓ_p-regularity. □

Example 3.3.2. Motivated by the model given by (3.3.2) we consider the fractional difference equation

$$\Delta^\alpha x(n) = (1-q)x(n) - rx(n-1) + f(n-1), \quad 1 < \alpha \leq 2, \tag{3.3.4}$$

with initial conditions $x(-1) = x(0) = x(1) = 0$. We illustrate the validity of the condition in the previous corollary plotting the graph of the complex function

$$f_{\alpha,-r,1}(z) = z^{2-\alpha}(z-1)^\alpha + rz^{-1},\, |z| = 1,$$

for different values of $1 < \alpha \leq 2$ and $r \in \mathbb{R}$. Observe that given α and r there are cases where exists q such that Corollary 3.3.1 is satisfied. Moreover, the graphs show that $\omega_{\alpha,-r,1} \to 0$ as $\alpha \to 1$ for some values of r (for instance when $r = 0.6$).

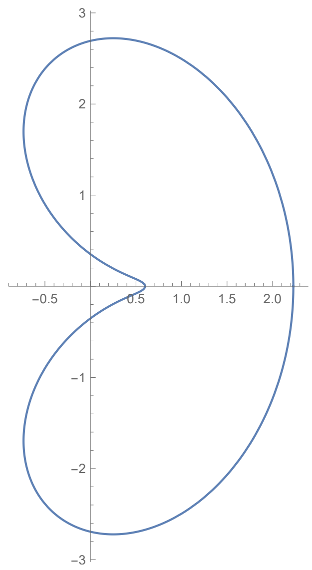

Figure 3.2: $\alpha = 1.5$ and $r = 0.6$. Observe that the minimum value $\omega_{1.5,-0.6,1}$ is attained approximately at 0.5 and consequently $0.5 < q < 1.5$.

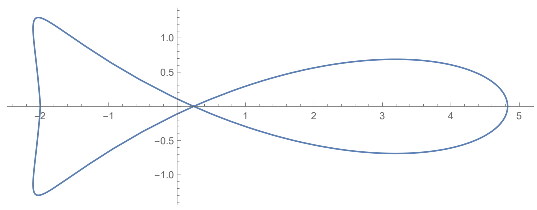

Figure 3.3: $\alpha = 1.5$ and $r = -2$. Observe that the minimum value $\omega_{1.5,-0.6,1}$ is attained approximately at 0.2 and consequently $0.8 < q < 1.2$.

Chapter 4

Existence of weighted bounded solutions for nonlinear discrete-time fractional equations

4.1 Introduction

In this chapter, we study the existence of weighted bounded solutions for a time-discrete nonlinear fractional equations of the form

$$\Delta^\alpha u(n) = Tu(n) + f(n, u(n)),\ n \in \mathbb{N}_0, \tag{4.1.1}$$

with initial conditions $u(0) = u_0$ and $u(1) = u_1$ whenever $1 < \alpha \leq 2$. Here Δ^α corresponds to the fractional difference operator of order $\alpha > 0$ in sense of Riemann-Liouville, T is a bounded linear operator defined on a Banach space X and $f : \mathbb{N}_0 \times X \to X$ is a function satisfying suitable conditions.

More specifically, by using operator-theoretical methods and fixed point theory, we show the existence of solutions of such class of equations on the vector-valued weighted space of sequences

$$l_f^\infty(\mathbb{N}_2; X) = \left\{ \eta : \mathbb{N}_2 \to X / \sup_{n \geq 2} \frac{\|\eta(n)\|}{nn!} < \infty \right\}.$$

4.2 Resolvent families: $1 < \alpha \leq 2$

In this section, we recall an operator theoretical method introduced in [52] in order to study the following linear fractional difference equation

$$\Delta^{\alpha} u(n) = Tu(n) + f(n),\ n \in \mathbb{N}_0 \tag{4.2.1}$$

with initial conditions $u(0) = u_0$ and $u(1) = u_1 \in X$, $1 < \alpha \leq 2$ and $T \in \mathcal{B}(X)$. We recall the following definition given in [52, Definition 3.1].

Definition 4.2.1. Let T be a bounded linear operator defined on a Banach space X and $1 < \alpha \leq 2$. We say that T is a generator of an α-resolvent sequence if there exists a sequence of bounded linear operators $\{S_{\alpha}(n)\}_{n \in \mathbb{N}_0}$ satisfying the following properties:

i) $S_{\alpha}(0) = I$,

ii) $S_{\alpha}(1) = I$,

iii) $S_{\alpha}(n+2) - S_{\alpha}(n+1) = T(S_{\alpha} * k^{\alpha-1})(n) + k^{\alpha-1}(n+2)I - (\alpha-1)k^{\alpha-1}(n+1)I$, $n \in \mathbb{N}_0$.

Remark 4.2.2. Note that if T generates an α-resolvent family, then it is unique (see [52, Lemma 3.3]).

Example 4.2.3. In the border case $\alpha = 2$, we have the recurrence relation

$$\begin{aligned} S_2(0) &= S_2(1) = I \\ S_2(n+2) - S_2(n+1) &= T \sum_{j=0}^{n} S_2(j). \end{aligned}$$

Thus, $S_2(n) = \displaystyle\sum_{j=0}^{[n/2]} \binom{n}{2j} T^j = \frac{(I + T^{1/2})^n + (I - T^{1/2})^n}{2}$.

Suppose that for all $z \in \mathbb{C}$ with $|z| = 1$ we have $z^{2-\alpha}(z-1)^{\alpha} \in \rho(T)$. Then, the following holds:

$$\tilde{S}_\alpha(z) = z(z-(\alpha-1))(z^{2-\alpha}(z-1)^\alpha - T)^{-1}. \tag{4.2.2}$$

In the border case $\alpha = 2$, we obtain from [6, Proposition 1.4.2] that

$$S_2(n) = \mathcal{C}(n),\ n \in \mathbb{N}_0,$$

where $\mathcal{C}$ is the discrete time cosine operator sequence generated by T introduced by Chojnacki in [27]. It follows from [6, Corollary 1.4.6] that $\mathcal{C}$ satisfies

$$\mathcal{C}(n+m) + \mathcal{C}(n-m) = 2\mathcal{C}(n)\mathcal{C}(m),\ n, m \in \mathbb{Z}.$$

We conclude this section with an important result concerning in a qualitative property of α-resolvent sequences.

Theorem 4.2.4. *Let* $1 < \alpha < 2$, $T \in \mathcal{B}(X)$ *and* $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ *be the* α*-resolvent sequence generated by* T*. If* $\|T\| < \dfrac{\alpha^\alpha(2-\alpha)^{2-\alpha}}{4}$ *then* $\|S_\alpha(n)\| \to 0$ *as* $n \to \infty$ *and the following estimate*

$$\sup_{n\in\mathbb{N}_0} \|S_\alpha(n)\| \le \frac{3}{\alpha^\alpha(2-\alpha)^{2-\alpha} - 4\|T\|}$$

holds.

Proof. Let C a circle of radius $R = \dfrac{2-\alpha}{2}$ that encloses all singularities of $z^n(z-(\alpha-1))(z^{2-\alpha}(z-1)^\alpha - T)^{-1}$. From (4.2.2) and (1.2.2), we have that

$$\begin{aligned} S_\alpha(n) &= \frac{1}{2\pi i}\int_{|z|=R} z^n(z-(\alpha-1))(z^{2-\alpha}(z-1)^\alpha - T)^{-1}dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} R^n e^{int}(Re^{it}-(\alpha-1))(R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha - T)^{-1}Re^{it}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} R^{n+1} e^{i(n+1)t}(Re^{it}-(\alpha-1))(R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha - T)^{-1}dt, \end{aligned} \tag{4.2.3}$$

and then

$$\begin{aligned}\|S_\alpha(n)\| &\le \frac{1}{2\pi}\int_0^{2\pi} R^{n+1}\left|(Re^{it}-(\alpha-1))\right| \left\|(R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha - T)^{-1}\right\| dt \\ &\le \frac{1}{2\pi}\int_0^{2\pi} R^{n+1}(R+\alpha-1) \left\|(R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha - T)^{-1}\right\| dt.\end{aligned} \tag{4.2.4}$$

On the other hand, by [45, Theorem 7.3–4, p.377], we have that $R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it} - 1)^\alpha - T$ is invertible for each $t \in (0, 2\pi)$ and

$$(R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha - T)^{-1} = \sum_{j=0}^{\infty} \frac{T^j}{(R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha)^{j+1}}. \tag{4.2.5}$$

As a consequence, and since $1 - R < |1 - Re^{it}|$, we have

$$\begin{aligned}\left\|R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha - T)^{-1}\right\| &\le \frac{1}{|R^{2-\alpha}e^{i(2-\alpha)t}(Re^{it}-1)^\alpha| - \|T\|} \\ &\le \frac{1}{(1-R)^\alpha R^{2-\alpha} - \|T\|}.\end{aligned} \tag{4.2.6}$$

Thus,

$$\begin{aligned}\|S_\alpha(n)\| &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{R^{n+1}(R+\alpha-1)}{(1-R)^\alpha R^{2-\alpha} - \|T\|} dt \\ &\le \frac{R^{n+1}(R+\alpha-1)}{(1-R)^\alpha R^{2-\alpha} - \|T\|}.\end{aligned}$$

As a consequence, we have $\|S_\alpha(n)\| \to 0$ as $n \to \infty$ and the inequality

$$\sup_{n\in\mathbb{N}_0} \|S_\alpha(n)\| \le \frac{3}{\alpha^\alpha(2-\alpha)^{2-\alpha} - 4\|T\|}$$

is valid for $1 < \alpha < 2$. $\square$

The following picture illustrates the function $\alpha \mapsto \dfrac{\alpha^\alpha(2-\alpha)^{2-\alpha}}{4}$. Observe that

$$\inf_{1<\alpha<2} \frac{\alpha^\alpha(2-\alpha)^{2-\alpha}}{4} = \frac{1}{4}.$$

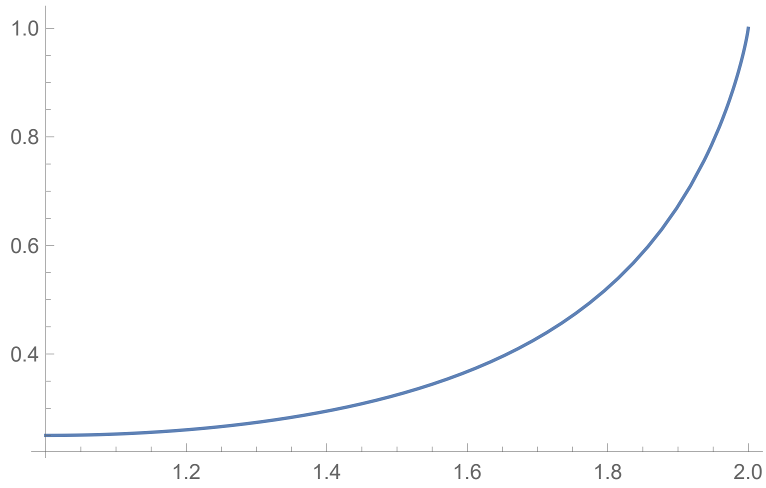

Figure 4.1: $\alpha \mapsto \dfrac{\alpha^{\alpha}(2-\alpha)^{2-\alpha}}{4}$

The importance of α-resolvent sequence of operators is that allows to obtain a representation of the solution of the equation (4.2.1) by means of a kind of variation of parameters formula, specifically we have the following theorem.

Theorem 4.2.5. [52, Theorem 3.8] *Let $1 < \alpha \leq 2$ and $f : \mathbb{N}_0 \to X$ be given. Then the unique solution of (4.2.1) with initial conditions $u(0) = x$ and $u(1) = y$ can be represented by*

$$u(n) = S_\alpha(n)x + (S_\alpha * h_\alpha)(n-1)[y-x] + (S_\alpha * h_\alpha * f)(n-2),\ n \in \mathbb{N}, n \geq 2, \quad (4.2.7)$$

where h_α is a function defined by

$$h_\alpha(n) = \begin{cases} (\alpha-1)^n & n \in \mathbb{N}_0, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Similar to Theorem 3.1.5 with $\beta = 0$ $\square$

4.3 A nonlinear fractional difference equation

In this section we study the nonlinear problem

$$\begin{cases} \Delta^\alpha u(n) = Tu(n) + f(n, u(n)), \quad n \in \mathbb{N}_0,\, 1 < \alpha \leq 2, \\ \quad u(0) = 0, \\ \quad u(1) = 0, \end{cases} \tag{4.3.1}$$

where T is a bounded linear operator defined on a Banach space X.

Definition 4.3.1. Let $T \in \mathcal{B}(X)$, $f : \mathbb{N}_0 \times X \to X$ and $1 < \alpha \leq 2$ be given. A sequence $u : \mathbb{N}_0 \to X$ is said to be a solution of (4.3.1) if u satisfies (4.3.1) for all $n \in \mathbb{N}_0$.

As a consequence of Theorem 4.2.5, we have the following result that gives an equivalent representation of the solution of (4.3.1) in terms of the family of operators $S_\alpha(n)$ generated by the operator T.

Theorem 4.3.2. *Let $T \in \mathcal{B}(X)$, $f : \mathbb{N}_0 \times X \to X$ and $1 < \alpha \leq 2$ be given. Let $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ be the α-resolvent sequence generated by T. Then, the following assertions are equivalent:*

i) u is a solution of (4.3.1).

ii) $u(0) = u(1) = 0$ and $u(n) = \displaystyle\sum_{j=0}^{n-2} (S_\alpha * h_\alpha)(n-2-j) f(j, u(j))$, $n \in \mathbb{N}$, $n \geq 2$.

Proof. First, we suppose that *ii*) is valid. Note that by definition of Δ^α, we have that

$$\begin{aligned}\Delta^\alpha u(n+2) &= \sum_{j=0}^{n} \Delta^2 (k^{2-\alpha} * S_\alpha * h_\alpha)(n-j) f(j,u(j)) \\ &+ (k^{2-\alpha} * S_\alpha * h_\alpha)(0) f(n+2, u(n+2)) \\ &+ (k^{2-\alpha} * S_\alpha * h_\alpha)(1) f(n+1, u(n+1)) \\ &- 2(k^{2-\alpha} * S_\alpha * h_\alpha)(0) f(n+1, u(n+1)) \\ &= \sum_{j=0}^{n} \Delta^\alpha (S_\alpha * h_\alpha)(n-j) f(j,u(j)) + f(n+2, u(n+2))\end{aligned}$$

and by [53, Lemma 3.6] we get that

$$\begin{aligned}\Delta^\alpha u(n+2) &= T \sum_{j=0}^{n} (S_\alpha * h_\alpha)(n-j) f(j,u(j)) + f(n+2, u(n+2)) \\ &= Tu(n+2) + f(n+2, u(n+2))\end{aligned}$$

for all $n \in \mathbb{N}_0$. Then, u is a solution of (4.3.1). Conversely suppose that u is a solution of (4.3.1). Then,

$$(S_\alpha * h_\alpha * f)(n-2) = (S_\alpha * h_\alpha * \Delta^\alpha u)(n-2) - T(S_\alpha * h_\alpha * u)(n-2). \qquad (4.3.2)$$

By [52, Lemma 3.6] we have

$$\begin{aligned}(S_\alpha * h_\alpha * \Delta^\alpha u)(n-2) &= \Delta^\alpha (S_\alpha * h_\alpha * u)(n-2) - (S_\alpha * h_\alpha)(n) u(0) \\ &\quad + \alpha (S_\alpha * h_\alpha)(n-1) u(0) + (S_\alpha * h_\alpha)(n-1) u(1)\end{aligned}$$

and

$$\begin{aligned}\Delta^\alpha (S_\alpha * h_\alpha * u)(n-2) &= (\Delta^\alpha S_\alpha * h_\alpha * u)(n-2) - S_\alpha(0)[(u * h_\alpha)(n) \\ &\quad + \alpha (u * h_\alpha)(n-1)] + S_\alpha(1)(u * h_\alpha)(n-1).\end{aligned}$$

Using the fact that $S_\alpha(0) = S_\alpha(1) = I$ and $u(0) = u(1) = 0$, we get that

$$(S_\alpha * h_\alpha * f)(n-2) = (\Delta^\alpha S_\alpha * h_\alpha * u)(n-2) - T(S_\alpha * h_\alpha * u)(n-2) + u(n).$$

By Theorem 4.2.5, we know that $\Delta^\alpha S_\alpha(n) = TS_\alpha(n)$. Then, $u(n) = (S_\alpha * h_\alpha * f)(n-2)$ is fulfilled. $\square$

We recall the following definition introduced in [57].

Definition 4.3.3. We call the factorial number system space *(fns-space) the vector-valued weighted space defined as follows*

$$l_f^\infty(\mathbb{N}_2; X) := \left\{ \eta : \mathbb{N}_2 \to X / \sup_{n\geq 2} \frac{\|\eta(n)\|}{nn!} < \infty \right\},$$

endowed with the norm $\|\eta\|_f = \sup_{n\geq 2} \dfrac{\|\eta(n)\|}{nn!}$.

The following result is a lemma that will be useful in the later results of this section.

Lemma 4.3.4. *Let $\{a_n\}$ and $\{b_n\}$ be sequences defined by*

$$a_n = \frac{1}{nn!}\left(\sum_{j=0}^{n-1}(j!-1)\right)$$

and

$$b_n = \frac{n-1}{nn!}.$$

Then $\sup_{n\in\mathbb{N}} a_n = \dfrac{1}{16}$, $\sup_{n\in\mathbb{N}} b_n = \dfrac{1}{4}$, $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = 0$.

Proof. Note that $\{a_n\}$ is a decreasing sequence for $n \geq 4$ and

$$a_{n+1} = \frac{n}{(n+1)^2}\left(a_n + \frac{n!-1}{nn!}\right).$$

If we asumme for a while that $a_n \leq \dfrac{1}{16}$, then by induction on $n \geq 4$ and [63, formula 33 p. 598] we get that

$$a_{n+1} \leq \frac{4}{25}\left(\frac{1}{16} + \frac{23}{600}\right) = \frac{3872}{240000} < \frac{1}{16}.$$

Also, we observe that

$$\frac{1}{n^2} - \frac{1}{n!} < a_n < \frac{1}{n} - \frac{1}{n!}.$$

Then, $\lim_{n\to\infty} a_n = 0$ is satisfied. On the other hand $\{b_n\}$ is a decreasing sequence for $n \geq 2$ and also since

$$b_{n+1} = \frac{n}{(n+1)^2}\left(b_n + \frac{1}{nn!}\right)$$

if we asumme for a while that $b_n \leq \frac{1}{4}$, then by induction on $n \geq 2$ we get that

$$b_{n+1} \leq \frac{2}{9}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{9} < \frac{1}{4}.$$

Also, it is trivial that $\lim_{n\to\infty} b_n = 0$. $\square$

Theorem 4.3.5. *Let $T \in \mathcal{B}(X)$ be the generator of a bounded α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$, for $1 < \alpha \leq 2$. Let $f : \mathbb{N}_0 \times X \to X$ be a function such that*

(F) $f(0,0) \neq 0$, $f(1,0) \neq 0$ and there exists a positive sequence $a \in \ell_1(\mathbb{N}_0)$ and constants $c \geq 0$, $b > 0$ such that $\|f(k,x)\| \leq a(k)(c\|x\| + b)$, for all $k \in \mathbb{N}_0$ and $x \in X$.

(L) The function f satisfies a Lipschitz condition in $x \in X$ uniformly in $k \in \mathbb{N}_0$, that is, there exists a constant $L > 0$ such that $\|f(k,x) - f(k,y)\| \leq L\|x - y\|$, for all $x, y \in X$, $k \in \mathbb{N}_0$, with $L < \dfrac{16}{\|S_\alpha\|_\infty}$.

Then the problem (4.3.1) has a unique non trivial solution in $l_f^\infty(\mathbb{N}_2; X)$.

Proof. Let us consider the operator $G : l_f^\infty(\mathbb{N}_2; X) \to l_f^\infty(\mathbb{N}_2; X)$ defined as follows

$$Gu(n) = \sum_{j=0}^{n-2} (S_\alpha * h_\alpha)(n-2-j) f(j, u(j)),\ n \geq 2.$$

We need to check the conditions of the Banach Fixed Point Theorem. First, we show that G is well defined. Indeed, let $u \in l_f^\infty(\mathbb{N}_2; X)$ be given. By assumption (F) and the boundedness of $S_\alpha(n)$ we have for each $n \in \mathbb{N}$, $n \geq 2$.

$$\begin{aligned}
\|Gu(n)\| &\leq \sum_{j=0}^{n-2} \|(S_\alpha * h_\alpha)(n-2-j)\| \, \|f(j, u(j))\| \\
&\leq \|S_\alpha\|_\infty \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} h_\alpha(n-2-j-k) \, \|f(k, u(k))\| \\
&\leq c \, \|S_\alpha\|_\infty \, \|a\|_\infty \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} \|u(k)\| + b \, \|S_\alpha\|_\infty \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} a(k) \\
&\leq c \, \|S_\alpha\|_\infty \, \|a\|_\infty \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} \|u(k)\| + b \, \|S_\alpha\|_\infty \, \|a\|_1 \, (n-1) \\
&\leq c \, \|S_\alpha\|_\infty \, \|a\|_\infty \, \|u\|_f \left(\sum_{j=0}^{n-1} (j! - 1) \right) + b \, \|S_\alpha\|_\infty \, \|a\|_1 \, (n-1).
\end{aligned}$$

Hence

$$\frac{\|Gu(n)\|}{nn!} \leq c \, \|S_\alpha\|_\infty \, \|a\|_\infty \, \|u\|_f \, \frac{1}{nn!} \left(\sum_{j=0}^{n-1} (j! - 1) \right) + b \, \|S_\alpha\|_\infty \, \|a\|_1 \left(\frac{n-1}{nn!} \right).$$

Then, taking supremum in both sides, we have $Gu \in l_f^\infty(\mathbb{N}_2; X)$. Next, we prove that G is a contraction on $l_f^\infty(\mathbb{N}_2; X)$. Let $u, v \in l_f^\infty(\mathbb{N}_2; X)$ be given. Then, using the assumption (L) we have

$$\begin{aligned}
\|Gu(n) - Gv(n)\| &\leq \sum_{j=0}^{n-2} \|(S_\alpha * h_\alpha)(n-2-j)\| \, \|f(j,u(j)) - f(j,v(j))\| \\
&\leq \|S_\alpha\|_\infty \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} \|f(j,u(j)) - f(j,v(j))\| \\
&\leq L \|S_\alpha\|_\infty \|u - v\|_f \left(\sum_{j=0}^{n-1} (j! - 1) \right).
\end{aligned}$$

Hence,

$$\frac{\|Gu(n) - Gv(n)\|}{nn!} \leq L \|S_\alpha\|_\infty \|u - v\|_f \frac{1}{nn!} \left(\sum_{j=0}^{n-1} (j! - 1) \right),$$

and consequently, taking supremum again

$$\|Gu - Gv\|_f \leq \frac{1}{16} L \|S_\alpha\|_\infty \|u - v\|_f .$$

Then, by the Banach fixed point Theorem we conclude that G has a unique fixed point in $l_f^\infty(\mathbb{N}_2; X)$. $\square$

As a consequence of the last theorem, the following result gives an explicit bound on the Lipschitz constant.

Corollary 4.3.6. *Let $T \in \mathcal{B}(X)$ and $1 < \alpha < 2$. Suppose that $4\|T\| < \alpha^\alpha(2-\alpha)^{2-\alpha}$. Let $f : \mathbb{N}_0 \times X \to X$ satisfying condition (F) and such that f is a Lipschitz function in $x \in X$ uniformly in $k \in \mathbb{N}_0$, with Lipschitz constant $L < \frac{16}{3}(\alpha^\alpha(2-\alpha)^{2-\alpha} - 4\|T\|)$. Then the problem (4.3.1) has a unique solution in $l_f^\infty(\mathbb{N}_2; X)$.*

The following lemma is needed in the next main result of this section. Its proof is based on a analogous lemma proved in [57, Lemma 4.1].

Lemma 4.3.7. *Let $U \subset l_f^\infty(\mathbb{N}_2; X)$ such that*

a) The set $H_n(U) = \left\{ \frac{u(n)}{nn!} : u \in U \right\}$ is relatively compact in X, for all $n \geq 2$.

b) $\lim_{n\to\infty} \frac{1}{nn!} \sup_{u\in U} \|u(n)\| = 0.$

Then U is relatively compact in $l_f^\infty(\mathbb{N}_2; X)$.

For $f : \mathbb{N}_2 \times X \to X$ we recall that the Nemytskii operator $\mathcal{N}_f : l_f^\infty(\mathbb{N}_2; X) \to l_f^\infty(\mathbb{N}_2; X)$ is defined by

$$\mathcal{N}_f(u)(n) := f(n, u(n)),\ n \in \mathbb{N},\ n \geq 2.$$

Finally, we conclude with the second main result for this section. It gives a useful criterion for existence of solutions without using Lipschitz conditions.

Theorem 4.3.8. *Let $T \in \mathcal{B}(X)$ be a compact operator and generator of an α-resolvent compact sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ for $1 < \alpha \leq 2$ and $f : \mathbb{N}_0 \times X \to X$ be a function. Assume that the Nemytskii operator is continuous in $l_f^\infty(\mathbb{N}_2; X)$ and condition (F) is satisfied. Then the problem (4.3.1) has a solution in $l_f^\infty(\mathbb{N}_2; X)$.*

Proof. We know that T is the generator of an α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$. We define the operator $G : l_f^\infty(\mathbb{N}_2; X) \to l_f^\infty(\mathbb{N}_2; X)$ as follows

$$Gu(n) = \sum_{j=0}^{n-2} (S_\alpha * h_\alpha)(n-2-j) f(j, u(j)),\ n \geq 2.$$

We need to check that the conditions of Leray-Schauder alternative theorem are fulfilled.

- G is well defined: It follows from the proof of Theorem 4.3.5.

- G is continuous: Let $\epsilon > 0$ and $u, v \in l_f^\infty(\mathbb{N}_2; X)$. Then for each $n \geq 2$,

$$\begin{aligned}
\|Gu(n) - Gv(n)\| &\leq \sum_{j=0}^{n-2} \|(S_\alpha * h_\alpha)(n-2-j)\| \, \|f(j, u(j)) - f(j, v(j))\| \\
&\leq \|S_\alpha\|_\infty \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} \|\mathcal{N}_f(u)(j) - \mathcal{N}_f(v)(j)\| \\
&\leq \|S_\alpha\|_\infty \|\mathcal{N}_f u - \mathcal{N}_f v\|_f \left(\sum_{j=0}^{n-1} (j! - 1) \right).
\end{aligned}$$

Therefore,

$$\frac{\|Gu(n) - Gv(n)\|}{nn!} \leq \|S_\alpha\|_\infty \|\mathcal{N}_f u - \mathcal{N}_f v\|_f \frac{1}{nn!} \left(\sum_{j=0}^{n-1} (j! - 1) \right),$$

and consequently, taking supremum again

$$\|Gu - Gv\|_f \leq \frac{1}{16} \|S_\alpha\|_\infty \|\mathcal{N}_f u - \mathcal{N}_f v\|_f .$$

Then, we obtain $\|Gu - Gv\|_f < \epsilon$.

- G is compact: We fix $R > 0$, let $B(0; R)$ be an open unit ball in $l_f^\infty(\mathbb{N}_2; X)$. Let $V = G(B(0; R))$, we need to check the conditions in Lemma 4.3.7 in order to prove relatively compactness of V.

  a) Let $u \in B(0; R)$ and $v = Gu$. We have that

$$\frac{v(n)}{nn!} = \frac{1}{n!} \left(\frac{1}{n} \sum_{j=0}^{n-2} (S_\alpha * h_\alpha)(j) f(n-2-j, u(n-2-j)) \right). \tag{4.3.3}$$

  Therefore, $\dfrac{v(n)}{nn!} \in \dfrac{1}{n!} co(K_n)$, where $co(K_n)$ denotes the convex hull of a set K_n defined by

$$K_n = \bigcup_{j=0}^{n-2} \{(S_\alpha * h_\alpha)(j) f(k, x) : k \in \{0, 1, 2, \ldots, n-2\}, \|x\| \leq R\},\ n \geq 2.$$

  By condition (F) we have that for all $a \in \mathbb{N}_0$ and $\sigma > 0$, the set

$\{f(k,x) : 0 \le k \le a,\ \|x\| \le \sigma\}$ is bounded. Consequently, the set $\{(S_\alpha * h_\alpha)(n)f(k,x) : 0 \le k \le a,\ \|x\| \le \sigma\}$ is relatively compact in X for all $n \ge 2$ because $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ generated by T is compact. Then it follows that each set K_n is relatively compact. From the inclusions $H_n(V) = \left\{\frac{v(n)}{nn!} : v \in V\right\} \subseteq \frac{1}{n!}co(K_n) \subseteq \frac{1}{n!}co(\overline{K_n})$, we conclude that $H_n(V)$ is relatively compact in X for all $n \ge 2$.

b) Let $u \in B(0;R)$ and $v = Gu$. By condition (F), for each $n \in \mathbb{N}_2$ we get

$$\begin{aligned}\frac{\|v(n)\|}{nn!} &\le \frac{1}{nn!}\sum_{k=0}^{n-2}\|(S_\alpha * h_\alpha)(n-2-k)\|\,\|f(k,u(k))\| \\ &\le \frac{1}{nn!}\sum_{k=0}^{n-2}\|(S_\alpha * h_\alpha)(n-2-k)\|\,a(k)(c\,\|u(k)\| + b) \\ &\le cR\,\|S_\alpha\|_\infty\,\|a\|_\infty\,\frac{1}{nn!}\left(\sum_{k=0}^{n-1}(k!-1)\right) + b\,\|S_\alpha\|_\infty\,\|a\|_1\left(\frac{n-1}{nn!}\right).\end{aligned}$$

Then $\lim_{n\to\infty}\frac{\|v(n)\|}{nn!} = 0$ for an arbitrary $u \in B(0;R)$.

- The set $U := \{u \in l_f^\infty(\mathbb{N}_2;X) : u = \gamma Gu,\ 0 < \gamma < 1\}$ is bounded: Indeed, let $u \in l_f^\infty(\mathbb{N}_2;X)$ such that $u = \gamma Gu$, $0 < \gamma < 1$. By condition (F),

$$\begin{aligned}\|u(n)\| \le \|Gu(n)\| &\le \|S_\alpha\|_\infty \sum_{j=0}^{n-2}\sum_{k=0}^{n-2-j}\|f(k,u(k))\| \\ &\le c\,\|S_\alpha\|_\infty\,\|a\|_\infty\,\|u\|_f\left(\sum_{j=0}^{n-1}(j!-1)\right) + b\,\|S_\alpha\|_\infty\,\|a\|_1\,(n-1).\end{aligned} \tag{4.3.4}$$

Hence, for each $n \ge 2$ we get

$$\frac{\|u(n)\|}{nn!} \le c\,\|S_\alpha\|_\infty\,\|a\|_\infty\,\|u\|_f\,\frac{1}{nn!}\left(\sum_{j=0}^{n-1}(j!-1)\right) + b\,\|S_\alpha\|_\infty\,\|a\|_1\left(\frac{n-1}{nn!}\right).$$

Finally, taking supremum in both sides we deduce that $U \subset l_f^\infty(\mathbb{N}_2;X)$ is a bounded set.

Finally, by Leray-Schauder fixed point alternative theorem, we have that G has a fixed point in $l_f^\infty(\mathbb{N}_2; X)$. $\square$

4.4 An Application

In this section we show a concrete application of the results obtained in the previous sections of this chapter.

Example 4.4.1. Let us consider an integral operator P defined on the Banach space $C[0,1]$ as follows

$$Pf(x) = \int_0^1 k(x,s)f(s)ds.$$

First we suppose that P defines a bounded non compact operator and $\|P\| = 1$, see for instance [10] for a concrete example of kernel $k(x,s)$ such that P is non compact.

We study the existence of solutions of the following problem

$$\begin{cases} \Delta^\alpha u(n,x) &= \dfrac{1}{5}Pu(n,x) + \dfrac{1+u(n,x)}{1+\sup_{0\le x\le 1}|u(n,x)|}, \\ u(0,x) &= 0, \\ u(1,x) &= 0, \end{cases} \tag{4.4.1}$$

for $n \in \mathbb{N}_0$, $x \in [0,1]$, $1 < \alpha < 2$. Define $T = \dfrac{1}{5}P$. We will apply Corollary 4.3.6. We have $\|T\| < \dfrac{\alpha^\alpha(2-\alpha)^{2-\alpha}}{4}$. Then, by Theorem 4.2.4, T generates a bounded α-resolvent sequence $\{S_\alpha(n)\}_{n\in\mathbb{N}_0}$ on $C[0,1]$.

Define $v(n)(x) := u(n,x)$ and $f(n, v(n)) := \dfrac{1+v(n)}{1+\|v(n)\|_\infty}$. Then the problem (4.4.1) can be rewritten as

$$\Delta^\alpha v(n) = Tv(n) + f(n, v(n)),\ n \in \mathbb{N}_0,$$

with initial conditions $v(0)=v(1)=0$. Note that the function f satisfies condition (F) of Theorem 4.3.5 with $a(k)=\dfrac{1}{2^k}$, $c=0$ and $b=1$. On the other hand, note that for $u_1,u_2\in l_f^\infty(\mathbb{N}_2;C[0,1])$ and $n\in\mathbb{N}_0$, we obtain that

$$\begin{aligned}
\|f(n,u_1(n))-f(n,u_2(n))\| &\leq \left\|\frac{1+u_1(n,x)}{1+\|u_1(n)\|}-\frac{1+u_2(n,x)}{1+\|u_2(n)\|}\right\| \\
&\leq \left\|\frac{u_1(n)-u_2(n)}{1+\|u_1(n)\|}\right\|+\frac{(1+\|u_2(n)\|)\,\|u_1(n)-u_2(n)\|}{(1+\|u_1(n)\|)(1+\|u_2(n)\|)} \\
&\leq 2\,\|u_1(n)-u_2(n)\|\,.
\end{aligned}$$

Therefore f is a Lipschitz function with constant $L=2$. Since $\|T\|=\dfrac{1}{5}$, then condition (L) in Corollary 4.3.6 is fulfilled if and only if

$$2<\frac{64}{3}\left(\frac{\alpha^\alpha(2-\alpha)^{2-\alpha}}{4}-\frac{1}{5}\right)$$

if and only if

$$\frac{7}{5}<\alpha<2. \tag{4.4.2}$$

Then, by Corollary 4.3.6 we conclude that for all values α given in (4.4.2), we have that the fractional integro-difference equation

$$\Delta^\alpha u(n,x)=\frac{1}{5}\int_0^1 k(x,s)u(n,s)ds+\frac{1+u(n,x)}{1+\sup_{0\leq x\leq 1}|u(n,x)|},\ x\in[0,1],\ n\in\mathbb{N}_0$$

with initial conditions $u(0,x)=u(1,x)=0$ admits a unique solution in

$$l_f^\infty(\mathbb{N}_2;C[0,1])=\left\{\xi:\mathbb{N}_2\to C[0,1]:\sup_{n\geq 2}\frac{\|\xi(n)\|_\infty}{nn!}<\infty\right\}.$$

On the other hand, if we suppose that P is a bounded compact operator and $\|P\|=1$, then the conditions of Theorem 4.3.8 for the problem (4.4.1) are fulfilled for any $\alpha>1$. As a consequence, the problem (4.4.1) has a unique nontrivial solution $u\in l_f^\infty(\mathbb{N}_2;C[0,1])$.

Conclusions

In this thesis, we obtain new results concerning the existence and uniqueness of solutions for different classes of fractional difference equations in the setting of Banach spaces.

First, we obtain a characterization of the existence and uniqueness of solutions belonging to vector-valued space of sequences $\ell_p(\mathbb{Z};X)$ for a fractional difference equation with damping. For this, we use a method based on an operator-valued multiplier theorem due to S. Blünck (see [20]). Recall that this theorem and its converse establish an equivalence between ℓ_p-multipliers and R-bounded sets. Using Blünck's theorem we found a characterization of maximal ℓ_p-regularity to equation (2.1.1) in terms of the operator-valued symbol asociated to such equation. Also, we give a useful criterion for the existence of ℓ_p-solutions for the nonlinear equation (2.1.10) if the nonlinear term satisfies conditions of Fréchet differentiability at 0.

Next, we obtain a characterization for the existence and uniqueness of solutions belonging to the vector-valued space of sequences $\ell_p(\mathbb{N}_0;X)$ for a fractional difference equation with a delay term. For this, we use Blünck's theorem and we found a characterization of maximal ℓ_p-regularity of equation (3.2.1) in terms of the operator-valued symbol asociated to such equation. Note that the solution of the equation (3.2.1) is written in terms of a family of bounded operators called α^τ-resolvent. In addition, we complete the study of maximal regularity for fractional difference equations of order $1 < \alpha \leq 2$.

Finally, we obtain a useful criterion for the existence of solutions belonging to the vector-valued space of sequences $\ell_f^\infty(\mathbb{N}_2; X)$ for a nonlinear fractional difference equation. For this, we use suitable conditions on the nonlinear term in order to guarantee that the nonlinear equation (4.3.1) has a unique solution. On the other hand, when we consider compactness conditions on certain bounded operators, we get a criterion for the existence of solutions for equation (4.3.1). Note that the solution of the equation (3.2.1) is written in terms of a family of bounded operators called α-resolvent. In addition, we complete the study of the existence of weighted bounded solutions for nonlinear fractional difference equations of order $1 < \alpha \leq 2$.

Finally, we mention that there are still a few open problems about this investigation.

i) The study of ℓ_p-well posedness for a fractional difference equations with damping similar to (2.1.1), when the fixed real number λ is replaced by an operator.

ii) Find a representation of the family α-resolvent (resp. α^τ-resolvent) of bounded operators in the case $1 < \alpha \leq 2$ in terms of its generator.

iii) The study of a nonlinear equation similar to equation (4.3.1) when a delay term is considered.

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