Universidad de Santiago de Chile
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On p-adic L-functions of Bianchi modular forms: functional equations and non-cuspidal constructions

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Abstract

The aim of this thesis is to contribute to the understanding and construction of the p-adic L-function of Bianchi modular forms, concretely: a) we obtain the functional equation of the p-adic L-function of small slope cuspidal Bianchi modular forms constructed by Williams in [41] and then, using p-adic families of Bianchi modular forms, we extend our result to the p-adic L-function of Σ-smooth base change Bianchi modular forms constructed by Barrera and Williams in [3]; b) we introduce and study the notions of quasi-cuspidality, C-cuspidality and overconvergent partial Bianchi modular symbols generalizing to the Bianchi setting ideas developed by Bellaïche and Dasgupta in [4]; c) we construct the p-adic L-function of non-cuspidal base change Bianchi modular forms and finally, we factor such function as the product of two Katz p-adic L-functions.

Keywords: Automorphic forms, Bianchi modular forms, p-adic L-function, p-adic families, functional equation, base change, CM forms.

Resumen

El objetivo de esta tesis es contribuir a la comprensión y construcción de la función L p-ádica de las formas modulares de Bianchi, concretamente: a) obtenemos la ecuación funcional de la función L p-ádica de formas modulares de Bianchi cuspidales de pendiente pequeña construidas por Williams en [41] y luego, usando familias p-ádicas de formas modulares de Bianchi, extendemos nuestro resultado a la función L p-ádica de formas modulares de Bianchi Σ-suaves que son cambio de base construídas por Barrera y Williams en [3]; b) introducimos y estudiamos las nociones de casi-cuspidalidad, C-cuspidalidad y símbolos modulares parciales de Bianchi sobreconvergentes generalizando al contexto Bianchi ideas desarrolladas por Bellaïche y Dasgupta en [4]; c) construimos la función L p-ádica de formas modulares de Bianchi no cuspidales que son cambio de base y finalmente, factorizamos tal función L p-ádica como el producto de dos funciones L p-ádicas de Katz.

Palabras clave: Formas automorfas, formas modulares de Bianchi, función L p-ádica, familias p-ádicas, ecuación funcional, cambio de base, formas CM.

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Introduction

In recent years, p-adic L-functions have been a subject of considerable study. As an example, the 'main conjecture' of Iwasawa theory says that p-adic L-functions control the size of cohomology groups of Galois representations. They have also been used in the construction of Stark–Heegner points, which are conjecturally global points on non-CM elliptic curves over number fields. On the other hand, there is an increasing interest in the construction and study of p-adic families for automorphic forms attached to reductive groups over number fields, this interest is justified by the important applications obtained in for example, the Langlands Program or Bloch-Kato conjectures. In this thesis we study both, p-adic L-functions and p-adic families, for Bianchi modular forms

0.1 First notions and motivation

Let F be a number field, $\mathbb{A}_F$ be its ring of adeles and $\mathbf{G}$ be a reductive group, automorphic forms are functions on the quotient $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)$ with good growth and regularity properties. One gets an action of $\mathbf{G}(\mathbb{A}_F)$, by right translations, on automorphic forms. Automorphic representations are irreducible representations of $\mathbf{G}(\mathbb{A}_F)$ occurring in spaces of automorphic forms. In this thesis we are interested in the case $\mathbf{G} = \mathrm{GL}_2$ and F an imaginary quadratic extension of the rational numbers, the automorphic forms in this case are called Bianchi modular forms. We also are interested (just for purposes of base change to Bianchi modular forms) in the case when $\mathbf{G} = \mathrm{GL}_2$ and $F = \mathbb{Q}$, that is the case of classical modular forms, which are arguably the most studied automorphic forms.

An automorphic form has associated an invariant called the complex L-function, which is of archimedean nature (complex holomorphic function) and contains important information of the automorphic form. The study of such L-functions is related with important problems in mathematics such as the Birch and Swinnerton-Dyer (BSD) conjecture, which is one of the Millennium Problems or Hilbert's twelfth problem. In this work we treat non-archimedean aspects related with this ideas. Fix a prime number p, instead of working with complex numbers, it is natural to consider p-adic numbers. In this new paradigm, two objects has been shown to be useful:

- The p-adic L-functions, that are the p-adic analogue of the complex L-functions.
- The p-adic families, that are geometric objects that encode the idea of deformation of our object of study.

Some problems where these ideas has been successfully used are: The BSD conjecture and its generalisations (see the works of Kato [24]; Darmon-Rotger [12]) the Artin's conjecture proved by Pilloni [32], the Langlands Program (see Chenevier [8]; Emerton [16], Boxer-Calegari-Gee-Pilloni [6]) or Hilbert's twelfth problem (see Dasgupta-Kakde [14]; Darmon,-Pozzi-Vonk [13]).

0.2 Overview of the p-adic L-function of Bianchi modular forms

Fix K an imaginary quadratic field, the case of automorphic representations over K has importance by itself, because even when the Langlands program has evolved in many directions, this is the first case where many of the algebraic geometry techniques does not work, letting a lot of open questions. However, in recent years, largely due to insights from medalists Fields A. Venkatesh [39] and P. Scholze [35], this context has rejuvenated, appearing fertile and full of research directions to be carried out.

Additionally since there exists a well developed picture of the construction of the p-adic L-function of modular forms using overconvergent modular symbols and p-adic families, there is a natural interest in generalize those results and constructions to the Bianchi case.

0.2.1 The small slope cuspidal case (Williams' construction)

The construction of p-adic L-functions attached to cuspidal Bianchi modular forms by Williams in [41] is based in the ideas of Stevens [36] and Pollack-Stevens in [33], i.e., the generalization of classical modular symbols to the Bianchi case and using the so called overconvergent Bianchi modular symbols. To a cuspidal Bianchi modular form Φ of weight (k,k) and level $\Omega_0(\mathfrak{n})$, where $(p)|\mathfrak{n}$, we associate a collection of functions $\mathcal{F}^1,...,\mathcal{F}^h : \mathcal{H}_3 \to V_{2k+2}(\mathbb{C})$, where h is the class number of K and $V_n(\mathbb{C})$ is the space of homogeneous polynomials in two variables of degree $n \geqslant 0$ over $\mathbb{C}$. Each $\mathcal{F}^i$ satisfying an automorphy condition for some discrete subgroup $\Gamma_i(\mathfrak{n})$ of $\mathrm{SL}_2(K)$. To each of these $\mathcal{F}^i$, we associate a classical Bianchi modular symbol $\phi_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n})}(V^*_{k,k}(\mathbb{C})) := \mathrm{Hom}_{\Gamma_i(\mathfrak{n})}(\Delta_0, V^*_{k,k}(\mathbb{C}))$ where $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(K))$ and $V^*_{k,k}(\mathbb{C}) = \mathrm{Hom}(V_{k,k}(\mathbb{C}),\mathbb{C})$ for $V_{k,k}(\mathbb{C}) = V_k(\mathbb{C}) \otimes_{\mathbb{C}} V_k(\mathbb{C})$. In the analogous way to modular forms there is a link between values of this symbol and critical values of the part of the L-function corresponding to $\mathcal{F}^i$.

Generalising Stevens' idea Williams define the space of overconvergent Bianchi modular symbols to be the space of Bianchi modular symbols taking values in some p-adic distribution space; precisely, he fix a suitable finite extension $L/\mathbb{Q}_p$, and denote by $\mathcal{A}_{k,k}(L)$ the space of locally analytic functions $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$ endowed with a suitable action depending by k, then the distribution space is $\mathcal{D}_{k,k}(L) = \mathrm{Hom}_{cts}(\mathcal{A}_{k,k}(L), L)$. There is a specialisation map from overconvergent to classical Bianchi modular symbols by dualising the natural inclusion $V_{k,k}(L) \hookrightarrow \mathcal{A}_{k,k}(L)$.

Then in the same way to Pollack-Stevens in [33], Williams obtains a control theorem in the Bianchi setting and construct $L_p(\Phi,-)$, the p-adic L-function of a cuspidal Bianchi eigenform Φ of weight (k,k), level $\Omega_0(\mathfrak{n})$ with $(p)|\mathfrak{n}$ and with small slope -i.e, the p-adic valuation of the eigenvalue of the Hecke operator $U_{\mathfrak{p}}$ for each $\mathfrak{p}|p$ is 'sufficiently small'- by,

- associating to Φ a small slope eigensymbol $(\phi_1, ..., \phi_h)$ in a direct sum of symbol spaces,
- lifting uniquely $(\phi_1, ..., \phi_h)$ to an overconvergent symbol $(\Psi_1, ..., \Psi_h)$ using the control theorem,
- patching together the distributions $\Psi_i(\{0\} - \{\infty\})$ to a locally analytic distribution $L_p(\Phi, -)$ on the ray class group $\mathrm{Cl}_K(p^\infty)$.

0.2.2 The critical slope base change cuspidal case (Barrera-Williams construction)

William's construction of p-adic L-functions depend of the small slope condition of the Bianchi modular form Φ. It is natural to ask for the p-adic L-function when Φ does not have small slope, i.e., the critical slope case. Such function was constructed by Barrera and Williams in [3] for suitable critical slope base-change Bianchi modular forms using p-adic families of Bianchi modular forms. We briefly describe the construction.

Let $f \in S_{k+2}(\Gamma_0(N))$ be a finite slope eigenform, with $p|N$, new or p-stabilised of a newform, regular, non CM by K, decent and such that the base-change to K, denoted by $f_{/K}$, is Σ-smooth (see Conditions 2.11 and definitions 2.18 and 2.19 for more details) and let $V_{\mathbb{Q}}$ be a neighbourhood of f in the Coleman-Mazur eigencurve such that the weight map is étale except possibly at f. Then, after shrinking $V_{\mathbb{Q}}$, Barrera and Williams constructed the *three-variable p-adic L-function*

$$\mathcal{L}_p : V_{\mathbb{Q}} \times \mathfrak{X}(\mathrm{Cl}_K(p^\infty)) \to L$$

for sufficiently large $L \subset \overline{\mathbb{Q}}_p$, such that for any classical point $y \in V_{\mathbb{Q}}(L)$ corresponding to a small slope base-change $f_{y/K}$ we have $\mathcal{L}_p(y, -) = c_y L_p(f_{y/K}, -)$, where $c_y \in L^\times$ is a p-adic period at y and $L_p(f_{y/K}, -)$ is the p-adic L-function of $f_{y/K}$ described above.

Now, suppose that $f_{/K}$ has critical slope and is Σ-smooth; then the missing p-adic L-function of $f_{/K}$ is defined to be the specialisation

$$L_p(f_{/K}, -) \coloneqq \mathcal{L}_p(x_f, -),$$

where $x_f \in V_{\mathbb{Q}}$ is the point corresponding to f.

0.3 Structure of the thesis

In this thesis we accomplish three tasks:

- We provide functional equations for the p-adic L-functions of cuspidal Bianchi modular forms described above in sections 0.2.1 and 0.2.2; for the critical slope ones we also prove a functional equation for families of p-adic L-functions.

- We introduce the notions of:
  - quasi-cuspidal and C-cuspidal Bianchi modular forms;
  - partial Bianchi modular symbols and overconvergent partial Bianchi modular symbols;

  to replicate the construction of the p-adic L-function of cuspidal Bianchi modular forms to the non-cuspidal case.

- We construct the p-adic L-function of the base change to K of a modular form with CM by K by proving that is a quasi-cuspidal Bianchi modular form. Given the nature of such base change forms we also factor the p-adic L-function as the product of two Katz p-adic L-functions.

Regarding the items above:

- the results in the first item related with functional equations of p-adic L-functions in the cuspidal case are contained in [30], the first work of my PhD;
- the results in the second and third item related with the construction of the p-adic L-function in the non-cuspidal case are contained in [31], the second work of my PhD.

This work is divided into four chapters, which will be summarized in the rest of this introduction.

Chapter 1 treats about automorphic forms for GL_1 and GL_2 over an imaginary quadratic field K i.e., Hecke characters of K and Bianchi modular forms, respectively. In section 1.2 we treat Bianchi modular forms; we introduce and study the notions of quasi-cuspidality and C-cuspidality which satisfies

$$\{\text{cuspidal}\} \subset \{\text{quasi-cuspidal}\} \subset \{C-\text{cuspidal}\} \subset \{\text{Bianchi modular forms}\},$$

for Bianchi modular forms with level at p. We also prove minor results about p-stabilisations and twists of Bianchi modular forms as well as the functional equation of its L-function when K has class number 1. In section 1.3 we study Bianchi modular forms given as base change to K of modular forms with complex multiplication by K. It is known the non-cuspidality of such functions, but we show that they are quasi-cuspidal using the work of Friedberg in [17]. Moreover, since the L-function of base change forms can be factored as the product of two Hecke L-functions we obtain algebraicity at critical values of such L-function.

In Chapter 2 we introduce the notion of partial Bianchi modular symbols, inspired by the work of Bellaïche and Dasgupta in [4]. In section 2.1 we show how to attach such a partial symbol to a C-cuspidal Bianchi modular form, more specifically we have (see Proposition 2.3)

Proposition 0.1. *Let Φ be a C-cuspidal Bianchi modular form of weight (k,ℓ) with $k \geqslant \ell$ and level $\Omega_0(\mathfrak{n})$, then to each $\mathcal{F}^i$ with $i = 1,..,h$ we can attach an element*

$$\phi_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^*(\mathbb{C}))$$

In section 2.2 we develop the theory of overconvergent partial Bianchi modular symbols and we obtain the following lifting theorem (see Theorem 2.2)

Theorem 0.1. *(Partial Bianchi control theorem). For each prime $\mathfrak{p}$ above p, let $\lambda_{\mathfrak{p}} \in L^{\times}$. Suppose that $v(\lambda_{\mathfrak{p}}) < (\min\{k,\ell\}+1)/e_{\mathfrak{p}}$ when p is inert as $\mathfrak{p}$ or p ramifies as $\mathfrak{p}^2$ where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$, or $v(\lambda_{\mathfrak{p}}) < k+1$ and $v(\lambda_{\overline{\mathfrak{p}}}) < \ell+1$ when p splits as $\mathfrak{p}\overline{\mathfrak{p}}$, then the restriction of the specialisation map*

$$\rho : \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),\mathcal{C}_i}(\mathcal{D}_{k,\ell}(L))^{\{U_{\mathfrak{p}}=\lambda_{\mathfrak{p}}:\mathfrak{p}|p\}} \longrightarrow \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),\mathcal{C}_i}(V_{k,\ell}^{*}(L))^{\{U_{\mathfrak{p}}=\lambda_{\mathfrak{p}}:\mathfrak{p}|p\}}$$

to the simultaneous $\lambda_{\mathfrak{p}}$-eigenspaces of the $U_{\mathfrak{p}}$ operators is an isomorphism.

In section 2.3 we give a brief overview of p-adic families and the Bianchi eigenvariety that will be used in section 3.2 to prove our results on functional equation of p-adic L-functions in the critical slope case.

Chapter 3 is devoted to p-adic L-functions, in section 3.1 we obtain the functional equation of the p-adic L-function of a small slope cuspidal Bianchi modular form when K has number class 1 (see Theorem 3.2)

Theorem 0.2. *Let $\mathcal{F}_p$ be a small slope p-stabilisation of a Bianchi newform $\mathcal{F} \in S_{(k,k)}(\Gamma_0(\mathfrak{n}))$, with $\mathfrak{n}$ prime to (p). Then the p-adic L-function of $\mathcal{F}_p$, denoted by $L_p(\mathcal{F}_p,-)$, satisfies*

$$L_p(\mathcal{F}_p,\kappa) = (*)L_p(\mathcal{F}_p,\kappa^{-1}\sigma_p^{k,k}),$$

for all $\kappa \in \mathfrak{X}(\mathrm{Cl}_K(p^{\infty}))$, where $(*)$ is an explicit factor.

Motivated by Barrera and Williams construction's of the p-adic L-function of cuspidal base change Bianchi families, section 3.2 contains the functional equation satisfied by such p-adic L-function in families (see Theorem 3.4), which specialized to the critical slope case give us an analogous theorem to the small slope case (see Corollary 3.1)

Corollary 0.1. *Let $\mathcal{F}$ be a Σ-smooth base-change to K of a decent modular form satisfying conditions 2.11, let $\mathfrak{n}$ be the prime-to-p part of the level of $\mathcal{F}$, then for all $\kappa \in \mathfrak{X}(\mathrm{Cl}_K(p^{\infty}))$ the distribution $L_p(\mathcal{F},-)$ satisfies the following functional equation*

$$L_p(\mathcal{F},\kappa) = (*)L_p(\mathcal{F},\kappa^{-1}\sigma_p^{k,k}).$$

Regarding the p-adic L-function of non-cuspidal Bianchi modular forms, section 3.3 contains the construction of the p-adic L-function of the small slope non-cuspidal Bianchi modular form given as base change to K of a classical modular form with CM by K (see Theorem 3.5)

Theorem 0.3. *Let φ be a Hecke character of K with conductor $\mathfrak{m}$ coprime with p and infinity type $(-k-1,0)$ with $k \geqslant 0$, denote by f_{φ} and f_{φ}^{p} the elliptic CM modular form induced by φ and its ordinary p-stabilisation respectively. Let $f_{\varphi/K}^{p}$ be the base-change to K of f_{φ}^{p}. Then there exists a* *unique locally analytic measure* $L_p(f^p_{\varphi/K}, -)$ *on* $\mathrm{Cl}_K(p^\infty)$ *such that for any Hecke character* ψ *of* K *of conductor* $\mathfrak{f}|p^\infty$ *and infinity type* $0 \leqslant (q,r) \leqslant (k,k)$*, we have*

$$L_p(f^p_{\varphi/K}, \psi_{p-\mathrm{fin}}) = (*)\Lambda(f^p_{\varphi/K}, \psi).$$

At the end of section 3.3 we factor this p-adic L-function as the product of two Katz p-adic L-functions obtaining (see Theorem 3.7)

Theorem 0.4. *Under the hypothesis of Theorem 0.3 for all* $\kappa \in \mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ *we have*

$$L_p(f^p_{\varphi/K}, \kappa) = \frac{L_{p,\mathrm{Katz}}(\varphi^c_{p-\mathrm{fin}}\kappa\sigma_p^{1,1})}{\Omega_p(A)^{k+1}} \frac{L_{p,\mathrm{Katz}}(\varphi^c_{p-\mathrm{fin}}\kappa^c\sigma_p^{1,1})}{\Omega_p(A)^{k+1}},$$

where the character $\sigma_p^{1,1}$ *is defined in (1.1) and* $\Omega_p(A)$ *is the* p*-adic period in the interpolation of Katz* p*-adic* L*-functions.*

Chapter 4 contains current work and further directions regarding the construction of p-adic L-functions of non-cuspidal Bianchi modular forms and the theory of p-adic Bianchi families. Section 4.2 contains our current work related with the construction of p-adic L functions of small slope non-cuspidal Bianchi modular form. In section 4.3 we present ideas for the construction and study of p-adic families of Bianchi modular forms around non-cuspidal Bianchi modular forms and p-adic L-functions attached to these families. Moreover, we expect to apply these ideas to construct p-adic L-functions of critical slope non-cuspidal Bianchi modular forms.

Chapter 1

Automorphic forms

In this chapter, we develop the theory of automorphic forms for GL_1 *and* GL_2 *over an imaginary quadratic field* K*. We start by recalling Hecke characters and algebraic properties of its Hecke* L*-function. Next we move to Bianchi modular forms, reviewing the theory and focusing in the non-cuspidal case by introducing and studying the new notions of quasi-cuspidality and* C*-cuspidality. We conclude by studying the non-cuspidal Bianchi modular forms given as base change to* K *of CM forms by* K*.*

1.1 Automorphic forms for $\mathrm{GL}_1(K)$

Dirichlet characters are the automorphic forms of $\mathrm{GL}_1(\mathbb{Q})$, in analogous way *Hecke characters over* K will be the automorphic forms of $\mathrm{GL}_1(K)$.

1.1.1 Hecke characters

Definition 1.1. *A Hecke character of* K *is a continuous homomorphism* $\varphi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$.

By restriction, for each place v of K, we obtain a character $\varphi_v : K_v^\times \to \mathbb{C}^\times$, where K_v denotes the completion of K at v. We have $\varphi = \varphi_\infty\varphi_f$ where $\varphi_\infty := \prod_{v|\infty}\varphi_v = \varphi|_{\mathbb{C}^\times}$ is the *the infinite part* of φ and $\varphi_f := \prod_{v\nmid\infty}\varphi_v$ is the *the finite part* of φ.

Definition 1.2. *We say* φ *is algebraic if* $\varphi_\infty(z) = z^k\overline{z}^\ell$ *for* $k,\ell \in \mathbb{Z}$*, in that case, we say* (k,ℓ) *is its infinity type. If* $k = \ell = 0$*, we call* φ *a finite order character.*

Suppose v corresponds to a finite prime $\mathfrak{q}$ of K, let $K_\mathfrak{q}$ the completion of K at $\mathfrak{q}$, $\mathcal{O}_\mathfrak{q}$ its ring of integers and denote by $\varphi_\mathfrak{q}$ the restriction of φ to $K_\mathfrak{q}^\times$. It can be shown that $\varphi_\mathfrak{q}(\mathcal{O}_\mathfrak{q}^\times) = 1$ for almost all primes $\mathfrak{q}$ of $\mathcal{O}_K$ and for the remaining finite set of primes $\mathfrak{q}$, there exists a non-negative integer $e_\mathfrak{q} > 0$ such that $\varphi_\mathfrak{q}(1 + \mathfrak{q}^{e_\mathfrak{q}}) = 1$ and $e_\mathfrak{q}$ is minimal with this property.

Definition 1.3. *Define the conductor of* φ *to be the ideal* $\mathfrak{f} := \prod_\mathfrak{q}\mathfrak{q}^{e_\mathfrak{q}}$.

Note that φ naturally gives rise to a Dirichlet character over K with conductor $\mathfrak{f}$ via the isomorphism $\widehat{\mathcal{O}}_K^\times/(1 + \mathfrak{f}\widehat{\mathcal{O}}_K) \cong (\mathcal{O}_K/\mathfrak{f})^\times \to \mathbb{C}^\times$.

Definition 1.4. *Let $I \subset O_K$ be an ideal, we define the character $\varphi_I := \prod_{\mathfrak{q}|I} \varphi_{\mathfrak{q}}$. So $\varphi_{\mathfrak{f}}$ determines the Dirichlet character associated to φ.*

When considering Hecke L-functions in section 1.1.3, will be useful to think Hecke characters as characters on fractional ideals of K. For this, we use the fact that we can associate a well-defined fractional ideal $I(x) = \prod_{\mathfrak{q}} \mathfrak{q}^{v_{\mathfrak{q}}(x_{\mathfrak{q}})}$ to an idele $x \in \mathbb{A}_K^\times$.

Definition 1.5. *Let φ be a Hecke character of conductor $\mathfrak{f}$, define a function φ on fractional ideals I of K by*

$$\varphi(I) = \begin{cases} \varphi_f(x) & : \text{if } (I,\mathfrak{f}) = 1, \quad \text{where } x \in \mathbb{A}_K^\times \text{ is such that } I = I(x) \text{ and } x_{\mathfrak{q}} = 1 \text{ for all primes } \mathfrak{q}|\mathfrak{f}; \\ 0 & : \text{if } (I,\mathfrak{f}) \neq 1. \end{cases}$$

Note that the definition above does not depend of the election of x in the case where I is coprime with $\mathfrak{f}$.

Remark 1.1. *In future sections, specially when talking about twisted L-functions by characters, we will need to relate the three functions $\varphi_{\mathfrak{f}}$, φ_∞ and φ as a function on ideals. By Proposition 1.2.9 in [42], for $\alpha \in K^\times$ such that $((\alpha),\mathfrak{f}) = 1$ we have $\varphi_\infty(\alpha)\varphi_{\mathfrak{f}}(\alpha)\varphi((\alpha)) = 1$.*

1.1.2 p-adic Hecke characters

There is a relation between Hecke characters of K an p-adic characters that play an important role for p-adic L-functions, in this section we recall it.

Definition 1.6. *Let $\mathfrak{f} \subset \mathcal{O}_K$ be an ideal, define the ray class group of level $\mathfrak{f}$ to be the analytic group*

$$\mathrm{Cl}_K(\mathfrak{f}) := K^\times \backslash \mathbb{A}_K^\times / \mathbb{C}^\times U(\mathfrak{f}),$$

where $U(\mathfrak{f}) = 1 + \mathfrak{f}\widehat{\mathcal{O}}_K$. Moreover, define the ray class group of level p^∞ to be

$$\mathrm{Cl}_K(p^\infty) := K^\times \backslash \mathbb{A}_K^\times / \mathbb{C}^\times = \varprojlim_n \mathrm{Cl}_K(p^n) U(p^\infty).$$

Note that, by class field theory, $\mathrm{Cl}_K(p^\infty)$ is isomorphic to the Galois group of the maximal abelian extension of K unramified outside p and ∞.

Definition 1.7. *Define the space $\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ to be the two-dimensional rigid space of p-adic characters on $\mathrm{Cl}_K(p^\infty)$.*

Remark 1.2. *(i) $\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ is the natural domain of the p-adic L-functions of chapter 3.*

(ii) Note the space $\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ contains the cyclotomic and anti-cyclotomic directions that is why it is two dimensional.

There is a bijection between algebraic Hecke characters of conductor dividing p^∞ and locally algebraic characters of $\mathrm{Cl}_K(p^\infty)$ such that if φ corresponds to $\varphi_{p\text{-fin}}$, both are equal when we restrict to the adeles away from the infinite place and the primes above p.

If φ is an algebraic Hecke character of K of conductor $\mathfrak{f}|p^\infty$ and infinity type (q,r), then, fixing an isomorphism $\mathbb{C} \cong \mathbb{C}_p$, we associate to φ a $K^\times$-invariant function

$$\begin{aligned} \varphi_{p-\mathrm{fin}}(x) : (\mathbb{A}_K^\times)_\mathfrak{f} &\longrightarrow \mathbb{C}_p \\ x &\longmapsto \varphi_\mathfrak{f}(x)\sigma_p^{q,r}(x), \end{aligned}$$

where

$$\sigma_p^{q,r}(x) := \begin{cases} x_\mathfrak{p}^q x_{\overline{\mathfrak{p}}}^r & : p \text{ splits as } \mathfrak{p}\overline{\mathfrak{p}}, \\ x_\mathfrak{p}^q \overline{x_\mathfrak{p}^r} & : p \text{ inert or ramified.} \end{cases} \tag{1.1}$$

Remark 1.3. *If we define for $\alpha \in K^\times$ a p-adic idele $x_{\alpha,p}$ by*

$$(x_{\alpha,p})_\mathfrak{q} = \begin{cases} \alpha & \textit{when } \mathfrak{q}|(p) \\ 1 & \textit{otherwise}; \end{cases}$$

then we have $\varphi_{p-\mathrm{fin}}(x_{\alpha,p}) = (\varphi_{p-\mathrm{fin}})_{(p)}(\alpha) = \varphi_{(p)}(\alpha)\alpha^q\overline{\alpha}^r$.

This p-adic idele will be useful for the expression of the functional equation of the p-adic L-function of Bianchi modular forms (for example in proof of Proposition 3.1).

1.1.3 Hecke L-function

Recall that given a Hecke character ψ on $K^\times\backslash\mathbb{A}_K^\times$ of conductor $\mathfrak{f}$, we can define a character on the fractional ideals of K coprime to $\mathfrak{f}$ and then extend ψ to a function of all fractional ideals by defining $\psi(I) = 0$ for I non-prime to $\mathfrak{f}$.

We can attach to ψ a complex L-function built as a Dirichlet series of the values of ψ on non trivial ideals of $\mathcal{O}_K$.

Definition 1.8. *Define the Hecke L-function for ψ by*

$$L(\psi,s) := \sum_{\substack{0\neq\mathfrak{m}\subset\mathcal{O}_K \\ (\mathfrak{m},\mathfrak{f})=1}} \psi(\mathfrak{m})N(\mathfrak{m})^{-s}. \quad (s \in \mathbb{C}).$$

Using the convergence of the Riemann zeta function, can be shown that $L(\psi,s)$ is convergent absolutely and uniformly on the set $Re(s) > 1$ and has the Euler product

$$L(\psi,s) = \prod_{\mathfrak{q} \text{ prime}} \left(1 - \frac{\psi(\mathfrak{q})}{N(\mathfrak{q})^s}\right)^{-1}.$$

In particular, when ψ is the trivial character we obtain the Dedekind zeta function of K.

Hecke L-functions have a functional equation obtained by Hecke by generalizing the proof of the functional equation for the Riemann zeta function.

For p-adic interpolation of the Hecke L-functions we are interested in the algebraicity of its *critical values*.

Definition 1.9. *We say m is a critical value of the Hecke L-function of ψ if the Γ-factors that arise in the functional equation for $L(\psi,s)$ are nonvanishing and have no poles at $s=m$.*

Let $\Omega(A)$ be the complex period attached to an elliptic curve A with complex multiplication by K, defined over a finite extension F of K as in section 2C in [5]. For a Hecke character ψ let $\Omega(\psi^c)$ be the complex period attached to $\psi^c := \psi\circ c$ (for c the complex conjugation on ideals) as in the comment above Proposition 2.11 in [5]. For suitable Hecke characters ψ we can prove algebraicity of the critical values $L(\psi,m)$ using the two periods above.

Lemma 1.1. *Let ψ be a Hecke character of K with infinity type (a,b) such that $a>b$ then for m a critical value of $L(\psi,s)$ we have*

$$\frac{L(\psi,m)}{(2\pi i)^{m+b}\Omega(A)^{a-b}} \in \overline{\mathbb{Q}}.$$

Proof. Let ν be a Hecke character with infinity type $(-r,-s)$ then by Proposition 2.11 in [5] (and noting that ν has infinity type (r,s) in op. cit. because its definition of infinity type is different than ours) we have

$$\frac{\Omega(\psi^c)}{(2\pi i)^s\Omega(A)^{r-s}} \in \overline{\mathbb{Q}}. \tag{1.2}$$

On the other hand, by Theorem 2.12 in [5], if $r>s$ we have for m a critical value of $L(\nu^{-1},m)$

$$\frac{L(\nu^{-1},m)}{(2\pi i)^m\Omega(\psi^c)} \in E_\nu. \tag{1.3}$$

where E_ψ is the subfield of $\overline{\mathbb{Q}}$ generated by the values of ν.

Then by (1.2) and (1.3) for a Hecke character ν with infinity type $(-r,-s)$ with $r>s$ we have

$$\frac{L(\nu^{-1},m)}{(2\pi i)^{m+s}\Omega(A)^{r-s}} \in \overline{\mathbb{Q}}.$$

finally taking $\nu=\psi^{-1}$ then $(-r,-s)=(-a,-b)$ and we obtain the result. □

1.1.4 Gauss sums

We start by defining the Gauss sum of a Hecke character, this definition in general varies among references. Since most of the results when we use Gauss sums are related with the work in [41], we use the definition given there.

Definition 1.10. *Let φ be a Hecke character of K with conductor $\mathfrak{f}$ and let $\delta=\sqrt{-D}$ for $-D$ the discriminant of K. The Gauss sum for φ is defined to be*

$$\tau(\varphi) := \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})\varphi_\infty(a/\delta)e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a/\delta)}.$$

Remark 1.4. *In the definition of $\tau(\varphi)$, it appears φ as a function on ideals and φ_∞, we can rewrite $\tau(\varphi)$ just using the idelic definition of φ, in fact, by Lemma 6.3.2 in [42] we have for*

$a \in \mathfrak{f}^{-1}$ with $(a\mathfrak{f}, \mathfrak{f}) = 1$ that

$$\varphi(a\mathfrak{f})^{-1}\varphi_\infty(a)^{-1} = \varphi(x_\mathfrak{f})^{-1}\varphi_\mathfrak{f}(ax_\mathfrak{f}).$$

Then we have

$$\tau(\varphi) = \varphi(x_\mathfrak{f})\varphi_\mathfrak{f}(x_\mathfrak{f})^{-1}\varphi_\infty(\delta)^{-1} \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \varphi_\mathfrak{f}(a)^{-1} e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a/\delta)}. \tag{1.4}$$

Note that, in particular, if φ has finite order and conductor $\mathfrak{f} = (f)$ we obtain a similar expression to a Gauss sum of a Dirichlet character given by

$$\tau(\varphi) = \sum_{b\in(\mathcal{O}_K/\mathfrak{f})^\times} \varphi_\mathfrak{f}(b)^{-1} e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(b/(\delta f))}.$$

The Gauss sum of a Hecke character φ over K satisfies the following property

Proposition 1.1. *(i) For all $c \in \mathcal{O}_K$, we have*

$$\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})\varphi_\infty(a/\delta) e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(ac/\delta)} = \tau(\varphi)\varphi_\mathfrak{f}(c).$$

(ii) By replacing φ with φ^{-1}, we have

$$\frac{1}{\tau(\varphi^{-1})} \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})\varphi_\infty(a/\delta) e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(ac/\delta)} = \begin{cases} \varphi_\mathfrak{f}(c)^{-1} & : ((c)\mathfrak{f},\mathfrak{f}) = 1, \\ 0 & : \text{otherwise}. \end{cases}$$

Proof. Writing $\tau(\varphi)$ as in (1.4) for $a \in \mathfrak{f}^{-1}$ such that $(a\mathfrak{f}, \mathfrak{f}) = 1$ and $c \in \mathcal{O}_K$ with $((c), \mathfrak{f}) = 1$ we can easily obtain

$$\varphi(x_\mathfrak{f})\varphi_\mathfrak{f}(x_\mathfrak{f})^{-1}\varphi_\infty(\delta)^{-1} \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \varphi_\mathfrak{f}(a)^{-1} e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(ac/\delta)} = \tau(\varphi)\varphi_\mathfrak{f}(c).$$

The case when $((c), \mathfrak{f}) \neq 1$ is a little bit more involving since we have to prove the left hand side of the equality in (i) is 0, see Proposition 6.14 in [42] for more details. □

1.2 Automorphic forms for $\mathrm{GL}_2(K)$

In this section we introduce the Bianchi modular forms, that is, automorphic forms for GL_2 over the imaginary quadratic field K. This are the principal objects of interest of this thesis.

1.2.1 Bianchi modular forms

In this section we define Bianchi modular forms first adelically as automorphic forms over an imaginary quadratic field and later we focus on their formulation as modular forms on the hyperbolic 3-space.

Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_K$, first we fix a level that will be a subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}_K})$. Define

$$\Omega_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}_K}) : c \in \mathfrak{n}\widehat{\mathcal{O}_K} \right\}.$$

We now define the space where our automorphic forms for GL_2 over K takes values, such space will be an irreducible representation of a suitable compact subgroup of $\mathrm{GL}_2(\mathbb{C})$.

Definition 1.11. *Let n be a positive integer and $V_n(\mathbb{C})$ be the space of homogeneous polynomials in two variables X and Y of degree n with complex coefficients. There is an irreducible representation $\rho : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}(V_n(\mathbb{C}))$ induced by the action*

$$(P|u)\begin{pmatrix} X \\ Y \end{pmatrix} = P\left(u \cdot \begin{pmatrix} X \\ Y \end{pmatrix}\right).$$

Finally, we fix a Hecke character φ, with conductor dividing $\mathfrak{n}$ and infinity type $(-k-2v_1, -\ell-2v_2)$ for $k, \ell \geqslant 0, v_1, v_2$ integers. For $u_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Omega_0(\mathfrak{n})$ we set $\varphi_{\mathfrak{n}}(u_f) = \varphi_{\mathfrak{n}}(d) = \prod_{\mathfrak{q}|\mathfrak{n}} \varphi_{\mathfrak{q}}(d_{\mathfrak{q}})$.

Definition 1.12. *We say a function $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \to V_{k+\ell+2}(\mathbb{C})$ is an Bianchi modular form of weight $\lambda = [(k,\ell),(v_1,v_2)]$, level $\Omega_0(\mathfrak{n})$ and central action φ if it satisfies:*

(i) *Φ is left-invariant under $\mathrm{GL}_2(K)$;*

(ii) *$\Phi(zg) = \varphi(z)\Phi(g)$ for $z \in \mathbb{A}_K^\times \cong Z(\mathrm{GL}_2(\mathbb{A}_K))$, where $Z(G)$ denote the center of the group G;*

(iii) *$\Phi(gu) = \varphi_{\mathfrak{n}}(u_f)\Phi(g)\rho_{k+\ell+2}(u_\infty)$ for $u = u_f \cdot u_\infty \in \Omega_0(\mathfrak{n}) \times \mathrm{SU}_2(\mathbb{C})$;*

(iv) *Φ is an eigenfunction of the operators D_σ, for $\sigma \in \{i, c\}$ (the two embeddings of K into $\mathbb{C}$),*

$$D_i\Phi = (k^2/2 + k)\Phi, \qquad D_c\Phi = (\ell^2/2 + \ell)\Phi,$$

where $D_\sigma/4$ denotes a component of the Casimir operator in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$, and where we consider $\Phi(g_\infty g_f)$ as a function of $g_\infty \in \mathrm{GL}_2(\mathbb{C})$.

(v) *there exists an $N \geqslant 0$ such that for every compact subset S of $B = \left\{ \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A}_K) \right\}$ and for any norm $\|\cdot\|$ in the complex vector space $V_{k+\ell+2}(\mathbb{C})$, Φ satisfies*

$$\left\| \Phi\left[\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right] \right\| = O(|t|^N + |t|^{-N})$$

uniformly over $\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \in S$.

The space of such functions will be denoted by $\mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$. If Φ also satisfies the cuspidal condition:

(vi) for all $g \in \mathrm{GL}_2(\mathbb{A}_K)$

$$\int_{K\backslash\mathbb{A}_K} \Phi\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) du = 0,$$

where du *is the Lebesgue measure on* $\mathbb{A}_K$*,*

we say that Φ *is a cuspidal Bianchi modular form and we denote the space of such functions by* $S_\lambda(\Omega_0(\mathfrak{n}), \varphi)$*.*

Remark 1.5. *(i) When* $v_1 = v_2 = 0$*, we denote the weight* λ *just by* (k, ℓ) *and accordingly the corresponding spaces by* $\mathcal{M}_{(k,\ell)}(\Omega_0(\mathfrak{n}), \varphi)$ *and* $S_{(k,\ell)}(\Omega_0(\mathfrak{n}), \varphi)$*. If also the weight is parallel, we just write* (k, k) *instead* λ*.*

(ii) From [20, §2.5, Cor 2.2]) we have $S_\lambda(\Omega_0(\mathfrak{n}), \varphi) = 0$ *if* $k \neq \ell$ *i.e. all non-trivial cuspidal Bianchi modular forms have parallel weight* $\lambda = [(k, k), (v_1, v_2)]$*.*

Bianchi modular forms over K can be seen as the analogous of classical modular forms in a suitable hyperbolic space by the descent that we present in the following, for this, let $I_1, ..., I_h$ be a set of representatives for the class group of K, such that $I_1 = \mathcal{O}_K$ and each I_i for $2 \leqslant i \leqslant h$ is an integral and prime ideal, with each I_i coprime to $\mathfrak{n}$, (p) and $\mathcal{D}$, the different ideal of K.

Let π_i be an uniformiser in K_{I_i} and define $t_i := (1, ..., 1, \pi_i, 1, ...) \in \mathbb{A}_K^\times$, then by the *strong approximation theorem* we have

$$\mathrm{GL}_2(\mathbb{A}_K) = \coprod_{i=1}^{h} \mathrm{GL}_2(K) \cdot \begin{pmatrix}t_i & 0\\ 0 & 1\end{pmatrix} \cdot [\mathrm{GL}_2(\mathbb{C}) \times \Omega_0(\mathfrak{n})]$$

and therefore Φ descends to give a non-canonical (depending on choices of representatives for the class group of K) collection of h functions $F^i : \mathrm{GL}_2(\mathbb{C}) \to V_{k+\ell+2}(\mathbb{C})$ defined by

$$F^i(g) := \Phi\left(\begin{pmatrix}t_i & 0\\ 0 & 1\end{pmatrix} g\right). \tag{1.5}$$

We can descend even more and see the functions F_i of above as functions in the hyperbolic space defined by $\mathcal{H}_3 := \{(z, t) : z \in \mathbb{C}, t \in \mathbb{R}_{>0}\}$, this can be done by the following lemma:

Lemma 1.2. *There is a decomposition* $\mathrm{GL}_2(\mathbb{C}) = \mathrm{Z}(\mathrm{GL}_2(\mathbb{C})) \cdot \mathrm{B} \cdot \mathrm{SU}_2(\mathbb{C})$*, where*

$$\mathrm{B} = \left\{\begin{pmatrix}t & z\\ 0 & 1\end{pmatrix} : z \in \mathbb{C}, t \in \mathbb{R}_{>0}\right\}.$$

Explicity, for $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$*,*

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \zeta \begin{pmatrix}t & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}\overline{v} & -\overline{u}\\ u & v\end{pmatrix} \tag{1.6}$$

where

$$z = \frac{a\bar{c} + b\bar{d}}{|c|^2 + |d|^2}, \quad t = \frac{|ad - bc|}{|c|^2 + |d|^2}, \quad \zeta = \epsilon(|c|^2 + |d|^2)^{1/2}, \quad u = \frac{c}{\zeta}, \quad v = \frac{d}{\zeta}, \quad \epsilon = \left(\frac{ad - bc}{|ad - bc|}\right)^{\frac{1}{2}}.$$

Moreover, in the decomposition (1.6), t and z are uniquely determined, and ζ, u and v are uniquely up to choice of the sign of ϵ.

Proof. See [7], Corollary 43. □

By the decomposition of $\mathrm{GL}_2(\mathbb{C})$ state above and considering that $\mathrm{B} \cong \mathcal{H}_3$, then using properties ii) and iii) in Definition 1.12 we obtain h functions $\mathcal{F}^i : \mathcal{H}_3 \to V_{k+\ell+2}(\mathbb{C})$ defined by

$$\mathcal{F}^i(z,t) := t^{v_1+v_2-1} F^i \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \tag{1.7}$$

where the factor $t^{v_1+v_2-1}$ comes from [18] where each $\mathcal{F}_i$ is defined by

$$\mathcal{F}^i(z,t) := t^{-\frac{k+\ell}{2}-1} F^i \left(\frac{1}{\sqrt{t}} \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \right)$$

and since by property i) in Definition 1.12 we have

$$F^i \left(\frac{1}{\sqrt{t}} \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \right) = \varphi_\infty \left(\frac{1}{\sqrt{t}} \right) F^i \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} = t^{\frac{k}{2}+v_1} t^{\frac{\ell}{2}+v_2} F^i \left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \right)$$

then (1.7) follows.

The properties i), ii) and iii) satisfied by Φ in Definition 1.12 give us an automorphy condition satisfied by $\mathcal{F}^i$ with respect to the discrete subgroup of $\mathrm{SL}_2(K)$ defined by

$$\Gamma_i(\mathfrak{n}) := \mathrm{SL}_2(K) \cap \begin{pmatrix} t_i & 0 \\ 0 & 1 \end{pmatrix} \Omega_0(\mathfrak{n}) \begin{pmatrix} t_i & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathrm{GL}_2(\mathbb{C}). \tag{1.8}$$

Consider $\Phi \in \mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ with $\lambda = [(k,\ell),(v_1,v_2)]$ and a fix idele s with trivial components at infinity and $\mathfrak{n}$, then we obtain the functions $F^s(g) = \Phi\left(\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} g\right)$ on $\mathrm{GL}_2(\mathbb{C})$, $\mathcal{F}^s(z,t) := t^{v_1+v_2-1} F^s \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}$ on $\mathcal{H}_3$ and the group $\Gamma_s(\mathfrak{n}) := \mathrm{SL}_2(K) \cap \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \Omega_0(\mathfrak{n}) \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathrm{GL}_2(\mathbb{C})$, as before.

Definition 1.13. *Let $\mathcal{F} : \mathcal{H}_3 \to V_{k+\ell+2}(\mathbb{C})$ be a function:*

(i) We say that $\mathcal{F}$ is a Bianchi modular form of weight λ, level Γ and nebentypus χ if there exists an idele s with trivial components at infinity and $\mathfrak{n}$, such that $\mathcal{F} = \mathcal{F}^s$, $\Gamma = \Gamma_s(\mathfrak{n})$ and $\chi = \varphi_{\mathfrak{n}}^{-1}$ for some Bianchi modular form Φ of weight λ, level $\Omega_0(\mathfrak{n})$ and central action φ whose conductor divides $\mathfrak{n}$.

(ii) We say that $\mathcal{F}$ is a cusp form if the Bianchi modular form Φ is cuspidal.

We denote the spaces of such forms by $\mathcal{M}_\lambda(\Gamma,\chi)$ *and* $S_\lambda(\Gamma,\chi)$ *respectively.*

We can compute explicitly the automorphy condition satisfied by $\mathcal{F}$ with respect to Γ but before we need some definitions.

Definition 1.14. *There is an action of* $\mathrm{GL}_2(\mathbb{C})$ *on* $\mathcal{H}_3$ *when we consider it as a quotient* $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times\mathrm{SU}_2(\mathbb{C})$. *This action can be described explicitly as:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z,t) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}|t|^2}{|cz+d|^2+|ct|^2}, \frac{|ad-bc|t}{|cz+d|^2+|ct|^2} \right).$$

Definition 1.15. *Define the factor of automorphy* $J(\gamma;(z,t))$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ *and* $(z,t) \in \mathcal{H}_3$ *by*

$$J(\gamma;(z,t)) := \begin{pmatrix} cz+d & \bar{c}t \\ -ct & \overline{cz+d} \end{pmatrix}.$$

Note that $J(\gamma_1\gamma_2,(z,t)) = J(\gamma_2,(z,t))J(\gamma_1,\gamma_2\cdot(z,t))$ for all $\gamma_1,\gamma_2 \in \mathrm{SL}_2(\mathbb{C})$.

Lemma 1.3. *Let* $\mathcal{F}$ *be a Bianchi modular form of weight* $\lambda = [(k,\ell),(v_1,v_2)]$, *level* Γ *and nebentypus* χ, *then*

$$\mathcal{F}(\gamma\cdot(z,t)) = \chi(d)\mathcal{F}(z,t)\rho_{k+\ell+2}(J(\gamma;(z,t))). \tag{1.9}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Remark 1.6. *Note that we see* $\rho_{k+\ell+2}$ *as a representation of* $\mathrm{GL}_2(\mathbb{C})$.

Proof. (of Lemma 1.3) Recall by Definition 1.13 that exist $\Phi \in \mathcal{M}_\lambda(\Omega_0(\mathfrak{n}),\varphi)$ with $\lambda = [(k,\ell),(v_1,v_2)]$ and an idele s with trivial components at infinity and $\mathfrak{n}$ such that $\mathcal{F} = \mathcal{F}^s$, $\Gamma = \Gamma_s(\mathfrak{n})$ and $\chi = \varphi_{\mathfrak{n}}^{-1}$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $g_\infty = \left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots\right) \in \mathrm{GL}_2(\mathbb{A}_K)$ with $t \in \mathbb{R}_{>0}$ and $z \in \mathbb{C}$ and let γ' be the diagonal embedding of γ in $\mathrm{GL}_2(\mathbb{A}_K)$ then by property (i) of Definition 1.12 we have

$$\Phi\left(\gamma'\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} g_\infty\right) = \Phi\left(\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} g_\infty\right) = F^s\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} = t^{1-v_1-v_2}\mathcal{F}^s(z,t) = t^{1-v_1-v_2}\mathcal{F}(z,t). \tag{1.10}$$

On the other hand, if we denote $g'_\infty = \left(\gamma\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots\right) \in \mathrm{GL}_2(\mathbb{A}_K)$, then by property iii) of Definition 1.12 and Lemma 1.2 we have

$$\begin{aligned} \Phi\left(\gamma'\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} g_\infty\right) &= \varphi_{\mathfrak{n}}(d)\Phi\left(\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} g'_\infty\right) = \varphi_{\mathfrak{n}}(d)F^s\left(\gamma\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right) \\ &= \varphi_{\mathfrak{n}}(d)\varphi_\infty(\zeta)F^s\begin{pmatrix} t' & z' \\ 0 & 1 \end{pmatrix}\rho_{k+\ell+2}\begin{pmatrix} \overline{v} & -\overline{u} \\ u & v \end{pmatrix} \end{aligned}$$

with

$$\zeta = (|cz+d|^2 + |ct|^2)^{\frac{1}{2}}, \quad t' = \frac{t}{|cz+d|^2+|ct|^2}, \quad z' = \frac{(az+b)\overline{(cz+d)} + a\overline{c}|t|^2}{|cz+d|^2+|ct|^2}$$

$$u = \frac{ct}{(|cz+d|^2+|ct|^2)^{\frac{1}{2}}}, \; v = \frac{cz+d}{(|cz+d|^2+|ct|^2)^{\frac{1}{2}}}.$$

Now, since $\varphi_\infty(\zeta) = (|cz+d|^2+|ct|^2)^{-\frac{k+\ell}{2}-v_1-v_2}$ and

$$F^s\begin{pmatrix} t' & z' \\ 0 & 1\end{pmatrix} = \left(\frac{t}{|cz+d|^2+|ct|^2}\right)^{1-v_1-v_2} \mathcal{F}^s(z',t') = \left(\frac{t}{|cz+d|^2+|ct|^2}\right)^{1-v_1-v_2} \mathcal{F}^s(\gamma\cdot(z,t)),$$

we have

$$\Phi(\gamma' g_\infty) = \varphi_{\mathfrak{n}}(d) t^{1-v_1-v_2}(|cz+d|^2+|ct|^2)^{-\frac{k+\ell}{2}-1}\mathcal{F}^s(\gamma\cdot(z,t))\rho_{k+\ell+2}\begin{pmatrix} \overline{v} & -\overline{u} \\ u & v\end{pmatrix}$$

and noting that

$$\begin{aligned}(|cz+d|^2+|ct|^2)^{-\frac{k+\ell}{2}-1}\rho_{k+\ell+2}\begin{pmatrix} \overline{v} & -\overline{u} \\ u & v\end{pmatrix} &= \rho_{k+\ell+2}\left((|cz+d|^2+|ct|^2)^{-\frac{1}{2}}\begin{pmatrix} \overline{v} & -\overline{u} \\ u & v\end{pmatrix}\right) \\ &= \rho_{k+\ell+2}(J(\gamma;(z,t)))^{-1}\end{aligned}$$

Then we have

$$\begin{aligned}\Phi\left(\gamma'\begin{pmatrix} s & 0 \\ 0 & 1\end{pmatrix} g_\infty\right) &= \varphi_{\mathfrak{n}}(d)t^{1-v_1-v_2}\mathcal{F}^s(\gamma\cdot(z,t))\rho_{k+\ell+2}(J(\gamma;(z,t)))^{-1} \\ &= \chi(d)^{-1}t^{1-v_1-v_2}\mathcal{F}(\gamma\cdot(z,t))\rho_{k+\ell+2}(J(\gamma;(z,t)))^{-1} \qquad (1.11)\end{aligned}$$

Putting together (1.10) and (1.11) we obtain the result. □

In analogous way to modular forms we can define a slash operator for Bianchi modular forms that will be important in next sections to compute the explicit action by Hecke operators.

Definition 1.16. *Let* $\mathcal{F} \in \mathcal{M}_\lambda(\Gamma,\chi)$ *with* $\lambda = [(k,\ell),(v_1,v_2)]$*, then for every* $\gamma \in \mathrm{GL}_2(\mathbb{C})$ *define*

$$(\mathcal{F}|_\gamma)(z,t) := det(\gamma)^{-\frac{k}{2}-v_1}\overline{det(\gamma)}^{-\frac{\ell}{2}-v_2}\mathcal{F}(\gamma\cdot(z,t))\rho^{-1}_{k+\ell+2}\left(J\left(\frac{\gamma}{\sqrt{det(\gamma)}};(z,t)\right)\right). \qquad (1.12)$$

Remark 1.7. *1)* $\mathcal{F}|_g(0,1) = F(g)$ *for* $g \in \mathrm{GL}_2(\mathbb{C})$*, in particular*

$$\mathcal{F}|_{\begin{pmatrix} t & z \\ 0 & 1\end{pmatrix}}(0,1) = F\begin{pmatrix} t & z \\ 0 & 1\end{pmatrix}.$$

2) $(\mathcal{F}|_\gamma)(z,t) = \chi(d)\mathcal{F}(z,t)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma$.

1.2.2 Fourier expansion and cuspidal conditions

In this section we recall the Fourier expansion of a Bianchi modular form and the descent of such expansion at all cusps $\mathbb{P}^1(K) = K \cup \{\infty\}$.

Let $\lambda = [(k,\ell),(v_1,v_2)]$ as before, consider the set J_λ consisting of the four elements

$$j = (j_1, j_2) = (\pm(k+1), \pm(\ell+1))$$

and define

$$h(j) = \frac{k+\ell+2}{2} + \frac{j_2 - j_1}{2},$$
$$\iota(j) = \left(-v_1 + \frac{1}{2}[j_1 - (k+1)], -v_2 + \frac{1}{2}[j_2 - (\ell+1)]\right).$$

Let $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \to V_{k+\ell+2}(\mathbb{C})$ be a Bianchi modular form of weight $\lambda = [(k,\ell),(v_1,v_2)]$, level $\Omega_0(\mathfrak{n})$ and central action φ, by Theorem 6.7 in [20], Φ has a Fourier-Whittacker expansion given by

$$\Phi\left[\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right] = |t|_{\mathbb{A}_K}\left[\sum_{j\in J_\lambda} t_\infty^{\iota(j)}\binom{k+\ell+2}{h(j)} c_j(t\mathcal{D},\Phi)X^{k+\ell+2-h(j)}Y^{h(j)} + \sum_{\alpha\in K^\times} c(\alpha t\mathcal{D},\Phi)W(\alpha t_\infty)e_K(\alpha z)\right], \tag{1.13}$$

where:

i) The functions $c_j(\cdot,\Phi)$ on the zero Fourier coefficient and the Fourier coefficients $c(\cdot,\Phi)$ inside the sum, are functions on the fractional ideals of K, with $c_j(\cdot,\Phi) = 0$ and $c(I,\Phi) = 0$ for I not-integral,

iii) e_K is an additive character of $K\backslash\mathbb{A}_K$ defined by

$$e_K = \left(\prod_{\mathfrak{q}\text{prime}} (e_q \circ \mathrm{Tr}_{K_\mathfrak{q}/\mathbb{Q}_q})\right)\cdot(e_\infty \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}),$$

for

$$e_q\left(\sum_j d_j q^j\right) = e^{-2\pi i \sum_{j<0} d_j q^j} \quad \text{and} \quad e_\infty(r) = e^{2\pi i r};$$

and

iv) $W : \mathbb{C}^\times \to V_{k+\ell+2}(\mathbb{C})$ is the Whittaker function

$$W(s) := \sum_{n=0}^{k+\ell+2}\binom{k+\ell+2}{n}\frac{1}{s^{v_1}\bar{s}^{v_2}}\left(\frac{s}{i|s|}\right)^{\ell+1-n} K_{n-\ell-1}(4\pi|s|)X^{k+\ell+2-n}Y^n,$$

where $K_n(x)$ is (a modified Bessel function which is) a solution to the differential equation

$$\frac{d^2K_n}{dx^2} + \frac{1}{x}\frac{dK_n}{dx} - \left(1 + \frac{n^2}{x^2}\right)K_n = 0,$$

with the asymptotic behaviour

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}$$

as $x \to \infty$.

Remark 1.8. *Note that for all Bianchi modular form* $\Phi = \Phi_0 X^{k+\ell+2} + \ldots + \Phi_n X^{k+\ell+2-n} Y^n + \ldots + \Phi_{k+\ell+2} Y^{k+\ell+2}$*, the constant term in the Fourier expansion of* Φ_n *is trivial if* $n \notin \{h(j) | j \in J_\lambda\} = \{0, k+1, \ell+1, k+\ell+2\}$.

The Fourier expansion of Φ descends to $\mathcal{H}_3$ where we can describe it in the individual components $\mathcal{F}_i$ of Φ (removing references to adeles). In fact, following [18] we have

$$\mathcal{F}^i\left((z,t); \begin{pmatrix} X \\ Y \end{pmatrix}\right) = \sum_{n=0}^{k+\ell+2} \mathcal{F}^i_n(z,t) X^{k+\ell+2-n} Y^n,$$

where

$$\begin{aligned} \mathcal{F}^i_n(z,t) = {} & t^{\frac{j_1+j_2-k-\ell}{2}} \binom{k+\ell+2}{n} \delta_{h(j),n} c_j(t_i\mathcal{D}) \\ & + |t_i|_{\mathbb{A}_K} t \binom{k+\ell+2}{n} \sum_{\alpha \in K^\times} \left[c(\alpha t_i \mathcal{D}) \frac{1}{\alpha^{v_1} \overline{\alpha}^{v_2}} \left(\frac{\alpha}{i|\alpha|}\right)^{\ell+1-n} K_{n-\ell-1}(4\pi|\alpha|t) e^{2\pi i(\alpha z + \overline{\alpha z})} \right], \end{aligned} \tag{1.14}$$

where to ease notation we have written $c_j(t_i\mathcal{D})$ and $c(\alpha t_i\mathcal{D})$ instead $c_j(t_i\mathcal{D}, \Phi)$ and $c(\alpha t_i \mathcal{D}, \Phi)$.

For each $i = 1, \ldots, h$, equation (1.14) may be thought of as the Fourier expansion of $\mathcal{F}^i$ at the cusp of infinity which by Remark 1.8 satisfies that the constant term in the Fourier expansion of $\mathcal{F}^i_n$ is trivial if $n \notin \{h(j) | j \in J_\lambda\} = \{0, k+1, \ell+1, k+\ell+2\}$.

We must consider Fourier expansions at all the "K-rational" cusps $\mathbb{P}^1(K) = K \cup \{\infty\}$, for this, let $\sigma \in \mathrm{GL}_2(K)$ sending ∞ to the cusp s. For each $i = 1, \ldots, h$, since $\mathcal{F}^i \in \mathcal{M}_{(k,l)}(\Gamma_i(\mathfrak{n}), \varphi_{\mathfrak{n}}^{-1})$ then $\mathcal{F}^i|_\sigma \in \mathcal{M}_{(k,l)}(\sigma^{-1}\Gamma_i(\mathfrak{n})\sigma, \varphi_{\mathfrak{n}}^{-1})$ and hence $\mathcal{F}^i|_\sigma$ has a Fourier expansion as in (1.14).

Observe that, in particular, for cusps different than ∞ the constant term of $(\mathcal{F}^i|_\sigma)_n$ can be non-trivial for $n \notin \{h(j)|j \in J_\lambda\} = \{0, k+1, \ell+1, k+\ell+2\}$.

Definition 1.17. *We say that* $\mathcal{F}^i$ *vanishes at the cusp* s *if* $\mathcal{F}^i|_\sigma$ *has trivial constant term, and quasi-vanishes at the cusp* s *if* $(\mathcal{F}^i|_\sigma)_n$ *has trivial constant term for* $1 \leqslant n \leqslant k+\ell+1$.

Remark 1.9. *1) The property of vanishing and quasi-vanishing at the cusp* s *are well defined, i.e. are independent of the choice of* σ*, in fact, any other choice has the form* $\sigma' = \sigma\tau$*, where* $\tau \in \mathrm{GL}_2(K)$ *fixes* ∞ *and may thus be written* $\tau = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$*. Therefore, by 1.10 we have*

$$\mathcal{F}^i|_{\sigma'}(z,t) = (ad)^{-\frac{k}{2}} \overline{(ad)}^{-\frac{\ell}{2}} \mathcal{F}^i|_\sigma(\tau \cdot (z,t)) \rho_{k+\ell+2}^{-1}\left(J\left(\frac{\tau}{\sqrt{det(\tau)}}; (z,t)\right)\right)$$

and then for each $0 \leqslant n \leqslant k+\ell+2$ *we have*

$$(\mathcal{F}^i|_{\sigma'})_n(z,t) = (ad)^{-\frac{k}{2}} \overline{(ad)}^{-\frac{\ell}{2}} (\mathcal{F}^i|_\sigma)_n \left(\frac{az+b}{d}, \frac{|a|t}{|d|}\right) (ad^{-1})^{\frac{k+\ell}{2}+1-n} |ad^{-1}|^n$$

Showing that the constant term of $(\mathcal{F}^i|_\sigma)_n$ *is trivial if and only if the constant term of* $(\mathcal{F}^i|_{\sigma'})_n$ *is trivial*

2) Let $\Phi \in S_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ *be a cuspidal Bianchi modular form, then the cuspidal condition vi) in Definition 1.12 is equivalent to the vanishing of* $\mathcal{F}^i$ *at all cusps for each* $0 \leqslant i \leqslant h$ *(see Proposition 3.2 in [43]).*

Definition 1.18. *We say that a Bianchi modular form* Φ *is quasi-cuspidal if* $\mathcal{F}^i$ *quasi-vanishes at all cusps for* $0 \leqslant i \leqslant h$.

1.2.3 C-cuspidality

In this section suppose $\mathfrak{n} = (p)\mathfrak{m}$ with $((p), \mathfrak{m}) = 1$ and let $\Phi \in \mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ with $\mathcal{F}^i \in \mathcal{M}_\lambda(\Gamma_i(\mathfrak{n}), \varphi_{\mathfrak{n}}^{-1})$ its corresponding descent to $\mathcal{H}_3$ for $i = 1, \dots, h$. Define for each i the set of cusps

$$C_i := \Gamma_i(\mathfrak{m})\infty \cup \Gamma_i(\mathfrak{m})0,$$

the subset of $\mathbb{P}^1(K)$ containing ∞ and all $\frac{x}{y} \in K$ in lowest terms with $x \in I_i$ either $y \in \mathfrak{m}$ or $y \in (\mathcal{O}_K/\mathfrak{m})^\times$. Note that $\Gamma_i(\mathfrak{m})$ stabilices C_i and hence its subgroup $\Gamma_i(\mathfrak{n})$.

Definition 1.19. *We say that* $\mathcal{F}^i$ *is* C_i*-cuspidal if quasi-vanishes at all cusps in* C_i.

Note that the previous definition does not look like the natural generalisation of a modular form being C-cuspidal given in [4] because we are not asking for vanishing of $\mathcal{F}^i$ at the cusps of C_i, instead we just need quasi-vanishing, i.e., we do not care about the vanish of the functions $\mathcal{F}^i_0$ and $\mathcal{F}^i_{k+\ell+2}$. The motivation for this definition will become clear in Proposition 2.3 where we attach certain modular symbols to such C_i-cuspidal forms.

We also want to state C_i-cuspidality for all i as a property of Φ, if we write $C = (C_1, \dots, C_h)$ then

Definition 1.20. *We say that* Φ *is* C*-cuspidal if* $\mathcal{F}^i$ *is* C_i*-cuspidal for* $i = 1, \dots, h$.

Remark 1.10. *Note that for Bianchi modular forms with level at* p *we have*

$$\{\textit{cuspidal}\} \subset \{\textit{quasi-cuspidal}\} \subset \{C-\textit{cuspidal}\} \subset \{\textit{Bianchi modular forms}\}.$$

1.2.4 Hecke operators

Let $\mathfrak{q} \subset \mathcal{O}_K$ be a prime ideal and fix a uniformiser $\pi_\mathfrak{q}$ of $K_\mathfrak{q}$. Then the Hecke operator is given by the double coset operator $\left[\Omega_0(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \pi_\mathfrak{q}\end{pmatrix}\Omega_0(\mathfrak{n})\right]$ where we see $\begin{pmatrix}1 & 0\\ 0 & \pi_\mathfrak{q}\end{pmatrix} \in \mathrm{GL}_2(K_\mathfrak{q})$ as a matrix in $\mathrm{GL}_2(\mathbb{A}_K^f)$ putting the identity in the places outside $\mathfrak{q}$. More explicitly we can describe the action of the Hecke operators $T_\mathfrak{q}$ on $\mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ for $\mathfrak{q} \nmid \mathfrak{n}$ by

$$\Phi|_{T_\mathfrak{q}}(g) := \sum_{u \bmod \mathfrak{q}} \Phi\left(g\begin{pmatrix}\pi_\mathfrak{q} & u\\ 0 & 1\end{pmatrix}\right) + \Phi\left(g\begin{pmatrix}1 & 0\\ 0 & \pi_\mathfrak{q}\end{pmatrix}\right).$$

where as before we see $\begin{pmatrix}\pi_\mathfrak{q} & u\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & \pi_\mathfrak{q}\end{pmatrix} \in \mathrm{GL}_2(K_\mathfrak{q})$ as matrices in $\mathrm{GL}_2(\mathbb{A}_K^f)$ putting the identity in the places outside $\mathfrak{q}$.

If $\mathfrak{q}|\mathfrak{n}$, we write $U_\mathfrak{q}$ instead of $T_\mathfrak{q}$ and define

$$\Phi|_{U_\mathfrak{q}}(g) := \sum_{u \bmod \mathfrak{q}} \Phi\left(g\begin{pmatrix}\pi_\mathfrak{q} & u \\ 0 & 1\end{pmatrix}\right).$$

The $T_\mathfrak{q}$ and $U_\mathfrak{q}$ are all independent of choices of representatives (see [40, chap. VI]).

We can describe the action of Hecke operators on the Bianchi modular forms $\mathcal{F}^i$ but when the ideals are not principal we need to use the whole collection $(\mathcal{F}^1, ..., \mathcal{F}^h)$ to describe it.

Recall our fixed representatives $I_1, ..., I_h$ for the class group and consider the prime ideal $\mathfrak{q} \nmid \mathfrak{n}$, then for each $i \in \{1, ..., h\}$ there is a unique $j_i \in \{1, ..., h\}$ such that $\mathfrak{q}I_i = (\alpha_i)I_{j_i}$, for $\alpha_i \in K$. Then $T_\mathfrak{q}$ act on each component of $(\mathcal{F}^1, ..., \mathcal{F}^h)$ by double cosets in the following way by

$$\Phi|_{T_\mathfrak{q}} = (\mathcal{F}^1, ..., \mathcal{F}^h)|_{T_\mathfrak{q}} = \left(\mathcal{F}^{j_1}\middle|\left[\Gamma_{j_1}(\mathfrak{n})\begin{pmatrix}1 & 0 \\ 0 & \alpha_1\end{pmatrix}\Gamma_1(\mathfrak{n})\right], ..., \mathcal{F}^{j_h}\middle|\left[\Gamma_{j_h}(\mathfrak{n})\begin{pmatrix}1 & 0 \\ 0 & \alpha_h\end{pmatrix}\Gamma_h(\mathfrak{n})\right]\right). \quad (1.15)$$

We can compute this concretely by writing down explicit representatives of the double coset. Note that the $T_\mathfrak{q}$ operator permute the individual components, depending on the class of $\mathfrak{q}$ in the class group; indeed, this permutation corresponds to multiplication by $[\mathfrak{q}]$ in the class group. When $\mathfrak{q}|\mathfrak{n}$, the Hecke operator at $\mathfrak{q}$ is denoted by $U_\mathfrak{q}$ and defined in the same way as (1.15).

We can similarly define Hecke operators for each ideal $I \subset \mathcal{O}_K$. Indeed, let $I = \prod_\mathfrak{q} \mathfrak{q}^r$ where $\mathfrak{q}^r$ exactly divides I, then the Hecke operator T_I is totally determined by the Hecke operators $T_\mathfrak{q}$ for $\mathfrak{q}|I$.

Note that if Φ is quasi-cuspidal then $\Phi|_{T_\mathfrak{q}}$ is quasi-cuspidal for all Hecke operators $T_\mathfrak{q}$, but for C-cuspidal forms we have a more subtle result.

Proposition 1.2. *Let $\Phi \in \mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ be a C-cuspidal Bianchi modular form, with $\mathfrak{n} = (p)\mathfrak{m}$ and $(\mathfrak{m}, (p)) = 1$; then $\Phi|_{T_\mathfrak{q}}$ for all primes $\mathfrak{q} \nmid \mathfrak{n}$ and $\Phi|_{U_\mathfrak{p}}$ for all primes $\mathfrak{p}|(p)$ are C-cuspidal.*

Proof. By (1.15) we have to show that for all prime $\mathfrak{q} \nmid \mathfrak{m}$ with $\mathfrak{q}I_i = (\alpha_i)I_{j_i}$ the function $\mathcal{F}^{j_i}\left|\left[\Gamma_{j_i}(\mathfrak{n})\begin{pmatrix}1 & 0 \\ 0 & \alpha_i\end{pmatrix}\Gamma_i(\mathfrak{n})\right]\right.$ is C_i-cuspidal.

We first observe 2 facts:

1) $\Gamma_i(\mathfrak{n})$ stabilices C_i for all $i = 1, ..., h$.

Since $C_i := \Gamma_i(\mathfrak{m})\infty \cup \Gamma_i(\mathfrak{m})0$ then clearly $\Gamma_i(\mathfrak{m})$ stabilices C_i and hence its subgroup $\Gamma_i(\mathfrak{n})$.

2) We have $\begin{pmatrix}1 & 0 \\ 0 & \alpha_i\end{pmatrix} \cdot c_i \in C_{j_i}$ for all $c_i \in C_i$.

Since $(\mathfrak{q}, \mathfrak{m}) = 1$ there exists $y_\mathfrak{q} \in \mathfrak{q}$ such that $y_\mathfrak{q} \in (\mathcal{O}_K/\mathfrak{m})^\times$, analogously, $(I_i, \mathfrak{m}) = 1$ there exists $y_i \in I_i$ such that $y_i \in (\mathcal{O}_K/\mathfrak{m})^\times$. By the identity $\mathfrak{q}I_i = (\alpha_i)I_{j_i}$ there exist an element $t_{j_i} \in I_{j_i}$ such that $y_\mathfrak{q}y_i = \alpha_i t_{j_i}$ and we have

$$\alpha_i = \frac{y_\mathfrak{q} y_i}{t_{j_i}}. \quad (1.16)$$

Let $c_i = x/y$, then $x \in I_i$ and either $y \in \mathfrak{m}$ or $y \in (\mathcal{O}_K/\mathfrak{m})^\times$, then we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \cdot \frac{x}{y} = \frac{x}{\alpha_i y} = \frac{t_{j_i} x}{y_{\mathfrak{q}} y_i y}$$

with $t_{j_i} x \in I_{j_i}$ and either $y_{\mathfrak{q}} y_i y \in \mathfrak{m}$ or $y_{\mathfrak{q}} y_i y \in (\mathcal{O}_K/\mathfrak{m})^\times$ then $\begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \cdot c_i \in C_{j_i}$.

Now, back to the proof, let $s_i \in C_i$, $\sigma_{s_i} \in \mathrm{GL}_2(K)$ be such that $\sigma_{s_i} \cdot \infty = s_i$ and let $\gamma \in \Gamma_{j_i}(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \Gamma_i(\mathfrak{n})$, then we have to show that the constant term of $(\mathcal{F}^{j_i}|_{\gamma\sigma_{s_i}})_n$ vanish for $1 \leqslant n \leqslant k + \ell + 1$.

If $\gamma = \gamma_{j_i} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \gamma_i$ with $\gamma_j \in \Gamma_j(\mathfrak{n})$ then $\gamma_i \cdot s_i = s_i' \in C_i$ by 1), $\begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \cdot s_i' = s_{j_i} \in C_{j_i}$ by 2) and $\gamma_{j_i} \cdot s_{j_i} = s_{j_i}' \in C_{j_i}$ by 1), then we have

$$\gamma_{j_i} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \gamma_i \sigma_{s_i} \cdot \infty = \gamma_{j_i} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \gamma_i \cdot s_i = \gamma_{j_i} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_i \end{pmatrix} \cdot s_i' = \gamma_{j_i} \cdot s_{j_i} = s_{j_i}' \in C_{j_i}.$$

Since $\mathcal{F}^{j_i}$ is C_{j_i}-cuspidal then for all $s_{j_i}' \in C_{j_i}$ and $\sigma_{s_{j_i}'} \in \mathrm{GL}_2(K)$ such that $\sigma_{s_{j_i}'} \cdot \infty = s_{j_i}'$ we have that the constant term of $(\mathcal{F}^{j_i}|_{\sigma_{s_{j_i}'}})_n$ is trivial for $1 \leqslant n \leqslant k + \ell + 1$. In particular taking $\sigma_{s_{j_i}'} = \gamma\sigma_{s_i}$ we obtain the result. □

Definition 1.21. *A Bianchi modular form is called an eigenform if it is a simultaneous eigenvector for the Hecke operators.*

Aditionally to Hecke operators we have an important involution acting on the space $\mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$. Let $\nu \in \mathcal{O}_K$ be such that $\mathfrak{n}\mathcal{O}_{\mathfrak{n}} = \nu\mathcal{O}_{\mathfrak{n}}$ and define $\gamma_{\mathfrak{n}} \in \mathrm{GL}_2(\mathbb{A}_K^f)$ by

$$(\gamma_{\mathfrak{n}})_v = \begin{cases} \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} & : v|\mathfrak{n} \\ I & : \text{otherwise.} \end{cases}$$

Define the *Weil involution* by the double coset $W_{\mathfrak{n}} := [\Omega_0(\mathfrak{n})\gamma_{\mathfrak{n}}\Omega_0(\mathfrak{n})]$ that act on $\mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ by

$$\Phi|_{W_{\mathfrak{n}}}(g) := (-1)^{-\frac{k+\ell}{2}} \nu^{-\frac{k}{2} - v_1} \overline{\nu}^{-\frac{\ell}{2} - v_2} \Phi(g\gamma_{\mathfrak{n}}).$$

If K has class number 1 and $\lambda = (k, k)$, then supposing $\mathfrak{n} = (\nu)$ we get the following explicit form of $W_{\mathfrak{n}}$ in the descent $\mathcal{F}$ of $\Phi \in \mathcal{M}_{(k,k)}(\Omega_0(\mathfrak{n}), \varphi)$ given by

$$\mathcal{F}|_{W_{\mathfrak{n}}} := (-1)^k |\nu|^k \mathcal{F}|_{\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}}, \tag{1.17}$$

and we call it the *Fricke involution.*

We can also in this case compute easily the action of $W_{\mathfrak{n}}$ on the components of $\mathcal{F}$:

Lemma 1.4. *Let $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}), \varphi_{\mathfrak{n}}^{-1})$, then for $0 \leqslant n \leqslant 2k+2$ we have*

$$(\mathcal{F}|_{W_{\mathfrak{n}}})_{2k+2-n}(0,t) = t^{-2k-2}(-1)^{k+n}\nu^{n-k-1}|\nu|^{-n}\mathcal{F}_n\left(0, \frac{1}{|\nu|t}\right).$$

Proof. Note that if $\gamma = \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}$ then $\gamma \cdot (0,t) = (0, 1/(|\nu|t))$ and

$$\begin{aligned}
\rho_{2k+2}^{-1}\left(J\left(\frac{\gamma}{\sqrt{det(\gamma)}}; (0,t)\right)\right)\binom{X}{Y}^{2k+2} &= \left(J\left(\frac{\gamma}{\sqrt{det(\gamma)}}; (0,t)\right)^{-1}\binom{X}{Y}\right)^{2k+2} \\
&= \left(\begin{pmatrix} 0 & -\nu^{-1/2}t^{-1} \\ \overline{\nu}^{-1/2}t^{-1} & 0 \end{pmatrix}\binom{X}{Y}\right)^{2k+2} \\
&= \begin{pmatrix} -\nu^{-1/2}t^{-1}Y \\ \overline{\nu}^{-1/2}t^{-1}X \end{pmatrix}^{2k+2}
\end{aligned}$$

where $\binom{X}{Y}^{2k+2} = (X^{2k+2}, X^{2k+1}Y, ..., X^{2k+2-n}Y^n, ..., XY^{2k+1}, Y^{2k+2})^t$.

Then,

$$\begin{aligned}
(\mathcal{F}|_{W_{\mathfrak{n}}})(0,t)\binom{X}{Y}^{2k+2} &= (-1)^k|\nu|^k\mathcal{F}|_{\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}}(0,t)\binom{X}{Y}^{2k+2} \\
&= (-1)^k\mathcal{F}(0, 1/(|\nu|t)) \cdot \begin{pmatrix} -\nu^{-1/2}t^{-1}Y \\ \overline{\nu}^{-1/2}t^{-1}X \end{pmatrix}^{2k+2}.
\end{aligned}$$

□

We finish this section with an important definition

Definition 1.22. *A Bianchi modular form Φ of level $\mathfrak{n}$ is called* ***newform*** *if it is an eigenform that is not induced from a Bianchi modular form with level strictly dividing $\mathfrak{n}$.*

Remark 1.11. *Notice that a Bianchi newform $\mathcal{F}$ is an eigenvector for the Fricke involution $W_{\mathfrak{n}}$ with $\mathcal{F}|_{W_{\mathfrak{n}}} = \epsilon(\mathfrak{n})\mathcal{F}$ for $\epsilon(\mathfrak{n}) = \pm 1$ (see section 2 in [11]).*

1.2.5 p-stabilisations

For Bianchi modular forms without level at a certain prime $\mathfrak{q}$, there exists a process called $\mathfrak{q}$-stabilisation that construct new Bianchi modular forms with level at $\mathfrak{q}$. In this section we explicitly describe this process for the special case of Bianchi modular forms of parallel weight and central action of trivial conductor, this specific case will be important for us when finding the functional equation of p-adic L-functions.

Let Φ be a Bianchi modular form of weight (k,k), level $\Omega_0(\mathfrak{n})$ with $\mathfrak{p}\nmid\mathfrak{n}$ and central action φ with infinity type $(-k,-k)$ and trivial conductor and recall the Hecke operators defined in section 1.2.4. Let $\lambda_\mathfrak{p}$ denote the $T_\mathfrak{p}$ eigenvalue of Φ, and let $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ denote the roots of the Hecke polynomial $X^2-\lambda_\mathfrak{p}X+N(\mathfrak{p})^{k+1}$.

Definition 1.23. *Let $\pi_\mathfrak{p}\in\mathcal{O}_K$ be a fixed uniformizer of $K_\mathfrak{p}$. Then, define the $\mathfrak{p}$-stabilisations of Φ to be*

$$\Phi^{\alpha_\mathfrak{p}}(g):=\Phi(g)-\alpha_\mathfrak{p}^{-1}\Phi\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)\quad \textit{and}\quad \Phi^{\beta_\mathfrak{p}}(g):=\Phi(g)-\beta_\mathfrak{p}^{-1}\Phi\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right),$$

where we see $\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\in\mathrm{GL}_2(K_\mathfrak{p})$ *as a matrix in* $\mathrm{GL}_2(\mathbb{A}_K^f)$ *putting 1 in the places outside* $\mathfrak{p}$.

Lemma 1.5. *The Bianchi modular forms $\Phi^{\alpha_\mathfrak{p}}$ and $\Phi^{\beta_\mathfrak{p}}$ are eigenforms of level $\Omega_0(\mathfrak{pn})$ with $U_\mathfrak{p}$-eigenvalues $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ respectively.*

Proof. The level is clear so we prove that $\Phi^{\alpha_\mathfrak{p}}$ has $U_\mathfrak{p}$-eigenvalue $\alpha_\mathfrak{p}$.

Since

$$\lambda_\mathfrak{p}\Phi(g)=\Phi|_{T_\mathfrak{p}}(g)=\Phi|_{U_\mathfrak{p}}(g)+\Phi\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right),$$

we have

$$\Phi|_{U_\mathfrak{p}}(g)=\lambda_\mathfrak{p}\Phi(g)-\Phi\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right). \tag{1.18}$$

On the other hand we have

$$\begin{aligned}\Phi|_{U_\mathfrak{p}}\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)&=\sum_{u \bmod \mathfrak{p}}\Phi\left(g\begin{pmatrix}\pi_\mathfrak{p}&u\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)=\sum_{u \bmod \mathfrak{p}}\Phi\left(g\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}\pi_\mathfrak{p}&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)\\&=\varphi_\mathfrak{p}(\pi_\mathfrak{p})\sum_{u \bmod \mathfrak{p}}\Phi\left(g\begin{pmatrix}1&u\\0&1\end{pmatrix}\right)=N(\mathfrak{p})^k\sum_{u \bmod \mathfrak{p}}\Phi(g)=N(\mathfrak{p})^{k+1}\Phi(g)\end{aligned} \tag{1.19}$$

where third equality uses property (ii) in Definition 1.12 with the character $\varphi_\mathfrak{p}$. In forth equality we use invariance of Φ by $\Omega_0(\mathfrak{n})$ and $\varphi_\infty(\pi_\mathfrak{p})\varphi_\mathfrak{p}(\pi_\mathfrak{p})=1$ then $\varphi_\mathfrak{p}(\pi_\mathfrak{p})=\varphi_\infty(\pi_\mathfrak{p})^{-1}=|\pi_\mathfrak{p}|^{2k}=N(\mathfrak{p})^k$, with $|\pi_\mathfrak{p}|$ the archimedean norm of $\pi_\mathfrak{p}$ as an element of K.

Applying $U_\mathfrak{p}$ using 1.18 and 1.19 we have

$$\begin{aligned}\Phi^{\alpha_\mathfrak{p}}|_{U_\mathfrak{p}}(g)&=\Phi|_{U_\mathfrak{p}}(g)-\alpha_\mathfrak{p}^{-1}\Phi|_{U_\mathfrak{p}}\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)\\&=\lambda_\mathfrak{p}\Phi(g)-\Phi\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)-\alpha_\mathfrak{p}^{-1}|\pi_\mathfrak{p}|^{2k+2}\Phi(g)\\&=(\lambda_\mathfrak{p}-\beta_\mathfrak{p})\Phi(g)-\Phi\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)\\&=\alpha_\mathfrak{p}\Phi(g)-\Phi\left(g\begin{pmatrix}1&0\\0&\pi_\mathfrak{p}\end{pmatrix}\right)=\alpha_\mathfrak{p}\Phi^{\alpha_\mathfrak{p}}(g).\end{aligned}$$

For $\Phi^{\beta_\mathfrak{p}}$ is analogous. $\square$

We can describe explicitly the descent of the $\mathfrak{p}$-stabilisation $\Phi^{\alpha_\mathfrak{p}}$ to $\mathcal{H}_3$, for example, in the case when K has class number 1 we proceed as follows:

Taking $g_\infty = \left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots\right) \in \mathrm{GL}_2(\mathbb{A}_K)$ with $t \in \mathbb{R}_{>0}$ and $z \in \mathbb{C}$ we define

$$\Phi^{\alpha_\mathfrak{p}}(g_\infty) := \Phi(g_\infty) - \alpha_\mathfrak{p}^{-1}\Phi\left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix} \dots\right).$$

Note that

$$\begin{aligned}
\Phi\left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix} \dots\right) &= \Phi\left(\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix}, \dots\right) \\
&= \Phi\left(\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix}, \dots\right) \\
&= \varphi_\mathfrak{p}(\pi_\mathfrak{p})\Phi\left(\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \dots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots\right) \\
&= |\pi_\mathfrak{p}|^{2k} F\left(\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right) = t|\pi_\mathfrak{p}|^{2k}\mathcal{F}|_{\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}}(z,t)
\end{aligned}$$

Where first equality follows by the left $\mathrm{GL}_2(K)$ invariance of Φ (Property (i) in Definition 2.1) with the matrix $\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}$. In second equality since φ has trivial conductor, Property (iii) in Definition 1.12 is just right $\Omega_0(\mathfrak{n})$ invariance of Φ and we use $\left(\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}, \dots \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots\right) \in \Omega_0(\mathfrak{n})$ where the place $\mathfrak{p}$ have the identity matrix. Third equality uses property (ii) in Definition 1.12 with the character $\varphi_\mathfrak{p}$. In forth equality we use $\varphi_\infty(\pi_\mathfrak{p})\varphi_\mathfrak{p}(\pi_\mathfrak{p}) = 1$ then $\varphi_\mathfrak{p}(\pi_\mathfrak{p}) = \varphi_\infty(\pi_\mathfrak{p})^{-1} = |\pi_\mathfrak{p}|^{2k}$ and the definition of F. Finally in the last equality we use (1.7) and 1) in Remark 1.7. Recalling $\alpha_\mathfrak{p}\beta_\mathfrak{p} = |\pi_\mathfrak{p}|^{2k+2}$ we have

$$\alpha_\mathfrak{p}^{-1}\Phi\left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix} \dots\right) = t|\pi_\mathfrak{p}|^{2k}\alpha_\mathfrak{p}^{-1}\mathcal{F}|_{\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}}(z,t) = t\frac{\beta_\mathfrak{p}}{|\pi_\mathfrak{p}|^2}\mathcal{F}|_{\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}}(z,t).$$

Now, let $\mathcal{F}^{\alpha_\mathfrak{p}}$ denote the descent of $\Phi^{\alpha_\mathfrak{p}}$ to $\mathcal{H}_3$ then

$$t\mathcal{F}^{\alpha_\mathfrak{p}}(z,t) = F^{\alpha_\mathfrak{p}}\left(\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right) = \Phi^{\alpha_\mathfrak{p}}(g_\infty) = t\mathcal{F}(z,t) - t\frac{\beta_\mathfrak{p}}{|\pi_\mathfrak{p}|^2}\mathcal{F}|_{\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}}(z,t),$$

obtaining

$$\mathcal{F}^{\alpha_\mathfrak{p}}(z,t) = \mathcal{F}(z,t) - \frac{\beta_\mathfrak{p}}{|\pi_\mathfrak{p}|^2}\mathcal{F}|_{\begin{pmatrix}\pi_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}}(z,t) = \mathcal{F}(z,t) - \beta_\mathfrak{p}\mathcal{G}(z,t), \tag{1.20}$$

where

$$\mathcal{G}(z,t) = |\pi_\mathfrak{p}|^{-2}\mathcal{F}|_{\begin{pmatrix}\pi_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}}(z,t).$$

Then, considering the descent of the $\mathfrak{p}$-stabilisation of a Bianchi modular form we define the $\mathfrak{p}$-stabilisations of a Bianchi eigenform $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}))$ to be

$$\mathcal{F}^{\alpha_\mathfrak{p}}(z,t) := \mathcal{F}(z,t) - \beta_\mathfrak{p}\mathcal{G}(z,t), \quad \mathcal{F}^{\beta_\mathfrak{p}}(z,t) := \mathcal{F}(z,t) - \alpha_\mathfrak{p}\mathcal{G}(z,t).$$

The p-stabilisations $\mathcal{F}^{\alpha_\mathfrak{p}}$ and $\mathcal{F}^{\beta_\mathfrak{p}}$ are Bianchi eigenforms of level $\Gamma_0(\mathfrak{p}\mathfrak{n})$ and $U_\mathfrak{p}$ eigenvalues $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$.

Lemma 1.6. *Let $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}))$ be a quasi-cuspidal Bianchi modular form, then any $\mathfrak{p}$-stabilisation of $\mathcal{F}$ is quasi-cuspidal.*

Proof. Since the matrix $\begin{pmatrix}\pi_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}$ on Bianchi modular forms does not move, sum or permute the components $\mathcal{F}_n$, then the constant terms at the position $1 \leqslant n \leqslant 2k+1$ remains trivial for all cusps. □

1.2.6 Twists

Given a Bianchi modular form $\mathcal{F}$ and a Hecke character ψ, there exists an operator that allow us to *twist* Φ by ψ and obtain a new Bianchi modular form.

Definition 1.24. *Let $\Phi \in \mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ and ψ be a Hecke character of conductor $\mathfrak{f}$. Define the twisting operator $R(\psi)$ by*

$$\Phi|R(\psi)(g) := \psi(det(g)) \sum_{[a]\in(\mathfrak{f}^{-1}/\mathcal{O}_K)^\times} \psi_\mathfrak{f}(a)\Phi\left(g\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\right), \quad \textit{for } g \in \mathrm{GL}_2(\mathbb{A}_K).$$

where we consider the unipotent matrix $\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}$ as an element in $\mathrm{GL}_2(\mathbb{A}_K^f)$.

Proposition 1.3. *Let $\Phi \in \mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ with $\lambda = [(k,\ell),(v_1,v_2)]$ where φ has infinity type $(-k-2v_1, -\ell-2v_2)$ and conductor dividing $\mathfrak{n}$ and let ψ be a Hecke character of infinity type (q,r) and conductor $\mathfrak{f}$. Then $\Phi|R(\psi) \in \mathcal{M}_\iota(\Omega_0(\mathfrak{m}), \varphi\psi^2)$ where $\iota = [(k,\ell),(v_1-q, v_2-r)]$ and $\mathfrak{m} = \mathfrak{n}\cap\mathfrak{f}^2$.*

Proof. See [20, §6, (6.7)]. □

Notation: For the rest of this section suppose that K has class number 1.

Remark 1.12. *Suppose* $\mathfrak{f} = (f)$*, if* $\mathcal{F}$ *is the descent of a Bianchi modular form* $\Phi \in \mathcal{M}_\lambda(\Omega_0(\mathfrak{n}), \varphi)$ *and we denote by* $\mathcal{F}_\psi$ *the descent of* $\Phi|R(\psi)$ *then by [20, (6.9)], we have*

$$\mathcal{F}_\psi = \sum_{b \in (\mathcal{O}_K/\mathfrak{f})^\times} \psi_\mathfrak{f}(b/f)\mathcal{F}|_{\begin{pmatrix} 1 & b/f \\ 0 & 1 \end{pmatrix}} = \psi_\infty(f) \sum_{b \in (\mathcal{O}_K/\mathfrak{f})^\times} \psi_\mathfrak{f}(b)\mathcal{F}|_{\begin{pmatrix} 1 & b/f \\ 0 & 1 \end{pmatrix}},$$

which is analogous to the twist of a modular form by a Dirichlet character in [22, Thm 7.4 (7.30)] up to a factor of a Gauss sum (and ψ_∞*) which then appears in the Fourier expansion of* $\mathcal{F}$ *(see [20, (6.8)]).*

Recall the Fricke involution defined in (1.17), then

Lemma 1.7. *If* $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}), \varphi_\mathfrak{n}^{-1})$ *then* $\mathcal{F}|_{W_\mathfrak{n}} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}), \varphi_\mathfrak{n})$*.*

Proof. Note that $\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}$ normalizes the group $\Gamma_0(\mathfrak{n})$, explicitly $\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}\gamma = \gamma'\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}$ where $\gamma' = \begin{pmatrix} d & -c/\nu \\ -b\nu & a \end{pmatrix}$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence, for $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}), \varphi_\mathfrak{n}^{-1})$ and $\gamma \in \Gamma_0(\mathfrak{n})$ we have

$$(\mathcal{F}|_{W_\mathfrak{n}})|_\gamma = (-1)^k|\nu|^k\mathcal{F}|_{\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}\gamma} = (-1)^k|\nu|^k\mathcal{F}|_{\gamma'\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}} = \varphi_\mathfrak{n}(\gamma')^{-1}\mathcal{F}|_{W_\mathfrak{n}} = \varphi_\mathfrak{n}(\gamma)\mathcal{F}|_{W_\mathfrak{n}},$$

where in last equality we use that $ad \equiv 1 \mod \mathfrak{n}$. □

Consider $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}), \varphi_\mathfrak{n}^{-1})$ and $\mathcal{F}_\psi \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{m}), (\varphi^{-1}\psi^{-2})_\mathfrak{m})$ its twist by ψ, then we can relate $\mathcal{F}|_{W_\mathfrak{n}}$ with $\mathcal{F}_\psi|_{W_\mathfrak{m}}$ by the following result.

Proposition 1.4. *Let* $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}), \varphi_\mathfrak{n}^{-1})$ *be a Bianchi modular form and let* ψ *be a Hecke character of conductor* $\mathfrak{f}$ *with* $(\mathfrak{n}, \mathfrak{f}) = 1$*. Then*

$$\mathcal{F}_\psi|_{W_\mathfrak{m}} = \varphi_\mathfrak{n}(f)^{-1}\psi_\mathfrak{f}(-\nu)^{-1}\psi_\infty(f)^2\,(\mathcal{F}|_{W_\mathfrak{n}})_{\psi^{-1}}\,.$$

for $(m) = \mathfrak{m} = \mathfrak{n}\mathfrak{f}^2 = (\nu)(f)^2$*,* $m = \nu f^2$*.*

Proof. Since for any v we have the identity

$$\begin{pmatrix} 1 & \frac{b}{f} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}\begin{pmatrix} f & -v \\ -b\nu & \frac{1+bv\nu}{f} \end{pmatrix}\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}$$

and choosing v such that $bv\nu \equiv -1 \pmod{f}$ to bring $\begin{pmatrix} f & -v \\ -b\nu & \frac{1+bv\nu}{f} \end{pmatrix}$ into $\Gamma_0(\mathfrak{n})$, then

$$\begin{aligned}
\mathcal{F}|_{\begin{pmatrix} 1 & \frac{b}{f} \\ 0 & 1 \end{pmatrix} W_{\mathfrak{m}}} &= (-1)^k |m|^k \mathcal{F}|_{\begin{pmatrix} 1 & \frac{b}{f} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix}} \\
&= (-1)^k |m|^k \mathcal{F}|_{\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}\begin{pmatrix} f & -v \\ -b\nu & \frac{1+bv\nu}{f} \end{pmatrix}\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}} \\
&= (-1)^k |m|^k |f^2|^{-k} \mathcal{F}|_{\begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}\begin{pmatrix} f & -v \\ -b\nu & \frac{1+bv\nu}{f} \end{pmatrix}\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}} \\
&= |m|^k |f|^{-2k} |\nu|^{-k} \left(\mathcal{F}|_{W_{\mathfrak{n}}}\right)|_{\begin{pmatrix} f & -v \\ -b\nu & \frac{1+bv\nu}{f} \end{pmatrix}\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}} \\
&= \left(\mathcal{F}|_{W_{\mathfrak{n}}}\right)|_{\begin{pmatrix} f & -v \\ -b\nu & \frac{1+bv\nu}{f} \end{pmatrix}\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}} = \varphi_{\mathfrak{n}}(f)^{-1} \left(\mathcal{F}|_{W_{\mathfrak{n}}}\right)|_{\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}}.
\end{aligned} \tag{1.21}$$

Where in the last equality we use that $m = \nu f^2$ and $\mathcal{F}|_{W_{\mathfrak{n}}} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}), \varphi_{\mathfrak{n}})$ by Lemma 1.7. Since we have that $bv\nu \equiv -1 \pmod{f}$, then $\psi_{\mathfrak{f}}(b) = \psi_{\mathfrak{f}}(-\nu)^{-1}\psi_{\mathfrak{f}}(v)^{-1}$. Now multiplying (1.21) by the latter and summing over the reduced residue class of $(\mathcal{O}_K/\mathfrak{f})^\times$ we obtain

$$\sum_{b \in (\mathcal{O}_K/\mathfrak{f})^\times} \psi_{\mathfrak{f}}(b) \mathcal{F}|_{\begin{pmatrix} 1 & b/f \\ 0 & 1 \end{pmatrix} W_{\mathfrak{m}}} = \varphi_{\mathfrak{n}}(f)^{-1}\psi_{\mathfrak{f}}(-\nu)^{-1} \sum_{v \in (\mathcal{O}_K/\mathfrak{f})^\times} \psi_{\mathfrak{f}}(v)^{-1} \left(\mathcal{F}|_{W_{\mathfrak{n}}}\right)|_{\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}}.$$

Multiplying by $\psi_\infty(f)$ in both sides and using Remark 1.12 we have

$$\mathcal{F}_\psi|_{W_{\mathfrak{m}}} = \varphi_{\mathfrak{n}}(f)^{-1}\psi_{\mathfrak{f}}(-\nu)^{-1}\psi_\infty(f) \sum_{v \in (\mathcal{O}_K/\mathfrak{f})^\times} \psi_{\mathfrak{f}}(v)^{-1} \left(\mathcal{F}|_{W_{\mathfrak{n}}}\right)|_{\begin{pmatrix} 1 & \frac{v}{f} \\ 0 & 1 \end{pmatrix}}.$$

Finally, since $\psi_{\mathfrak{f}}(v)^{-1} = \psi_\infty(f)(\psi_\infty(f)^{-1}\psi_{\mathfrak{f}}(v)^{-1}) = \psi_\infty(f)\psi_{\mathfrak{f}}(v/f)^{-1}$ we obtain the result. □

1.2.7 L-function of Bianchi modular forms

In an analogous way to Hecke characters, we can attach a complex L-function to a Bianchi modular form Φ by taking a Dirichlet series but this time using the Fourier coefficients of $\mathcal{F}$.

Definition 1.25. *Let ψ be a Hecke character over K of conductor $\mathfrak{f}$. We define the L-function of a Bianchi modular form Φ twisted by ψ as*

$$L(\Phi, \psi, s) := \sum_{\substack{0 \neq \mathfrak{a} \subset \mathcal{O}_K \\ (\mathfrak{a}, \mathfrak{f}) = 1}} c(\mathfrak{a}, \Phi)\psi(\mathfrak{a})N(\mathfrak{a})^{-s}.$$

Note that the constant term of Φ is not included in the sum.

For every Bianchi modular form $\mathcal{F}^i$ corresponding to Φ respect to I_i we assign a "part" of the above L-function:

Definition 1.26. *Let $w = |\mathcal{O}_K^\times|$. Define*

$$L^i(\Phi, \psi, s) = L(\mathcal{F}^i, \psi, s) := w^{-1} \sum_{\alpha \in K^\times} c(\alpha\delta I_i, \Phi)\psi(\alpha\delta I_i) N(\alpha\delta I_i)^{-s}$$

where here we scale by w^{-1} as when we sum over element of $K^\times$, we include each ideal w times (once for each unit).

Note that

$$L(\Phi, \psi, s) = L^1(\Phi, \psi, s) + \cdots + L^h(\Phi, \psi, s).$$

Remark 1.13. *In [40], it is proved that each component of the twisted L-function is a holomorphic function on a right half-plane.*

For C-cuspidal Bianchi modular forms, each component of the L-function can be written in terms of an integral formula.

Proposition 1.5. *Let $\Phi \in \mathcal{M}_{(k,\ell)}(\Omega_0(\mathfrak{n}))$ be a C-cuspidal Bianchi modular form with $\mathfrak{n} = (p)\mathfrak{m}$ and $\mathfrak{m}$ coprime to (p), then for a Hecke character ψ of K of conductor $\mathfrak{f}$ such that $(\mathfrak{f}, \mathfrak{m})$=1 and $(\mathfrak{f}, I_i) = 1$ for each i and infinity type $(u, v) = (-\frac{\ell+1-n}{2}, \frac{\ell+1-n}{2})$ with $1 \leqslant n \leqslant k + \ell + 1$, we have*

$$L^i(\Phi, \psi, s) = A(i, n, \psi, s) \left[\sum_{\substack{[a] \in \mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f}, \mathfrak{f})=1}} \psi_{\mathfrak{f}}(a) \int_0^\infty t^{2s-2} \mathcal{F}_n^i(a, t) dt \right], \tag{1.22}$$

where

$$A(i, n, \psi, s) = \psi(t_i)|t_i|_f^{s-1} \frac{4(2\pi)^{2s} \sqrt{-1}^{\ell+1-n} \binom{k+\ell+2}{n}^{-1}}{|\delta|^{2s} \Gamma(s + \frac{n-\ell-1}{2}) \Gamma(s - \frac{n-\ell-1}{2}) w \tau(\psi^{-1})}.$$

Proof. This is a generalization for weight (k, ℓ) of [41, Thm.1.8] where the result is stated for cuspidal Bianchi modular forms, in this case after applying the changes for the weight we have to be careful with the elements $a \in K$ that we take in order to ensure the convergence of the integrals.

For each $b \pmod{\mathfrak{f}}$ we can take an element $d_b \in \mathcal{O}_K$ such that $d_b \in I_1, I_2, ..., I_h$ and $d_b \equiv b \pmod{\mathfrak{f}}$ using the Chinese Remainder Theorem.

Let $\alpha_i \in K$ such that $\mathfrak{f}I_i = (\alpha_i)I_{j_i}$, note that $\alpha_i^{-1} \in \mathfrak{f}^{-1}I_i^{-1}I_{j_i} \subset \mathfrak{f}^{-1}I_i^{-1}$, so in particular as b ranges over all classes of $(\mathcal{O}_K/\mathfrak{f})^\times$ and as $d_b \in I_i$, we see that d_b/α_i ranges over a full set of coset representatives $[a]$ for $\mathfrak{f}^{-1}/\mathcal{O}_K$ with $(a)\mathfrak{f}$ coprime to $\mathfrak{f}$ (this relies on the fact that we are taking invertible elements $\pmod{\mathfrak{f}}$. Accordingly, the ideal $(d_b/\alpha_i) = \mathfrak{f}^{-1}J$, where J is coprime to $\mathfrak{f}$, and hence $d_b/\alpha_i \notin \mathcal{O}_K$. It is clear that if $b \neq b' \pmod{\mathfrak{f}}$ then d_b/α_i and $d_{b'}/\alpha_i$ define different classes in $\mathfrak{f}^{-1}/\mathcal{O}_K$).

By (1.16) we have that $\alpha_i = y_{\mathfrak{f}} y_i / t_{j_i}$ with $y_{\mathfrak{f}} \in (\mathcal{O}_K/\mathfrak{m})^\times$, $y_i \in (\mathcal{O}_K/\mathfrak{m})^\times$ and $t_{j_i} \in I_{j_i}$.

Since $y_{\mathfrak{f}} y_i \in (\mathcal{O}_K/\mathfrak{m})^\times$ and $d_b \in I_i$ we obtain

$$\frac{d_b}{\alpha_i} = \frac{d_b t_{j_i}}{y_{\mathfrak{f}} y_i} \in C_i.$$

Since Φ is C-cuspidal, then for all i, $\mathcal{F}^i_n$ vanish for all cusps in C_i for $1 \leqslant n \leqslant k+\ell+1$; then taking $a = d_b/\alpha_i$ as above for the representatives $[a]$ for $\mathfrak{f}^{-1}/\mathcal{O}_K$ with $(a)\mathfrak{f}$ coprime to $\mathfrak{f}$ we conclude that the integrals in 1.22 converges. □

Via meromorphic continuation, the integrals above gives the definition of the L-function on all $\mathbb{C}$.

It is convenient to think the twisted L-function as a function on Hecke characters instead as a complex function of one variable. Then we put

Definition 1.27. *Let Φ be a Bianchi modular form, and let ψ a Hecke character. Then define*

$$L(\Phi,\psi) = L(\Phi,\psi,1).$$

In the case when the Bianchi modular form is an eigenform, its Fourier coefficients coincide with its Hecke eigenvalues, then, the L-function of an eigenform is related to its Hecke eigenvalues, so that in a sense, the L-functions are built from local data at the finite primes (much like in the classical case, where the L-function of a Hecke eigenform has an Euler product). We *complete* the L-function by adding the appropriate factors at infinity.

Definition 1.28. *Let ψ be a Hecke character of infinity type (q,r). Define*

$$\Lambda(\Phi,\psi) := \frac{\Gamma(q+1)\Gamma(r+1)}{(2\pi i)^{q+1}(2\pi i)^{r+1}} L(\Phi,\psi).$$

where Γ is the usual Gamma function. This is the L-function renormalised by Deligne's Γ-factors at infinity.

Theorem 1.1. *Let $\Phi \in \mathcal{M}_{(k,\ell)}(\Omega_0(\mathfrak{n}))$ be a C-cuspidal Bianchi modular form with $\mathfrak{n} = (p)\mathfrak{m}$ and $\mathfrak{m}$ coprime to (p), then for a Hecke character ψ of K of conductor $\mathfrak{f}$ such that $(\mathfrak{f},\mathfrak{m}) = 1$ and $(\mathfrak{f}, I_i) = 1$ for each i and infinity type $0 \leqslant (q,r) \leqslant (k,\ell)$, we have*

$$\Lambda(\Phi,\psi) = \frac{(-1)^{q+1}4\psi_{\mathfrak{f}}(x_{\mathfrak{f}})\binom{k+\ell+2}{\ell+q-r+1}^{-1}}{\psi(x_{\mathfrak{f}})Dw\tau(\psi^{-1})} \sum_{i=1}^{h} \psi(t_i) \left[\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \psi_{\mathfrak{f}}(a) \int_0^\infty t^{q+r} \mathcal{F}^i_{\ell+q-r+1}(a,t)dt \right], \quad (1.23)$$

Proof. This is a generalization for weight (k,ℓ) of Theorem 2.11 in [41] for cuspidal Bianchi modular forms.

For the proof we first prove the analogue of Proposition 2.10 in [41] for C-cuspidal forms of weight (k,ℓ) using Proposition 1.5 and then we generalise section 2.6 in [41] to obtain the result. □

In the case when we are dealing with cuspidal Bianchi modular forms, the "critical" values of this L-function can be controlled; in particular, we have the following (See Theorem 8.1 in [20]):

Proposition 1.6. *There exists a period $\Omega_\Phi \in \mathbb{C}^\times$ and a number field E such that, if ψ is a Hecke character of infinity type $0 \leqslant (q,r) \leqslant (k,k)$, with $q,r \in \mathbb{Z}$, we have*

$$\frac{\Lambda(\Phi,\psi)}{\Omega_\Phi} \in E(\psi),$$

where $E(\psi) \subset \overline{\mathbb{Q}}$ is the extension of E generated by the values of ψ.

In general, we do not have this result for non-cuspidal Bianchi modular forms, but finding such a period in the non-cuspidal case is related with our current work (see Remark 4.1). Nevertheless depending on the specific Bianchi modular forms we are interested we can prove algebraicity of critical L-values "by hand" (as in Proposition 1.8 in next sections).

1.2.8 Functional equation of the L-function

In this section we obtain the functional equation of the L-function of quasi-cuspidal Bianchi automorphic newforms of parallel weight, also K has class number 1.

By Remark 1.10, Theorem 1.1 holds for quasi-cuspidal Bianchi modular forms with level at (p), in fact, we can obtain a more general result for L-functions of quasi-cuspidal Bianchi modular forms twisted by a character ψ without restrictions on the level of $\mathcal{F}$ and the conductor of ψ. We state the result for parallel weight and class number 1.

Theorem 1.2. *Let $\mathcal{F}$ be a quasi-cuspidal Bianchi modular form of weight (k,k) and level $\Gamma_0(\mathfrak{n})$ then for a Hecke character ψ of conductor $\mathfrak{f} = (f)$ and infinity type $0 \leqslant (q,r) \leqslant (k,k)$ we have*

$$\Lambda(\mathcal{F},\psi) = \frac{(-1)^{q+1}4}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}} \sum_{b \in (\mathcal{O}_K/f)^\times} \psi_{\mathfrak{f}}(b/f) \int_0^\infty t^{q+r} \mathcal{F}_{k+q-r+1}(b/f,t)dt$$

Using the integral form of the L function of a Bianchi modular form and the Fricke involution we can obtain the functional equation of the L-function.

Theorem 1.3. *Let $\mathcal{F} \in \mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}))$ be a quasi-cuspidal newform with $\mathfrak{n} = (\nu)$ and let ψ be a Hecke character of K of conductor $\mathfrak{f} = (f)$ with $(f,\nu) = 1$ and infinity type $0 \leqslant (q,r) \leqslant (k,k)$, the L-function of $\mathcal{F}$ satisfies the following functional equation*

$$\Lambda(\mathcal{F},\psi) = \frac{(-1)^{k+1}\epsilon(\mathfrak{n})|\nu|^k\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})}{\psi_{\mathfrak{f}}(-\nu)\psi_\infty(-\nu)\tau(\psi^{-1})}\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}).$$

Proof. By Theorem 1.2 we know that for a Hecke character ψ of K of conductor $\mathfrak{f} = (f)$ and infinity type $0 \leqslant (q,r) \leqslant (k,k)$, we have

$$\begin{aligned}\Lambda(\mathcal{F},\psi) &= \frac{(-1)^{q+1}4}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}} \sum_{b \in (\mathcal{O}_K/f)^\times} \psi_{\mathfrak{f}}(b/f) \int_0^\infty t^{q+r} \mathcal{F}_{k+q-r+1}(b/f,t)dt \\ &= \frac{(-1)^{q+1}4}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}} \int_0^\infty t^{q+r} \left[\sum_{b \in (\mathcal{O}_K/f)^\times} \psi_{\mathfrak{f}}(b/f)\mathcal{F}_{k+q-r+1}(b/f,t)\right]dt\end{aligned}$$

$$= \frac{(-1)^{q+1}4}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}} \int_0^\infty t^{q+r}\mathcal{F}_{\psi,k+q-r+1}(0,t)dt,$$

where last equality comes from Remark 1.12.
Changing variable $t \to 1/(|m|t)$ we have

$$\Lambda(\mathcal{F},\psi) = \frac{(-1)^{q+1}4|m|^{-q-r-1}}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}} \int_0^\infty t^{-q-r-2}\mathcal{F}_{\psi,k+q-r+1}(0,1/(|m|t))dt.$$

Recall that $\mathcal{F}_\psi$ has weight $[(k,k),(-q,-r)]$ then by Lemma 1.4

$$(\mathcal{F}_\psi|_{W_\mathfrak{m}})_{k-q+r+1}(0,t) = t^{-2k-2}(-1)^{q-r+1}m^{q-r}|m|^{-(k+q-r+1)}\mathcal{F}_{\psi,k+q-r+1}(0,1/(|m|t))$$

and replacing $\mathcal{F}_{\psi,k+q-r+1}(0,1/(|m|t))$ above we have

$$\begin{aligned}\Lambda(\mathcal{F},\psi) &= \frac{(-1)^{q+1}4|m|^{-q-r-1}}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}} \\ &\times \int_0^\infty t^{-q-r-2}\left[t^{2k+2}(-1)^{q-r+1}m^{-q+r}|m|^{k+q-r+1}(\mathcal{F}_\psi|_{W_\mathfrak{m}})_{k-q+r+1}(0,t)\right]dt. \\ &= \frac{(-1)^{r}4m^{-q+r}|m|^{k-2r}}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}} \int_0^\infty t^{2k-q-r}(\mathcal{F}_\psi|_{W_\mathfrak{m}})_{k-q+r+1}(0,t)dt.\end{aligned}$$

By Proposition 1.4 since $(\nu,f)=1$ we have $\mathcal{F}_\psi|_{W_\mathfrak{m}} = \psi_\mathfrak{f}(-\nu)^{-1}\psi_\infty(f)^2(\mathcal{F}|_{W_\mathfrak{n}})_{\psi^{-1}}$, also $\mathcal{F}$ is a newform, then $\mathcal{F}_\psi|_{W_\mathfrak{m}} = \epsilon(\mathfrak{n})\psi_\mathfrak{f}(-\nu)^{-1}\psi_\infty(f)^2\mathcal{F}_{\psi^{-1}}$ and we have

$$\begin{aligned}\Lambda(\mathcal{F},\psi) &= \frac{(-1)^{r}4m^{-q+r}|m|^{k-2r}\epsilon(\mathfrak{n})\psi_\infty(f)^2}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}\psi_\mathfrak{f}(-\nu)} \int_0^\infty t^{2k-q-r}\mathcal{F}_{\psi^{-1},k-q+r+1}(0,t)dt \\ &= \frac{(-1)^{r}4m^{-q+r}|m|^{k-2r}\epsilon(\mathfrak{n})\psi_\infty(f)^2}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}\psi_\mathfrak{f}(-\nu)} \int_0^\infty t^{2k-q-r}\left[\sum_{b\in(\mathcal{O}_K/f)^\times}\psi_\mathfrak{f}^{-1}(b/f)\mathcal{F}_{k-q+r+1}(b/f,t)\right]dt \\ &= \frac{(-1)^{r}4m^{-q+r}|m|^{k-2r}\epsilon(\mathfrak{n})\psi_\infty(f)^2}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}\psi_\mathfrak{f}(-\nu)} \sum_{b\in(\mathcal{O}_K/f)^\times}\psi_\mathfrak{f}^{-1}(b/f)\int_0^\infty t^{2k-q-r}\mathcal{F}_{k-q+r+1}(b/f,t)dt.\end{aligned}$$

Now note that the integral above is exactly the integral appearing on the coefficient $c_{k-q,k-r}(b/f)$, more explicitly

$$c_{k-q,k-r}(b/f) = \frac{2(-1)^{r+1}}{\binom{2k+2}{k-q+r+1}} \int_0^\infty t^{2k-q-r}\mathcal{F}_{k-q+r+1}(b/f,t)dt$$

and since $\binom{2k+2}{k-q+r+1} = \binom{2k+2}{k+q-r+1}$, we have

$$\Lambda(\mathcal{F},\psi) = \frac{(-1)2m^{-q+r}|m|^{k-2r}\epsilon(\mathfrak{n})\psi_\infty(f)^2}{Dw\tau(\psi^{-1})\psi_\mathfrak{f}(-\nu)} \sum_{b\in(\mathcal{O}_K/f)^\times}\psi_\mathfrak{f}^{-1}(b/f)c_{k-q,k-r}(b/f).$$

The Hecke character $\psi^{-1}|\cdot|_{\mathbb{A}_K}^k$ has conductor $\mathfrak{f}$ and infinity type $(k-q,k-r)$ and we know by Theorem 1.1 that

$$\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^k) = \frac{(-1)^{k+q+r}2}{Dw\tau((\psi^{-1}|\cdot|_{\mathbb{A}_K}^k)^{-1})} \sum_{b\in(\mathcal{O}_K/f)^\times}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^k)_\mathfrak{f}(b/f)c_{k-q,k-r}(b/f)$$

$$= \frac{(-1)^{k+q+r}2|f|^{2k}}{Dw\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})} \sum_{b\in(\mathcal{O}_K/f)^\times} \psi_{\mathfrak{f}}^{-1}(b/f)c_{k-q,k-r}(b/f).$$

Finally we obtain

$$\begin{aligned}\Lambda(\mathcal{F},\psi) &= \frac{(-1)2m^{-q+r}|m|^{k-2r}\epsilon(\mathfrak{n})\psi_\infty(f)^2}{Dw\tau(\psi^{-1})\psi_{\mathfrak{f}}(-\nu)}\left[\frac{(-1)^{k+q+r}2|f|^{2k}}{Dw\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})}\right]^{-1}\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\\ &= \frac{(-1)^{k+q+r+1}m^{-q+r}|m|^{k-2r}\epsilon(\mathfrak{n})\psi_\infty(f)^2\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})}{\tau(\psi^{-1})\psi_{\mathfrak{f}}(-\nu)|f|^{2k}}\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\\ &= \frac{(-1)^{k+1}\epsilon(\mathfrak{n})|\nu|^k\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})}{\psi_{\mathfrak{f}}(-\nu)\psi_\infty(-\nu)\tau(\psi^{-1})}\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}),\end{aligned}$$

where we used that

$$\begin{aligned}\frac{m^{-q+r}|m|^{k-2r}\psi_\infty(f)^2}{|f|^{2k}} &= \frac{|m|^k\psi_\infty^{-1}(m)\psi_\infty(f^2)}{|f|^{2k}} = \frac{|\nu f^2|^k\psi_\infty^{-1}(\nu f^2)\psi_\infty(f^2)}{|f|^{2k}}\\ &= |\nu|^k\psi_\infty^{-1}(\nu)\end{aligned}$$

and $(-1)^{q+r}\psi_\infty^{-1}(\nu) = \psi_\infty^{-1}(-\nu)$. □

Remark 1.14. *The theorem above generalizes the functional equation obtained in [11, Prop 2.1] for the untwisted L-function of a cuspidal Bianchi modular form of weight* $(0,0)$*. Also note that Theorem 1.3 is not a new result, but rather a reformulation of a classical result in [23].*

For the p-adic setting we work with the L-function of a Bianchi modular form with level at p twisted by Hecke characters ψ with conductor $\mathfrak{f}|p^\infty$, then Theorem 1.3 no longer holds.

Fortunately, the L-function of a Bianchi modular form $\mathcal{F}$ of level coprime to a prime $\mathfrak{p}$ and the L-function of a $\mathfrak{p}$-stabilisation $\mathcal{F}^{\alpha_{\mathfrak{p}}}$ are related, in fact, if we define for a Hecke character χ of conductor $\mathfrak{f}$ and $\epsilon \in \mathbb{C}^\times$ the factor

$$Z_{\mathfrak{p}}^{\epsilon}(\chi) := \begin{cases} 1-\epsilon^{-1}\chi(\mathfrak{p})^{-1} & : \mathfrak{p}\nmid\mathfrak{f},\\ 1 & : \text{otherwise},\end{cases} \tag{1.24}$$

we have the following:

Lemma 1.8. *Let ψ be a Hecke character with conductor $\mathfrak{f}$. We have for $\epsilon \in \{\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}}\}$*

$$\Lambda(\mathcal{F}^{\epsilon},\psi) = Z_{\mathfrak{p}}^{\epsilon}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\Lambda(\mathcal{F},\psi),$$

Proof. Recall the definition of the $\mathfrak{p}$-stabilisation of $\mathcal{F}$ in (1.20).

(i) We first compute $\mathcal{G}_n$: note that if $\gamma = \begin{pmatrix}\pi_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}$ then $\gamma\cdot(z,t) = (\pi_{\mathfrak{p}}z, |\pi_{\mathfrak{p}}|t)$ and analogously to Lemma 1.4 we have

$$\rho_{2k+2}^{-1}\left(J\left(\frac{\gamma}{\sqrt{det(\gamma)}};(0,t)\right)\right)\begin{pmatrix}X\\ Y\end{pmatrix}^{2k+2} = \begin{pmatrix}\pi_{\mathfrak{p}}^{1/2}X\\ \overline{\pi}_{\mathfrak{p}}^{1/2}Y\end{pmatrix}^{2k+2}, \quad \text{then}$$

$$\mathcal{G}(z,t)\binom{X}{Y}^{2k+2} = |\pi_{\mathfrak{p}}|^{-2}\mathcal{F}|_{\begin{pmatrix}\pi_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}}(z,t)\binom{X}{Y}^{2k+2} = |\pi_{\mathfrak{p}}|^{-k-2}\mathcal{F}(\pi_{\mathfrak{p}}z,|\pi_{\mathfrak{p}}|t)\begin{pmatrix}\pi_{\mathfrak{p}}^{1/2}X\\ \overline{\pi_{\mathfrak{p}}}^{1/2}Y\end{pmatrix}^{2k+2}.$$

Computing the n-th component we have

$$\mathcal{G}_n(z,t) = |\pi_{\mathfrak{p}}|^{-1}\mathcal{F}_n(\pi_{\mathfrak{p}}z,|\pi_{\mathfrak{p}}|t)\left(\frac{\pi_{\mathfrak{p}}}{|\pi_{\mathfrak{p}}|}\right)^{k+1-n}. \tag{1.25}$$

(ii) Now, we relate $\Lambda(\mathcal{G},\psi)$ and $\Lambda(\mathcal{F},\psi)$ when $\mathfrak{p}\nmid\mathfrak{f}$ ($\Lambda(\mathcal{G},\psi)=0$ if $\mathfrak{p}|\mathfrak{f}$): by Theorem 1.1 and (1.25) if ψ has infinity type $0\leqslant(q,r)\leqslant(k,k)$ we have

$$\begin{aligned}\Lambda(\mathcal{G},\psi) &= (*)\sum_{b\in(\mathcal{O}_K/f)^\times}\psi_{\mathfrak{f}}(b/f)\int_0^\infty t^{q+r}\mathcal{G}_{k+q-r+1}(b/f,t)dt\\ &= (*)\sum_{b\in(\mathcal{O}_K/f)^\times}\psi_{\mathfrak{f}}(b/f)\int_0^\infty t^{q+r}\left[|\pi_{\mathfrak{p}}|^{-1}\mathcal{F}_{k+q-r+1}(\pi_{\mathfrak{p}}b/f,|\pi_{\mathfrak{p}}|t)\left(\frac{\pi_{\mathfrak{p}}}{|\pi_{\mathfrak{p}}|}\right)^{-q+r}\right]dt\end{aligned}$$

where $(*)=\frac{(-1)^{q+1}4}{Dw\tau(\psi^{-1})\binom{2k+2}{k+q-r+1}}$.

Changing variable $t\to|\pi_{\mathfrak{p}}|^{-1}t$ we obtain

$$\begin{aligned}\Lambda(\mathcal{G},\psi) &= \psi_\infty(\pi_{\mathfrak{p}})^{-1}N(\mathfrak{p})^{-1}(*)\sum_{b\in(\mathcal{O}_K/f)^\times}\psi_{\mathfrak{f}}(b/f)\int_0^\infty t^{q+r}\mathcal{F}_{k+q-r+1}(\pi_{\mathfrak{p}}b/f,t)dt\\ &= \psi_{\mathfrak{f}}(\pi_{\mathfrak{p}})^{-1}\psi_\infty(\pi_{\mathfrak{p}})^{-1}N(\mathfrak{p})^{-1}(*)\sum_{b\in(\mathcal{O}_K/f)^\times}\psi_{\mathfrak{f}}(b/f)\int_0^\infty t^{q+r}\mathcal{F}_{k+q-r+1}(b/f,t)dt\\ &= \psi(\mathfrak{p})N(\mathfrak{p})^{-1}\Lambda(\mathcal{F},\psi),\end{aligned}$$

where in second equality we change $\pi_{\mathfrak{p}}b\to b$ since $\pi_{\mathfrak{p}}\nmid f$ and in last equality we use $\psi_{\mathfrak{f}}(\pi_{\mathfrak{p}})^{-1}\psi_\infty(\pi_{\mathfrak{p}})^{-1}=\psi_{\mathfrak{p}}(\pi_{\mathfrak{p}})=\psi(\mathfrak{p})$. Finally we obtain

$$\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}}},\psi)=\Lambda(\mathcal{F},\psi)-\frac{N(\mathfrak{p})^{k+1}}{\alpha_{\mathfrak{p}}}\Lambda(\mathcal{G},\psi)=\begin{cases}\left(1-\frac{\psi(\mathfrak{p})N(\mathfrak{p})^k}{\alpha_{\mathfrak{p}}}\right)\Lambda(\mathcal{F},\psi) & \text{if } \mathfrak{p}\nmid\mathfrak{f}\\ \Lambda(\mathcal{F},\psi) & \text{otherwise.}\end{cases}$$

Noting that

$$\psi(\mathfrak{p})N(\mathfrak{p})^k=(\psi^{-1}(\mathfrak{p})N(\mathfrak{p})^{-k})^{-1}=(\psi^{-1}(\mathfrak{p})|x_{\mathfrak{p}}|_{\mathbb{A}_K}^k)^{-1}.$$

Where $x_{\mathfrak{p}}$ is the idele associated to $\mathfrak{p}$. We have

$$\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}}},\psi)=Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^k)\Lambda(\mathcal{F},\psi)$$

obtaining the result for the Λ-function of $\mathcal{F}^{\alpha_{\mathfrak{p}}}$; for $\mathcal{F}^{\beta_{\mathfrak{p}}}$ is analogous. □

Let $\mathcal{F}_p$ be a Bianchi modular form obtained by successively stabilising at each different prime $\mathfrak{p}$ above p a newform $\mathcal{F}\in\mathcal{M}_{(k,k)}(\Gamma_0(\mathfrak{n}))$, with $\mathfrak{n}=(\nu)$ prime to (p).

Lemma 1.9. *For any Hecke character ψ of conductor $\mathfrak{f}=(f)$ with $(f,\nu)=1$ and infinity type*

$0 \leqslant (q,r) \leqslant (k,k)$ *we have*

$$\left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)\right)\Lambda(\mathcal{F}_p,\psi) = \varepsilon(\mathcal{F},\psi)\left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\right)\Lambda(\mathcal{F}_p,\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}),$$

where $\varepsilon(\mathcal{F},\psi) = \left[\frac{(-1)^{k+1}\epsilon(\mathfrak{n})|\nu|^k\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})}{\psi_{\mathfrak{f}}(-\nu)\psi_\infty(-\nu)\tau(\psi^{-1})}\right]$ *and* $\alpha_{\mathfrak{p}}$ *are the* $U_{\mathfrak{p}}$*-eigenvalues of* $\mathcal{F}_p$ *for each* $\mathfrak{p}|p$*.*

Proof. By Theorem 1.3 we have

$$\Lambda(\mathcal{F},\psi) = \varepsilon(\mathcal{F},\psi)\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{-k}) \tag{1.26}$$

Now let $\mathfrak{p}$ be a prime over p, if we define $\mathcal{F}^{\alpha_{\mathfrak{p}}}$ as a $\mathfrak{p}$–stabilisation of $\mathcal{F}$, we obtain by Lemma 1.8 the following relations between the Λ-function of $\mathcal{F}^{\alpha_{\mathfrak{p}}}$ and $\mathcal{F}$

$$\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}}},\psi) = Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\Lambda(\mathcal{F},\psi), \tag{1.27}$$

$$\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}}},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}) = Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}). \tag{1.28}$$

Putting (1.27), (1.26) and (1.28) together

$$\begin{aligned} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}}},\psi) &= Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\Lambda(\mathcal{F},\psi) \\ &= Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\varepsilon(\mathcal{F},\psi)\Lambda(\mathcal{F},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}) \\ &= \varepsilon(\mathcal{F},\psi)Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}}},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}). \end{aligned}$$

Note that if p is inert or ramified we are done and $\mathcal{F}_p = \mathcal{F}^{\alpha_{\mathfrak{p}}}$. If p split we have to do one more stabilisation, let $\bar{\mathfrak{p}}$ be the other prime above p.

If we define $\mathcal{F}^{\alpha_{\mathfrak{p}},\alpha_{\bar{\mathfrak{p}}}}$ as the $\bar{\mathfrak{p}}$–stabilisation of $\mathcal{F}^{\alpha_{\mathfrak{p}}}$ and doing the same process above, we obtain

$$\left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)\right)\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}},\alpha_{\bar{\mathfrak{p}}}},\psi) = \varepsilon(\mathcal{F},\psi)\left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\right)\Lambda(\mathcal{F}^{\alpha_{\mathfrak{p}},\alpha_{\bar{\mathfrak{p}}}},\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}).$$

Putting $\mathcal{F}_p = \mathcal{F}^{\alpha_{\mathfrak{p}},\alpha_{\bar{\mathfrak{p}}}}$ when p split we obtain the result. □

1.3 Automorphic forms of $\mathrm{GL}_2(K)$ from $\mathrm{GL}_2(\mathbb{Q})$

Let K be an imaginary quadratic extension of $\mathbb{Q}$ as usual, then the *base change lifting* gives a map from automorphic forms for $\mathrm{GL}_2(\mathbb{Q})$ to automorphic forms for $\mathrm{GL}_2(K)$. This base change lifting is a special case of Langlands functoriality.

The behaviour of base change Bianchi modular forms regarding cuspidality depend of a property called *complex multiplication* of the classical modular forms to be lifted.

1.3.1 CM modular forms

Recall that $-D$ is the discriminant of K and let χ_K be the Kronecker character of K, that is, the associated quadratic character of conductor D.

Definition 1.29. *A cuspidal newform $f = \sum_{n\geqslant 1} a_n q^n$ is said to have complex multiplication (CM) by K if it admits a self twist by the Kronecker character χ_K of K, that is, if $a_q = \chi_K(q)a_q$ for all but finitely many primes q.*

There is a connection between modular forms with complex multiplication (CM) and Hecke characters of an imaginary quadratic field.

Let φ be a Hecke character of K of conductor $\mathfrak{m}$ and infinity type $(-k-1,0)$ with $k \geqslant 0$, then the inverse Mellin transform f_φ of $L(\varphi, s)$ is known to be an eigenform:

$$f_\varphi(z) = \sum_{n=1}^{\infty} a_n q^n = \sum_{\mathfrak{a} \text{ integral}} \varphi(\mathfrak{a}) q^{N(\mathfrak{a})}, \text{where } q = e^{2\pi i z},\ z \in \mathbb{C},\ Im(z) > 0.$$

In particular, if we define $\eta_{\mathbb{Z}}$ to be the Dirichlet character modulo $M = N(\mathfrak{m})$ given by

$$\eta_{\mathbb{Z}} : a \mapsto \frac{\varphi(a\mathcal{O}_K)}{a^{k+1}}, \text{with } a \in \mathbb{Z}, (a, M) = 1. \tag{1.29}$$

then:

Theorem 1.4. ***(Hecke, Shimura)*** *f_φ is a newform of weight $k+2$, level DM and nebentypus character $\chi_K \eta_{\mathbb{Z}}$.*

$$f_\varphi \in S_{k+2}(\Gamma_0(DM), \chi_K \eta_{\mathbb{Z}}).$$

On the other hand, any newform with CM comes from a Hecke character by [34, Prop 4.4, Thm 4.5].

1.3.2 Base change Bianchi modular forms

Let f be a classical cuspidal newform of weight $k+2$, level $\Gamma_0(N)$ and nebentypus ϵ_f, as was mentioned previously, there is a process of *base change lift* from $\mathbb{Q}$ to K which constructs from f, a Bianchi modular form $f_{/K}$. Lifting may be described in the language of automorphic representations, or in the more classical language of automorphic forms.

Let π be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f and let $\mathrm{BC}(\pi)$ be the base change of π to $\mathrm{GL}_2(\mathbb{A}_K)$ (see [26]). The *base change* of f to K is the normalised new vector $f_{/K}$ in $\mathrm{BC}(\pi)$ which is a Bianchi modular form of weight (k,k) and level $\mathfrak{n} \subset \mathcal{O}_K$ with $\frac{N}{(N,D)}\mathcal{O}_K | \mathfrak{n} | N\mathcal{O}_K$ (see section 2.3 in [17]).

Remark 1.15. *The Hecke eigenvalues of $f_{/K}$ are determined from those of f. For every prime q not dividing the level of f, if the eigenvalue of T_q on f is a_q then the eigenvalue $a_{\mathfrak{q}}$ for $\mathfrak{q} \mid q$ of its base-change are given by the following:*

i) if q splits in K as $q\mathcal{O}_K = \mathfrak{q}\overline{\mathfrak{q}}$ then $a_{\mathfrak{q}} = a_{\overline{\mathfrak{q}}} = a_q$;

ii) if q ramifies in K as $q\mathcal{O}_K = \mathfrak{q}^2$ then $a_{\mathfrak{q}} = a_q$;

iii) if q is inert in K with $q\mathcal{O}_K = \mathfrak{q}$ then $a_{\mathfrak{q}} = a_q^2 - 2q^{k+1}$.

When f does not have CM by K, its base change to K is cuspidal; on the other hand when f has CM by K, then the base change to K is known to be non-cuspidal. In the non-cuspidal case we need information about the constant terms at the cusps, to obtain such information we use the classical lifting using automorphic forms described in [17].

Although the base change to K of a CM modular form is non-cuspidal, using the classical lifting stated above, we can still prove a nice property of vanishing for the Bianchi modular forms resulting.

Proposition 1.7. *Let $f_{/K}$ be the base change to K of a modular form f of weight $k+2$ with CM by K, then $f_{/K}$ is quasi-cuspidal.*

Proof. We give briefly the details of how to check the C-cuspidality of $f_{/K}$, all the ingredients required are contained in Section 3.1 in [17]. First note that by (1.5), the Bianchi modular form $f_{/K}$ give us h forms $F^i : \mathrm{GL}_2(\mathbb{C}) \to V_{2k+2}(\mathbb{C})$ and since the argument to prove the proposition works for each i, we pick one of these functions $F^i = (F_0^i, ..., F_{2k+2}^i)$, henceforth denoting it simply by $F = (F_0, ..., F_{2k+2})$. In Theorem 3.1 in [17] is stated the Fourier expansion at ∞ of each element $F_n(t^{-1/2}\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right))$ for $0 \leqslant n \leqslant 2k+2$, in such expansion only the constant term of F_0 and F_{2k+2} can be non-trivial and therefore denoting by $\mathcal{F}$ the descent of F to $\mathcal{H}_3$, we have by (1.7) that $\mathcal{F}(z,t) = t^{1-k}F(t^{-1/2}\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right))$ quasi-vanishes at the cusp ∞.

In general, the paragraph before Corollary 3.3 in [17] give us the Fourier expansion of F_n at any cusp, noting that in notation of op.cit. we have $F(\gamma g, V, \mathcal{L}_1) = F(g, V^\gamma, \mathcal{L}_1^\gamma)$. We observe that in Theorem 3.1 in [17] the constant term does not depend of V or $\mathcal{L}$, then we have the same situation as before in the cusp of ∞ for every cusp, i.e., only the constant term of F_0 and F_{2k+2} can be non-trivial. Then $\mathcal{F}$ quasi-vanishes for all cusps. Then for each i we have that $\mathcal{F}^i$ quasi-vanishes for all cusps, then $f_{/K}$ is quasi-cuspidal. $\square$

1.3.3 Base change of p-stabilisations

The natural object to attach p-adic L-functions are p-stabilisations, in this short section we suppose p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$, and we are interested in a p-stabilisation of $f_{\varphi/K}$ that satisfies $v(\lambda_\mathfrak{p}) < k+1$ for $\lambda_\mathfrak{p}$ the eigenvalue of $U_\mathfrak{p}$ for all $\mathfrak{p}|p$ (this will become clear with Definition 2.12), such p-stabilisation can be explicitly described from the p-stabilisations of f_φ as follows.

Suppose f_φ does not has level at p, and consider its Hecke polynomial at p given by $x^2 - a_p(f_\varphi)x + \epsilon_{f_\varphi}(p)p^{k+1}$.

Since $p = \mathfrak{p}\overline{\mathfrak{p}}$ we have

$$a_p(f_\varphi) = \varphi(\mathfrak{p}) + \varphi(\overline{\mathfrak{p}})$$

and

$$\epsilon_{f_\varphi}(p) = \chi_K(p)\eta_\mathbb{Z}(p) = \varphi(p\mathcal{O}_K)/p^{k+1} = \varphi(\mathfrak{p}\overline{\mathfrak{p}})/p^{k+1}.$$

Then the roots are $\alpha_p = \varphi(\mathfrak{p})$ and $\beta_p = \varphi(\overline{\mathfrak{p}})$.

On the other hand, since by Remark 1.15 we have

$$a_{\mathfrak{p}}(f_{\varphi/K}) = a_{\overline{\mathfrak{p}}}(f_{\varphi/K}) = a_p(f_\varphi)$$

and for all $\mathfrak{p}|p$ the nebentypus of $f_{\varphi/K}$ satisfies

$$\epsilon_{f_{\varphi/K}}(\mathfrak{p}) = \chi_K(N(\mathfrak{p}))\eta_{\mathbb{Z}}(N(\mathfrak{p})) = \varphi(p\mathcal{O}_K)/p^{k+1} = \epsilon_{f_\varphi}(p)$$

then the Hecke polynomials of $f_{\varphi/K}$ at $\mathfrak{p}|p$ and f_φ at p are equal.

Then for all $\mathfrak{p}|p$ we can take the roots of the Hecke polynomial of $f_{\varphi/K}$ at $\mathfrak{p}$ to be $\alpha_{\mathfrak{p}} = \alpha_p = \varphi(\mathfrak{p})$ and $\beta_{\mathfrak{p}} = \beta_p = \varphi(\overline{\mathfrak{p}})$.

If f_φ^α (resp. f_φ^β) is the p-stabilisation of f_φ corresponding to α_p (resp. β_p), we define its base change to K to be the p-stabilisation $f_{\varphi/K}^{\alpha\alpha}$ (resp. $f_{\varphi/K}^{\beta\beta}$) of $f_{\varphi/K}$ corresponding to $\alpha_{\mathfrak{p}}$ (resp. $\beta_{\mathfrak{p}}$) for all $\mathfrak{p}|p$.

Remark 1.16. *Note that*

$$\varphi(\mathfrak{p}) := \varphi_{\mathfrak{p}}(\pi_{\mathfrak{p}}) = \varphi_{\mathfrak{m}}(\pi_{\mathfrak{p}})^{-1}\varphi_\infty(\pi_{\mathfrak{p}})^{-1} = \varphi_{\mathfrak{m}}(\pi_{\mathfrak{p}})^{-1}\pi_{\mathfrak{p}}^{k+1},$$

then we have that $v_p(\alpha_{\mathfrak{p}}) = v_p(\alpha_{\overline{\mathfrak{p}}}) = k+1$*, on the other hand,* $\varphi(\overline{\mathfrak{p}}) = \varphi_{\mathfrak{m}}(\overline{\pi_{\mathfrak{p}}})^{-1}\overline{\pi_{\mathfrak{p}}}^{k+1}$ *then* $v_p(\beta_{\mathfrak{p}}) = v_p(\beta_{\overline{\mathfrak{p}}}) = 0$*. When constructing the* p*-adic* L*-function in section 3.3 we will need the condition* $v(\lambda_{\mathfrak{p}}) < k+1$ *for* $\lambda_{\mathfrak{p}}$ *the eigenvalue of* $U_{\mathfrak{p}}$ *for all* $\mathfrak{p}|p$*, henceforth we will work with* $f_{\varphi/K}^{\beta\beta}$ *and denote it by*

$$f_{\varphi/K}^p := f_{\varphi/K}^{\beta\beta}.$$

1.3.4 L-function of base change Bianchi modular forms

Let f be a modular form with CM by K of weight $k+2$, then $f = f_\varphi$ where φ is a Hecke character of K of infinity type $(-k-1, 0)$ and conductor $\mathfrak{m}$, henceforth we work with $f_{\varphi/K}$, the base change to K of f_φ.

Lemma 1.10. *Let* ψ *be a Hecke character of* K *and recall the definition of* $\psi^c(\mathfrak{q}) := \psi(\overline{\mathfrak{q}})$ *where* $\mathfrak{q}$ *is a fractional ideal of* K *and* $\overline{\mathfrak{q}}$ *is the conjugate ideal of* $\mathfrak{q}$*. Then*

$$L(f_{\varphi/K}, \psi, s) = L(\varphi^c\psi, s)L(\varphi^c\psi^c\lambda_K, s) = L(\varphi^c\psi\lambda_K, s)L(\varphi^c\psi^c, s)$$

where $\lambda_K = \chi_K \circ N$*.*

Proof. The Fourier coefficient of f_φ at a prime number q is:

$$a_q = \begin{cases} \varphi(\mathfrak{q}) + \varphi(\overline{\mathfrak{q}}) & : q\mathcal{O}_K = \mathfrak{q}\overline{\mathfrak{q}}, \\ \varphi(\mathfrak{q}) & : q\mathcal{O}_K = \mathfrak{q}^2, \\ 0 & : q\mathcal{O}_K = \mathfrak{q}. \end{cases}$$

Then the Fourier coefficient of $f_{\varphi/K}$ at prime ideals $\mathfrak{q}|q$ are:

1. $a_{\mathfrak{q}} = a_{\overline{\mathfrak{q}}} = a_q = \varphi(\mathfrak{q}) + \varphi(\overline{\mathfrak{q}})$, if $q\mathcal{O}_K = \mathfrak{q}\overline{\mathfrak{q}}$;

2. $a_{\mathfrak{q}} = a_q = \varphi(\mathfrak{q})$ if $q\mathcal{O}_K = \mathfrak{q}^2$;

3. $a_{\mathfrak{q}} = a_q^2 - 2\chi_K(q)\eta_{\mathbb{Z}}(q)q^{k+1} = 2\varphi(\mathfrak{q})$ if $q\mathcal{O}_K = \mathfrak{q}$.

Finally comparing the Euler factors at each prime $\mathfrak{q} \subset \mathcal{O}_K$ we obtain the result. □

Remark 1.17. *There are 6 more ways to factor $L(f_{\varphi/K}, \psi, s)$ as product of two Hecke L-functions, this comes from the fact that for a Hecke character ν we have $L(\nu, s) = L(\nu^c, s)$. The analogue factorizations in the p-adic setting not necessarily hold, because if ν has infinity type (q, r) the involution $\nu \to \nu^c$ corresponds to the map $(q, r) \to (r, q)$ on weight space and therefore does not preserve the lower right quadrant of weights of Hecke characters that lie in the range of classical interpolation of the Katz p-adic L-functions (see Theorem 3.6).*

Proposition 1.8. *There exists a period $\Omega_{f_{\varphi/K}} \in \mathbb{C}^\times$ such that for all Hecke character ψ of K with infinity type $0 \leqslant (q, r) \leqslant (k, k)$, we have $\Lambda(f_{\varphi/K}, \psi)/\Omega_{f_{\varphi/K}} \in \overline{\mathbb{Q}}$.*

Proof. By Lemma 1.10

$$L(f_{\varphi/K}, \psi, 1) = L(\varphi^c\psi, 1)L(\varphi^c\psi^c\lambda_K, 1) = L(\varphi^c\psi|\cdot|_{\mathbb{A}_K}, 0)L(\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K}, 0).$$

Recalling that φ has infinity type $(-k-1, 0)$ in particular we have that $\varphi^c\psi|\cdot|_{\mathbb{A}_K}$ has infinity type $(q+1, r-k)$ and $\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K}$ has infinity type $(r+1, q-k)$, since $q+1 > r-k$ and $r+1 > q-k$ by Lemma 1.1 we have

$$\frac{L(\varphi^c\psi|\cdot|_{\mathbb{A}_K}, 0)}{(2\pi i)^{r-k}\Omega(A)^{q+1-r+k}} \in \overline{\mathbb{Q}} \quad \text{and} \quad \frac{L(\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K}, 0)}{(2\pi i)^{q-k}\Omega(A)^{r+1-q+k}} \in \overline{\mathbb{Q}}.$$

We obtain that

$$\frac{L(f_{\varphi/K}, \psi, 1)}{(2\pi i)^{q+r-2k}\Omega(A)^{2k+2}} \in \overline{\mathbb{Q}}.$$

Finally since

$$\Lambda(f_{\varphi/K}, \psi) = \frac{\Gamma(q+1)\Gamma(r+1)}{(2\pi i)^{q+r+2}}L(f_{\varphi/K}, \psi, 1),$$

then taking

$$\Omega_{f_{\varphi/K}} = \left(\frac{\Omega(A)}{2\pi i}\right)^{2k+2}$$

we obtain the result. □

Remark 1.18. *Since we are interested in algebraicity, we normalise the period $\Omega_{f_{\varphi/K}}$ for convenience to*

$$\Omega'_{f_{\varphi/K}} = \frac{2D^{k+1}}{w}\Omega_{f_{\varphi/K}}.$$

Suppose $f_{\varphi/K}$ does not have level at p and consider its p-stabilisation $f^p_{\varphi/K}$ as in the previous section.

Lemma 1.11. *Let $\Omega'_{f_{\varphi/K}}$ be as in Proposition 1.8, then for all Hecke character ψ of K with infinity type $0 \leqslant (q, r) \leqslant (k, k)$, we have $\Lambda(f^p_{\varphi/K}, \psi)/\Omega'_{f_{\varphi/K}} \in \overline{\mathbb{Q}}$.*

Proof. First note by Proposition 1.7 we have that $f_{\varphi/K}$ is quasi-cuspidal, then by Lemma 1.6 $f^p_{\varphi/K}$ is quasi-cuspidal. The L-functions of $f^p_{\varphi/K}$ and $f_{\varphi/K}$ at Hecke characters ψ are related (see Lemma 1.8 for $f^p_{\varphi/K}$ with trivial nebentypus and K with class number 1) by

$$L(f^p_{\varphi/K}, \psi) = \left(1 - \frac{\varphi(\mathfrak{p})\psi(\mathfrak{p})}{N(\mathfrak{p})}\right)\left(1 - \frac{\varphi(\mathfrak{p})\psi(\overline{\mathfrak{p}})}{N(\overline{\mathfrak{p}})}\right) L(f_{\varphi/K}, \psi).$$

Then for ψ with infinity type $0 \leqslant (q, r) \leqslant (k, k)$, we have

$$\Lambda(f^p_{\varphi/K}, \psi) = \left(1 - \frac{\varphi(\mathfrak{p})\psi(\mathfrak{p})}{N(\mathfrak{p})}\right)\left(1 - \frac{\varphi(\mathfrak{p})\psi(\overline{\mathfrak{p}})}{N(\overline{\mathfrak{p}})}\right) \Lambda(f_{\varphi/K}, \psi)$$

and by Proposition 1.8 we obtain the result. □

Chapter 2

Partial modular symbols, eigenvarieties and p-adic families

In this chapter we introduce partial Bianchi modular symbols, which are algebraic analogues of C-cuspidal Bianchi modular that are easier to study p-adically. In last section we recall some facts about eigenvarieties and p-adic families.

2.1 Partial modular symbols

2.1.1 Abstract partial modular symbols over K

Let Γ be a discrete subgroup of $\mathrm{SL}_2(K)$, let $\mathcal{C}$ be a non-empty Γ-invariant subset of $\mathbb{P}^1(K)$.

Definition 2.1. *Define by $\Delta_{\mathcal{C}}$ the abelian group of divisors on $\mathcal{C}$, i.e.*

$$\Delta_{\mathcal{C}} = \left\{ \sum_{c\in\mathcal{C}} n_c\{c\} : n_c \in \mathbb{Z},\ n_c = 0 \textit{ for almost all } c \right\},$$

moreover let $\Delta_{\mathcal{C}}^0$ be the subgroup of divisors of degree 0 (i.e., such that $\sum_{c\in\mathcal{C}} n_c = 0$). Note that $\Delta_{\mathcal{C}}^0$ has a left action by the group Γ (and indeed, of $\mathrm{SL}_2(K)$) by fractional linear transformations. Concretely, this is the action induced linearly by

$$\gamma \cdot r = \frac{ar+b}{cr+d},\ \textit{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K),\ \textit{and } r \in \mathbb{P}^1(K).$$

Remark 2.1. *(i) We define the completed upper half-space $\mathcal{H}_3^* = \mathcal{H}_3 \cup \mathbb{P}^1(K)$, in particular, we say $r \in \mathbb{P}^1(K)$ is a cusp of $\mathcal{H}_3^*$ and we see it as an element $(r,0)$ on the boundary of $\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$.*

(ii) Note that $\Delta_{\mathcal{C}}^0$ is spanned by elements of the form $\{r\} - \{s\}$ for $r, s \in \mathcal{C}$. One should view this element as representing a path between the cusps r and s.

Let V a right Γ-module, then there is a right action of Γ in $\mathrm{Hom}(\Delta_{\mathcal{C}}^0, V)$ given by

$$\phi_{|\gamma}(D) := \phi(\gamma \cdot D)_{|\gamma}.$$

Definition 2.2. *We define the space of partial modular symbols on $\mathcal{C}$ for Γ with values in V, as the subgroup of $\mathrm{Hom}(\Delta_{\mathcal{C}}^0, V)$ fixed by the action of Γ:*

$$\mathrm{Symb}_{\Gamma,\mathcal{C}}(V) = \mathrm{Hom}(\Delta_{\mathcal{C}}^0, V)^{\Gamma}$$

Remark 2.2. *When $\mathcal{C} = \mathbb{P}^1(K)$ we drop $\mathcal{C}$ from the notation and call* $\mathrm{Symb}_\Gamma(V)$ *the space of modular symbols for Γ with values in V recovering Definition 2.3 in [41].*

2.1.2 Partial Bianchi modular symbols

Recall from section 1.2 the group $\Omega_0(\mathfrak{n})$ and from (1.8) its twist $\Gamma_i(\mathfrak{n})$ for each $i = 1, ..., h$. Taking in Definition 2.2 the group $\Gamma = \Gamma_i(\mathfrak{n})$ and suitable modules V to be defined below we can obtain more concrete partial modular symbols.

For a ring R recall that $V_k(R)$ denote the space of homogeneous polynomials over R in two variables of degree k. Moreover, for integers $k, \ell \geqslant 0$ we put

$$V_{k,\ell}(R) := V_k(R) \otimes_R V_\ell(R).$$

We can identify $V_{k,\ell}(R)$ with the space of polynomials that are homogeneous of degree k in two variables X, Y and homogeneous of degree ℓ in two further variables $\overline{X}, \overline{Y}$.

Definition 2.3. *We have a left-action of $\Gamma_i(\mathfrak{n})$ on $V_k(\mathbb{C})$ defined by*

$$\gamma \cdot P\begin{pmatrix} X \\ Y \end{pmatrix} = P\begin{pmatrix} dX + bY \\ cX + aY \end{pmatrix}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We then obtain a left-action of $\Gamma_i(\mathfrak{n})$ on $V_{k,\ell}(\mathbb{C})$ by

$$\gamma \cdot P\left[\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}\right] = P\left[\begin{pmatrix} dX + bY \\ cX + aY \end{pmatrix}, \begin{pmatrix} \overline{d}\overline{X} + \overline{b}\overline{Y} \\ \overline{c}\overline{X} + \overline{a}\overline{Y} \end{pmatrix}\right].$$

The left-action of $\Gamma_i(\mathfrak{n})$ described above translates to a *right*-action on the dual space $V^*_{k,\ell}(\mathbb{C})$ setting

$$\mu|_\gamma(P) = \mu(\gamma \cdot P). \tag{2.1}$$

Let $\mathcal{C} = (\mathcal{C}_1, ..., \mathcal{C}_h)$ with $\mathcal{C}_i$ a non-empty $\Gamma_i(\mathfrak{n})$-invariant subset of $\mathbb{P}^1(K)$, then

Definition 2.4. *(i) Define the space of partial Bianchi modular symbols on $\mathcal{C}_i$ of weight (k, ℓ) and level $\Gamma_i(\mathfrak{n})$ to be the space* $\mathrm{Symb}_{\Gamma_i(\mathfrak{n}),\mathcal{C}_i}(V^*_{k,\ell}(\mathbb{C}))$.
(ii) Define the space of partial Bianchi modular symbols on $\mathcal{C}$ of weight (k, ℓ) and level $\Omega_0(\mathfrak{n})$ to be the space

$$\mathrm{Symb}_{\Omega_0(\mathfrak{n}),\mathcal{C}}(V^*_{k,\ell}(\mathbb{C})) := \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),\mathcal{C}_i}(V^*_{k,\ell}(\mathbb{C})).$$

Remark 2.3. *When $\mathcal{C}_i = \mathbb{P}^1(K)$ for all i, we drop $\mathcal{C}$ from the notation and we recover the space* $\mathrm{Symb}_{\Omega_0(\mathfrak{n})}(V^*_{k,\ell}(\mathbb{C}))$ *of Bianchi modular symbols of Definition 2.4 in [41].*

In the same way as with Bianchi modular forms, we can define Hecke operators on the space of Bianchi modular symbols. To do so, consider our fixed representatives $I_1, ..., I_h$ for the class group as in section 1.2.1. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_K$, for each $i \in \{1, ..., h\}$ there is a unique $j_i \in \{1, ..., h\}$ such that $\mathfrak{q} I_i = (\alpha_i) I_{j_i}$, for some $\alpha_i \in K$.

Definition 2.5. *Let $\mathfrak{q} \nmid \mathfrak{n}$ be a prime ideal, then the Hecke operator $T_\mathfrak{q}$ is defined on the space of Bianchi modular symbols* $\mathrm{Symb}_{\Omega_0(\mathfrak{n})}(V_{k,\ell}^*(\mathbb{C}))$ *by*

$$(\phi_1, ..., \phi_h)|_{T_\mathfrak{q}} = \left(\phi_{j_1}\left|\left[\Gamma_{j_1}(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \alpha_1\end{pmatrix}\Gamma_1(\mathfrak{n})\right], ..., \phi_{j_h}\right|\left[\Gamma_{j_h}(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \alpha_h\end{pmatrix}\Gamma_h(\mathfrak{n})\right]\right).$$

If $\mathfrak{q}|\mathfrak{n}$ we denote the Hecke operator by $U_\mathfrak{q}$.

We can also define a Hecke operator at an ideal $I \subset \mathcal{O}_K$ from the Hecke operators at prime ideals dividing I.

Note that in the case of partial Bianchi modular symbols, since there could be some conditions regarding every set of cusps $\mathcal{C}_i$, not all the Hecke operators act in $\mathrm{Symb}_{\Omega_0(\mathfrak{n}),\mathcal{C}}(V_{k,\ell}^*(\mathbb{C}))$.

In order to relate partial Bianchi modular symbols with C-cuspidal Bianchi modular forms of level $\Omega_0(\mathfrak{n})$ for $\mathfrak{n} = (p)\mathfrak{m}$ with $((p), \mathfrak{m}) = 1$ and weight (k, ℓ) we henceforth take $\mathcal{C} = C = (C_1, ..., C_h)$ where $C_i = \Gamma_i(\mathfrak{m})\infty \cup \Gamma_i(\mathfrak{m})0$ and consequently work with the space of *partial Bianchi modular symbols* on $C = (C_1, ..., C_h)$ of weight (k, ℓ) and level $\Omega_0(\mathfrak{n})$.

Proposition 2.1. *The Hecke operators $T_\mathfrak{q}$ for all primes $\mathfrak{q} \nmid \mathfrak{m}$ and $U_\mathfrak{p}$ for all primes $\mathfrak{p}|(p)$ act on* $\mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(\mathbb{C}))$.

Proof. Let $(\phi_1, ..., \phi_h) \in \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(\mathbb{C}))$, we have to show that for all prime $\mathfrak{q} \nmid \mathfrak{m}$ with $\mathfrak{q} I_i = (\alpha_i) I_{j_i}$ and for each $i = 1, ...h$ then

$$\phi_{j_i}\left|\left[\Gamma_{j_i}(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \alpha_i\end{pmatrix}\Gamma_i(\mathfrak{n})\right]\right. \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^*(\mathbb{C})).$$

It suffices to prove $\gamma \cdot s_i \in C_{j_i}$ for all $\gamma = \gamma_{j_i}\begin{pmatrix}1 & 0\\ 0 & \alpha_i\end{pmatrix}\gamma_i \in \left[\Gamma_{j_i}(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \alpha_i\end{pmatrix}\Gamma_i(\mathfrak{n})\right]$ and $s_i \in C_i$.

Following the proof of Proposition 1.2 , for all $s_i \in C_i$ we have $\gamma_i \cdot s_i = s_i' \in C_i$ by fact 1), $\begin{pmatrix}1 & 0\\ 0 & \alpha_i\end{pmatrix} \cdot s_i' = s_{j_i} \in C_{j_i}$ by fact 2) and $\gamma_{j_i} \cdot s_{j_i} = s_{j_i}' \in C_{j_i}$ again by fact 1) and we obtain the result since $\phi_{j_i} \in \mathrm{Symb}_{\Gamma_{j_i}(\mathfrak{n}),C_{j_i}}(V_{k,\ell}^*(\mathbb{C}))$. □

2.1.3 Relation with C-cuspidal Bianchi modular forms

Let Φ be a C-cuspidal Bianchi modular form on $\mathrm{GL}_2(\mathbb{A}_K)$ of weight (k, ℓ) and level $\Omega_0(\mathfrak{n})$ with $\mathfrak{n} = (p)\mathfrak{m}$ and $((p), \mathfrak{m}) = 1$, and write $\mathcal{F}^1, ..., \mathcal{F}^h$ for the associated C_i-cuspidal Bianchi modular forms on $\mathcal{H}_3$ (under our fixed set of class group representatives) of level $\Gamma_i(\mathfrak{n})$. We pick one of these functions $\mathcal{F}^i$, henceforth denoting it simply by $\mathcal{F}$, and describe how to attach a modular symbol $\phi_\mathcal{F}$ to it. Throughout, we write $\Gamma = \Gamma_i(\mathfrak{n})$ for the level of $\mathcal{F}$ and $C^\bullet := C_i$.

In Definition 1.11 we defined a right action of $\mathrm{SU}_2(\mathbb{C})$ on $V_k(\mathbb{C})$, here we use the corresponding left action defined by $u \cdot P := P|u^{-1}$.

The Clebsch-Gordan formula says that, for $k \geqslant \ell$,

$$V_k(\mathbb{C}) \otimes_{\mathbb{C}} V_\ell(\mathbb{C}) = V_{k+\ell}(\mathbb{C}) \oplus V_{k+\ell-2}(\mathbb{C}) \oplus \cdots \oplus V_{k-\ell}(\mathbb{C})$$

as (left or right) $\mathrm{SU}_2(\mathbb{C})$-modules. Since $V_{k,\ell}(\mathbb{C}) \cong V_k(\mathbb{C}) \otimes_{\mathbb{C}} V_\ell(\mathbb{C})$, we have

$$V_{k,\ell}(\mathbb{C}) \otimes_{\mathbb{C}} V_2(\mathbb{C}) = V_{k+\ell+2}(\mathbb{C}) \oplus V_{k+\ell}(\mathbb{C})^2 \oplus \cdots \oplus V_0(\mathbb{C})$$

as right $\mathrm{SU}_2(\mathbb{C})$, and hence that there is an injection of (left) $\mathrm{SU}_2(\mathbb{C})$-modules

$$V_{k+l+2}(\mathbb{C}) \hookrightarrow V_{k,\ell}(\mathbb{C}) \otimes_{\mathbb{C}} V_2(\mathbb{C}).$$

Let F be the function $\mathrm{GL}_2(\mathbb{C}) \to V_{k+\ell+2}(\mathbb{C})$ corresponding to $\mathcal{F}$, we compose F with the map defined above to give

$$\sigma \circ F : \mathrm{SL}_2(\mathbb{C}) \to V_{k,\ell}(\mathbb{C}) \otimes_{\mathbb{C}} V_2(\mathbb{C}).$$

This associates to F and each $g \in \mathrm{SL}_2(\mathbb{C})$ a polynomial that is homogeneous of degree k in variables X and Y, homogeneous of degree ℓ in variables $\overline{X}$ and $\overline{Y}$ and homogeneous of degree 2 in variables A and B. We then use Proposition 2.5 in [41] to pass from $V_2(\mathbb{C})$ to differentials; namely, we replace A^2 with dz, AB with $-dt$ and B^2 with $-d\overline{z}$ to obtain a differential 1-form on $\mathrm{SL}_2(\mathbb{C})$ with values in $V_{k,\ell}(\mathbb{C})$. To obtain a differential on the quotient $\mathcal{H}_3 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C})$, we scale by the action of $\mathrm{SL}_2(\mathbb{C})$.

Definition 2.6. *Define a differential ω_F on $\mathrm{SL}_2(\mathbb{C})$ by*

$$\omega_F(g) = g \cdot (\sigma \circ F(g)), \quad g \in \mathrm{SL}_2(\mathbb{C}).$$

Here $\mathrm{SL}_2(\mathbb{C})$ acts on $V_{k,\ell}(\mathbb{C}) \otimes_{\mathbb{C}} V_2(\mathbb{C})$ by

$$\gamma \cdot P\left[\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}, \begin{pmatrix} A \\ B \end{pmatrix}\right] = P\left[\gamma^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}, \overline{\gamma}^{-1}\begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}, \frac{1}{|a|^2 + |c|^2}\begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}\right], \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since for $u \in \mathrm{SU}_2(\mathbb{C})$ we have

$$\omega_F(gu) = gu \cdot (\sigma \circ F(gu)) = gu \cdot u^{-1}(\sigma(F(g))) = g \cdot (\sigma \circ F(g)) = \omega_F(g),$$

Then the differential ω_F is invariant under right multiplication by $\mathrm{SU}_2(\mathbb{C})$ and gives a well-defined differential on $\mathcal{H}_3$.

The above construction gives a $V_{k,\ell}(\mathbb{C})$-valued differential $\omega_{\mathcal{F}} = \omega_F$ on $\mathcal{H}_3$ that is invariant under the action of Γ, this differential is harmonic from the definition of automorphic forms, and hence we can integrate it between cusps of $C^\bullet$ in a path-independent manner.

Proposition 2.2. *Let $\mathcal{F}^i : \mathcal{H}_3 \to V_{k+\ell+2}(\mathbb{C})$ be a C_i-cuspidal Bianchi modular form of weight (k, ℓ)*

with $k \geqslant \ell$ and level $\Gamma_i(\mathfrak{n})$. Then the map $\phi'_{\mathcal{F}^i} : \Delta^0_{C_i} \to V_{k,\ell}(\mathbb{C})$ given by

$$\phi'_{\mathcal{F}^i} := \int_r^s \omega_{\mathcal{F}^i}$$

defines a partial Bianchi modular symbol with values in $V_{k,\ell}(\mathbb{C})$.

Since we require partial Bianchi modular symbols with values in $V^*_{k,\ell}(\mathbb{C})$ we use Proposition 2.6 in [41] for non-parallel weight and obtain an $\mathrm{SL}_2(\mathbb{C})$-equivariant isomorphism

$$V_{k,\ell}(\mathbb{C}) \longrightarrow V^*_{k,\ell}(\mathbb{C}) \tag{2.2}$$

Definition 2.7. *The partial Bianchi modular symbol attached to $\mathcal{F}^i$ is the element*

$$\phi_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V^*_{k,\ell}(\mathbb{C}))$$

given by composing $\phi'_{\mathcal{F}^i}$ with the map of (2.2).

2.1.4 Link with L-values of C-cuspidal Bianchi modular forms

We can describe explicitly the partial Bianchi modular symbol $\phi_{\mathcal{F}^i}$ of the previous section.

Proposition 2.3. *Let Φ be a C-cuspidal Bianchi modular form of weight (k,ℓ) with $k \geqslant \ell$ and level $\Omega_0(\mathfrak{n})$, then for each $i = 1,..,h$ and $a \in C_i$ we have:*

$$\phi_{\mathcal{F}^i}(\{a\} - \{\infty\}) = \sum_{q,r=0}^{k} c^i_{q,r}(a)(\mathcal{Y} - a\mathcal{X})^{k-q}\mathcal{X}^q(\overline{\mathcal{Y}} - \overline{a}\overline{\mathcal{X}})^{\ell-r}\overline{\mathcal{X}}^r, \tag{2.3}$$

where

$$c^i_{q,r}(a) := 2\binom{k+\ell+2}{\ell+q-r+1}^{-1}(-1)^{\ell-r+1}\int_0^\infty t^{q+r}\mathcal{F}^i_{\ell+q-r+1}(a,t)dt.$$

Proof. This is an adaptation to weight (k,ℓ) and the C_i-cuspidal situation of Proposition 2.9 in [41] noting first that $c^i_{q,r}$ is defined by an integral of $\mathcal{F}^i_{\ell+q-r+1}$ which is convergent by C_i-cuspidality; and second the necessary condition of $k \geqslant \ell$ coming from Clebsch-Gordan formula in section 2.1.3. □

Note that for $a \in C^i$ the integral

$$\int_0^\infty t^{q+r}\mathcal{F}^i_{\ell+q-r+1}(a,t)dt$$

appears both in $\phi_{\mathcal{F}^i}$ and in the integral form of the L-function of Φ in Theorem 1.1, this allow us to link $\phi_\Phi = (\phi_{\mathcal{F}^1},...,\phi_{\mathcal{F}^h}) \in \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V^*_{k,\ell}(\mathbb{C}))$ with the critical L-values of Φ in exactly the same way as in Section 2.5 and 2.6 in [41] obtaining by similar proofs the following analogous to Theorem 2.11 in [41].

Theorem 2.1. *We have*

$$\Lambda(\Phi,\psi) = \left[\frac{(-1)^{\ell+q+r}2\psi_{\mathfrak{f}}(x_{\mathfrak{f}})}{\psi(x_{\mathfrak{f}})Dw\tau(\psi^{-1})}\right]\sum_{i=1}^{h}\left[\psi(t_i)\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\((a)\mathfrak{f},\mathfrak{f})=1}}\psi_{\mathfrak{f}}(a)c^i_{q,r}(a)\right],$$

Here:

- Φ *is a* C*-cuspidal Bianchi modular form of weight* (k,ℓ) *with* $k \geqslant \ell$*, level* $\Omega_0(\mathfrak{n})$ *for* $\mathfrak{n} = (p)\mathfrak{m}$ *with* $((p),\mathfrak{m}) = 1$ *and central action* φ *of trivial conductor and infinity type* $(-k,-\ell)$*,*
- ψ *is a Hecke character of* K *of conductor* $\mathfrak{f}$ *such that* $(\mathfrak{f},\mathfrak{m}) = 1$ *and* $(\mathfrak{f}, I_i) = 1$ *and infinity type* $0 \leqslant (q,r) \leqslant (k,\ell)$*,*
- $-D$ *is the discriminant of* K*,*
- w *is the size of the unit group of* $\mathcal{O}_K$*,*
- $\tau(\psi^{-1})$ *is the Gauss sum of* ψ^{-1} *from Definition 1.10,*
- h *is the class number of* K*,*
- t_i *is an idele corresponding to the* $i-th$ *representative* I_i *of the class group, which is coprime to* $\mathfrak{n}$*,*
- $c^i_{q,r}(a)$ *is the coefficient of* $(\mathcal{Y} - a\mathcal{X})^{k-q}\mathcal{X}^q(\overline{\mathcal{Y}} - \overline{a}\overline{\mathcal{X}})^{\ell-r}\overline{\mathcal{X}}^r$ *in* $\phi_{\mathcal{F}^i}(\{a\} - \{\infty\})$*, where*
- $\phi_{\mathcal{F}^i}$ *is the partial Bianchi modular symbol attached to the* C_i *cuspidal Bianchi modular form* $\mathcal{F}^i$ *on* $\mathcal{H}_3$ *induced by* Φ*.*

2.2 Overconvergent partial symbols and control theorem

In this section we develop the theory of overconvergent partial Bianchi modular symbols in order to prove a partial Bianchi control theorem. To do so, we consider partial Bianchi modular symbols with values in a space of p-adic distributions and closely follow and adapt the results in sections 2 and 3 in [41].

2.2.1 Overconvergent partial Bianchi modular symbols

In this section we define the relevant space of p-adic distributions and introduce the space of overconvergent partial Bianchi modular symbols.

Since we are in the p-adic setting, we work with the space $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which is the analogous of the p-adic integers for our setting.

We fix embeddings

$$\mathrm{inc} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p, \qquad \sigma = (\sigma_1, \sigma_2) : K \otimes_{\mathbb{Q}} \mathbb{Q}_p \to \overline{\mathbb{Q}}_p \times \overline{\mathbb{Q}}_p, \tag{2.4}$$

and work with L a finite extension of $\mathbb{Q}_p$ such that the image of σ lies in L^2. We equip L with a valuation v, normalised so that $v(p) = 1$, and denote the ring of integers in L by $\mathcal{O}_L$, with uniformiser π_L.

Now, we proceed to define the p-adic distribution spaces.

Definition 2.8. *Define $\mathcal{A}(L)$ to be the space of locally analytic functions $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$ and $\mathcal{D}(L) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}(L), L)$ to be the space of locally analytic distributions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$.*

To define partial Bianchi modular symbols with values in $\mathcal{D}(L)$ we consider the semigroup

$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) : v_{\mathfrak{p}}(c) > 0\ \forall \mathfrak{p}|p,\ a \in (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times, ad - bc \neq 0 \right\}. \tag{2.5}$$

To endow $\mathcal{D}(L)$ with an action of $\Sigma_0(p)$ we define some more general objects which will also appear in section 2.3 when defining the Bianchi eigenvariety.

Definition 2.9. *The Bianchi weight space is the rigid analytic space $\mathcal{W}_K$ whose L-points are*

$$\mathcal{W}_K(L) = \mathrm{Hom}_{cts}((\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \mathcal{O}_K^\times, L^\times).$$

Definition 2.10. *We say that a weight $\lambda \in \mathcal{W}_K(L)$ is classical if $\lambda = \epsilon\lambda^{\mathrm{alg}}$, with ϵ a finite order character and $\lambda^{\mathrm{alg}}(z) = z^k \overline{z}^\ell$ for $k, \ell \in \mathbb{Z}$.*

For each $\lambda \in \mathcal{W}_K(L)$ we have a left weight λ action of $\Sigma_0(p)$ on $\mathcal{A}(L)$ defined by

$$(\gamma \cdot_\lambda f)(z) = \lambda(a + cz) f\left(\frac{b + dz}{a + cz}\right), \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.6}$$

which induces a right weight λ action on $\mathcal{D}(L)$ defined by

$$(\mu|_\lambda \gamma)(f) = \mu(\gamma \cdot_\lambda f). \tag{2.7}$$

We denote by $\mathcal{A}_\lambda(L)$ (resp. $\mathcal{D}_\lambda(L)$) the space of locally analytic functions (resp. locally analytic distributions) equipped with the above weight λ action of $\Sigma_0(p)$.

Remark 2.4. *If $\lambda = \epsilon\lambda^{\mathrm{alg}} \in \mathcal{W}_K(L)$ is a classical weight such that ϵ is trivial, i.e., $\lambda(z) = z^k \overline{z}^\ell$; we denote by $\mathcal{A}_{k,\ell}(L)$ (resp. $\mathcal{D}_{k,\ell}(L)$) the spaces $\mathcal{A}_\lambda(L)$ (resp. $\mathcal{D}_\lambda(L)$).*

For the rest of the section 2.2 we work with the spaces $\mathcal{A}_{k,\ell}(L)$ and $\mathcal{D}_{k,\ell}(L)$).

Recall the groups $\Gamma_i(\mathfrak{n})$ from (1.8) and note that to define partial Bianchi modular symbols with values in $\mathcal{D}_{k,\ell}(L)$ we need an action of $\Gamma_i(\mathfrak{n})$ on that space.

Using the embedding (2.4) and that all $\mathfrak{p}|p$ divides the lower left entry of a matrix in $\Gamma_i(\mathfrak{n})$ (because $(p)|\mathfrak{n}$), we obtain that $\Gamma_i(\mathfrak{n}) \subset \Sigma_0(p)$. Then we can equip $\mathcal{D}_{k,\ell}(L)$ with an action of $\Gamma_i(\mathfrak{n})$ and define:

Definition 2.11. *(i) Define the space of overconvergent partial Bianchi modular symbols on C_i of weight (k, ℓ) and level $\Gamma_i(\mathfrak{n})$ with coefficients in L to be the space $\mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}(\mathcal{D}_{k,\ell}(L))$.*
(ii) Define the space of overconvergent partial Bianchi modular symbols on C of weight (k, ℓ) and level $\Omega_0(\mathfrak{n})$ with coefficients in L to be the space

$$\mathrm{Symb}_{\Omega_0(\mathfrak{n}), C}(\mathcal{D}_{k,\ell}(L)) := \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}(\mathcal{D}_{k,\ell}(L)).$$

2.2.2 Partial Bianchi control theorem

Once we have defined overconvergent Bianchi modular symbols we want to relate them with the partial Bianchi modular symbols of seccion 2.1, that is, relate $\mathcal{D}_{k,\ell}(L)$ with $V_{k,\ell}^*(L)$.

Note that an element of $V_{k,\ell}(L)$ can be seen as a function on L^2 that is polynomial of degree at most k in the first variable and polynomial of degree at most ℓ in the second variable, moreover, since $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ embed in L^2, we can think such function $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$.

There is a natural inclusion $V_{k,\ell}(L) \hookrightarrow \mathcal{A}_{k,\ell}(L)$, which we can dualise and obtain a surjection from $\mathcal{D}_{k,l}(L)$ to the dual space $V_{k,\ell}^*(L)$,

Remark 2.5. *Note that the right action on $V_{k,\ell}^*(L)$ inherit from $\mathcal{D}_{k,\ell}(L)$ and the action defined in (2.1) are compatible.*

The map $\mathcal{D}_{k,l}(L) \to V_{k,\ell}^*(L)$ induces an specialisation map

$$\mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(\mathcal{D}_{k,\ell}(L)) \to \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^*(L)),$$

which give rises to a map

$$\rho : \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathcal{D}_{k,\ell}(L)) \to \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(L)).$$

In Proposition 2.1 we proved the action of Hecke operators in partial Bianchi modular symbols. Note that the matrices defining the $U_\mathfrak{p}$ operator for $\mathfrak{p}|p$ can be seen in $\Sigma_0(p)$, then the $U_\mathfrak{p}$ operator acts on the space of overconvergent partial Bianchi modular symbols and we have the following:

Theorem 2.2. *(Partial Bianchi control theorem)*

Let p be prime with $p\mathcal{O}_K = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e_\mathfrak{p}}$. For each prime $\mathfrak{p}$ above p, let $\lambda_\mathfrak{p} \in L^\times$. Suppose that $v(\lambda_\mathfrak{p}) < (\min\{k,\ell\}+1)/e_\mathfrak{p}$ for p inert or ramified, or $v(\lambda_\mathfrak{p}) < k+1$ and $v(\lambda_{\overline{\mathfrak{p}}}) < \ell+1$ for p split, then the restriction of the specialisation map

$$\rho : \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathcal{D}_{k,\ell}(L))^{\{U_\mathfrak{p}=\lambda_\mathfrak{p}:\mathfrak{p}|p\}} \longrightarrow \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(L))^{\{U_\mathfrak{p}=\lambda_\mathfrak{p}:\mathfrak{p}|p\}}$$

to the simultaneous $\lambda_\mathfrak{p}$-eigenspaces of the $U_\mathfrak{p}$ operators is an isomorphism.

The following sections will be devoted to prove Theorem 2.2, but before embarking us in such a proof we conclude this section introducing the notion of a *small slope* Bianchi modular form.

Definition 2.12. *Let Φ be an eigenform of weight (k,ℓ) with eigenvalues λ_I, we say Φ has small slope if $v(\lambda_\mathfrak{p}) < (\min\{k,\ell\}+1)/e_\mathfrak{p}$ when p is inert as $\mathfrak{p}$ or p ramifies as $\mathfrak{p}^2$; or if $v(\lambda_\mathfrak{p}) < k+1$ and $v(\lambda_{\overline{\mathfrak{p}}}) < \ell+1$ when p splits as $\mathfrak{p}\overline{\mathfrak{p}}$. We say Φ has critical slope if does not have small slope.*

2.2.3 Partial Bianchi control theorem (rigid analytic distributions)

In this section we introduce the space of rigid analytic distributions, such space have nice descriptions and it is easier to work than the space of locally analytic distributions. The idea is to use rigid analytic distributions to obtain a equivalent statement to Theorem 2.2 that will be easier to prove.

Rigid analytic distributions

Let R be either a p-adic field or the ring of integers in a finite extension of $\mathbb{Q}_p$.

Definition 2.13. *Let $\mathbb{A}(R)$ be the ring of rigid analytic functions on the closed unit disc defined over R. We write $\mathbb{A}_2(R)$ for the completed tensor product $\mathbb{A}(R)\hat{\otimes}_R\mathbb{A}(R)$. We let $\mathbb{D}_2(R) = \mathrm{Hom}_{cts}(\mathbb{A}_2(R), R)$ be the topological dual of $\mathbb{A}_2(R)$.*

We define

$$\Sigma_0(\mathcal{O}_L) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_L) : p \mid c, (a,p) = 1, ad - bc \neq 0 \right\}$$

and endow $\mathbb{A}_2(L)$ with a left weight (k,ℓ) action of $\Sigma_0(\mathcal{O}_L)^2$ given by

$$((\gamma_1,\gamma_2)\cdot_{(k,\ell)} f)(x,y) = (c_1x+a_1)^k(c_2y+a_2)^\ell f\left(\frac{d_1x+b_1}{c_1x+a_1}, \frac{d_2y+b_2}{c_2y+a_2}\right), \quad \gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}. \tag{2.8}$$

This gives rise to a right weight (k,ℓ) action of $\Sigma_0(\mathcal{O}_L)^2$ on $\mathbb{D}_2(L)$ defined by

$$\mu|_{(k,\ell)}(\gamma_1,\gamma_2)(f) = \mu((\gamma_1,\gamma_2)\cdot_{(k,\ell)} f).$$

When talking about these spaces equipped with the weight (k,ℓ) action, we denote them by $\mathbb{A}_{k,\ell}(R)$ and $\mathbb{D}_{k,\ell}(R)$ respectively.

Using the embedding σ from (2.4) to obtain $\Gamma_i(\mathfrak{n}) \hookrightarrow \Sigma_0(\mathcal{O}_L)^2$, we see that $\Gamma_i(\mathfrak{n})$ act on the right on $\mathbb{D}_{k,\ell}(L)$ and consequently we can define partial Bianchi modular symbols with values on the space $\mathbb{D}_{k,\ell}(L)$. Note that, also the matrices defining the $U_\mathfrak{p}$ operator for $\mathfrak{p}|p$ can be seen in $\Sigma_0(\mathcal{O}_L)^2$ via de embedding σ, then the $U_\mathfrak{p}$ operator acts on the space of overconvergent partial Bianchi modular symbols with values on $\mathbb{D}_{k,\ell}(L)$.

Connection between locally and rigid analytic distributions

We now define spaces of locally analytic distributions in suitable neighbourhoods on $\mathbb{C}_p^2$ using the embedding $\sigma = (\sigma_1,\sigma_2)$ of (2.4). Such spaces will allow us to relate the space of locally analytic distributions with the space of rigid analytic distributions.

Let $r, s > \mathbb{R}_{>0}$, define the (r,s)-neighbourhood on $\mathbb{C}_p^2$ to be

$$B(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p, r, s) = \{(x,y) \in \mathbb{C}_p^2 : \exists u \in \mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p \text{ such that } |x - \sigma_1(u)| \leqslant r, |y - \sigma_2(u)| \leqslant s\}.$$

Definition 2.14. *Define the space of locally analytic functions of radius (r,s) over L, denoted by $\mathbb{A}[L,r,s]$, to be the space of rigid analytic functions $B(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p, r, s) \to L$.*

Denote by $\mathbb{A}_{k,\ell}[L,r,s]$ the space $\mathbb{A}[L,r,s]$ endowed with the weight (k,ℓ)-action of the semigroup $\Sigma(p)$ identical to the action defined in (2.6) on $\mathcal{A}_{k,\ell}(L)$.

Consider $r \leqslant r'$ and $s \leqslant s'$, since $B(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p, r, s) \subset B(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p, r', s')$, we have a natural and completely continuous injection $\mathbb{A}_{k,\ell}[L,r',s'] \hookrightarrow \mathbb{A}_{k,\ell}[L,r,s]$. Note that in particular, we have that

$$\mathcal{A}_{k,\ell}(L) := \varinjlim \mathbb{A}_{k,\ell}[L,r,s] = \bigcup_{r,s} \mathbb{A}_{k,\ell}[L,r,s].$$

Definition 2.15. *Define the space of locally analytic distributions of order (r,s) over L to be*

$$\mathbb{D}_{k,\ell}[L,r,s] := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}_{k,\ell}[L,r,s],L)$$

There is a canonical $\Sigma_0(p)$-equivariant isomorphism

$$\mathcal{D}_{k,\ell}(L) \cong \varprojlim \mathbb{D}_{k,\ell}[L,r,s] = \bigcap_{r,s} \mathbb{D}_{k,\ell}[L,r,s].$$

Note that the spaces of locally analytic functions of order $(1,1)$ and rigid analytic functions are equal, i.e., $\mathbb{A}[L,1,1] = \mathbb{A}_2(L)$.

On the other hand, the weight (k,ℓ) action of $\Sigma_0(p)$ on $\mathbb{A}[L,1,1]$ induced by the action on $\mathcal{A}(L)$ of (2.6) is compatible with the action $\Sigma_0(\mathcal{O}_L)^2$ on $\mathbb{A}_2(L)$ of (2.8) via $\Sigma_0(p) \hookrightarrow \Sigma_0(\mathcal{O}_L)^2$ induced by the embedding σ in (2.4).

Then we have $\mathbb{A}_{k,\ell}[L,1,1] = \mathbb{A}_{k,\ell}(L)$ and consequently $\mathbb{D}_{k,\ell}[L,1,1] = \mathbb{D}_{k,\ell}(L)$.

Partial Bianchi control theorem for rigid analytic distributions

Proposition 2.4. *Let $\Psi \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(\mathbb{D}_{k,\ell}(L))$,*

(i) Suppose p is inert in K and Ψ is a U_p-eigensymbol with non-zero eigenvalue, then Ψ is an element of $\mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(\mathcal{D}_{k,\ell}(L))$.

(ii) Suppose p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$ and Ψ is simultaneously a $U_{\mathfrak{p}}^n$- and $U_{\overline{\mathfrak{p}}}^n$-eigensymbol with non-zero eigenvalues, then Ψ is an element of $\mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(\mathcal{D}_{k,\ell}(L))$.

Proof. Both parts of the proposition are proved in exactly the same way as Propositions 5.8 and 6.12 in [41]: by using that the corresponding Hecke operator acts invertibly in its eigenspace and that the action of the matrices defining such Hecke operator, moves in the inverse system of locally analytic distributions of order (r,s). □

By Proposition 2.4, to prove Theorem 2.2 it suffices to prove the following partial Bianchi control theorem of overconvergent partial Bianchi modular symbols with values in rigid analytic distributions.

Theorem 2.3. *(Partial Bianchi control theorem, rigid analytic distributions)*

Let p be prime with $p\mathcal{O}_K = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e_{\mathfrak{p}}}$. For each prime $\mathfrak{p}$ above p, let $\lambda_{\mathfrak{p}} \in L^\times$. Suppose that $v(\lambda_{\mathfrak{p}}) < (\min\{k,\ell\}+1)/e_{\mathfrak{p}}$ for p inert or ramified, or $v(\lambda_{\mathfrak{p}}) < k+1$ and $v(\lambda_{\overline{\mathfrak{p}}}) < \ell+1$ for p split, then the restriction of the specialisation map

$$\rho : \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathbb{D}_{k,\ell}(L))^{\{U_{\mathfrak{p}}=\lambda_{\mathfrak{p}}:\mathfrak{p}|p\}} \longrightarrow \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(L))^{\{U_{\mathfrak{p}}=\lambda_{\mathfrak{p}}:\mathfrak{p}|p\}}$$

to the simultaneous $\lambda_{\mathfrak{p}}$-eigenspaces of the $U_{\mathfrak{p}}$ operators is an isomorphism.

Remark 2.6. *Our strategy to prove Theorem 2.3 is by separating it in three cases regarding the splitting behaviour of p in K. Such separation obeys the fact that the proof for p ramified and split in K can be obtained by adapting the proof in the inert case. Moreover, in the case when p ramifies in K we just have to do an small modification. Henceforth, we focus in the cases of p inert and split in K.*

Following the previous remark, to prove Theorem 2.3 it suffices to prove the following two theorems:

Theorem 2.4. *(Partial Bianchi control theorem for p inert)*

Let $\lambda \in L^\times$. Then, when $v_p(\lambda) < \min\{k,\ell\}+1$, the restriction of the specialisation map

$$\rho : \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathbb{D}_{k,\ell}(L))^{\{U_p=\lambda\}} \longrightarrow \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(L))^{\{U_p=\lambda\}}$$

(where the superscript $\{U_p=\lambda\}$ denotes the λ-eigenspace for U_p) is an isomorphism.

Theorem 2.5. *(Partial Bianchi control theorem for p split as $\mathfrak{p}\overline{\mathfrak{p}}$)*

Take $\lambda_1,\lambda_2 \in L^\times$ with $v(\lambda_1) < k+1, v(\lambda_2) < \ell+1$. Then the restriction of the specialisation map

$$\rho : \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathbb{D}_{k,\ell}(L))^{U_{\mathfrak{p}}=\lambda_1,U_{\overline{\mathfrak{p}}}=\lambda_2} \longrightarrow \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(L))^{U_{\mathfrak{p}}=\lambda_1,U_{\overline{\mathfrak{p}}}=\lambda_2}$$

(where the superscript denotes the simultaneous λ_1-eigenspace of $U_{\mathfrak{p}}$ and λ_2-eigenspace of $U_{\overline{\mathfrak{p}}}$) is an isomorphism.

Before start the proof of Theorems 2.4 and 2.5 in sections 2.2.4 and 2.2.5, we give an alternative description of rigid analytic distributions which will help us in the proofs.

Aside on integral rigid analytic distributions

In the next two sections, we would prove Theorems 2.4 and 2.5 using integral distributions, accordingly in this section we introduce the necessary terminology.

Consider R to be either a p-adic field or the ring of integers in a finite extension of $\mathbb{Q}_p$.

Definition 2.16. *Let $\mu \in \mathbb{D}_2(R)$ be a two variable distribution. Define the* moments *of μ to be the values $(\mu(x^iy^j))_{i,j\geqslant 0}$, noting that these values totally determine the distribution since the span of the x^iy^j is dense in $\mathbb{A}_2(R)$.*

Remark 2.7. *Note that we can identify $\mathbb{D}_2(L)$ with the set of doubly indexed bounded sequences in L obtaining*

$$\mathbb{D}_2(L) \cong \mathbb{D}_2(\mathcal{O}_L) \otimes_{\mathcal{O}_L} L,$$

where $\mathbb{D}_2(\mathcal{O}_L)$ is the subspace of $\mathbb{D}_2(L)$ consisting of distributions with integral moments.

To obtain the concrete link with overconvergent partial modular symbols with values on integral distributions we have first the following:

Lemma 2.1. *Let $\mathfrak{n} = (p)\mathfrak{m}$ be an ideal of $\mathcal{O}_K$ with $((p),\mathfrak{m}) = 1$, let $C_i = \Gamma_i(\mathfrak{m})\infty \cup \Gamma_i(\mathfrak{m})0$ be our usual $\Gamma_i(\mathfrak{n})$-invariant subset of cusps in $\mathbb{P}^1(K)$. Then $\Delta_{C_i}^0$, the subgroup of divisors in C_i of degree 0 is a finitely generated $\mathbb{Z}[\Gamma_i(\mathfrak{n})]$-module.*

Proof. This is the analogous result for $\Delta^0_{C_i}$ of Lemma 3.8 in [41] and it is proved in the same way. $\square$

Proposition 2.5. *Suppose D have the structure of both a $\mathcal{O}_L$-module and a right $\Gamma_i(\mathfrak{n})$-module. Then we have*

$$\mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(D \otimes_{\mathcal{O}_L} L) \cong \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(D) \otimes_{\mathcal{O}_L} L.$$

Proof. Let $\phi \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(D \otimes_{\mathcal{O}_L} L)$. Using Lemma 2.1, take a finite set of generators $\beta_1, ..., \beta_n$ for $\Delta^0_{C_i}$ as a $\mathbb{Z}[\Gamma_i(\mathfrak{n})]$-module. We can find some element $x \in \mathcal{O}_L$ such that $x\phi(\beta_j) \in D$ for each j and we obtain that $x\phi \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(D)$. $\square$

2.2.4 Proof of partial Bianchi control theorem (p inert)

In this section we prove Theorem 2.4.

In section 4 in [41] it is proved with a general machinery a lifting theorem (Theorem 4.1). Replacing such machinery with the suitable one coming from our setting, we can prove our partial Bianchi control theorems.

We now define the analogous objects in our setting to the ones defined in Section 4 in [41], we follow the same order as introduced there.

(i) The monoid $\Sigma_0(\mathcal{O}_L)^2$.

(ii) The $\mathcal{O}_L$-module $\mathbb{D}_{k,\ell}(\mathcal{O}_L)$ with the right action of $\Sigma_0(\mathcal{O}_L)^2$.

(iii) Define:

  (a) $\mathcal{F}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) := \{\mu \in \mathbb{D}_{k,\ell}(\mathcal{O}_L) : \mu(x^iy^j) \in \pi_L^{N-i-j}\mathcal{O}_L\}$.

  (b) $\mathbb{D}^0_{k,\ell}(\mathcal{O}_L) := \{\mu \in \mathbb{D}_{k,\ell}(\mathcal{O}_L) : \mu(x^iy^j) = 0$ for $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant \ell\}$.

  (c) $F^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) := \mathcal{F}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) \cap \mathbb{D}^0_{k,\ell}(\mathcal{O}_L)$.

Then we have the $\Sigma_0(\mathcal{O}_L)^2$-stable filtration of $\mathbb{D}_{k,\ell}(\mathcal{O}_L)$

$$\mathbb{D}_{k,\ell}(\mathcal{O}_L) \supset F^0\mathbb{D}_{k,\ell}(\mathcal{O}_L) \supset F^1\mathbb{D}_{k,\ell}(\mathcal{O}_L) \supset \cdots$$

(where $\Sigma_0(\mathcal{O}_L)^2$-stability is proven in the same way as Proposition 3.12 in [41]). If we define

$$A^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) := \frac{\mathbb{D}_{k,\ell}(\mathcal{O}_L)}{F^N\mathbb{D}_{k,\ell}(\mathcal{O}_L)},$$

then we have

$$\mathbb{D}_{k,\ell}(\mathcal{O}_L) \cong \varprojlim A^N\mathbb{D}_{k,\ell}(\mathcal{O}_L),$$

and where the $F^N\mathbb{D}_{k,\ell}(\mathcal{O}_L)$ have trivial intersection.

(iv) Let $\lambda \in L^*$, define the right $\Sigma_0(\mathcal{O}_L)^2$-stable submodule

$$V_{k,\ell}^{\lambda}(\mathcal{O}_L) := \left\{ f \in V_{k,\ell}^{*}(\mathcal{O}_L) : f(x^i y^j) \in \lambda p^{-(i+j)}\mathcal{O}_L, 0 \leqslant i+j \leqslant \lfloor v(\lambda) \rfloor \right\},$$

of $A^0\mathbb{D}_{k,\ell}(\mathcal{O}_L) \cong V_{k,\ell}^{*}(\mathcal{O}_K)$ and denote

$$D = \{\mu \in \mathbb{D}_{k,\ell}(\mathcal{O}_L) : \mu(\mathrm{mod}\ F^0(\mathbb{D}_{k,\ell}(\mathcal{O}_L))) \in V_{k,\ell}^{\lambda}(\mathcal{O}_L)\}.$$

(v) The operator

$$U_p := \sum_{[\alpha]\in\mathcal{O}_K/(p)} \begin{pmatrix} 1 & \alpha \\ 0 & p \end{pmatrix}.$$

noting that for each α we can see $\begin{pmatrix} 1 & \alpha \\ 0 & p \end{pmatrix}$ in $\Sigma_0(\mathcal{O}_L)^2$.

(vi) For each $i = 1, ..., h$ we have the subgroup $\Gamma_i(\mathfrak{n}) \subset \Sigma_0(\mathcal{O}_L)^2$ that for all $\begin{pmatrix} 1 & \eta \\ 0 & p \end{pmatrix}$ appearing in the U_p operator satisfies,

$$\Gamma_i(\mathfrak{n}) \begin{pmatrix} 1 & \eta \\ 0 & p \end{pmatrix} \Gamma_i(\mathfrak{n}) = \coprod_{i=0}^{r} \Gamma_i(\mathfrak{n}) \begin{pmatrix} 1 & \alpha \\ 0 & p \end{pmatrix};$$

(vii) let $C_i = \Gamma_i(\mathfrak{m})\infty \cup \Gamma_i(\mathfrak{m})0$ be our usual $\Gamma_i(\mathfrak{n})$-invariant subset of cusps in $\mathbb{P}^1(K)$, then we have the (countable) left $\mathbb{Z}[\Gamma_i(\mathfrak{n})]$-module $\Delta_{C_i}^0$.

Note that we have $\Sigma_0(\mathcal{O}_L)^2$-equivariant projection maps π^N from $\mathbb{D}_{k,\ell}(\mathcal{O}_L)$ to $A^N\mathbb{D}_{k,\ell}(\mathcal{O}_L)$, and we see that π_0 is in fact the map

$$\pi^0 : \mathbb{D}_{k,\ell}(\mathcal{O}_L) \longrightarrow V_{k,\ell}^{*}(\mathcal{O}_L)$$

that gives rise to the ($\Sigma_0(\mathcal{O}_L)^2$-equivariant) specialisation map

$$\rho^0 : \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(\mathbb{D}_{k,\ell}(\mathcal{O}_L)) \longrightarrow \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^{*}(\mathcal{O}_L)).$$

We now state the analogous to Lemma 3.15 in [41].

Lemma 2.2. *(i) Let $\mu \in \mathbb{D}_{k,\ell}(\mathcal{O}_L)$ be such that $\pi^0(\mu) \in V_{k,\ell}^{\lambda}(\mathcal{O}_L)$. Then, for $a_i \in \mathcal{O}_L$, we have*

$$\mu \left| \left[\begin{pmatrix} 1 & a_1 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & p \end{pmatrix} \right] \right. \in \lambda\mathbb{D}_{k,\ell}(\mathcal{O}_L).$$

(ii) Let $\mu \in F^N\mathbb{D}_{k,\ell}(\mathcal{O}_L)$, and suppose $v(\lambda) < \min\{k,\ell\} + 1$. Then

$$\mu \left| \left[\begin{pmatrix} 1 & a_1 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & p \end{pmatrix} \right] \right. \in \lambda F^{N+1}\mathbb{D}_{k,\ell}(\mathcal{O}_L).$$

Proof. The proof is the same as the proof of Lemma 3.15 in [41] adapting the corresponding objects of our setting. □

The above Lemma describes the action of the matrices appearing in the definition of U_p on our above filtrations. We highlight the change in the condition on the valuation of λ, namely, $v(\lambda) < \min\{k,\ell\}+1$ compared to the parallel weight situation.

Remark 2.8. *By Theorem 4.1 in [41] and Lemma 2.2, for $\lambda \in L^\times$ such that $v_p(\lambda) < \min\{k,\ell\}+1$ the map*

$$\rho^0 : \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(D) \to \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^{\lambda}(\mathcal{O}_L))$$

restricted λ-eigenspaces of the U_p operator is an isomorphism.

Theorem 2.6. *Let $\lambda \in L^\times$. Then, when $v_p(\lambda) < \min\{k,\ell\}+1$, the restriction of the specialisation map*

$$\rho^0 : \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(\mathbb{D}_{k,\ell}(L))^{\{U_\mathfrak{p}=\lambda\}} \longrightarrow \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^*(L))^{\{U_\mathfrak{p}=\lambda\}}$$

(where the superscript $\{U_p=\lambda\}$ denotes the λ-eigenspace for U_p) is an isomorphism.

Proof. By Remark 2.8 we have that the map

$$\rho^0 : \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(D)^{\{U_p=\lambda\}} \to \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^{\lambda}(\mathcal{O}_L))^{\{U_p=\lambda\}}$$

is an isomorphism. The result now follows by right-exactness of tensor product and Proposition 2.5, since $D \otimes_{\mathcal{O}_L} L \cong \mathbb{D}_{k,\ell}(L)$ and $V_{k,\ell}^{\lambda} \otimes_{\mathcal{O}_L} L \cong V_{k,\ell}^*(L)$. □

Note that Theorem 2.4 follows from Theorem 2.6 since the U_p operator acts separately on each component.

2.2.5 Proof of partial Bianchi control theorem (p split)

In this section we prove Theorem 2.5. The idea is to lift a partial Bianchi modular symbol to a space of partial Bianchi modular symbols that are overconvergent in one variable, and then again from this space to the space of fully overconvergent partial Bianchi modular symbols we considered previously.

To do so, let R to be either a p-adic field or the ring of integers in a finite extension of $\mathbb{Q}_p$ and consider the space

$$[\mathbb{D}_k \otimes V_\ell^*](R) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}_k(R) \otimes_R V_\ell(R), R),$$

with the appropriate action of $\Sigma_0(\mathcal{O}_L)^2$ (where this makes sense) induced from the action on $\mathbb{A}_{k,\ell}(L)$. This gives us

$$V_{k,\ell}(R) \subset [\mathbb{D}_k \otimes V_\ell^*](R) \subset \mathbb{D}_{k,\ell}(R).$$

To lift the operators $U_\mathfrak{p}$ and $U_{\overline{\mathfrak{p}}}$ we have to use two times the Theorem 4.1 in [41], in the same way as section 6.1 in op. cit. As before, we just describe the objects from section 4 in [41] but this time since we will use the theorem twice there will be two filtrations and spaces, one on the first variable, denoted by 1) and one in the second variable, denoted by 2).

Here we describe what is every point of the theorem in our context:

(i) The monoid $\Sigma_0(\mathcal{O}_L)^2$.

(ii) The $\mathcal{O}_L$-module $\mathbb{D}_{k,\ell}(\mathcal{O}_L)$ with the right action of $\Sigma_0(\mathcal{O}_L)^2$.

(iii) 1) Define:

(a) $\mathcal{F}^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) = \{\mu \in [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) : \mu(x^i y^j) \in \pi_L^{N-i}\mathcal{O}_L \text{ for all } j\}$,

(b) $[\mathbb{D}_k \otimes V_\ell^*]^0(\mathcal{O}_L) = \{\mu \in [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) : \mu(x^i y^j) = 0 \text{ for all } 0 \leqslant i \leqslant k\}$,

(c) $F^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) = \mathcal{F}^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \cap [\mathbb{D}_k \otimes V_\ell^*]^0(\mathcal{O}_L)$.

2) Define:

(i) $\mathcal{F}_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) = \{\mu \in \mathbb{D}_{k,\ell}(\mathcal{O}_L) : \mu(x^i y^j) \in \pi_L^{N-j}\mathcal{O}_L \text{ for all } i\}$,

(ii) $\mathbb{D}^0_{k,\ell,\mathfrak{p}}(\mathcal{O}_L) = \{\mu \in \mathbb{D}_{k,\ell}(\mathcal{O}_L) : \mu(x^i y^j) = 0 \text{ for all } 0 \leqslant j \leqslant \ell\}$,

(iii) $F_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) = \mathcal{F}_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) \cap \mathbb{D}^0_{k,\ell,\mathfrak{p}}(\mathcal{O}_L)$.

Then we have the $\Sigma_0(\mathcal{O}_L)^2$-stable filtrations of $\mathbb{D}_{k,\ell}(\mathcal{O}_L)$

$$\mathcal{F}^0[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \subset \cdots \subset \mathcal{F}^M[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \subset \cdots \subset [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \subset \\ \cdots \subset \mathcal{F}_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) \subset \cdots \subset \mathbb{D}_{k,\ell}(\mathcal{O}_L).$$

(where $\Sigma_0(\mathcal{O}_L)^2$-stability is proven in the same way as Proposition 6.3 in [41]).

1) If we define

$$A^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) := \frac{[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L)}{F^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L)},$$

then we have

$$[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \cong \varprojlim A^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L),$$

and where the $F^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L)$ have trivial intersection.

2) If we define

$$A_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L) := \frac{\mathbb{D}_{k,\ell}(\mathcal{O}_L)}{F_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L)},$$

then we have

$$\mathbb{D}_{k,\ell}(\mathcal{O}_L) \cong \varprojlim A_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L),$$

and where the $F_{\mathfrak{p}}^N\mathbb{D}_{k,\ell}(\mathcal{O}_L)$ have trivial intersection.

(iv) 1) Let $\lambda_1 \in L^*$, define the right $\Sigma_0(\mathcal{O}_L)^2$-stable submodule

$$V^{\lambda_1}_{k,\ell,\mathfrak{p}}(\mathcal{O}_L) := \left\{f \in V^*_{k,\ell}(\mathcal{O}_L) : f(x^i y^j) \in \lambda_1 p^{-i}\mathcal{O}_L, 0 \leqslant i \leqslant \lfloor v(\lambda_1)\rfloor\right\},$$

of $A^0[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \cong V^*_{k,\ell}(\mathcal{O}_K)$.

2) Let $\lambda_2 \in L^*$, define the right $\Sigma_0(\mathcal{O}_L)^2$-stable submodule

$$[\mathbb{D}_k \otimes V_\ell^*]^{\lambda_2}(\mathcal{O}_L) := \left\{ f \in [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) : f(x^i y^j) \in \lambda_2 p^{-j} \mathcal{O}_L, 0 \leqslant j \leqslant \lfloor v(\lambda_2) \rfloor \right\},$$

of $[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L)$.

(v) Choose n such that $\mathfrak{p}^n = (\beta)$ is principal (noting that this also forces $\overline{\mathfrak{p}}^n$ to be principal). Then we define $U_{\mathfrak{p}^n} = U_{\mathfrak{p}}^n$ as

$$\sum_{a \bmod \mathfrak{p}^n} \begin{pmatrix} 1 & a \\ 0 & \beta \end{pmatrix}.$$

noting that for each a we can see $\begin{pmatrix} 1 & a \\ 0 & \beta \end{pmatrix}$ in $\Sigma_0(\mathcal{O}_L)^2$.

(vi) For each $i = 1, ..., h$ we have the subgroup $\Gamma_i(\mathfrak{n}) \subset \Sigma_0(\mathcal{O}_L)^2$ that for all $\begin{pmatrix} 1 & \eta \\ 0 & \beta \end{pmatrix}$ appearing in the $U_{\mathfrak{p}^n}$ operator satisfies,

$$\Gamma_i(\mathfrak{n}) \begin{pmatrix} 1 & \eta \\ 0 & \beta \end{pmatrix} \Gamma_i(\mathfrak{n}) = \coprod_{i=0}^{r} \Gamma_i(\mathfrak{n}) \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix};$$

(vii) let $C_i = \Gamma_i(\mathfrak{m})\infty \cup \Gamma_i(\mathfrak{m})0$ be our usual $\Gamma_i(\mathfrak{n})$-invariant subset of cusps in $\mathbb{P}^1(K)$, then we have the (countable) left $\mathbb{Z}[\Gamma_i(\mathfrak{n})]$-module $\Delta^0_{C_i}$.

The filtrations above lead to $\Sigma_0(\mathcal{O}_L)^2$-equivariant projection maps

$$\pi_1^N : [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \longrightarrow A^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L)$$

and

$$\pi_2^N : \mathbb{D}_{k,\ell}(\mathcal{O}_L) \longrightarrow A_{\mathfrak{p}}^N \mathbb{D}_{k,\ell}(\mathcal{O}_L),$$

which again give maps ρ_1^N and ρ_2^N on the corresponding symbol spaces.

We see that π_1^0 and π_2^0 are in fact the maps:

$$\pi_1^0 : [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \to V_{k,\ell}^*(\mathcal{O}_L) \supset V_{k,\ell,\mathfrak{p}}^\lambda(\mathcal{O}_L),$$

$$\pi_2^0 : \mathbb{D}_{k,\ell}(\mathcal{O}_L) \to [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L) \supset [\mathbb{D}_k \otimes V_\ell^*]^\lambda(\mathcal{O}_L).$$

The following two lemmas have the same idea of Lemma 2.2 in our current setting.

Lemma 2.3. *Let $a_1, a_2 \in \mathcal{O}_L$ and $\mathfrak{p}^n = (\beta)$.*

(i) Suppose $\mu \in [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L)$ with $\pi_1^0(\mu) \in V_{k,k,\mathfrak{p}}^\lambda(\mathcal{O}_L)$. Then

$$\mu \Big| \left[\begin{pmatrix} 1 & a_1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \overline{\beta} \end{pmatrix} \right] \in \lambda [\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L).$$

(ii) Suppose $v(\lambda) < n(k+1)$. Then for $\mu \in F^N[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L)$, we have

$$\mu \Bigg| \left[\begin{pmatrix} 1 & a_1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \overline{\beta} \end{pmatrix} \right] \in \lambda F^{N+1}[\mathbb{D}_k \otimes V_\ell^*](\mathcal{O}_L).$$

Lemma 2.4. *Let $a_1, a_2 \in \mathcal{O}_L$ and $\overline{\mathfrak{p}}^n = (\delta)$.*

(i) Suppose $\mu \in \mathbb{D}_{k,\ell}(\mathcal{O}_L)$ with $\pi_2^0(\mu) \in [\mathbb{D}_k \otimes V_\ell^*]^\lambda(\mathcal{O}_L)$. Then

$$\mu \Bigg| \left[\begin{pmatrix} 1 & a_1 \\ 0 & \overline{\delta} \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \delta \end{pmatrix} \right] \in \lambda \mathbb{D}_{k,\ell}(\mathcal{O}_L).$$

(ii) Suppose $v(\lambda) < n(k+1)$. Then for $\mu \in F_{\mathfrak{p}}^N \mathbb{D}_{k,\ell}(\mathcal{O}_L)$, we have

$$\mu \Bigg|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & \overline{\delta} \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \delta \end{pmatrix} \right] \in \lambda F_{\mathfrak{p}}^{N+1} \mathbb{D}_{k,\ell}(\mathcal{O}_L).$$

The proof of both Lemmas above is identical to the proof of Lemmas 6.6 and 6.7 in [41] respectively, as before just by replacing the objects there for the objects in our setting.

Lemmas 2.3 and 2.4 together with Theorem 4.1 in [41] applied twice and some details regarding tensor products (which are analogous to the proof of Theorem 2.6) give us the following Lemma:

Lemma 2.5. *Let $\lambda_1, \lambda_2 \in L^\times$. Then, when $v(\lambda_1) < n(k+1)$ and $v(\lambda_2) < n(\ell+1)$, with n an integer such that $\mathfrak{p}^n$ is principal we have:*

(i) The restriction of the specialisation map

$$\rho_1^0 : \mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}([\mathbb{D}_k \otimes V_\ell^*](L))^{U_{\mathfrak{p}}^n = \lambda_1} \longrightarrow \mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}(V_{k,\ell}^*(L))^{U_{\mathfrak{p}}^n = \lambda_1}$$

(where the superscript $(U_{\mathfrak{p}} = \lambda_1)$ denotes the λ_1-eigenspace for $U_{\mathfrak{p}}$) is an isomorphism.

(ii) The restriction of the specialisation map

$$\rho_2^0 : \mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}(\mathbb{D}_{k,\ell}(L))^{U_{\overline{\mathfrak{p}}}^n = \lambda_2} \longrightarrow \mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}([\mathbb{D}_k \otimes V_\ell^*](L))^{U_{\overline{\mathfrak{p}}}^n = \lambda_2}$$

is an isomorphism.

Theorem 2.7. *Take $\lambda_1, \lambda_2 \in L^*$ with $v(\lambda_1) < k+1, v(\lambda_2) < \ell+1$. Then the restriction of the specialisation map*

$$\rho^0 : \mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}(\mathbb{D}_{k,\ell}(L))^{U_{\mathfrak{p}}^n = \lambda_1^n, U_{\overline{\mathfrak{p}}}^n = \lambda_2^n} \longrightarrow \mathrm{Symb}_{\Gamma_i(\mathfrak{n}), C_i}(V_{k,\ell}^*(L))^{U_{\mathfrak{p}}^n = \lambda_1^n, U_{\overline{\mathfrak{p}}}^n = \lambda_2^n}$$

(where the superscript denotes the simultaneous λ_1^n-eigenspace of $U_{\mathfrak{p}}^n$ and λ_2^n-eigenspace of $U_{\overline{\mathfrak{p}}}^n$) is an isomorphism.

Proof. The proof follows exactly in the same way as Theorem 6.10 (i) in [41], by using Lemma 2.5 to lift a simultaneous $U_{\mathfrak{p}}^n$- and $U_{\overline{\mathfrak{p}}}^n$-eigensymbol ϕ^0, with eigenvalues λ_1^n, λ_2^n respectively to some overconvergent eigensymbol. □

We conclude by noting that the proof of Theorem 2.5 is analogous to the proof of Theorem 6.10 (ii) in [41]. Such proof is done by using Lemma 6.9 in [41] to obtain the result for the $U_{\mathfrak{p}}$ operator instead $U_{\mathfrak{p}}^n$ and then just express the direct sum over $1 \leqslant i \leqslant h$.

2.2.6 Admissible distributions

For each pair r, s, the space $\mathbb{D}_{k,\ell}[L,r,s]$ admits an operator norm $||\cdot||_{r,s}$ via

$$||\mu||_{r,s} = \sup_{0\neq f\in\mathbb{A}_{k,\ell}[L,r,s]} \frac{|\mu(f)|_p}{|f|_{r,s}},$$

where $|\cdot|_p$ is the usual p-adic absolute value on L and $|\cdot|_{r,s}$ is the sup norm on $\mathbb{A}_{k,\ell}[L,r,s]$. Note that if $r \leqslant r'$, $s \leqslant s'$, then $||\mu||_{r,s} \geqslant ||\mu||_{r',s'}$ for $\mu \in \mathbb{D}_{k,\ell}[L,r',s']$.

These norms give rise to a family of norms on the space of locally analytic functions that allow us to classify locally analytic distributions by growth properties as we vary in this family, motivating the definition of admissible distributions (see Definitions 5.10 and 6.14 in [41]).

Proposition 2.6. *Let* $\Psi \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(\mathcal{D}_{k,\ell}(L))$,

(i) Suppose p is inert in K and Ψ is a U_p-eigensymbol with eigenvalue λ and slope $h = v(\lambda)$. Then, for every $D \in \Delta^0_{C_i}$, the distribution $\Psi(D)$ is h-admissible.

(ii) Suppose p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$ and Ψ is simultaneously a $U_{\mathfrak{p}}^n$- and $U_{\overline{\mathfrak{p}}}^n$-eigensymbol with non-zero eigenvalues λ_1^n and λ_2^n with slopes $h_1 = v(\lambda_1)$ and $h_2 = v(\lambda_2)$. Then, for every $D \in \Delta^0_{C_i}$, the distribution $\Psi(D)$ is (h_1,h_2)-admissible.

Proof. This proposition is the adaptation to the C_i-cuspidal case of Propositions 5.12 y 6.15 in [41] and is proved in the same way. □

2.2.7 Mellin transform of overconvergent partial Bianchi modular symbols

In this short section we define the Mellin transform of a partial Bianchi modular symbol, which will be useful to define the p-adic L-function of a Bianchi modular form in Chapter 4.

Recall from section 1.1.2 the definition of the ray class group $\mathrm{Cl}_K(p^\infty)$, which can be written as

$$\mathrm{Cl}_K(p^\infty) = \bigcup_{i\in\mathrm{Cl}_K} \mathrm{Cl}_K^i(p^\infty),$$

where $\mathrm{Cl}_K^i(p^\infty)$ is the fibre of i under the canonical surjection $\mathrm{Cl}_K(p^\infty) \twoheadrightarrow \mathrm{Cl}_K$ to the class group of K.

The choice of representative $t_i \in \mathbb{A}_K^{f,\times}$ identifies $\mathrm{Cl}_K^i(p^\infty)$ non-canonically with $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times/\mathcal{O}_K^\times$.

Let $\Psi = (\Psi_1, ..., \Psi_h) \in \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathcal{D}_{k,\ell}(L))$ be an overconvergent partial Bianchi modular symbol, then we define for $i,j \in \mathrm{Cl}_K$ a distribution $\mu_i(\Psi_j) \in \mathcal{D}(\mathrm{Cl}_K^i(p^\infty), L)$ as follows.

Since $\{0\} - \{\infty\} \in \Delta^0_{C_i}$ for all i, we have a distribution $\Psi_j(\{0\} - \{\infty\})|_{(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times}$ on $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$. This restricts to a distribution on $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \mathcal{O}_K^\times$, which gives the distribution $\mu_i(\Psi_j)$ on $\mathrm{Cl}^i_K(p^\infty)$ under the identification above.

Definition 2.17. *The Mellin transform of* $\Psi \in \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathcal{D}_{k,\ell}(L))$ *is the (L-valued) locally analytic distribution on* $\mathrm{Cl}_K(p^\infty)$ *given by*

$$\mathrm{Mel}(\Psi) := \sum_{i \in \mathrm{Cl}_K} \mu_i(\Psi_i) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty), L).$$

Remark 2.9. *The distribution* $\mathrm{Mel}(\Psi)$ *is independent of the choice of class group representatives.*

2.3 Eigenvarieties and p-adic families

As mentioned in the introduction, p-adic families describe the variation of automorphic forms (and in general of automorphic representations) as their weight varies p-adic analytically. They are important tools in the Langlands program and the Bloch–Kato conjectures for example. Their behaviour is captured geometrically in the theory of eigenvarieties. The objective of this section is to introduce the background necessary about eigenvarieties to recall the construction of the p-adic L-function of critical slope base change cuspidal Bianchi modular forms in section 3.2.

2.3.1 Coleman-Mazur eigencurve

The Coleman-Mazur eigencurve is a p-adic rigid analytic curve which parametrizes overconvergent elliptic modular eigenforms of finite slope.

To describe it in more detail, consider a positive integer N and let $\mathcal{W}_{\mathbb{Q}}$ be the weight space for $\mathrm{GL}_2/\mathbb{Q}$, that is, the rigid analytic space whose L-points are $\mathcal{W}_{\mathbb{Q}}(L) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, L^\times)$ for $L \subset \mathbb{C}_p$. For $N = 1$, Coleman and Mazur proved the following theorem, Buzzard treating the case of arbitrary N coprime with p.

Theorem 2.8. *(Coleman-Mazur, Buzzard) There is a reduced, equidimensional rigid analytic curve* $\mathcal{C}_N$ *together with a morphism* $w : \mathcal{C}_N \to \mathcal{W}_{\mathbb{Q}}$ *and global sections* $U_p \in \mathcal{O}(\mathcal{C}_N)$, $T_l \in \mathcal{O}(\mathcal{C}_N)$ *for all* $l \nmid Np$ *, such that:*

i) The morphism w *has discrete fibers.*

ii) The points $x \in w^{-1}(\lambda)$ *in the fiber over a fixed weight* $\lambda \in \mathcal{W}_{\mathbb{Q}}$ *are in bijection with overconvergent modular eigenforms of weight* λ.

iii) If f_x *is an overconvergent modular eigenform corresponding to a point* $x \in \mathcal{C}_N$, *the eigenvalue of* T_l *(resp.* U_p*) acting on* f_x *equals the image* $T_l(x)$ *(resp.* $U_p(x)$*) of* T_l *(resp.* U_p*) in the residue field of the stalk* $\mathcal{O}_{\mathcal{C}_N,x}$.

iv) *If f is a classical normalized eigenform of weight $k+2$, with $x_f \in \mathcal{C}_N$ the associated point, then $w(x_f) = k+2$. If furthermore $v_p(a_f(p)) < k+1$, then there is an open neighborhood U of $x_f \in \mathcal{C}$ such that $w|_U : U \to \mathcal{W}_{\mathbb{Q}}$ is a homeomorphism onto its image and $T_l|_U = a_l \circ w$.*

This construction and the ideas driving it have been vastly generalized by a number of authors, resulting in a whole universe of eigenvarieties (see for example [19], [38], [1]).

2.3.2 The Bianchi eigenvariety

Define the locally symmetric space

$$Y_0(\mathfrak{n}) := \mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)/\mathbb{C}^\times\mathrm{SU}_2(\mathbb{C})\Omega_0(\mathfrak{n})$$

which is the Bianchi analogue of the modular curve.

Let $L \subset \mathbb{C}_p$ any sufficiently large extension of $\mathbb{Q}_p$ and recall from section 2.2 the definitions of:

- the Bianchi weight space $\mathcal{W}_K$;
- the space of locally analytic distributions $\mathcal{D}(L)$ with values in L;
- the semi-group $\Sigma_0(p)$.

Also recall that for $\lambda \in \mathcal{W}_K$ we denoted by $\mathcal{D}_\lambda(L)$ the space $\mathcal{D}(L)$ equipped with the weight λ right action of $\Sigma_0(p)$ given in (2.7). Note that $\mathcal{D}_\lambda(L)$ gives rise to a local system on $Y_0(\mathfrak{n})$, which we denote by $\mathscr{D}_\lambda(L)$.

Let $\mathbb{H}_{\mathfrak{n},p}$ denote the $\mathbb{Z}_p$-algebra generated by the Hecke operators $\{T_{\mathfrak{q}} : (\mathfrak{q},\mathfrak{n}) = 1\}$ and $\{U_{\mathfrak{p}} : \mathfrak{p}|p\}$ and write H_c^* for total cohomology.

Theorem 2.9. *(Hansen) There exists a separated rigid analytic space $\mathcal{E}_{\mathfrak{n}}$, and a morphism $w : \mathcal{E}_{\mathfrak{n}} \to \mathcal{W}_K$, such that for each finite extension L of $\mathbb{Q}_p$, the L-points y of $\mathcal{E}_{\mathfrak{n}}$ with $w(y) = \lambda \in \mathcal{W}_K(L)$ are in bijection with systems $\psi_y : \mathbb{H}_{\mathfrak{n},p} \to L$ of Hecke eigenvalues occurring in $H_c^*(Y_0(\mathfrak{n}), \mathscr{D}_\lambda(L))$.*

A point $y \in \mathcal{E}_{\mathfrak{n}}$ (resp. $\mathcal{C}_N$) is *classical* if there is a Bianchi (resp. classical) eigenform Φ_y (resp. f_y) of weight $w(y)$ such that $t\Phi_y = \psi_y(t)\Phi_y$ (resp. $tf_y = \psi_y(t)f_y$) for all $t \in \mathbb{H}_{\mathfrak{n},p}$ (resp. for all classical Hecke operators t).

The Bianchi eigenform has curious properties as for example that classical points in the Bianchi eigenvariety are not Zariski-dense. In [3] is constructed a 'parallel weight' eigenvariety using degree 1 overconvergent cohomology over the parallel weight line in the Bianchi weight space, this eigenvariety is better behaved as for example the desirable property that the classical points are Zariski-dense is satisfied, and by p-adic Langlands functoriality it contains all classical points corresponding to base change forms.

More specifically, if $\mathcal{W}_{K,\mathrm{par}}$ is the parallel weight line in $\mathcal{W}_K$, i.e. the image of the closed immersion $\mathcal{W}_{\mathbb{Q}} \hookrightarrow \mathcal{W}_K$ induced by the norm map $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to \mathbb{Z}_p^\times$ then we have

Theorem 2.10. *(Barrera-Williams) There exists a separated rigid analytic space $\mathcal{E}_{\mathfrak{n},\mathrm{par}}$, and a morphism $w : \mathcal{E}_{\mathfrak{n},\mathrm{par}} \to \mathcal{W}_{K,\mathrm{par}}$, such that for each finite extension L of $\mathbb{Q}_p$, the L-points y of $\mathcal{E}_{\mathfrak{n},\mathrm{par}}$ with $w(y) = \lambda \in \mathcal{W}_{K,\mathrm{par}}(L)$ are in bijection with systems $\psi_y : \mathbb{H}_{\mathfrak{n},p} \to L$ of Hecke eigenvalues occurring in $H^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))$.*

2.3.3 Base change functoriality from the Coleman-Mazur eigencurve to the Bianchi eigenvariety

We are interested in classical modular forms satisfying some conditions that we present as follows

Conditions 2.11. *Let N be divisible by p. Fix $f \in S_{k+2}(\Gamma_0(N))$ such that:*

(C1') (finite slope eigenform) f is an eigenform, and $U_p f = \lambda_p f$ with $\lambda_p \neq 0$;

(C2') (p-stabilised newform) f is new or the p-stabilisation of a newform f_{new} of level prime to p;

(C3') (regular) if f is the p-stabilisation of f_{new}, then the Hecke polynomial at p of f_{new} has two different roots, and if p is inert in K, $a_p(f_{\mathrm{new}}) \neq 0$;

(C4') (non CM) f does not have CM by K;

Conditions 2.12. *Let $\Phi \in S_{(k,k)}(\Omega_0(\mathfrak{n}))$ with $\mathfrak{n}$ divisible by each $\mathfrak{p}|p$, be a finite slope p-regular p-stabilised newform, in the sense that:*

(C1) Φ is an eigenform, and for each $\mathfrak{p}|p$, we have $U_{\mathfrak{p}}\Phi = \alpha_{\mathfrak{p}}\Phi$ with $\alpha_{\mathfrak{p}} \neq 0$;

(C2) there exist $S \subset \{\mathfrak{p}|p\}$, $\mathfrak{m}$ prime to S, and a newform $\Phi_{\mathrm{new}} \in S_{(k,k)}(\Omega_0(\mathfrak{m}))$ such that $\mathfrak{n} = \mathfrak{m}\prod_{\mathfrak{p}\in S}\mathfrak{p}$ and Φ is obtained from Φ_{new} by $\mathfrak{p}$-stabilising for $\mathfrak{p} \in S$;

(C3) for each $\mathfrak{p} \in S$, the Hecke polynomial of Φ_{new} at $\mathfrak{p}$ has distinct roots.

Note newforms of level $\mathfrak{n}$ themselves satisfy (C2),(C3) with $S = \varnothing$. We say a classical point y satisfies Conditions 2.12 if Φ_y does (resp. Conditions 2.11 if f_y does).

Theorem 3.4 in [3] show that there is a finite morphism $\mathrm{BC}_N : \mathcal{C}_N \to \mathcal{E}_{N\mathcal{O}_K}$ of rigid spaces such that if $y \in \mathcal{C}_N(L)$ corresponds to a classical modular form f, then $\mathrm{BC}_N(y) \in \mathcal{E}_{\mathfrak{n}}(L)$ corresponds to the (stabilisation to level $N\mathcal{O}_K$ of the) system of eigenvalues attached to the base change to K of f.

Remark 2.10. *As in Remark 3.6 in [3] we will assume that if $x \in \mathcal{C}_N$ satisfies Conditions 2.11, then there is a neighbourhood $V_{\mathbb{Q}}$ of x in $\mathcal{C}_N$ such that every classical point of $\mathrm{BC}_N(V_{\mathbb{Q}}) \subset \mathcal{E}_{N\mathcal{O}_K}$ satisfies Conditions 2.12, since the results in section 3.2 are locally, again by Remark 3.6 in [3] we will work with $\mathcal{E}_{\mathfrak{n}}$ for some $\mathfrak{n}|N\mathcal{O}_K$.*

By Theorem 5.1.6 in [19] and Remark 2.10 the map BC_N factors through $\mathcal{C}_N \to \mathcal{E}_{\mathfrak{n},\mathrm{par}}$. Let

$$\mathcal{E}_{\mathfrak{n},\mathrm{bc}} := \mathrm{BC}_N(\mathcal{C}_N) \subset \mathcal{E}_{\mathfrak{n},\mathrm{par}}$$

denote the image.

We finish this section with two definitions that we will use in section 3.2

Definition 2.18. *We say f is* decent *if f is non-critical (see [33]), or f has vanishing adjoint Selmer group $\mathrm{H}^1_f(\mathbb{Q}, \mathrm{ad}\rho_f) = 0$, where $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(L)$ is the p-adic Galois representation attached to f.*

Definition 2.19. *A point $x \in \mathcal{E}_{\mathfrak{n},bc}$ is Σ-smooth if every irreducible component $I \subset \mathcal{E}_{\mathfrak{n},par}$ through x is contained in $\mathcal{E}_{\mathfrak{n},bc}$ (equivalently, if the natural inclusion $\mathcal{E}_{\mathfrak{n},bc} \subset \mathcal{E}_{\mathfrak{n},par}$ is locally an isomorphism at x).*

Chapter 3

p-adic L-functions

In this chapter we obtain the functional equation of p-adic L-functions attached to cuspidal Bianchi modular forms and we construct the p-adic L-function of non-cuspidal base change Bianchi modular forms. The results in sections 3.1 and 3.2 are contained in the first paper of the author [30] and are under the assumption of class number 1. The results in section 3.3 are contained in the second paper of the author [31] and there is no restriction on the class number.

3.1 Functional equation of the p-adic L-function of small slope cuspidal Bianchi modular forms

The p-adic L-function of a Bianchi modular form Φ is defined as a locally analytic distribution on $\mathrm{Cl}_K(p^\infty) := K^\times\backslash\mathbb{A}_K^\times/\mathbb{C}^\times\prod_{v\nmid p}\mathcal{O}_v^\times$ that interpolates the classical L-values of Φ.

3.1.1 Construction of the p-adic L-function (Williams)

In [41], Williams constructs the p-adic L-function of a small slope Bianchi eigenform $\mathcal{F}$ by developing the theory of overconvergent Bianchi modular symbols of parallel weight. In such construction he attaches to $\mathcal{F}$ a classical Bianchi eigensymbol $\phi_{\mathcal{F}}$ (this is done in the same way as we did for C-cuspidal forms in section 2.1.3) with coefficients in a p-adic field L, and lifts it to its corresponding unique overconvergent Bianchi eigensymbol $\Psi_{\mathcal{F}}$ (again, in the same way as we did in sections 2.2.4 and 2.2.5). Then the p-adic L-function of $\mathcal{F}$ is the locally analytic distribution on $\mathrm{Cl}_K(p^\infty)$ defined by

$$L_p(\mathcal{F},-) = \mathrm{Mel}(\Psi_{\mathcal{F}}),$$

and is it proved that satisfies the interpolation and admissibility properties desired (see Definitions 5.10 and 6.14 in [41]). More precisely he obtains in Theorem 7.4 in [41]:

Theorem 3.1. *(Williams) Let $\mathcal{F}$ be a cuspidal Bianchi modular eigenform of weight (k,k) and level $\Gamma_0(\mathfrak{n})$, where $(p)|\mathfrak{n}$, with $U_{\mathfrak{p}}$-eigenvalues $\lambda_{\mathfrak{p}}$, where $v(\lambda_{\mathfrak{p}}) < (k+1)/e_{\mathfrak{p}}$ for all $\mathfrak{p}|p$. Let $\Omega_{\mathcal{F}}$ be the complex period in Proposition 1.6. Then there exists a locally analytic distribution $L_p(\mathcal{F},-)$ on $\mathrm{Cl}_K(p^\infty)$ such that for any Hecke character ψ of K of conductor $\mathfrak{f}|(p^\infty)$ and infinity type*

$0 \leqslant (q,r) \leqslant (k,k)$, *we have*

$$L_p(\mathcal{F},\psi_{p-\mathrm{fin}}) = \left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\lambda_{\mathfrak{p}}}(\psi)\right)\left[\frac{Dw\tau(\psi^{-1})}{(-1)^{k+q+r}2\lambda_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]\Lambda(\mathcal{F},\psi), \tag{3.1}$$

with $Z_{\mathfrak{p}}^{\lambda_{\mathfrak{p}}}(\psi)$ *as in (1.24).*

The distribution $L_p(\mathcal{F},-)$ *is* $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$*-admissible, where* $h_{\mathfrak{p}} = v_p(\lambda_{\mathfrak{p}})$*, and hence is unique.*

Remark 3.1. *When* p *ramifies, in the Theorem above it suffices* $\mathfrak{p}|\mathfrak{n}$ *instead* $(p)|\mathfrak{n}$*.*

3.1.2 Functional equation

In this section we obtain the functional equation of the p-adic L-function of a small slope p-stabilisation of a cuspidal Bianchi modular eigenform.

Let $\mathcal{F}_p$ be a Bianchi modular form obtained by successively stabilising at each different prime $\mathfrak{p}$ above p a newform $\mathcal{F} \in S_{(k,k)}(\Gamma_0(\mathfrak{n}))$, with $\mathfrak{n} = (\nu)$ prime to (p). Recall that $\mathcal{F}$ is an eigenform for the Fricke involution $W_{\mathfrak{n}}$, with $\mathcal{F}|_{W_{\mathfrak{n}}} = \epsilon(\mathfrak{n})\mathcal{F}$ with $\epsilon(\mathfrak{n}) = \pm 1$ (see section 2 in [11]).

Proposition 3.1. *If* $\mathcal{F}_p$ *has small slope, then for any Hecke character* ψ *of conductor* $\mathfrak{f}|(p^\infty)$ *with* $\mathfrak{f} = (f)$ *and infinity type* $0 \leqslant (q,r) \leqslant (k,k)$*, the distribution* $L_p(\mathcal{F}_p,-)$ *satisfies*

$$L_p(\mathcal{F}_p,\psi_{p-\mathrm{fin}}) = (-1)^{k+1}\epsilon(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{k/2}\psi_{p-\mathrm{fin}}^{-1}(x_{-\nu,p})L_p(\mathcal{F}_p,\psi_{p-\mathrm{fin}}^{-1}\sigma_p^{k,k}),$$

where $x_{-\nu,p}$ *is the idele associated to* $-\nu$ *defined in Remark 1.3 and* $\sigma_p^{k,k}$ *as in equation (1.1).*

Proof. By Theorem 3.1 we have the following interpolations

$$L_p(\mathcal{F}_p,\psi_{p-\mathrm{fin}}) = \left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)\right)\left[\frac{Dw\tau(\psi^{-1})}{(-1)^{k+q+r}2\lambda_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]\Lambda(\mathcal{F}_p,\psi). \tag{3.2}$$

$$L_p(\mathcal{F}_p,(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})_{p-\mathrm{fin}}) = \left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\right)\left[\frac{Dw\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})}{(-1)^{k+q+r}2\lambda_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]\Lambda(\mathcal{F}_p,\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k}). \tag{3.3}$$

By (3.2), Lemma 1.9 and (3.3) we have

$$\begin{aligned}
L_p(\mathcal{F}_p,\psi_{p-\mathrm{fin}}) &= \left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi)\right)\left[\frac{Dw\tau(\psi^{-1})}{(-1)^{k+q+r}2\lambda_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]\Lambda(\mathcal{F}_p,\psi)\\
&= \left[\frac{Dw\tau(\psi^{-1})}{(-1)^{k+q+r}2\lambda_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]\varepsilon(\mathcal{F},\psi)\left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}^{\alpha_{\mathfrak{p}}}(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\right)\Lambda(\mathcal{F}_p,\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})\\
&= \left[\frac{Dw\tau(\psi^{-1})}{(-1)^{k+q+r}2\lambda_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]\varepsilon(\mathcal{F},\psi)\left[\frac{Dw\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})}{(-1)^{k+q+r}2\lambda_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]^{-1} L_p(\mathcal{F}_p,(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})_{p-\mathrm{fin}})\\
&= \varepsilon(\mathcal{F},\psi)\tau(\psi^{-1})\tau(\psi|\cdot|_{\mathbb{A}_K}^{-k})^{-1}L_p(\mathcal{F}_p,(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})_{p-\mathrm{fin}})\\
&= (-1)^{k+1}\epsilon(\mathfrak{n})|\nu|^{k}\psi_{\mathfrak{f}}^{-1}(-\nu)\psi_{\infty}^{-1}(-\nu)L_p(\mathcal{F}_p,(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})_{p-\mathrm{fin}}).
\end{aligned}$$

By Remark 1.3 we have

$$\begin{aligned}\psi_{\mathfrak{f}}^{-1}(-\nu)\psi_{\infty}^{-1}(-\nu) &= \psi_{(p)}^{-1}(-\nu)(-\nu)^{-q}(-\overline{\nu})^{-r}\\ &= (\psi_{p-\mathrm{fin}}^{-1})_{(p)}(-\nu)\\ &= \psi_{p-\mathrm{fin}}^{-1}(x_{-\nu,p}),\end{aligned}$$

and noting that for a finite idele x we have

$$(\psi^{-1}|\cdot|_{\mathbb{A}_K}^{k})_{p-\mathrm{fin}}(x) = \psi_{p-\mathrm{fin}}^{-1}(x)(|\cdot|_{\mathbb{A}_K}^{k})_{p-\mathrm{fin}}(x) = \psi_{p-\mathrm{fin}}^{-1}(x)\sigma_p^{k,k}(x) = (\psi_{p-\mathrm{fin}}^{-1}\sigma_p^{k,k})(x),$$

we obtain the result. □

Theorem 3.2. *For $\mathcal{F}_p$ as above with small slope, the distribution $L_p(\mathcal{F}_p,-)$ satisfies the following functional equation*

$$L_p(\mathcal{F}_p,\kappa) = (-1)^{k+1}\epsilon(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{k/2}\kappa(x_{-\nu,p})^{-1}L_p(\mathcal{F}_p,\kappa^{-1}\sigma_p^{k,k}),$$

for all $\kappa\in\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$.

Proof. Define a new distribution $L_p'(\mathcal{F}_p,-)$ by

$$L_p'(\mathcal{F}_p,\kappa) := L_p(\mathcal{F}_p,\kappa) + (-1)^{k}\epsilon(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{k/2}\kappa(x_{-\nu,p})^{-1}L_p(\mathcal{F}_p,\kappa^{-1}\sigma_p^{k,k})$$

for any $\kappa\in\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$.

Since $v_p(\alpha_{\mathfrak{p}}) < (k+1)/e_{\mathfrak{p}}$ for all $\mathfrak{p}|p$, the distribution $L_p(\mathcal{F}_p,-)$ is $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible, where $h_{\mathfrak{p}} = v_p(\alpha_{\mathfrak{p}})$. Then $L_p'(\mathcal{F}_p,-)$ is $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible.

In [27] it is proved that a distribution $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible like $L_p'(\mathcal{F}_p,-)$ is uniquely determined by its values on the p-adic characters $\psi_{p-\mathrm{fin}}\in\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ that arise from Hecke characters ψ of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $0\leqslant(q,r)\leqslant(k,k)$. By Proposition 3.1 we have that $L_p'(\mathcal{F}_p,\psi_{p-\mathrm{fin}})=0$ for all $\psi_{p-\mathrm{fin}}$, then $L_p'(\mathcal{F}_p,-)=0$ and the functional equation of $L_p(\mathcal{F}_p,-)$ follows. □

Remark 3.2. *In the case when p split, the property that $L_p'(\mathcal{F}_p,-)$ is uniquely determined by its values on the p-adic characters $\psi_{p-\mathrm{fin}}$; is proved in Theorem 3.11 in [27] in the case where $v_p(\alpha_{\mathfrak{p}})<1$ for $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, which he assumes merely for simplicity. For a more detailed example of the general situation in the one variable case, see [9].*

Example: Suppose p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$. Let $\mathcal{F}$ be a newform with weight (k,k) and level $\mathfrak{n}$ prime to p with $\lambda_{\mathfrak{p}}=\lambda_{\overline{\mathfrak{p}}}=0$. Then the Hecke polynomials at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ coincide, and their roots α, β both have p-adic valuation $(k+1)/2$. Assuming $\alpha\neq\beta$ there are four choices of stabilisations of level $(p)\mathfrak{n}$ and each is small slope, giving rise to four p-adic L-functions attached to $\mathcal{F}$, each one satisfying the corresponding p-adic functional equation of Theorem 3.2.

3.2 Functional equation of critical slope base change cuspidal Bianchi modular forms

The construction of the p-adic L-function in the previous section and consequently the functional equation in Theorem 3.2 depend of the small slope of the Bianchi modular form $\mathcal{F}$.

In this section we generalize the functional equation of Theorem 3.2 for Σ-smooth base-change Bianchi modular forms, in particular, making no assumption about the slope. To this end we use the *three-variable p-adic L-function* constructed in [3] that specialises to $L_p(f_{/K}, -)$, the p-adic L-function of a base-change Bianchi modular form $f_{/K}$.

We first recall briefly the definitions and construction of such p-adic L-function.

3.2.1 Construction of the p-adic L-function (Barrera-Williams)

Let N be divisible by p. Fix $f \in S_{k+2}(\Gamma_0(N))$ satisfying conditions 2.11, i.e., f is a finite slope eigenform, p-stabilised newform, regular and non-CM. Also suppose f is decent and Σ-smooth (see definitions 2.18 and 2.19).

Then there exists a neighbourhood $V_{\mathbb{Q}}$ (as in Remark 2.10) of f in the Coleman-Mazur eigencurve such that the weight map w is étale except possibly at f.

Theorem 3.3. *(Barrera-Williams) Up to shrinking $V_{\mathbb{Q}}$, and for sufficiently large $L \subset \overline{\mathbb{Q}}_p$, there exists a unique rigid-analytic function*

$$\mathcal{L}_p : V_{\mathbb{Q}} \times \mathfrak{X}(\mathrm{Cl}_K(p^\infty)) \to L,$$

such that for any classical point $y \in V_{\mathbb{Q}}(L)$ with small slope base-change $f_{y/K}$ we have $\mathcal{L}_p(y, -) = c_y L_p(f_{y/K}, -)$, where $c_y \in L^\times$ is a p-adic period at y and $L_p(f_{y/K}, -)$ is the p-adic L-function of $f_{y/K}$ of Theorem 3.1.

Note that in [3], $\mathcal{L}_p$ depends of ϕ, a finite order Hecke character of K of conductor prime to $p\mathcal{O}_K$, and is denoted by $\mathcal{L}_p^\phi$. Here we take ϕ to be trivial.

3.2.2 Functional equation

We can transfer the functional equation in Theorem 3.2 to $\mathcal{L}_p$.

Remark 3.3. *Shrinking $V_{\mathbb{Q}}$ we can suppose that there exists a Zariski-dense set $S \subset V_{\mathbb{Q}}$ of classical points such that for every $y \in S$ we have*

(i) $f_{y/K}$ is a successive p-stabilisation at each prime $\mathfrak{p}$ above p of a Bianchi newform of level $\Gamma_0(\mathfrak{n})$ where $\mathfrak{n}$ is the prime-to-p part of the level of $f_{/K}$.

(ii) $f_{y/K}$ has small slope.

(iii) The weight (k_y, k_y) of $f_{y/K}$ satisfies $k_y \equiv k \pmod{p-1}$.

Condition (iii) means that we are working in one of the $(p-1)$ discs in the weight space.

For purposes of p-adic variation of the weight we have to give meaning to p-adic exponents.

Definition 3.1. *Let $\mathfrak{p}|p$ and $s \in \mathcal{O}_\mathfrak{p}$, define the function $\langle\cdot\rangle^s := \exp(s \cdot \log_p(\langle\cdot\rangle))$ in $\mathcal{O}_\mathfrak{p}^\times$, where $\log_p$ denotes the p-adic logarithm and $\langle\cdot\rangle$ is the projection of $\mathcal{O}_\mathfrak{p}^\times$ to $1+\mathfrak{p}^{r_\mathfrak{p}}\mathcal{O}_\mathfrak{p}$ for $r_\mathfrak{p}$ the smallest positive integer such that the usual p-adic exponential map converges on $\mathfrak{p}^{r_\mathfrak{p}}\mathcal{O}_\mathfrak{p}$. Define by $\langle\cdot\rangle^s = \prod_{\mathfrak{p}|p}\langle\cdot\rangle^{s_\mathfrak{p}}$ with $\boldsymbol{s} = (s_\mathfrak{p})_{\mathfrak{p}|p} \in \mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p \cong \prod_{\mathfrak{p}|p}\mathcal{O}_\mathfrak{p}$ the corresponding function in $(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p)^\times$. Let $w_{\mathrm{Tm},\mathfrak{p}} : \mathcal{O}_\mathfrak{p}^\times \to (\mathcal{O}_\mathfrak{p}/\mathfrak{p}^{r_\mathfrak{p}})^\times \subset \mathcal{O}_\mathfrak{p}^\times$ denote the Teichmüller character at $\mathfrak{p}$, so that for $z \in \mathcal{O}_\mathfrak{p}^\times$, we have $z = w_{\mathrm{Tm},\mathfrak{p}}(z)\langle z\rangle$. Also let $w_{\mathrm{Tm}} := \prod_{\mathfrak{p}|p} w_{\mathrm{Tm},\mathfrak{p}}$ be the corresponding character of $(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p)^\times$.*

Recall the definition of $\sigma_p^{k,k}(x)$ in equation (1.1) and note that, for example, for $x \in (\mathbb{A}_K^\times)_f$ we have $\sigma_p^{k,k}(x) = [\langle x_p\rangle w_{\mathrm{Tm}}(x_p)]^k$, where $x_p = (x_\mathfrak{p})_{\mathfrak{p}|p}$.

Theorem 3.4. *Let $V_\mathbb{Q}$ as in Remark 3.3, then for every $y \in V_\mathbb{Q}$ and $\kappa \in \mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ we have*

$$\mathcal{L}_p(y,\kappa) = (-1)^{k+1}\epsilon(\mathfrak{n})w_{\mathrm{Tm}}(\mathrm{N}(\mathfrak{n}))^{k/2}\langle\mathrm{N}(\mathfrak{n})\rangle^{k_y/2}\kappa(x_{-\nu,p})^{-1}\mathcal{L}_p(y,\kappa^{-1}w_{\mathrm{Tm}}^k\langle\cdot\rangle^{k_y}),$$

where $\epsilon(\mathfrak{n}) = \pm 1$ is the eigenvalue of $f_{y/K}$ for the Fricke involution $W_\mathfrak{n}$ for all y, $x_{-\nu,p}$ is the idele associated to $-\nu$ defined in Remark (1.3).

Proof. Consider $S \subset V_\mathbb{Q}$ as in Remark 3.3 and note that for $y \in S$, by Theorem 3.2, the distribution $L_p(f_{y/K}, -)$ satisfies for all $\kappa \in \mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ the following functional equation

$$L_p(f_{y/K},\kappa) = (-1)^{k+1}\epsilon(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{k_y/2}\kappa(x_{-\nu,p})^{-1}L_p(f_{y/K},\kappa^{-1}\sigma_p^{k_y,k_y}),$$

multiplying both sides by the p-adic period c_y, we have

$$\mathcal{L}_p(y,\kappa) = (-1)^{k+1}\epsilon(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{k_y/2}\kappa(x_{-\nu,p})^{-1}\mathcal{L}_p(y,\kappa^{-1}\sigma_p^{k_y,k_y}).$$

Note that $N(\mathfrak{n}) = w_{\mathrm{Tm}}(\mathrm{N}(\mathfrak{n}))\langle\mathrm{N}(\mathfrak{n})\rangle$ where each factor is well defined because $\mathfrak{p} \nmid \mathfrak{n}$ for all $\mathfrak{p}|p$, also, since $k_y \equiv k \pmod{p-1}$ we have $w_{\mathrm{Tm}}(\mathrm{N}(\mathfrak{n}))^{k_y/2} = w_{\mathrm{Tm}}(\mathrm{N}(\mathfrak{n}))^{k/2}$, then for all $y \in S$

$$\mathcal{L}_p(y,\kappa) = (-1)^{k+1}\epsilon(\mathfrak{n})w_{\mathrm{Tm}}(\mathrm{N}(\mathfrak{n}))^{k/2}\langle\mathrm{N}(\mathfrak{n})\rangle^{k_y/2}\kappa(x_{-\nu,p})^{-1}\mathcal{L}_p(y,\kappa^{-1}w_{\mathrm{Tm}}^k\langle\cdot\rangle^{k_y}).$$

Finally, since S is Zariski-dense on $V_\mathbb{Q}$, then the functional equation hold for every $y \in V_\mathbb{Q}$. ☐

Corollary 3.1. *Let $\mathcal{F}$ be a Σ-smooth base-change to K of a decent modular form satisfying conditions 2.11, let $\mathfrak{n} = (\nu)$ be the prime-to-p part of the level of $\mathcal{F}$, then for all $\kappa \in \mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ the distribution $L_p(\mathcal{F}, -)$ satisfies the following functional equation*

$$L_p(\mathcal{F},\kappa) = (-1)^{k+1}\epsilon(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{k/2}\kappa(x_{-\nu,p})^{-1}L_p(\mathcal{F},\kappa^{-1}\sigma_p^{k,k}).$$

Proof. Let x be the classic point in the Coleman-Mazur eigencurve such that $\mathcal{F} = f_{x/K}$, then specialise the functional equation in Theorem 3.4 at x. ☐

Notice that corollary above generalises Theorem 3.2 with no non-critical assumption on the Σ-smooth base-change Bianchi modular form. In particular, we obtain the functional equation of $L_p(\mathcal{F},-)$ for a Σ-smooth small slope base-change $\mathcal{F}$ new at p, this case is interesting, considering for example, that when p is split in K, small slope is automatic.

3.3 p-adic L-function of base change non-cuspidal Bianchi modular forms

Let K be an imaginary quadratic field with arbitrary class number and suppose p splits in K, then in this section we present our construction of the p-adic L-function of small slope non-cuspidal Bianchi modular forms given as base change to K of classical modular forms.

3.3.1 Construction of the p-adic L-function

Since we want to work with the base change to K of a modular form with CM by K, then recall from section 1.3.2 that if f is such a CM form of weight $k+2$ and level coprime to p, then $f = f_\varphi$ where φ is a Hecke character of K of infinity type $(-k-1,0)$ and conductor $\mathfrak{m}$ for some ideal $\mathfrak{m} \subset \mathcal{O}_K$ coprime to p. Also for the p-adic setting we need to p-stabilise f_φ and work with $f^p_{\varphi/K}$ as in Remark 1.16.

Note that $f_{\varphi/K}$ is quasi-cuspidal by Proposition 1.7 and then $f^p_{\varphi/K}$ is quasi-cuspidal by Lemma 1.6. Since $f^p_{\varphi/K}$ is a quasi-cuspidal Bianchi modular form of parallel weight we can attach a full Bianchi modular symbol, exactly in the same way as in Proposition 2.9 in [41] for cuspidal Bianchi modular forms.

Denote by $\mathcal{F}^i$ for $i = 1,..,h$, the collection of descents to $\mathcal{H}_3$ of $f^p_{\varphi/K}$, then the Bianchi modular symbol attached to $f^p_{\varphi/K}$ is defined as $\phi_{f^p_{\varphi/K}} = (\phi_{\mathcal{F}^1},...,\phi_{\mathcal{F}^h})$ with $\phi_{\mathcal{F}^i}$ defined by

$$\phi_{\mathcal{F}^i}(\{a\}-\{\infty\}) = \sum_{q,r=0}^{k} c^i_{q,r}(a)(\mathcal{Y}-a\mathcal{X})^{k-q}\mathcal{X}^q(\overline{\mathcal{Y}}-\overline{a}\overline{\mathcal{X}})^{k-r}\overline{\mathcal{X}}^r,$$

with

$$c^i_{q,r}(a) := 2\binom{2k+2}{k+q-r+1}^{-1}(-1)^{k+r+1}\int_0^\infty t^{q+r}\mathcal{F}^i_{k+q-r+1}(a,t)dt.$$

for $a \in K$.

Remark 3.4. *Recall that the integrals defining such symbol are convergent since by Proposition 1.7 we have that $f^p_{\varphi/K}$ is quasi-cuspidal, then for each i, $\mathcal{F}^i$ quasi-vanishes for all cusps.*

Proposition 3.2. *Let $\Omega'_{f_{\varphi/K}}$ be the period in Remark 1.18, then the Bianchi modular symbol $\phi'_{f^p_{\varphi/K}} := \phi_{f^p_{\varphi/K}}/\Omega'_{f_{\varphi/K}}$ takes values in $V^*_{k,l}(E)$ for some number field E.*

Proof. Let ψ be a Hecke character of K of infinity type $0 \leqslant (q,r) \leqslant (k,k)$ then by Theorem 2.11

in [41] we have

$$\Lambda(f^p_{\varphi/K},\psi)=\left[\frac{(-1)^{k+q+r}2\psi_{\mathfrak{f}}(x_{\mathfrak{f}})}{\psi(x_{\mathfrak{f}})Dw\tau(\psi^{-1})}\right]\sum_{i=1}^{h}\left[\psi(t_i)\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\((a)\mathfrak{f},\mathfrak{f})=1}}\psi_{\mathfrak{f}}(a)C^i_{q,r}(a)\right],\tag{3.4}$$

where

$$C^i_{q,r}(a)=\phi_{f^p_{\varphi/K}}(\{a\}-\{\infty\})\left[(X+aY)^qY^{k-q}(\overline{X}+\overline{a}\overline{Y})^r\overline{Y}^{k-r}\right].$$

By Lemma 1.11, the quotient $\Lambda(f^p_{\varphi/K},\psi)/\Omega'_{f_{\varphi/K}}$ is algebraic for all ψ with infinity type $0\leqslant(q,r)\leqslant(k,k)$, then by (3.4) the symbol $\phi'_{f^p_{\varphi/K}}:=\phi_{f^p_{\varphi/K}}/\Omega'_{f_{\varphi/K}}\in V^*_{k,k}(\overline{\mathbb{Q}})$. Finally, since Δ^0 is finitely generated as a $\mathbb{Z}[\Gamma_i(\mathfrak{n})]$ (see Lemma 3.8 in [41]), we have $\phi'_{\mathcal{F}^i}\in\mathrm{Symb}_{\Gamma_i(\mathfrak{n})}(V^*_{k,k}(E_i))$ for some number field E_i for each $i=1,..h$, then there exists a number field E such that $\phi'_{f^p_{\varphi/K}}\in\mathrm{Symb}_{\Omega_0(\mathfrak{n})}(V^*_{k,k}(E))$. □

Proposition 3.2 allows us to see the partial Bianchi modular symbol $\phi'_{f^p_{\varphi/K}}$ as having values in $V^*_{k,k}(L)$ for a sufficiently large p-adic field L. Then, since $f^p_{\varphi/K}$ has small slope, using the Theorem 6.10 (control theorem) in [41] we can lift $\phi'_{f^p_{\varphi/K}}$ to its corresponding unique overconvergent Bianchi eigensymbol $\Psi=(\Psi_1,...,\Psi_h)$ and then obtain.

Theorem 3.5. *Let φ be a Hecke character of K with conductor $\mathfrak{m}$ coprime with p and infinity type $(-k-1,0)$ with $k\geqslant 0$, denote by f_φ and f^p_φ the CM modular form induced by φ and its ordinary p-stabilisation respectively. Let $f^p_{\varphi/K}$ be the base-change to K of f^p_φ, let $\Omega'_{f_{\varphi/K}}$ be a complex period as in Proposition 1.8. Then there exists a unique locally analytic measure $L_p(f^p_{\varphi/K},-)$ on $\mathrm{Cl}_K(p^\infty)$ such that for any Hecke character ψ of K of conductor $\mathfrak{f}=\mathfrak{p}^t\overline{\mathfrak{p}}^s$ and infinity type $0\leqslant(q,r)\leqslant(k,k)$, we have*

$$L_p(f^p_{\varphi/K},\psi_{p\text{-fin}})=E_p(f^p_{\varphi/K})'\left[\frac{DwG(\psi)}{(-1)^{k+q+r}2\varphi(\overline{\mathfrak{p}})^{t+s}\Omega'_{f_{\varphi/K}}}\right]\Lambda(f^p_{\varphi/K},\psi),\tag{3.5}$$

where

$$G(\psi):=\psi_\infty(\delta)\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\((a)\mathfrak{f},\mathfrak{f})=1}}\psi_{\mathfrak{f}}(a)e^{2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(a/\delta)},$$

$$E_p(f^p_{\varphi/K})'=\left(1-\frac{1}{\varphi(\overline{\mathfrak{p}})\psi(\mathfrak{p})}\right)\left(1-\frac{1}{\varphi(\overline{\mathfrak{p}})\psi(\overline{\mathfrak{p}})}\right).$$

Proof. By the above construction of the overconvergent modular eigensymbol Ψ associated to $f^p_{\varphi/K}$ and by [41, Thm. 7.4] there exist a unique locally analytic distribution $L_p(f^p_{\varphi/K},-):=\mathrm{Mel}(\Psi)$ on $\mathrm{Cl}_K(p^\infty)$ such that for any Hecke character ψ of K of conductor $\mathfrak{f}=\mathfrak{p}^t\overline{\mathfrak{p}}^s$ and infinity type $0\leqslant(q,r)\leqslant(k,k)$ satisfies

$$L_p(f^p_{\varphi/K},\psi_{p\text{-fin}})=\left(\prod_{\mathfrak{p}|p}(1-[a_{\mathfrak{p}}(f^p_{\varphi/K})\psi(\mathfrak{p})]^{-1})\right)\left[\frac{\psi(x_{\mathfrak{f}})Dw\tau(\psi^{-1})}{(-1)^{k+q+r}2\psi_{\mathfrak{f}}(x_{\mathfrak{f}})a_{\mathfrak{f}}((f^p_{\varphi/K}))\Omega'_{f_{\varphi/K}}}\right]\Lambda(f^p_{\varphi/K},\psi).$$

Also note that

i) $a_{\mathfrak{p}}(f^p_{\varphi/K}) = a_{\overline{\mathfrak{p}}}(f^p_{\varphi/K}) = \varphi(\overline{\mathfrak{p}})$ and

ii) $a_{\mathfrak{f}}(f^p_{\varphi/K}) = a_{\mathfrak{p}^t}(f^p_{\varphi/K}) a_{\overline{\mathfrak{p}}^s}(f^p_{\varphi/K}) = a_{\mathfrak{p}}(f^p_{\varphi/K})^t a_{\overline{\mathfrak{p}}}(f^p_{\varphi/K})^s = \varphi(\overline{\mathfrak{p}})^t \varphi(\overline{\mathfrak{p}})^s = \varphi(\overline{\mathfrak{p}})^{t+s}$;

iii) By remark 1.4 we have

$$\tau(\psi^{-1}) = \psi(x_{\mathfrak{f}})^{-1}\psi_{\mathfrak{f}}(x_{\mathfrak{f}})G(\psi)$$

Then, by $i)$, $ii)$ and $iii)$ we obtain the interpolation desired. □

Remark 3.5. *In order to compare $L_p(f^p_{\varphi/K}, -)$ to Katz p-adic L-functions, will be useful to relate the interpolation property in (3.5) with $\Lambda(f_{\varphi/K}, -)$. Given the relation between $\Lambda(f^p_{\varphi/K}, -)$ and $\Lambda(f_{\varphi/K}, -)$ in proof of Lemma 1.11, then for any Hecke character ψ of K of conductor $\mathfrak{f} = \mathfrak{p}^t\overline{\mathfrak{p}}^s$ and infinity type $0 \leqslant (q,r) \leqslant (k,k)$, we have*

$$L_p(f^p_{\varphi/K}, \psi_{p\text{-fin}}) = E_p(f^p_{\varphi/K}) \left[\frac{DwG(\psi)}{(-1)^{k+q+r} 2\varphi(\overline{\mathfrak{p}})^{t+s}\Omega'_{f_{\varphi/K}}} \right] \Lambda(f_{\varphi/K}, \psi),$$

where

$$E_p(f^p_{\varphi/K}) = \prod_{\mathfrak{q}|p} \left(1 - \frac{\varphi(\mathfrak{p})\psi(\mathfrak{q})}{N(\mathfrak{p})}\right)\left(1 - \frac{1}{\varphi(\overline{\mathfrak{p}})\psi(\mathfrak{q})}\right).$$

3.3.2 Katz p-adic L-function

Let ψ be a Hecke character of K of suitable infinity type and conductor $\mathfrak{m}\mathfrak{f}$ with $\mathfrak{m}$ prime to p and $\mathfrak{f}|p^\infty$, then Katz constructed in [25] the p-adic L-function of ψ when $\mathfrak{m}$ is trivial. Later, Hida and Tilouine in [21] extended the construction of Katz for non-trivial $\mathfrak{m}$.

In this section we just state the interpolation property of the Katz measure, for more details the reader can see [5].

Recall the definition of the Gauss sum of ψ in section 1.1.4, we now define the local Gauss sum of ψ at prime ideals $\mathfrak{q}$ dividing the conductor of ψ by

$$\tau_{\mathfrak{q}}(\psi) := \psi(\pi_{\mathfrak{q}}^{-t}) \sum_{u \in (\mathcal{O}_{\mathfrak{q}}/\mathfrak{q}^t)^\times} \psi_{\mathfrak{q}}(u) e_K(u/\pi_{\mathfrak{q}}^t d_{\mathfrak{q}})$$

where e_K is the character in section 1.2.2, $\pi_{\mathfrak{q}}$ is a prime element in $\mathcal{O}_{\mathfrak{q}}$, $t = t(\mathfrak{q})$ is the exponent of $\mathfrak{q}$ in the conductor of ψ and $d_{\mathfrak{q}}$ is the $\mathfrak{q}$ component of the idele d associated to the different ideal of K. Outside the conductor of ψ, we simply put $\tau_{\mathfrak{q}}(\psi) = 1$.

Recall the construction of the period $\Omega(A)$ in section 2C in [5] and properties regarding the algebraicity of critical values of Hecke L-functions proved in Lemma 1.1; analogously there exists a p-adic period $\Omega_p(A)$ constructed in section 2D in [5] which is obtained by considering the base change $A_{\mathbb{C}_p}$ of the elliptic curve A of section 1.1.3 to $\mathbb{C}_p$.

Theorem 3.6. *(Katz, Hida-Tilouine) Let ψ be a Hecke character of K of infinity type (a,b) with $a \geqslant 1$ and $b \leqslant 0$ and conductor $\mathfrak{m}\mathfrak{f}$ with $\mathfrak{m}$ prime to p and $\mathfrak{f} = \mathfrak{p}^t\overline{\mathfrak{p}}^s$. Then there is a unique locally* *analytic measure $L_p(-)$ on the ray class group $\mathrm{Cl}_K(\mathfrak{m}p^\infty)$ satisfying:*

$$\frac{L_p(\psi_{p\text{-}fin})}{\Omega_p(A)^{a-b}} = \frac{w}{2}W_{\mathfrak{p}}(\psi)\frac{(-1)^{a+b}\Gamma(a)\sqrt{D}^b}{(2\pi)^b\Omega(A)^{a-b}}(1-\psi(\overline{\mathfrak{p}}))\left(1-\frac{1}{\psi(\mathfrak{p})N(\mathfrak{p})^t}\right)L(\psi,0). \tag{3.6}$$

where $W_{\mathfrak{p}}(\psi) = N(\mathfrak{p})^{-t}\tau_{\mathfrak{p}}(\psi)$.

3.3.3 Relation between the p-adic L-function of non-cuspidal base change and Katz p-adic L-function

In section 1.3.4 we factor the complex L function of the base change to K of a CM modular form as the products of two Hecke L-functions, such factorization suggests a factorization of p-adic L-functions, in fact, this is the case.

Theorem 3.7. *Let φ be a Hecke character of K with conductor $\mathfrak{m}$ coprime with p and infinity type $(-k-1,0)$ with $k \geqslant 0$. Denote by f_φ the CM modular form induced by φ and f_φ^p its ordinary p-stabilisation. Let $f_{\varphi/K}^p$ be the base-change to K of f_φ^p, then for all $\kappa \in \mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ we have*

$$L_p(f_{\varphi/K}^p,\kappa) = \frac{L_{p,\mathrm{Katz}}(\varphi_{p\text{-fin}}^c\kappa\sigma_p^{1,1})}{\Omega_p(A)^{k+1}}\frac{L_{p,\mathrm{Katz}}(\varphi_{p\text{-fin}}^c\kappa^c\sigma_p^{1,1})}{\Omega_p(A)^{k+1}},$$

where the character $\sigma_p^{1,1}$ is defined in (1.1) and $\Omega_p(A)$ is the p-adic period in Theorem 3.6.

Remark 3.6. *The p-adic L-function of $f_{\varphi/K}^p$ is a function in $\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ but the Katz p-adic L-functions in Theorem 3.7 are functions on $\mathfrak{X}(\mathrm{Cl}_K(\overline{\mathfrak{m}}p^\infty)$. To relate them, we see the later as functions on $\mathfrak{X}(\mathrm{Cl}_K(p^\infty))$ via the map*

$$\mathrm{Cl}_K(\overline{\mathfrak{m}}p^\infty) \to \mathrm{Cl}_K(p^\infty).$$

Proof. (*of Theorem 3.7*) Since the Bianchi modular form $f_{\varphi/K}^p$ is ordinary at every prime $\mathfrak{p}|p$ then $L_p(f_{\varphi/K}^p,-)$ is $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible, where $h_{\mathfrak{p}} = v_p(\lambda_{\mathfrak{p}}) = 0$ for each $\mathfrak{p}|p$, then $L_p(f_{\varphi/K}^p,-)$ is a bounded distribution, i.e. a measure. To obtain the equality of measures in the Theorem, we only need the equality on p-adic characters $\psi_{p\text{-fin}}$ coming from Hecke characters ψ of finite order and conductor $\mathfrak{f} = \mathfrak{p}^t\overline{\mathfrak{p}}^s$. For such characters we have from Corolary 3.5 and Theorem 3.6 the following interpolations:

$$L_p(f_{\varphi/K}^p,\psi_{p\text{-fin}}) = E_p(f_{\varphi/K}^p)\left[\frac{DwG(\psi)}{(-1)^{k+1}2\varphi(\overline{\mathfrak{p}})^{t+s}\Omega_{f_{\varphi/K}}(2\pi)^2}\right]L(f_{\varphi/K},\psi,1),$$

$$\frac{L_{p,\mathrm{Katz}}((\varphi^c\psi|\cdot|_{\mathbb{A}_K})_{p\text{-fin}})}{\Omega_p(A)^{k+1}} = \frac{(-1)^{k-1}wW_{\mathfrak{p}}(\varphi^c\psi|\cdot|_{\mathbb{A}_K})(2\pi)^k}{2\sqrt{D}^k\Omega(A)^{k+1}E_p(\varphi^c\psi|\cdot|_{\mathbb{A}_K})^{-1}}L(\varphi^c\psi,1)$$

where

$$E_p(\varphi^c\psi|\cdot|_{\mathbb{A}_K}) = \left(1-\frac{\varphi(\mathfrak{p})\psi(\overline{\mathfrak{p}})}{N(\mathfrak{p})}\right)\left(1-\frac{1}{\varphi(\overline{\mathfrak{p}})\psi(\mathfrak{p})}\right) \tag{3.7}$$

and
$$\frac{L_{p,\mathrm{Katz}}((\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K})_{p\text{-fin}})}{\Omega_p(A)^{k+1}} = \frac{(-1)^{k-1}wW_{\mathfrak{p}}(\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K})(2\pi)^k}{2\sqrt{D}^k\Omega(A)^{k+1}E_p(\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K})^{-1}}L(\varphi^c\psi^c\lambda_K,1)$$
where
$$E_p(\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K}) = \left(1-\frac{\varphi(\mathfrak{p})\psi(\mathfrak{p})}{N(\mathfrak{p})}\right)\left(1-\frac{1}{\varphi(\overline{\mathfrak{p}})\psi(\overline{\mathfrak{p}})}\right). \tag{3.8}$$

In order to obtain the equality desired we simplify the Euler products, the Gauss sums and study the periods on those interpolations.

1) Euler factors: The product of the Euler factors in equations (3.7) and (3.8) is equal to the Euler factor in the interpolation of $L_p(f^p_{\varphi/K},\psi_{p\text{-fin}})$, i.e.
$$E_p(\varphi^c\psi|\cdot|_{\mathbb{A}_K})E_p(\varphi^c\psi^c\lambda_K|\cdot|_{\mathbb{A}_K}) = E_p(f^p_{\varphi/K}).$$

2) Gauss sums: The Gauss sums in the interpolations of the Katz p-adic L-functions are

$$\begin{aligned} W_p(\varphi^c\psi|\cdot|_{\mathbb{A}_K}) &= N(\mathfrak{p}^{-t})\tau_{\mathfrak{p}}(\varphi^c\psi|\cdot|_{\mathbb{A}_K}) \\ &= (\varphi^c\psi)(\pi_{\mathfrak{p}}^{-t})\sum_{u\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^t)^\times}\psi_{\mathfrak{p}}(u)e_K(\pi_{\mathfrak{p}}^{-t}u) \\ &= (\varphi^c\psi)(\pi_{\mathfrak{p}}^{-t})\sum_{u\in(\mathcal{O}_K/\mathfrak{p}^t)^\times}\psi_{\mathfrak{p}}(u)e^{-2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(u/\pi_{\mathfrak{p}}^t)} \\ &= (\varphi^c\psi)(\pi_{\mathfrak{p}}^{-t})\psi_{\mathfrak{p}}(-\delta)^{-1}\sum_{u\in(\mathcal{O}_K/\mathfrak{p}^t)^\times}\psi_{\mathfrak{p}}(u)e^{2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(u/\pi_{\mathfrak{p}}^t\delta)}, \end{aligned}$$

$$\begin{aligned} W_p(\varphi^c\psi^c\lambda|\cdot|_{\mathbb{A}_K}) &= N(\mathfrak{p}^{-s})\tau_{\mathfrak{p}}(\varphi^c\psi^c\lambda|\cdot|_{\mathbb{A}_K}) \\ &= (\varphi^c\psi^c)(\pi_{\mathfrak{p}}^{-s})\sum_{u\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^s)^\times}\psi^c_{\mathfrak{p}}(u)e_K(\pi_{\mathfrak{p}}^{-s}u) \\ &= (\varphi^c\psi^c)(\pi_{\mathfrak{p}}^{-s})\sum_{u\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^s)^\times}\psi_{\mathfrak{p}}(\overline{u})e^{-2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(u/\pi_{\mathfrak{p}}^s)} \\ &= (\varphi^c\psi^c)(\pi_{\mathfrak{p}}^{-s})\sum_{u\in(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^s)^\times}\psi_{\mathfrak{p}}(\overline{u})e^{-2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(\overline{u}/\overline{\pi_{\mathfrak{p}}^s})} \\ &= (\varphi^c\psi^c)(\pi_{\mathfrak{p}}^{-s})\psi_{\overline{\mathfrak{p}}}(-\delta)^{-1}\sum_{v\in(\mathcal{O}_K/\overline{\mathfrak{p}}^s)^\times}\psi_{\overline{\mathfrak{p}}}(v)e^{2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(v/\overline{\pi_{\mathfrak{p}}}^s\delta)}. \end{aligned}$$

Then, can be shown using for example ii) in Proposition 2.14 in [29] that the product of both Gauss sums is related with $G(\psi)$ as follows
$$\begin{aligned} W_p(\varphi^c\psi|\cdot|_{\mathbb{A}_K})W_p(\varphi^c\psi^c\lambda|\cdot|_{\mathbb{A}_K}) &= \varphi^c(\pi_{\mathfrak{p}}^{-t-s})\psi^{-1}(\pi_{\mathfrak{f}})\psi_{\mathfrak{f}}(-\delta)^{-1}\sum_{b\in(\mathcal{O}_K/\mathfrak{f})^\times}\psi_{\mathfrak{f}}(b)e^{2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(b/\pi_{\mathfrak{f}}\delta)} \\ &= \varphi(\overline{\mathfrak{p}})^{-t-s}G(\psi) \end{aligned}$$
where in last equality since ψ has finite order we have $\psi^{-1}(\pi_{\mathfrak{f}}) = \psi_\infty(\pi_{\mathfrak{f}}) = 1$ and $\psi_{\mathfrak{f}}(-\delta)^{-1} = 1$ because $-\delta$ is a $\mathfrak{p}$-adic unit for all $\mathfrak{p}$ above p.

3) The periods: Recall the normalisation of the period $\Omega'_{f_{\varphi/K}}$ in Remark 1.18 given by

$$\Omega'_{f_{\varphi/K}} = \frac{2D^{k+1}}{w}\Omega_{f_{\varphi/K}} = \frac{2D^{k+1}\Omega(A)^{2k+2}}{w(2\pi i)^{2k+2}}.$$

Then putting together 1), 2) and 3) we obtain

$$L_p(f^p_{\varphi/K}, \psi_{p\text{-fin}}) = \frac{1}{\Omega_p(A)^{2k+2}} L_{p,\mathrm{Katz}}((\varphi^c\psi)_{p\text{-fin}}\sigma_p^{1,1}) L_{p,\mathrm{Katz}}((\varphi^c\psi^c\lambda_K)_{p\text{-fin}}\sigma_p^{1,1}) \tag{3.9}$$

for all Hecke character ψ of finite order and conductor dividing p^∞, since there are infinitely many such characters and both sides of (3.9) are bounded functions. The result follows since a non-zero bounded analytic function on an open ball has at most finitely many zeros. □

Remark 3.7. *In Theorem 3.7 the the p-adic L-function of $f^p_{\varphi/K}$ can be modified to be a function on $\mathfrak{X}(\mathrm{Cl}_K(\overline{\mathfrak{m}}p^\infty))$ by the same method as in Section 3.4 in [2].*

Chapter 4

Work in progress and further directions

In this chapter we present our current work and further directions, which have the objective of in some sense "complete" the work done by Williams in [41], Barrera and Williams in [3] and this thesis constructing p-adic L-functions attached to Bianchi modular forms.

4.1 Motivation

In sections 3.1 and 3.2 we recalled the construction of the p-adic L-function of:

- small slope cuspidal Bianchi modular forms;
- critical slope Σ-smooth base change cuspidal Bianchi modular;

done by Williams in [41] and Barrera-Williams in [3] respectively.

In section 3.3 we construct the p-adic L-function of:

- small slope base change non-cuspidal Bianchi modular forms.

Given those constructions our purpose is to construct the p-adic L-function of Bianchi modular forms in some of the remaining cases specifically:

(I) small slope non-cuspidal Bianchi modular forms (Section 4.2);

(II) critical slope base change non-cuspidal Bianchi modular forms (Section 4.3);

(III) critical slope non-cuspidal Bianchi modular forms (Section 4.2).

4.2 Work in progress

The purpose of this section is to explain our currect work in the construction of the p-adic L-function of small slope non-cuspidal Bianchi modular forms.

With the theory of partial Bianchi modular symbols developed in chapter 2 we can construct the p-adic L-function attached to a small slope C-cuspidal Bianchi modular form following the same steps of Williams in [41]. Such construction in the C-cuspidal case (which is currently in a writing process), together with our results in chapter 2 and section 3.3 will be contained in [31], the second work of my PhD.

More precisely, let Φ be a small slope C-cuspidal Bianchi eigenform of level $\Omega_0(\mathfrak{n})$ with $(p)|\mathfrak{n}$ and weight (k,ℓ) with $k \geqslant \ell$, then we can construct its p-adic L-function following the next steps:

(i) Attach to Φ a classical partial Bianchi modular eigensymbol:

By Proposition 2.3 we can attach an element

$$\phi_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^*(\mathbb{C}))$$

to each $\mathcal{F}^i$ with $i = 1,..,h$ descent to $\mathcal{H}_3$ from Φ, then we define

$$\phi_\Phi = (\phi_{\mathcal{F}^1}, ...\phi_{\mathcal{F}^h}) \in \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(\mathbb{C})).$$

(ii) Attach to Φ a p-adic valued partial Bianchi modular eigensymbol:

By fixing an isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$ as in [28] we can obtain a symbol ϕ'_Φ with values in $V_{k,\ell}^*(\overline{\mathbb{Q}}_p)$ from the partial Bianchi modular eigensymbol ϕ_Φ. Note that by the type finiteness of $\Delta_{C_i}^0$ as a $\mathbb{Z}[\Gamma_i(\mathfrak{n})]$-module it follows that for each i we have that $\phi'_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_i(\mathfrak{n}),C_i}(V_{k,\ell}^*(E_i))$ for some number field E_i, then $\phi'_\Phi \in \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(V_{k,\ell}^*(E))$ for some number field E (see the paragraph before section 4 in [4] for the analogous situation for modular forms).

(iii) Lift ϕ'_Φ to its unique overconvergent partial Bianchi modular symbol:

Using the embedding $\mathrm{inc} : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ fixed in (2.4) we view the coefficients as living in $V_{k,\ell}^*(L)$, where L is the finite extension of $\mathbb{Q}_p$ generated by E. Since Φ has small slope, we can use the partial Bianchi control theorem (Theorem 2.2) to lift ϕ'_Φ to $\Psi_\Phi \in \mathrm{Symb}_{\Omega_0(\mathfrak{n}),C}(\mathcal{D}_{k,\ell}(L))$.

(iv) Construct the p-adic L-function of Φ:

Recall the definition of the Mellin tranform of an overconvergent partial Bianchi modular symbol in Definition 2.17, we define the p-adic L-function of Φ by

$$L_p(\Phi, -) := \mathrm{Mel}(\Psi_\Phi).$$

Then the distribution $L_p(\Phi,-)$ is $(h_\mathfrak{p})_{\mathfrak{p}|p}$-admissible, where $h_\mathfrak{p} = v_p(\lambda_\mathfrak{p})$, and hence is unique (see section 2.2.6).

Remark 4.1. *1) Note that we start with a C-cuspidal Bianchi modular form, but currently we are working in a generalization for non-cuspidal Bianchi modular forms. The key idea (which is inspired by the work of Bellaïche and Dasgupta in [4]) is to turn the non-cuspidal Bianchi modular form into C-cuspidal by producing the vanishing of suitable constant terms constructing linear combinations. The feasibility of that idea lies in the special form of the constant term in* *the Fourier expansion of Bianchi modular forms (see Remark 1.8, that come from Theorem 6.7 in [20]).*

2) Another relevant aspect that we are currently working, is the algebraicity of the partial Bianchi modular symbol obtained in (i). It would be desirable to not depend of the isomorphism ι in (ii). The ideal situation would be the existence of a period as in Proposition 2.12 in [41] for cuspidal Bianchi modular forms.

4.3 Further directions

In this section we try to establish our next projects constructing the p-adic L-function of critical slope non-cuspidal Bianchi modular forms.

Let G be a connected reductive group over $\mathbb{Q}$, and suppose $G_{/\mathbb{Q}_p}$ is quasi-split. In this case Hansen in [19] constructed eigenvarieties for G using overconvergent cohomology groups; his work generalises earlier constructions of Ash and Stevens [1] and Urban [38]. In [3], Barrera and Williams specializes Hansen's main results to the Bianchi setting and work with the Bianchi eigenvariety, in particular, they study one dimensional p-adic Bianchi families for cuspidal Bianchi modular forms that are base change of non CM modular forms. It is interesting to observe that we also can have the possibility of two dimensional families (over non-parallel weight space) having intersection with a 1-dimensional p-adic family at a single non-cuspidal classical point.

- Project 1: Suppose p splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in K, let φ be a Hecke character of K with conductor coprime to p, denote by f_φ the CM modular form induced by φ and denote by $f_{\varphi/K}^{\mathrm{crit}}$ the critical slope p-stabilisation of the base change $f_{\varphi/K}$, then our first objective is to construct the p-adic L-function of $f_{\varphi/K}^{\mathrm{crit}}$ by generalizing the methods of Barrera and Williams in [3] in the cuspidal base change case using p-adic Bianchi families and using our construction in the small slope case in section 3.3. Since the p-adic L-function of $f_{\varphi/K}^{\mathrm{crit}}$ will not be a measure, then, when trying to factorise it as the product of two Katz p-adic L-functions (as in section 3.3.3 for the p-adic L-function of the small slope p-stabilisation) we expect the additional appearance of a 'two dimensional' logarithmic factor analogous to the 'one dimensional' logarithmic factor appearing in Theorem 1.1 in [4] for Kubota-Leopoldt functions.

- Project 2: Our second objective is the construction of the p-adic L-function of critical slope non-cuspidal Bianchi modular forms beyond the base change, which is the natural next step after the construction in 4.2 for the small slope non-cuspidal case. This is an ambitious project given that mixes the techniques of C-cuspidality and partial modular symbols developed in this thesis for the non-cuspidal case with eigenvarieties and p-adic families applied to the critical slope situation (to be developed in project 1 above).

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