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Laura Rezzieri Gambera

**Zero-one law for (a,k) -regularized resolvent
families and the Blackstock-Crighton-Westervelt
equation on Banach spaces**

São José do Rio Preto
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Tesis presentada al Departamento de Matemática y Ciencia de la Computación de la Facultad de Ciencia de la Universidad de Santiago de Chile, para optar al grado de Doctor en Ciencia Mención Matemática, en cotutela con la Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

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To my family

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“Para triunfar es necesario vencer, para vencer es necesario luchar, para luchar es necesario estar preparado, para estar preparado es necesario proveerse de una gran entereza de ánimo y una paciencia a toda prueba.”

(Carlos Bernardo González Pecotche, 1957, p. 133 [56])

ABSTRACT

This work presents some results of the theory of the (a,k) -regularized resolvent families, that are the main tool used in this thesis. Related with this families, one result proved in this work is the zero-one law, providing new insights on the structural properties of the theory of (a,k) -regularized resolvent families including strongly continuous semigroups, strongly continuous cosine families, integrated semigroups, among others. Moreover, an abstract nonlinear degenerate hyperbolic equation is considered, that includes the semilinear Blackstock-Crighton-Westervelt equation. By proposing a new approach based on strongly continuous semigroups and resolvent families of operators, it is proved an explicit representation of the strong and mild solutions for the linearized model by means of a kind of variation of parameters formula. In addition, under nonlocal initial conditions, a mild solution of the nonlinear equation is established.

Keywords: (a,k) -regularized resolvent families. Zero-one law. Blackstock-Crighton-Westervelt equation. Well-posedness.

RESUMO

Este trabalho apresenta alguns resultados da teoria de famílias resolventes (a,k) -regularizadas, que é a principal ferramenta utilizada nesta tese. Relacionado com estas famílias, um resultado provado neste trabalho é a lei zero-um, que fornece novas percepções de propriedades estruturais da teoria de famílias resolventes (a,k) -regularizadas, incluindo os semigrupos fortemente contínuos, as famílias cosseno fortemente contínuas, os semigrupos integrados, entre outras. Além disso, uma equação hiperbólica degenerada não-linear abstrata é considerada, a qual inclui a equação semilinear de Blackstock-Crighton-Westervelt. Propondo uma nova abordagem baseada em semigrupos fortemente contínuos e famílias resolvente, é demonstrada uma representação explícita das soluções forte e branda para a linearização do modelo por uma espécie de método de variação dos parâmetros. Por fim, sob condições iniciais não-locais, uma solução branda da equação não-linear é estabelecida.

Palavras-chave: Famílias resolventes (a,k) -regularizadas. Lei zero-um. Equação de Blackstock-Crighton-Westervelt. Boa colocação.

RESUMEN

Este trabajo presenta algunos resultados de la teoría de las familias resolventes (a,k) -regularizadas, que es la principal herramienta utilizada en esta tesis. Relacionado con estas familias, uno de los resultados demostrados en este trabajo es la ley cero-uno, proveyendo nuevas percepciones de propiedades estructurales de la teoría de las familias resolventes (a,k) -regularizadas, incluyendo los semigrupos fuertemente continuos, las familias coseno fuertemente continuas, los semigrupos integrados, entre otras. Además, una ecuación degenerada no lineal abstracta es considerada, la cual incluye la ecuación semilineal de Blackstock-Crighton-Westervelt. Proponiendo un nuevo enfoque basado en semigrupos fuertemente continuos y familias resolvente, es demostrada una representación explícita de las soluciones fuerte y débil de la linealización del modelo por una especie de método de variación de parámetros. Por fin, bajo condiciones iniciales no locales, una solución débil de la ecuación no lineal es establecida.

Palabras clave: Familias resolventes (a,k) -regularizadas. Ley cero-uno. Ecuación de Blackstock-Crighton-Westervelt. Buen planteamiento.

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Introduction

Among the most used tools for the study of existence of solutions and qualitative properties of partial differential equations is the theory of operators. This approach consists in transform a partial differential equation into an abstract problem by means of the transform theory (Laplace, Fourier), providing a more viable and transparent view of the original problem. The first and most famous tool that uses such idea is the theory of strongly continuous semigroup.

Later, the theory of strongly continuous cosine family appears. Concerning these two families of operators, there are uniqueness and qualitative properties of solutions for a large number of partial differential equations, but a lot of evolution equations are not well-posed. For this reason, many alternative tools have been in development, consisting of families of operators with some regularity.

The theory of (a, k) -regularized resolvent families unify many of the theories of bounded linear operators, such as semigroups, cosine families, integrated semigroups, etc. One of its advantages is that it is an useful tool for many evolution equations, such as integral equations, differential equations and integrodifferential equations. This theory was introduced in the literature by Lizama in 2000, and it still is an ongoing research theme, which may provide many future works.

In this work the theory of (a, k) -regularized resolvent families is used to study two problems that up to the present moment, by our knowledge, are not considered in the literature. The first one is related to a specific property of this family of bounded linear operators, and the other one consisting in apply this theory to study a particular differential equation in a Banach space.

This thesis is composed of four themed chapters. The first chapter presents some of the notation and results used throughout the work.

The second chapter provides an overview of the theory of the (a, k) -regularized resolvent families, including definitions and classical theorems stated in the literature.

In the third chapter an intriguing structural property familiar of these families is studied. This property is known as zero-one law. The results of the first section of this chapter were published in [29].

The fourth chapter is concerned with the study of the Blackstock-Crighton-Westervelt equation in a generalized abstract form defined in a Banach space. This equation is an example of nonlinear acoustic models, which solutions describes the acoustic velocity potential in some fluid. Using the families presented on the second chapter it is possible to prove well-posedness and strong solution for this equation in Banach spaces, a local mild solutions to the semilinear problem, and a mild solution to this problem with nonlocal initial conditions. The results of this chapter were published in [30].

1 Preliminaries

This chapter contains some of the notations used throughout this thesis and some results scattered in the literature which are important to the development of the theory in this work but is also of independent interest.

1.1 Notation

Most notation used throughout this thesis are fairly standard in the modern mathematical literature. For instance, \mathbb{N} , \mathbb{R} and \mathbb{C} denote the sets of natural, real and complex numbers respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_{+\circ} = [0, \infty)$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\}$. If (M, d) is a metric space and $N \subset M$, then $\overset{\circ}{N}$ and \overline{N} designate the interior and the closure of N respectively.

The capital letters X , Y and Z will denote Banach spaces endowed with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$, which the subscripts being dropped when there is no possibility of confusion.

If A is a linear operator defined on X , $D(A)$ denotes the domain of A , while $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A respectively. The spectrum can be decomposed into the point spectrum, the residual spectrum and the continuous spectrum of A , which are denoted by $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$, respectively.

Even when A is a closed linear operator, its domain $D(A) \subset X$ is not necessarily closed when inherits the topology of X , but it is a Banach space when equipped with the graph norm of A , $\|x\|_A = \|x\| + \|Ax\|$, henceforward denoted by X_A . Moreover, if A is, in addition, bounded, it admits a natural and norm preserving extension to the closure of $D(A)$, i.e, there exists $\tilde{A} : \overline{D(A)} \rightarrow X$ such that $\|A\| = \|\tilde{A}\|$. A proof of this result can be found in [14, Corollary 2.3, p. 33]

If a sequence $(x_n)_{n=0}^{\infty} \subset X$ converges to $x \in X$, the convergence is denoted by $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$.

The space of bounded linear operators from X to Y will be denoted by $\mathcal{L}(X, Y)$ ($\mathcal{L}(X) = \mathcal{L}(X, X)$ for short) and will be endowed with the uniform operator norm which makes it a Banach space. It follows from the definition of $\mathcal{L}(X, Y)$ and its norm that the uniform limit of a sequence of bounded operators is a bounded operator. However, the classic result mentioned below, a direct consequence of the Banach Steinhaus Theorem, states that pointwise limit of a sequence of bounded operators also defines a bounded operator (see [14, Theorem 2.2, p. 33]).

If (M, d) is a metric space, with M compact and X is a Banach space, then $C(M; X)$ denotes the space of all continuous functions $f : M \rightarrow X$. This space becomes a Banach space when endowed with the sup-norm

$$\|f\|_0 = \sup_{t \in M} \|f(t)\|.$$

The space of all functions $f : M \rightarrow X$ which are uniformly Lipschitz-continuous is denoted by $\text{Lip}(M; X)$, and

$$\|f\|_{\text{Lip}} = \sup_{t \neq s} \frac{\|f(t) - f(s)\|}{d(t, s)}.$$

If (Ω, Σ, μ) is a measure space then $L^p(\Omega, \Sigma, \mu; X)$, $1 \leq p < \infty$, denotes the space of equivalence classes of all Bochner-measurable functions $f : \Omega \rightarrow X$ such that $\|f(\cdot)\|^p$ is integrable, where the equivalence relation is given by $f \sim g$, that is, f equal g almost everywhere. This space is also a well-known Banach space when equipped with the norm

$$\|f\|_p = \left(\int_{\Omega} \|f(t)\|^p d\mu(t) \right)^{\frac{1}{p}}.$$

Similarly, $L^\infty(\Omega, \Sigma, \mu; X)$ denotes the space of equivalence classes of Bochner-measurable essentially bounded functions $f : \Omega \rightarrow X$, and its norm is defined by

$$\|f\|_\infty = \text{ess sup}_{t \in \Omega} \|f(t)\|.$$

In particular, for $\Omega \subset \mathbb{R}^n$ open, Σ the Lebesgue σ -algebra and μ the Lebesgue measure, $L_p(\Omega; X)$ denotes the abbreviation of $L^p(\Omega, \Sigma, \mu; X)$.

Also, for $\Omega \subset \mathbb{R}^n$ open, $C^n(\overline{\Omega}; X)$ denotes the space of all functions $f : \overline{\Omega} \rightarrow X$ with continuous partial derivatives $\partial^m f$ in Ω that can be extended continuously to $\overline{\Omega}$, for each $|m| < n$, and $C^\infty(\overline{\Omega}; X)$ is the space $\bigcap_{n \geq 1} C^n(\overline{\Omega}; X)$.

Given an interval I in the real line \mathbb{R} , a function $f : I \rightarrow \mathbb{R}$ is absolutely continuous on I if for every positive number ε , there is a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I with $x_k, y_k \in I$ satisfies

$$\sum_k (y_k - x_k) < \delta \quad \text{then} \quad \sum_k |f(y_k) - f(x_k)| < \varepsilon.$$

The collection of all absolutely continuous functions on I is denoted by $AC(I)$.

Another space of interest is the space of all functions $f : [a, b] \rightarrow X$ of strongly bounded variation, i.e. the supremum

$$\text{Var } f|_{[a,b]} = \sup \left\{ \sum_{j=1}^N \|f(t_j) - f(t_{j-1})\|; t_0 < t_1 < \dots < t_n, t_j \in [a, b] \right\}$$

is finite. This space is denoted by $BV([a, b]; X)$ and is of fundamental importance as it gives a sufficient condition to the almost everywhere differentiability and integrability of the respective derivative function. A proof of this fact can be found in [61, Corollary 6, p. 118].

The subscript *loc* sometimes assigned to any of the function spaces above stands for *locally* and it is used to drop the requirement that the membership property must be satisfied globally. Rigorously, if W is any of the function spaces mentioned above, then $f : \Omega \rightarrow X$ belongs to W_{loc} if and only if $f|_K \in W$ for each compact subsets K of Ω . It is clear that $W \subset W_{loc}$.

Moreover, if $X = \mathbb{K}$ is the underlying scalar field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, the second set in the function space notation introduced above will be dropped. For example $L^1_{loc}(\mathbb{R})$ denotes the space of all measurable scalar-valued functions which are integrable over each compact interval.

As usual, the star $*$ is employed for indicate the convolution of functions defined on the line and on the half-line, that is

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)ds, \quad t \in \mathbb{R}, \quad (1.1)$$

e.g. for $f, g \in L^1(\mathbb{R})$, and

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds, \quad t \in \mathbb{R}_+, \quad (1.2)$$

e.g. for $f, g \in L^1(\mathbb{R}_+)$. Observe that (1.1) and (1.2) are equivalent for functions which vanish for $t < 0$.

If f, g are integrable functions such that

$$\int_0^t f(t-s)g(s)ds = 0$$

almost everywhere in the interval $0 < s < \kappa$, then there exist $\lambda \geq 0$ and $\mu \geq 0$ satisfying $\lambda + \mu \geq \kappa$ such that $f(t) = 0$ almost everywhere in $0 < t < \lambda$ and $g(t) = 0$ almost everywhere in $0 < t < \mu$. This result is called the Titchmarsh convolution Theorem.

1.2 Laplace transform

A function $f \in L^1_{loc}(\mathbb{R}_+; X)$ is said to be of exponential growth or Laplace transformable if there exists $\omega \in \mathbb{R}_+$ such that $\int_0^{\infty} e^{-\omega t} |f(t)| dt < \infty$. The Laplace transform is then defined by

$$\widehat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t) dt, \quad \operatorname{Re} \lambda \geq \omega.$$

Theorem 1.1. [46, Theorem 1.2] Let $M, \omega \geq 0$ and $f_n : [0, \infty) \rightarrow \mathcal{L}(X)$ be a sequence of functions such that $f_n(0) = 0$ and

$$\|f_n(t+h) - f_n(t)\| \leq M e^{\omega(t+h)},$$

for $t, h \geq 0, n \in \mathbb{N}_0$. Then the following assertions are equivalent:

- (i) $\lim_{n \rightarrow \infty} \int_0^{\infty} e^{-\lambda t} f_n(t) x dt = \int_0^{\infty} e^{-\lambda t} f_0(t) x dt$ for all $x \in X$ and $\lambda > \omega$;
- (ii) $\lim_{n \rightarrow \infty} f_n(t) x = f_0(t) x$ for all $x \in X$ and $t \geq 0$.

Theorem 1.2. [58, Theorem 0.2, p.6] Let $f : (0, \infty) \rightarrow X$. The following are equivalent:

- (i) There exists $u \in \operatorname{Lip}(\mathbb{R}_+; X)$, $u(0) = 0$, such that $f(\lambda) = \widehat{u}'(\lambda)$ for all $\lambda > 0$;
- (ii) $f \in C^\infty((0, \infty); X)$ and

$$\sup \left\{ \frac{\lambda^{n+1} \|f^{(n)}(\lambda)\|}{n!}; \lambda > 0, n \in \mathbb{Z}_+ \right\} =: m_\infty(f) < \infty.$$

In this case $\|u\|_{\operatorname{Lip}} = m_\infty(f)$.

1.3 Mittag-Leffler

The special functions

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \quad \alpha, \beta \in \mathbb{C}, \quad \operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0, \quad z \in \mathbb{C}, \quad (1.3)$$

are called Mittag-Leffler functions. They were introduced by Mittag-Leffler in connection with his method of summation of some divergent series. The main properties of these functions can be found in the book by Erdélyi [25, Section 18.1]. The Mittag-Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, and especially in the investigation of the fractional generalization of the kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems.

From the asymptotic expansion of the Mittag-Leffler function (see [31, eq. (6)] for details) one obtains that for $0 < \alpha < 2$ and arbitrary β :

$$E_{\alpha,\beta}(z) \sim \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} e^{z^{\frac{1}{\alpha}}} \quad |z| \rightarrow \infty, \quad |\arg z| < \frac{\alpha\pi}{2}, \quad (1.4)$$

and this expression is used in the main result of Section 3.1 of Chapter 3.

Now, observe that

$$b_{\alpha,\beta}(\lambda, t) := \int_0^t (t-\tau)^{\alpha-1} \tau^{\beta-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) E_{\alpha,\beta}(\lambda\tau^\alpha) d\tau = t^{\alpha+\beta-1} \frac{d}{dt} E_{\alpha,\beta}(\lambda t^\alpha), \quad (1.5)$$

which can be checked using the Laplace transform on both sides of the above equality and taking into account the following identity [57, p.21]:

$$\int_0^\infty e^{-\mu t} t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm \omega t^\alpha) dt = \frac{k! \mu^{\alpha-\beta}}{(\mu^\alpha \mp \omega)^{k+1}}, \quad \operatorname{Re}(\mu) > |\omega|^{1/\alpha}. \quad (1.6)$$

Moreover, from [31, equations (38) and (43)] it follows that for $|z| > q$ (q is a fixed number)

$$E'_{\alpha,\beta}(z) = \frac{1}{\alpha z} [E_{\alpha,\beta-1}(z) - (\beta-1)E_{\alpha,\beta}(z)]. \quad (1.7)$$

Theorem 1.3. [57, Theorem 1.6, p.35] If $\alpha < 2$, β is an arbitrary positive real number, μ is such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$ and C is a real constant, then

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi, \quad |z| \geq 0. \quad (1.8)$$

1.4 Hausdorff measure of noncompactness

The next results and definitions are related to the Hausdorff measure of noncompactness, and are very much used in Section 4.4 of Chapter 4.

Definition 1.4. Let Z be a bounded subset of a normed space Y . The Hausdorff measure of noncompactness of Z is defined by

$$\eta(Z) = \inf\{\varepsilon > 0 : Z \text{ has a finite cover by balls of radius } \varepsilon\}.$$

This measure has some useful properties. For more general information of measure of noncompactness, the reader can consult [7, 8, 3].

Lemma 1.5. [8, Lemma 5.1, p.222] Let X be a real Banach space and B_1, B_2 be bounded subsets of X . Then

- (i) $\eta(B_1) = 0$ if and only if B_1 is totally bounded;
- (ii) $\eta(B_1) \leq \eta(B_2)$ if $B_1 \subseteq B_2$;
- (iii) $\eta(B_1) = \eta(\overline{B_1}) = \eta(\overline{\text{co}}(B_1))$, where $\overline{B_1}$ denotes the closure of B_1 and $\overline{\text{co}}(B_1)$ is the closed convex hull of B_1 ;
- (iv) $\eta(B_1 \cup B_2) = \max\{\eta(B_1), \eta(B_2)\}$;
- (v) $\eta(\lambda B_1) = |\lambda|\eta(B_1)$, with $\lambda \in \mathbb{R}$;
- (vi) $\eta(B_1 + B_2) \leq \eta(B_1) + \eta(B_2)$, where $B_1 + B_2 = \{b_1 + b_2; b_1 \in B_1, b_2 \in B_2\}$.

In what follows, ξ denotes the Hausdorff measure of noncompactness defined in X and γ denotes the Hausdorff measure of noncompactness on $C(I; X)$. Moreover, η denotes the Hausdorff measure of noncompactness for general Banach spaces Y .

Lemma 1.6. [68, Property 1.1, p.10] Let $W \subseteq C(I; X)$ be a subset of continuous functions. If W is bounded and equicontinuous, then the set $\overline{\text{co}}(W)$ is also bounded and equicontinuous.

Let W be a set of functions from I to X and $t \in I$ fixed, and denote $W(t) = \{w(t) : w \in W\}$.

Lemma 1.7. [8, Lemma 5.3, p.224] Let $W \subseteq C(I; X)$ be a bounded set. Then $\xi(W(t)) \leq \gamma(W)$ for all $t \in I$. Furthermore, if W is equicontinuous on I , then $\xi(W(t))$ is continuous on I , and

$$\gamma(W) = \sup\{\xi(W(t)) : t \in I\}.$$

A set of functions $W \subseteq L^1(I; X)$ is said to be uniformly integrable if there exists a positive function $\kappa \in L^1(I; \mathbb{R})$ such that $\|w(t)\| \leq \kappa(t)$ a.e. for all $w \in W$.

Lemma 1.8. [8, Lemma 5.4, p.224] If $\{u_n\}_{n=1}^\infty \subseteq L^1(I; X)$ is uniformly integrable, then for each $n \in \mathbb{N}$ the function $t \mapsto \xi(\{u_n(t)\}_{n=1}^\infty)$ is measurable and

$$\xi\left(\left\{\int_0^t u_n(s) ds\right\}_{n=1}^\infty\right) \leq 2 \int_0^t \xi(\{u_n(s)\}_{n=1}^\infty) ds.$$

The proof of the next result is in [13, Theorem 2].

Lemma 1.9. Let Y be a Banach space. If $W \subseteq Y$ is a bounded subset, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^\infty \subseteq W$ such that

$$\eta(W) \leq 2\eta(\{u_n\}_{n=1}^\infty) + \varepsilon.$$

Lemma 1.10. [34, Lemma 2.4] Let W be a closed and convex subset of a complex Banach space Y , and let $F : W \rightarrow W$ be a continuous operator such that $F(W)$ is a bounded set. Define

$$F^1(W) = F(W), \quad F^n(W) = F(\overline{\text{co}}(F^{n-1}(W))), \quad n = 2, 3, \dots$$

If there exists a constant $0 \leq r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\eta(F^{n_0}(W)) \leq r\eta(W), \tag{1.9}$$

then F has a fixed point in W .

The following lemma is used in Section 4.4 of Chapter 4 and its proof can be found in [34, Theorem 3.1].

Lemma 1.11. For all $0 \leq m \leq n$, denote $C_m^n = \binom{n}{m}$. If $0 < \epsilon < 1$, $h > 0$ and

$$S_n = \epsilon^n + C_1^n \epsilon^{n-1} h + C_2^n \epsilon^{n-2} \frac{h^2}{2!} + \dots + C_{n-1}^n \epsilon^{n-(n-1)} \frac{h^{n-1}}{(n-1)!} + C_n^n \frac{h^n}{n!}, \quad n \in \mathbb{N}, \quad (1.10)$$

then $\lim_{n \rightarrow \infty} S_n = 0$.

2 (a,k)-regularized resolvent families

Here is provided an overview of the theory of the (a, k) -regularized resolvent families, including definitions and classical theorems. This theory was introduced by Lizama in [47] and has been extensively studied in [49, 48]. Moreover, the (a, k) -regularized resolvent families include well-known families, such as C_0 -semigroups, cosine and resolvent families of bounded linear operators.

This chapter is organized in the following way: the first section deals with the definitions and properties of these families as well as their relationship with the results of the other families present in the literature. Sections two, three, four and five contain classical results of (a, k) -regularized resolvent families and their comparison with similar results for other classes of operators.

2.1 Definitions and properties

This section presents the definitions and properties of the theory of the (a, k) -regularized resolvent families.

Definition 2.1. Let $k \in C(\mathbb{R}_+)$, $k \not\equiv 0$, $a \in L^1_{loc}(\mathbb{R}_+)$, $a \not\equiv 0$, and let $A : D(A) \subset X \rightarrow X$ be a closed and densely defined operator. A strongly continuous family $\{R_{a,k}(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ is called an (a, k) -regularized resolvent family on X having the operator A as a generator if the following properties hold:

- (i) $\lim_{t \rightarrow 0^+} \frac{R_{a,k}(t)}{k(t)}x = x$ for all $x \in X$;
- (ii) $R_{a,k}(t)x \in D(A)$ and $R_{a,k}(t)Ax = AR_{a,k}(t)x$ for all $x \in D(A)$ and $t \geq 0$;
- (iii) $R_{a,k}(t)x = k(t)x + \int_0^t a(t-s)AR_{a,k}(s)x ds$, $t \geq 0$, $x \in D(A)$.

Example 2.2. The choice of the pair (a, k) classifies different families of bounded linear operators.

- (1) If $k \equiv 1$ and $a \equiv 1$, $\{R_{a,k}(t)\}_{t \geq 0}$ is a C_0 -semigroup denoted as $\{T(t)\}_{t \geq 0}$ (see [6, Section 3.1]).
- (2) If $k \equiv 1$ and $a(t) = t$, $\{R_{a,k}(t)\}_{t \geq 0}$ is a strongly continuous cosine family, denoted by $\{C(t)\}_{t \geq 0}$ (see [6, Section 3.4]).
- (3) If $k(t) = t$ and $a(t) = t$, $\{R_{a,k}(t)\}_{t \geq 0}$ is a sine family.

- (4) The case when $k \equiv 1$ and $a(t) \in L^1_{loc}(\mathbb{R}_+)$ was studied in [58] and $\{R_{a,k}(t)\}_{t \geq 0}$ is called a resolvent family, denoted by $\{S(t)\}_{t \geq 0}$.
- (5) Bazhlekova in [9] studied the case when $k \equiv 1$ and $a(t) = g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\alpha > 0$ and Γ denotes the Gamma function. This family is called α -resolvent family (also called fractional resolvent families [45] or solution families [44]) and it is denoted by $\{S_\alpha(t)\}_{t \geq 0}$.
- (6) If $a(t) = g_\alpha(t)$ and $k(t) = g_\beta(t)$, $\alpha > 0$, $\beta > 0$, then $\{R_{a,k}(t)\}_{t \geq 0}$ is an (α, β) -resolvent of bounded linear operators, and it is denoted by $\{R_{\alpha,\beta}(t)\}_{t \geq 0}$.

Lemma 2.3. Let $\{R_{a,k}(t)\}_{t \geq 0}$ be an (a, k) -regularized resolvent family generated by A . Then, $(a * R_{a,k})(t)x \in D(A)$ for all $x \in X$ and $t \geq 0$, and

$$R_{a,k}(t)x = k(t)x + A \int_0^t a(t-s)R_{a,k}(s)x ds, \quad x \in X, t \geq 0. \quad (2.1)$$

Proof. Let $x \in X$ and define $y = (\lambda - A)^{-1}x \in D(A)$, where $\lambda \in \rho(A)$ is fixed. Consider $z = (a * R_{a,k})(t)x$, $t \geq 0$. From (ii) and (iii) of Definition 2.1,

$$\begin{aligned} z &= (\lambda - A)(a * R_{a,k})(t)y = \lambda(a * R_{a,k})(t)y - (a * AR_{a,k})(t)y \\ &= \lambda(a * R_{a,k})(t)y - (R_{a,k}(t)y - k(t)y), \end{aligned}$$

that is, $z \in D(A)$ and

$$(\lambda - A)z = \lambda(a * R_{a,k})(t)x - (R_{a,k}(t)x - k(t)x) = \lambda z - (R_{a,k}(t)x - k(t)x),$$

which gives (2.1). \square

Corollary 2.4. If R_{a,k_1} is an (a, k_1) -regularized resolvent family and R_{a,k_2} is an (a, k_2) -regularized resolvent family, then $(k_1 * R_{a,k_2})(t) = (k_2 * R_{a,k_1})(t)$ for all $t \geq 0$.

Proof. Let $x \in D(A)$. From (ii) and (iii) of Definition 2.1 it follows that

$$\begin{aligned} (k_2 * R_{a,k_1})(t)x &= ((R_{a,k_2} - (a * AR_{a,k_2})) * R_{a,k_1})(t)x \\ &= (R_{a,k_2} * R_{a,k_1})(t)x - (a * R_{a,k_2} * AR_{a,k_1})(t)x \\ &= R_{a,k_2} * (R_{a,k_1} - (a * AR_{a,k_1}))(t)x \\ &= (R_{a,k_2} * k_1)(t)x = (k_1 * R_{a,k_2})(t)x. \end{aligned}$$

Now, let $\lambda \in \rho(A)$ and $y \in X$. Define $x = (\lambda - A)^{-1}y$. Since

$$(\lambda - A)(k_2 * R_{a,k_1})(t)x = (\lambda - A)(k_1 * R_{a,k_2})(t)x,$$

it follows that $(k_2 * R_{a,k_1})(t)y = (k_1 * R_{a,k_2})(t)y$ for each $y \in X$ and $t \geq 0$. \square

Remark 2.5. (1) As consequence of the previous corollary and Titchmarsh convolution Theorem (see Section 1.1), the (a, k) -resolvent family is unique.

- (2) If $\{R_{a,k_1}(t)\}_{t \geq 0}$ is an (a, k_1) -regularized resolvent family and also an (a, k_2) -regularized resolvent family, then $k_1 = k_2$.

(3) Let $\{R_{a,k_i}(t)\}_{t \geq 0}$ be an (a, k_i) -regularized resolvent family, $i = 1, 2$.

Then $\{(R_{a,k_1} + R_{a,k_2})(t)\}_{t \geq 0}$ is an $(a, k_1 + k_2)$ -regularized resolvent family.

Definition 2.6. $\{R_{a,k}(t)\}_{t \geq 0}$ is called **exponentially bounded** if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|R_{a,k}(t)\| \leq Me^{\omega t}, \text{ for all } t \geq 0; \quad (2.2)$$

or more precisely (M, ω) is called a **type** of $\{R_{a,k}(t)\}_{t \geq 0}$.

Remark 2.7. The boundedness condition (2.2) can be proved for some families, see, for example [55, Theorem 2.2, p.4] in the case of C_0 -semigroups and [28, Theorem 1.1, p.25] in the case of cosine and sine families.

The next result characterizes an (a, k) -regularized resolvent family.

Proposition 2.8. Let $\{R(t)\}_{t \geq 0}$ be an exponentially bounded and strongly continuous operator family in $\mathcal{L}(X)$ of type (M, ω) such that the Laplace transform $\widehat{R}(\lambda)$ exists for $\lambda > \omega$. Then, $\{R(t)\}_{t \geq 0}$ is an (a, k) -regularized resolvent family of type (M, ω) if and only if, for every $\lambda > \omega$, $(I - \widehat{a}(\lambda)A)^{-1}$ exists in $\mathcal{L}(X)$ and

$$\widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}x = \int_0^\infty e^{-\lambda s} R(s)x ds, \quad \forall x \in X. \quad (2.3)$$

Proof. Suppose that $\{R(t)\}_{t \geq 0}$ is an (a, k) -regularized resolvent family. By assumption, the Laplace transform $H(\lambda) = \widehat{R}(\lambda)$ of the (a, k) -regularized resolvent family exists for $\lambda > \omega$. Then, $\frac{1}{a} \in \rho(A)$ and $H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$.

In fact, from (ii) and (iii) of Definition 2.1 and Lemma 2.3 and the convolution Theorem, for $\operatorname{Re} \lambda > \omega$, it follows that

$$H(\lambda)x = \widehat{k}(\lambda)x + \widehat{a}(\lambda)H(\lambda)Ax, \quad (2.4)$$

for each $x \in D(A)$, and

$$H(\lambda)x = \widehat{k}(\lambda)x + A\widehat{a}(\lambda)H(\lambda)x, \quad (2.5)$$

for each $x \in X$. For each $x \in D(A)$,

$$H(\lambda)[I - \widehat{a}(\lambda)A]x = \widehat{k}(\lambda)x$$

can be obtained from (2.4), and for each $x \in X$

$$[I - \widehat{a}(\lambda)A]H(\lambda)x = \widehat{k}(\lambda)x$$

can be obtained from (2.5).

Thus the operators $I - \widehat{a}(\lambda)A$ are invertible for all $\lambda \in \mathbb{C}$, with $\operatorname{Re} \lambda > \omega$ and

$$H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}, \text{ for } \operatorname{Re} \lambda > \omega.$$

In particular, $\frac{1}{\widehat{a}(\lambda)} \in \rho(A)$ for all such λ , provided $\widehat{a}(\lambda) \neq 0$. To prove this fact assume that $\widehat{a}(\lambda_0) = 0$ for some λ_0 with $\operatorname{Re} \lambda_0 > \omega$. Since $\widehat{a}(\lambda)$ is holomorphic, λ_0 is an isolated zero of finite multiplicity, and $H(\lambda_0) = \widehat{k}(\lambda_0)$. Choose a small circle Γ around λ_0 which is entirely contained in the half-plane $\operatorname{Re} \lambda > \omega$ such that $\widehat{a}(\lambda) \neq 0$ on Γ .

Notice that $\frac{H(\lambda)}{\widehat{k}(\lambda)}$ is holomorphic on Γ , then

$$\frac{H(\lambda_0)}{\widehat{k}(\lambda_0)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{H(\lambda)}{(\lambda - \lambda_0)\widehat{k}(\lambda)} d\lambda. \quad (2.6)$$

Now, since A is closed, it follows that

$$A = A \frac{1}{\widehat{k}(\lambda_0)} H(\lambda_0) = A \frac{1}{2\pi i} \int_{\Gamma} \frac{H(\lambda)}{(\lambda - \lambda_0)\widehat{k}(\lambda)} d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \frac{AH(\lambda)}{(\lambda - \lambda_0)\widehat{k}(\lambda)} d\lambda.$$

Moreover,

$$AH(\lambda) = \frac{(H(\lambda) - \widehat{k}(\lambda))}{\widehat{a}(\lambda)}$$

is well-defined and holomorphic on Γ . Then

$$A = \frac{1}{2\pi i} \int_{\Gamma} \frac{H(\lambda) - \widehat{k}(\lambda)}{(\lambda - \lambda_0)\widehat{a}(\lambda)\widehat{k}(\lambda)} d\lambda$$

is obtained by Cauchy Integral Formula, and so A is bounded, which is a contradiction to the standing hypothesis. Thus $\widehat{a}(\lambda) \neq 0$ for all $\text{Re } \lambda > \omega$.

Conversely, let $\mu, \lambda > \omega$ and $x \in D(A)$. Then $x = (I - \widehat{a}(\mu)A)^{-1}y$ for some $y \in X$. Since $(I - \widehat{a}(\mu)A)^{-1}$ and $(I - \widehat{a}(\lambda)A)^{-1}$ commute and A is closed, it follows that

$$\begin{aligned} \int_0^{\infty} e^{-\lambda t} R(t)x dt &= \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}(I - \widehat{a}(\mu)A)^{-1}y \\ &= (I - \widehat{a}(\mu)A)^{-1}\widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}y \\ &= (I - \widehat{a}(\mu)A)^{-1}\widehat{R}(\lambda)y \\ &= \int_0^{\infty} e^{-\lambda t}(I - \widehat{a}(\mu)A)^{-1}R(t)y dt. \end{aligned}$$

Hence, by uniqueness of the Laplace transform,

$$R(t)x = (I - \widehat{a}(\mu)A)^{-1}R(t)(I - \widehat{a}(\mu)A)x$$

for almost all $t \geq 0$, and then, $R(t)x \in D(A)$. Further, since $\widehat{a}(\mu) \neq 0$ it follows from the above equality that $AR(t)x = R(t)Ax$ for every $t \geq 0$, $x \in D(A)$.

Now let $\lambda > \omega$ and $x \in D(A)$. From the convolution theorem,

$$\begin{aligned} \int_0^{\infty} e^{-\lambda t} k(t)x dt &= \widehat{k}(\lambda)x = \widehat{R}(\lambda)(I - \widehat{a}(\lambda)A)x = \widehat{R}(\lambda)x - \widehat{R}(\lambda)\widehat{a}(\lambda)Ax \\ &= \int_0^{\infty} e^{-\lambda t} \left[R(t)x - \int_0^t a(t-s)R(s)Axs ds \right] dt. \end{aligned}$$

The uniqueness of the Laplace transform and the strong continuity of $R(t)$ yield that

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Axs ds.$$

From the proof of Lemma 2.3, $R(0) = k(0)I$ and the proof is complete. \square

The next result presents a property of the generator A . For this, following assumption on $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ is considered.

$(H_{a,k})$ There exists $\epsilon_{a,k} > 0$ and $t_{a,k} > 0$ such that for all $0 < t \leq t_{a,k}$

$$\left| \int_0^t a(t-s)k(s)ds \right| \geq \epsilon_{a,k} \int_0^t |a(t-s)k(s)|ds.$$

Theorem 2.9. Suppose A is the generator of an (a, k) -regularized resolvent family $\{R_{a,k}(t)\}_{t \geq 0}$ such that $t \mapsto |k(t)|$ is a nondecreasing function and satisfies

$$\limsup_{t \rightarrow 0^+} \frac{\|R_{a,k}(t)\|}{|k(t)|} < \infty. \quad (2.7)$$

Assume $(H_{a,k})$. Then

- (i) $D(A) = \left\{ x \in X; \lim_{t \rightarrow 0^+} \frac{R_{a,k}(t)x - k(t)x}{(k * a)(t)} \text{ exists} \right\};$
- (ii) $\lim_{t \rightarrow 0^+} \frac{R_{a,k}(t)x - k(t)x}{(k * a)(t)} = Ax, \text{ for all } x \in D(A).$

Proof. Let $z \in D(A)$. Then, items (ii) and (iii) of Definition 2.1, the strong continuity of $R_{a,k}$ and the fact that $|k(t)|$ is nondecreasing implies that

$$\begin{aligned} \left\| \frac{R_{a,k}(t)z}{k(t)} - z \right\| &= \frac{1}{|k(t)|} \left\| \int_0^t a(t-s)AR_{a,k}(s)zds \right\| \\ &\leq \left(\int_0^t |a(t-s)| \frac{\|R_{a,k}(s)\|}{|k(s)|} ds \right) \|Az\|. \end{aligned}$$

Hence, for all $z \in D(A)$

$$\lim_{t \rightarrow 0^+} \left\| \frac{R_{a,k}(t)z}{|k(t)|} - z \right\| = 0.$$

The denseness of $D(A)$ and (2.7) imply that this actually holds for all $z \in X$. Thus, for every $z \in X$ and $\epsilon > 0$, there exists $0 < t_{\epsilon,z} < \min\{t_{a,k}, 1\}$ such that

$$\left\| \frac{R_{a,k}(t)z}{k(t)} - z \right\| < \epsilon, \quad \forall t \in (0, t_{\epsilon,z}). \quad (2.8)$$

Now define the set

$$\tilde{D}(A) := \left\{ x \in X; \lim_{t \rightarrow 0^+} \frac{R_{a,k}(t)x - k(t)x}{(k * a)(t)} \text{ exists} \right\}.$$

Let $x \in D(A)$ and define $z = Ax$. For all $t \in (0, t_{\epsilon, Ax})$,

$$\left\| \frac{R_{a,k}(t)Ax}{k} - Ax \right\| < \epsilon,$$

can be obtained from (2.8). Therefore, using (iii) of Definition 2.1 and $(H_{a,k})$ for all $\tau \in (0, t_{\varepsilon, Ax})$, it follows that

$$\begin{aligned} \left\| \frac{R_{a,k}(\tau)x - k(\tau)x}{(k * a)(\tau)} - Ax \right\| &= \frac{1}{|(k * a)(\tau)|} \left\| \int_0^\tau a(\tau - s)k(s) \left[\frac{R_{a,k}(s)Ax}{k(s)} - Ax \right] ds \right\| \\ &\leq \frac{1}{|(k * a)(\tau)|} \int_0^\tau |a(\tau - s)k(s)| \varepsilon ds = \frac{\varepsilon}{\epsilon_{a,k}}. \end{aligned}$$

Thus, $x \in \tilde{D}(A)$, that is, $D(A) \subset \tilde{D}(A)$ and condition (ii) holds. On the other hand, let $x \in \tilde{D}(A)$. Then,

$$\lim_{t \rightarrow 0^+} \frac{R_{a,k}(t)x - k(t)x}{(k * a)(t)} = y$$

for some $y \in X$. For given $\varepsilon > 0$ and all $t \in (0, t_{\varepsilon, x})$, by using (2.8) and $(H_{a,k})$ it follows that

$$\begin{aligned} \left\| \frac{1}{(k * a)(t)} \int_0^t a(t - s)R_{a,k}(s)x ds - x \right\| &= \frac{1}{|(k * a)(t)|} \left\| \int_0^t a(t - s)k(s) \left[\frac{R_{a,k}(s)x}{k(s)} - x \right] ds \right\| \\ &\leq \frac{\varepsilon}{|(k * a)(t)|} \int_0^t |a(t - s)k(s)| ds \leq \frac{\varepsilon}{\epsilon_{a,k}}. \end{aligned}$$

Then

$$\lim_{t \rightarrow 0^+} \frac{1}{(k * a)(t)} \int_0^t a(t - s)R_{a,k}(s)x ds = x.$$

Now, from (iii) of Definition 2.1, observe that

$$\begin{aligned} \left\| A \left[\frac{1}{(k * a)(t)} \int_0^t a(t - s)R_{a,k}(s)x ds \right] - y \right\| &= \left\| \frac{1}{(k * a)(t)} \int_0^t a(t - s)AR_{a,k}(s)x ds - y \right\| \\ &= \left\| \frac{R_{a,k}(t)x - k(t)x}{(k * a)(t)} - y \right\|, \end{aligned}$$

where the right hand side goes to zero as $t \rightarrow 0^+$. Then, since A is closed, $x \in D(A)$ and $Ax = y$, proving the theorem. \square

Example 2.10. It is not difficult to find examples of functions a and k that satisfy assumption $(H_{a,k})$.

(1) If a and k are positive functions, then

$$\left| \int_0^t a(t - s)k(s) ds \right| = \int_0^t |a(t - s)k(s)| ds.$$

That is, assumption $(H_{a,k})$ is satisfied with $\epsilon_{a,k} = 1$.

(2) If $a(t) = -b - c^2t$ and $k \equiv 1$, then

$$\begin{aligned} |(a * k)(t)| &= \left| \int_0^t -b - c^2(t - s) ds \right| = \left| \int_0^t -b - c^2t + c^2s ds \right| \\ &= \left| -bt - c^2t^2 + c^2 \frac{t^2}{2} \right| = bt + c^2 \frac{t^2}{2} = \int_0^t |-b - c^2(t - s)| ds, \end{aligned}$$

that is, assumption $(H_{a,k})$ is satisfied with $\epsilon_{a,k} = 1$.

2.2 The Generation Theorem

This section presents a theorem known as the Generation Theorem.

Some general properties are well-known. For example, to relate different α -resolvent families and its generators, Li, Chen and Li [45] shown that if the operator $-A$ generates a bounded α -times resolvent family then, with some suitable β , $-A^\beta$ also generates an α -times resolvent family. There also exists a principle of subordination (see [9, Chapter 3]). For instance, if an operator A generates a cosine operator family then it also generates an α -resolvent family for any $0 < \alpha < 2$, but the converse is not true. Moreover, considering for any $\alpha \in (0, 2)$ and $\theta \in [0, \pi)$ the differential operator $B_\theta = e^{i\theta} \partial_{xx}^2$, with $D(B_\theta) = \{g \in W^{2,2}(0, 1), g(0) = g(1) = 0\}$ on $X = L^2(0, 1)$, then B_θ generates a bounded α -resolvent family if and only if $|\theta| \leq (1 - \frac{\alpha}{2}) \pi$. However, for $\frac{\pi}{2} < |\theta| \leq (1 - \frac{\alpha}{2}) \pi$, the operator B_θ does not generates any C_0 -semigroup [9, Section 2.2].

In the literature, this theorem was presented also in the particular cases of families of bounded linear operators:

- (1) In [55, Theorem 3.1, p. 8] there is the Hille-Yosida theorem of generation of C_0 -semigroups.
- (2) For cosine families, [28, Theorem 2.1, p. 28]
- (3) [58, Theorem 1.3, p. 43] for resolvent families.
- (4) [9, Theorem 2.8, p. 23] for α -resolvents.

Theorem 2.11. Let $A : D(A) \subset X \rightarrow X$ be a closed linear densely defined operator in a Banach space X . Then A is the generator of a (a, k) -regularized resolvent family $\{R_{a,k}(t)\}_{t \geq 0}$ of type (M, ω) if and only if the following conditions hold:

- (i) $\widehat{a}(\lambda) \neq 0$ and $\frac{1}{\widehat{a}(\lambda)} \in \rho(A)$ for all $\lambda \in \mathbb{R}, \lambda > \omega$;
- (ii) $H(\lambda) := \widehat{R}_{a,k}(\lambda)$ satisfies $H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$ and

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \quad \lambda > \omega, \quad n \in \mathbb{Z}_+. \quad (2.9)$$

Proof. If $\{R_{a,k}(t)\}_{t \geq 0}$ is an (a, k) -regularized resolvent family of type (M, ω) , its Laplace transform

$$H(\lambda) = \widehat{R}_{a,k}(\lambda) = \int_0^\infty e^{-\lambda s} R_{a,k}(s) ds, \quad \operatorname{Re} \lambda > \omega,$$

is well-defined and holomorphic for $\operatorname{Re} \lambda > \omega$, and satisfies

$$\begin{aligned} \|H(\lambda)\| &\leq \int_0^\infty \|e^{-\lambda s} R_{a,k}(s)\| ds \leq M \int_0^\infty \|e^{-\lambda s}\| e^{\omega s} ds \\ &\leq M \int_0^\infty e^{-(\operatorname{Re} \lambda - \omega)s} ds = \frac{M}{(\operatorname{Re} \lambda - \omega)}. \end{aligned}$$

Also, note that for $\operatorname{Re}(\lambda) > \omega$,

$$\|H'(\lambda)\| = \left\| \int_0^\infty -s e^{-\lambda s} R_{a,k}(s) ds \right\| \leq \int_0^\infty s e^{-\operatorname{Re} \lambda s} M e^{\omega s} ds$$

$$\leq \int_0^\infty \frac{M e^{-(\operatorname{Re} \lambda - \omega)s}}{(\operatorname{Re} \lambda - \omega)} ds \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^2}.$$

Now, suppose

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\operatorname{Re} \lambda - \omega)^{n+1}}, \operatorname{Re} \lambda > \omega. \quad (2.10)$$

Then,

$$\begin{aligned} \|H^{(n+1)}(\lambda)\| &= \left\| \int_0^\infty (-s)^{n+1} e^{-\lambda s} R_{a,k}(s) ds \right\| \\ &= \left\| \left[\frac{(-s)^{n+1}}{(-\lambda)} e^{-\lambda s} R_{a,k}(s) \right]_0^\infty + \frac{n+1}{\lambda} \int_0^\infty (-s)^n e^{-\lambda s} R_{a,k}(s) ds \right\| \\ &\leq \frac{n+1}{|\lambda|} \|H^{(n)}(\lambda)\|. \end{aligned}$$

Using (2.10), it can be concluded the following inequality

$$\|H^{(n+1)}(\lambda)\| \leq M(n+1)!(\operatorname{Re} \lambda - \omega)^{-(n+2)}.$$

Then the following estimates holds

$$\|H^{(n)}(\lambda)\| \leq Mn!(\operatorname{Re} \lambda - \omega)^{-(n+1)}, \operatorname{Re} \lambda > \omega, n \in \mathbb{Z}_+. \quad (2.11)$$

Now, from (ii) and (iii) of Definition 2.1, Lemma 2.3 and the convolution theorem, for $\operatorname{Re} \lambda > \omega$, it follows that

$$H(\lambda)x = \widehat{k}(\lambda)x + \widehat{a}(\lambda)H(\lambda)Ax, \quad (2.12)$$

for each $x \in D(A)$, and

$$H(\lambda)x = \widehat{k}(\lambda)x + A\widehat{a}(\lambda)H(\lambda)x, \quad (2.13)$$

for each $x \in X$. Now, for each $x \in D(A)$

$$H(\lambda)[I - \widehat{a}(\lambda)A]x = \widehat{k}(\lambda)x$$

can be obtained from (2.12), and for each $x \in X$,

$$[I - \widehat{a}(\lambda)A]H(\lambda)x = \widehat{k}(\lambda)x$$

can be obtained from (2.13).

Thus the operators $I - \widehat{a}(\lambda)A$ are invertible for all $\lambda \in \mathbb{C}$, with $\operatorname{Re} \lambda > \omega$, and

$$H(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}, \text{ for } \operatorname{Re} \lambda > \omega.$$

In particular, $\frac{1}{\widehat{a}(\lambda)} \in \rho(A)$ for all such λ , provided $\widehat{a}(\lambda) \neq 0$. In fact, assuming that $\widehat{a}(\lambda_0) = 0$ for some λ_0 with $\operatorname{Re} \lambda_0 > \omega$.

Proceeding as in the proof of Proposition 2.8, it follows that A is bounded, which is a contradiction to the standing hypothesis. Thus $\widehat{a}(\lambda) \neq 0$ for all $\operatorname{Re} \lambda > \omega$.

On the other hand, assuming conditions (i) and (ii), by Theorem 1.2 there exists a Lipschitz family $\{U_\omega(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ with $U_\omega(0) = 0$ and $\widehat{U}'_\omega(\lambda) = H(\lambda + \omega)$, $\lambda > 0$. Then

$$\lambda \widehat{U}_\omega(\lambda) = \lambda \widehat{U}_\omega(\lambda) - U_\omega(0) = \widehat{U}'_\omega(\lambda) = H(\lambda + \omega),$$

that is,

$$\widehat{U}_\omega(\lambda) = \frac{H(\lambda + \omega)}{\lambda}.$$

Define $\{U(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ by

$$U(t) = e^{\omega t}U_\omega(t) - \omega \int_0^t e^{\omega s}U_\omega(s)ds, \quad t \geq 0. \quad (2.14)$$

Given $\gamma > 0$, let $s, t \in (0, \gamma)$ such that $s < t$.

Using the Mean Value Theorem for integrals, note that

$$\begin{aligned} \|U(\gamma) - U(0)\| &= \left\| e^{\omega \gamma}U_\omega(\gamma) - \omega \int_0^\gamma e^{\omega \tau}U_\omega(\tau)d\tau \right\| \\ &= \left\| \int_0^\gamma e^{\omega \tau} \frac{d}{d\tau}U_\omega(\tau)d\tau \right\| \\ &\leq \gamma e^{\omega \eta} \left\| \frac{d}{d\tau}U_\omega(\eta) \right\|, \end{aligned}$$

for some $\eta \in (0, \gamma)$.

Then, since $U_\omega(t)$ is Lipschitz and differentiable, its derivative is bounded, and using that $\eta < \gamma$, it follows that

$$\|U(\gamma) - U(0)\| \leq \gamma e^{\omega \gamma} M_\omega.$$

Then,

$$\begin{aligned} \|U(t) - U(s)\| &= \left\| \int_s^t e^{\omega \tau} \frac{d}{d\tau}U_\omega(\tau)d\tau \right\| \\ &\leq |t - s| e^{\omega \eta} \left\| \frac{d}{d\tau}U_\omega(\eta) \right\| \\ &\leq |t - s| e^{\omega \gamma} M_\omega = |t - s| M_\gamma, \end{aligned}$$

that is, $U(t)$ is locally Lipschitz.

Now, by First Shifting Property,

$$\begin{aligned} \widehat{U}(\lambda) &= \widehat{U}_\omega(\lambda - \omega) - \frac{\omega}{\lambda} \widehat{U}_\omega(\lambda - \omega) \\ &= \frac{H(\lambda)}{\lambda - \omega} - \frac{\omega H(\lambda)}{(\lambda - \omega)\lambda} \\ &= H(\lambda) \left(\frac{1}{\lambda - \omega} - \frac{\omega}{(\lambda - \omega)\lambda} \right) \\ &= \frac{H(\lambda)}{\lambda}. \end{aligned}$$

The definition of $H(\lambda)$ shows that $U(t)$ commutes with A and yields the identity

$$\widehat{U}(\lambda) = \lambda^{-1} \widehat{\mathbf{k}}(\lambda) + \widehat{\mathbf{a}}(\lambda) \widehat{U}(\lambda) A,$$

i.e.

$$U(t)x = \int_0^t \mathbf{k}(s)x ds + (\mathbf{a} * U)(t)Ax, \quad \text{for all } x \in D(A), t \geq 0.$$

For each $x \in D(A)$ let $g(t) = \int_0^t k(s)ds$, then $g'(t) = k(t)x$. Given $[a, b] \subset \mathbb{R}_+$, it follows that

$$\|g(t)\| = \left\| \int_0^t k(s)x ds \right\| \leq \int_0^t |k(s)| \|x\| ds \leq \infty,$$

that is, $g \in L_{loc}^\infty(\mathbb{R}_+; X)$. Similarly, $\|g'(t)\| = \|k(t)x\|$ and since $k \in C(\mathbb{R}_+)$, it follows that $g' \in L_{loc}^\infty(\mathbb{R}_+)$. Therefore, $g \in W_{loc}^{1,\infty}(\mathbb{R}_+; X)$.

Now, for each $x \in D(A)$, $f(t) = U(t)Ax$ is locally Lipschitz, i.e. $f \in BV_{loc}(\mathbb{R}_+; X)$. Then f is continuous and, given $[a, b] \subset \mathbb{R}_+$, there exists $K = K_{a,b}$ such that $\|f(t)\|_X \leq K_{a,b}$, that is, $f \in L_{loc}^\infty(\mathbb{R}_+; X)$.

Note that f is differentiable almost everywhere on (a, b) for all $a, b \in \mathbb{R}_+$, $a < b$ because $f \in BV_{loc}(\mathbb{R}_+; X)$ (see Section 1.1 of Chapter 1).

So, since f is locally Lipschitz, there exists $\tilde{K} = \tilde{K}_{a,b}$ such that $\|f'(t)\|_X \leq \tilde{K}_{a,b}$. Then, $f' \in L_{loc}^\infty(\mathbb{R}_+; X)$. Therefore, $f \in W_{loc}^{1,\infty}(\mathbb{R}_+; X)$.

Finally, let $t \in \mathbb{R}_+$ and $b > t$, then

$$\begin{aligned} \|(a * f)(t)\|_X &= \left\| \int_0^t a(t-s)f(s)ds \right\|_X \leq \int_0^b \|a(t-s)f(s)\|_X ds \\ &\leq \|f\|_{L^\infty([a,b];X)} \int_0^b \|a(t-s)x\| ds < \infty, \end{aligned}$$

and, similarly,

$$\|(a * f')(t)\| \leq \|f'\|_{L^\infty([a,b];X)} \int_0^b \|a(t-s)\| ds < \infty.$$

Therefore, $U(\cdot)x \in W_{loc}^{1,\infty}(\mathbb{R}_+; X)$ and

$$\frac{d}{dt}U(t)x = k(t)x + \frac{d}{dt}(a * f)(t) = k(t)x + a * \frac{d}{dt}f(t), \text{ for almost all } t \geq 0.$$

Notice that $a * df(t) = a * \frac{d}{dt}(U(t)Ax)$ and $U(t)Ax$ is only defined and consistent provided $x \in D(A^2)$ because

$$U(t)Ax = \int_0^t k(s)Axs ds + (a * U)(t)A^2x.$$

This implies that $\left(a * \frac{d}{dt}f\right)(t)$ is even continuous, from which in turn $U(t)x$ is continuously differentiable on \mathbb{R}_+ for each $x \in D(A^2)$.

Then, $\frac{d}{dt}U(t)$ is uniformly bounded for t bounded, so

$$T_h := \frac{U(t+h) - U(t)}{h}$$

is uniformly bounded for $0 < h \leq 1$ and t bounded. That is, $T_h \in \mathcal{L}(D(A^2); X)$ and $\sup_{0 < h \leq 1} \|T_h\|_{\mathcal{L}(D(A^2); X)} < \infty$.

Since $T_h x \rightarrow \frac{d}{dt}U(t)x$, it follows that $\frac{d}{dt}U(t) \in \mathcal{L}(D(A^2); X)$ and

$$\left\| \frac{d}{dt}U(t) \right\|_{\mathcal{L}(D(A^2); X)} < \infty.$$

Now, once $D(A^2)$ is dense and $\frac{d}{dt}U(t)$ is bounded, it is possible to get a bounded extension of $\frac{d}{dt}U(t)$ in X , denoted as $\frac{d}{dt}U(t)$ (see Section 1.1 of Chapter 1). Then $\frac{d}{dt}U(t) : X \rightarrow X$ with

$$\left\| \frac{d}{dt}U(t) \right\|_{\mathcal{L}(X)} = \left\| \frac{d}{dt}U(t) \right\|_{\mathcal{L}(D(A^2); X)} < \infty.$$

Note that $\frac{d}{dt}U(t)x$ is continuous for t bounded if $x \in D(A^2)$. From the fact that $D(A^2)$ is dense in X , for $y \in X$, ξ close to t and x close to y it is possible to obtain

$$\begin{aligned} & \left\| \frac{d}{dt}U(t+\xi)y - \frac{d}{dt}U(t)y \right\| \\ & \leq \left\| \frac{d}{dt}U(t+\xi)(y-x) + \frac{d}{dt}U(t+\xi)x - \frac{d}{dt}U(t)x - \frac{d}{dt}U(t)(y-x) \right\| \\ & < \varepsilon. \end{aligned}$$

So, $\frac{d}{dt}U(t)x$ exists for all $x \in X$ and $\frac{d}{dt}(U(t)x) \in C(\mathbb{R}_+; X)$. That is, $U(t)x$ is continuously differentiable on \mathbb{R}_+ for each $x \in X$.

Define $R_{a,k}(t)x = \frac{d}{dt}U(t)x$, $t \geq 0$, $x \in X$. Then $R_{a,k}(t)x$ is strongly continuous. Moreover, $R_{a,k}(t)$ commutes with A for all $x \in D(A)$ because

$$\frac{U(t+h) - U(t)}{h}Ax = A \left(\frac{U(t+h) - U(t)}{h} \right)x,$$

for $x \in D(A)$. Since $U(t)$ commutes with A and using the fact that A is closed, it follows that

$$\frac{d}{dt}U(t)Ax = \lim_{h \rightarrow 0} \left(\frac{U(t+h) - U(t)}{h} \right)Ax = A \left(\lim_{h \rightarrow 0} \frac{U(t+h) - U(t)}{h} \right)x = A \frac{d}{dt}U(t)x.$$

Finally, $\widehat{R}_{a,k}(\lambda) = H(\lambda)$ for $\operatorname{Re} \lambda > \omega$. By assumption (ii),

$$\widehat{R}_{a,k}(\lambda) = \widehat{k}(\lambda) + \widehat{a}(\lambda)\widehat{R}_{a,k}A,$$

that is,

$$R_{a,k}(t)x = k(t)x + \int_0^t a(t-s)R_{a,k}(s)Ax ds,$$

then the resolvent equation of Definition 2.1, item (iii), is satisfied.

Moreover,

$$\|\widehat{R}_{a,k}(\lambda)x\| = \|H(\lambda)x\| \leq \frac{M}{\lambda - \omega} \|x\|,$$

which implies that

$$\|R_{a,k}(t)x\| \leq Me^{\omega t} \|x\|,$$

that is, $R_{a,k}$ is of type (M, ω)

Then the proof is complete. \square

2.3 Approximation Theorem

This section presents a theorem of approximation for the (a, k) -regularized resolvent families. This result appeared in the literature for semigroups (see [55, Theorem 4.2, p. 85]), resolvent families (see [58, Theorem 6.3, p. 167]) and α -resolvent families (see [9, Theorem 2.21, p. 30]).

Theorem 2.12. Let $(k_n)_{n=0}^\infty \subset L^1_{loc}(\mathbb{R}_+)$ and $(a_n)_{n=0}^\infty \subset AC_{loc}(\mathbb{R}_+)$ of type (M, ω) , $\omega \geq 0$, such that $\widehat{a}(\mu) \neq 0$ for $\mu > \omega$ and $\int_0^\infty e^{-\lambda s} \left| \frac{d}{ds} a_n(s) \right| ds < \infty$. Let $(A_n)_{n=0}^\infty$ be closed and linear operators in X such that A_0 is densely defined. Assume each A_n generates an (a_n, k_n) -regularized resolvent family $\{R_{a_n, k_n}(t)\}_{t \geq 0}$ in X for each $n \in \mathbb{N}$ and assume that

$$\sup_{n \in \mathbb{N}} \|R_{a_n, k_n}(t)\| \leq M e^{\omega t}, \quad t \in \mathbb{R}_+. \quad (2.15)$$

Suppose also $a_n(t) \rightarrow a_0(t)$ and $k_n \rightarrow k_0$ as $n \rightarrow \infty$. Then, the following statements are equivalent:

- (i) $\lim_{n \rightarrow \infty} \widehat{k}_n(\lambda)(I - \widehat{a}_n(\lambda)A_n)^{-1}x = \widehat{k}_0(\lambda)(I - \widehat{a}_0(\lambda)A_0)^{-1}x$, for all $\lambda > \omega$, $x \in X$.
- (ii) $\lim_{n \rightarrow \infty} R_{a_n, k_n}(t)x = R_{a_0, k_0}(t)x$, for all $x \in X$, $t \geq 0$. Moreover, the convergence is uniform in t on every compact subset of \mathbb{R}_+ .

Proof. Notice that

$$\widehat{k}_n(\lambda)(I - \widehat{a}_n(\lambda)A_n)^{-1}x = \int_0^\infty e^{-\lambda s} R_{a_n, k_n}(s)x ds.$$

Then, assuming (ii), by Lebesgue's Dominated Convergence Theorem, (i) holds. Conversely, define $K_n(t) := (a_n * R_{a_n, k_n})(t)$. From the hypothesis, it follows that

$$\begin{aligned} \|K_n(t+h) - K_n(t)\| &= \left\| \int_t^{t+h} \frac{d}{ds} K_n(s) ds \right\| \\ &\leq \int_t^{t+h} \left\| a_n(0)R_{a_n, k_n}(s) + \left(\left(\frac{d}{ds} a_n \right) * R_{a_n, k_n} \right)(s) \right\| ds \\ &\leq |a_n(0)| \int_t^{t+h} \|R_{a_n, k_n}(s)\| ds + \int_t^{t+h} \left\| \left(\left(\frac{d}{ds} a_n \right) * R_{a_n, k_n} \right)(s) \right\| ds \\ &\leq C h e^{\omega(t+h)}, \quad \forall t, h \geq 0. \end{aligned}$$

Denote $H_n(\lambda) := \widehat{k}_n(\lambda)(I - \widehat{a}_n(\lambda)A)^{-1}$. By Proposition 2.8 and assumption (i) it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{K}_n(\lambda)x &= \lim_{n \rightarrow \infty} \widehat{a}_n(\lambda) \widehat{R}_{a_n, k_n}(\lambda)x = \lim_{n \rightarrow \infty} \widehat{a}_n H_n(\lambda)x \\ &= \widehat{a}_0(\lambda) H_0(\lambda)x = \widehat{a}_0(\lambda) \widehat{R}_{a_0, k_0}(\lambda)x = \widehat{K}_0(\lambda)x, \end{aligned}$$

for all $\lambda > \omega$ and $x \in X$.

Therefore, by Theorem 1.1

$$\lim_{n \rightarrow \infty} K_n(t)x = k_0(t)x, \quad \forall x \in X,$$

where the convergence is uniform in t on compact subsets of \mathbb{R}_+ , for a fixed $x \in X$.

Now let $y \in D(A_0)$ be fixed and let $x = [H_0(\lambda)]^{-1}y$, where $\lambda > \omega$. Then,

$$\begin{aligned} \|R_{a_n, k_n}(t)y - R_{a_0, k_0}(t)y\| &\leq \|R_{a_n, k_n}(t)(H_0(\lambda)x - H_n(\lambda)x)\| \\ &\quad + \|R_{a_n, k_n}(t)H_n(\lambda)x - R_{a_0, k_0}(t)H_0(\lambda)x\| \end{aligned} \quad (2.16)$$

and, since (2.15) and (i) hold,

$$\lim_{n \rightarrow \infty} \|R_{a_n, k_n}(t)(H_0(\lambda)x - H_n(\lambda)x)\| = 0.$$

By the fact that $H_n(\lambda) - \widehat{k}_n(\lambda) = \widehat{a}_n(\lambda)A_nH_n(\lambda)$, from (iii) of Definition 2.1 it follows that

$$\begin{aligned} R_{a_n, k_n}(t)H_n(\lambda)x &= k_n(t)H_n(\lambda)x + \int_0^t a_n(t-s)R_{a_n, k_n}(s)H_n(\lambda)x ds \\ &= k_n(t)H_n(\lambda)x + K_n(t)A_nH_n(\lambda)x \\ &= k_n(t)H_n(\lambda)x + \frac{1}{\widehat{a}_n(\lambda)}K_n(t)[H_n(\lambda) - \widehat{k}_n(\lambda)] \\ &= k_n(t)H_n(\lambda)x + \frac{1}{\widehat{a}_n(\lambda)}K_n(t)H_n(\lambda)x - \frac{\widehat{k}_n(\lambda)}{\widehat{a}_n(\lambda)}K_n(t)x, \end{aligned}$$

for $x \in X$, $\lambda > \omega$, $t \geq 0$ and $n \in \mathbb{N}$.

Thus,

$$\lim_{n \rightarrow \infty} \|R_{a_n, k_n}(t)H_n(\lambda)x - R_{a_0, k_0}(t)H_0(\lambda)x\| = 0.$$

Once $D(A_0)$ is dense, the assertion follows. \square

2.4 Perturbation Theorem

Let $B : D(A) \rightarrow X$ be a linear operator. The objective of this section is to study conditions in order to guarantee the existence of an (a, k) -regularized resolvent family generated by $A + B$. This result is called the Perturbation Theorem.

This theorem appeared in the literature for C_0 -semigroups (see [59], [55, Theorem 1.1, p. 76]), cosine families (see [53, Theorem 1]), resolvent and sine families (see [60]) and α -resolvent families [9, Theorem 2.25, p.35].

Theorem 2.13. Let $A : D(A) \subset X \rightarrow X$ be the generator of an (a, k) -regularized resolvent family $\{R_{a, k}(t)\}_{t \geq 0}$ of type (M, ω) . Suppose that

- (i) there exists $b \in L^1_{loc}(\mathbb{R}_+)$ such that $(b * k)(t) = a(t)$, for all $t \geq 0$;
- (ii) there exist constants $\mu > \omega$ and $\gamma \in [0, 1)$ such that

$$\int_0^\infty e^{-\mu\tau} \left\| B \int_0^\tau b(\tau-s)R_{a, k}(s)x ds \right\| d\tau \leq \gamma \|x\|, \quad \forall x \in D(A).$$

Then, $A + B$ generates an (a, k) -regularized resolvent family $\{S_{a, k}(t)\}_{t \geq 0}$ on X such that

$$\|S_{a, k}(t)\| \leq \frac{M}{1-\gamma} e^{\mu t}.$$

In addition,

$$S_{a,k}(t)x = R_{a,k}(t)x + \int_0^t S_{a,k}(t-\tau)B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds d\tau, \quad x \in D(A).$$

Proof. Let $T_0(t) := R_{a,k}(t)$, then $T_0(t) \in \mathcal{L}(X)$, $t \mapsto T_0(t)$ is strongly continuous and

$$\|T_0(t)\| \leq M e^{\omega t} \leq \gamma^0 M e^{\mu t}, \quad t \geq 0.$$

Suppose that there exists operators $T_j(t) \in \mathcal{L}(X)$, $j = 0, 1, \dots, n$, $n \in \mathbb{N}$, with the following properties:

- (a) $t \mapsto T_j(t)$ is strongly continuous;
- (b) $\|T_j(t)\| \leq \gamma^j M e^{\mu t}$, $t \geq 0$.

For $x \in D(A)$ define

$$T_{n+1}(t)x := \int_0^t T_n(t-\tau)B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds d\tau.$$

Then, $t \mapsto T_{n+1}(t)x$ is continuous and, since $T_n(t)$ satisfies (b), it follows that

$$\begin{aligned} \|T_{n+1}(t)x\| &= \left\| \int_0^t T_n(t-\tau)B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds d\tau \right\| \\ &\leq \int_0^t \|T_n(t-\tau)\| \left\| B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds \right\| d\tau \\ &\leq \int_0^t \gamma^n M e^{\mu(t-\tau)} \left\| B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds \right\| d\tau \\ &\leq \gamma^n M e^{\mu t} \int_0^t e^{-\mu\tau} \left\| B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds \right\| d\tau, \end{aligned}$$

and, by hypothesis (ii),

$$\|T_{n+1}(t)x\| \leq \gamma^n M e^{\mu t} \gamma \|x\| = \gamma^{n+1} M e^{\mu t} \|x\|.$$

Since $D(A) \subset X$ is dense, $T_{n+1}(t)$ can be uniquely extended to an operator $\tilde{T}_{n+1}(t)$, also denoted as $T_{n+1}(t)$, and which satisfies (a) and (b).

Then, there exists operators $T_n(t) \in \mathcal{L}(X)$, $n = 0, 1, 2, \dots$, $t \geq 0$, with the properties (a) and (b).

Let $S_{a,k}(t) := \sum_{n=0}^{\infty} T_n(t)$. Note that $S_{a,k}$ is well defined because

$$\sum_{n=0}^{\infty} \|T_n(t)\| \leq \sum_{n=0}^{\infty} \gamma^n M e^{\mu t} = \frac{M}{1-\gamma} e^{\mu t}.$$

Moreover,

$$\|S_{a,k}\| \leq \frac{M}{1-\gamma} e^{\mu t}.$$

For each $x \in D(A)$, using (a) and (b), it follows that the map $t \mapsto S_{a,k}(t)x$ is continuous and

$$\begin{aligned} S_{a,k}(t)x &= \sum_{n=0}^{\infty} T_n(t)x = T_0(x) + \sum_{n=1}^{\infty} T_n(t)x = R_{a,k}(t)x + \sum_{n=0}^{\infty} T_{n+1}(t)x \\ &= R_{a,k}(t)x + \sum_{n=0}^{\infty} \left(\int_0^t T_n(t-\tau)B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds d\tau \right) \\ &= R_{a,k}(t)x + \int_0^t \sum_{n=0}^{\infty} T_n(t-\tau)B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds d\tau \\ &= R_{a,k}(t)x + \int_0^t S_{a,k}(t-\tau)B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds d\tau. \end{aligned}$$

In particular, $S_{a,k}(0)x = R_{a,k}(0)x = k(0)x$ for all $x \in D(A)$, and as $D(A)$ is dense, $S_{a,k}(0) = k(0)I$.

Now, let $x \in X$ and define

$$H(\lambda)x = \int_0^{\infty} e^{-\lambda t} S_{a,k}(t)x dt =: \widehat{S}_{a,k}(\lambda)x$$

and

$$H(\lambda; A)x = \int_0^{\infty} e^{-\lambda t} R_{a,k}(t)x dt =: \widehat{R}_{a,k}(\lambda)x = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}x.$$

Then, consider $H_k(\lambda)x = \frac{1}{\lambda \widehat{k}(\lambda)} H(\lambda)x$ and $H_k(\lambda; A)x = \frac{1}{\lambda \widehat{k}(\lambda)} H(\lambda; A)x$.

Note that $H_k(\lambda)$ is a bounded operator and

$$\begin{aligned} \|H_k(\lambda)\| &= \frac{1}{\lambda |\widehat{k}(\lambda)|} \left\| \int_0^{\infty} e^{-\lambda t} S_{a,k}(t) dt \right\| \leq \frac{1}{\lambda |\widehat{k}(\lambda)|} \int_0^{\infty} e^{-\lambda t} \|S_{a,k}(t)\| dt \\ &\leq \frac{1}{\lambda |\widehat{k}(\lambda)|} \frac{1}{1-\gamma} \int_0^{\infty} e^{-(\lambda-\mu)t} dt = \frac{M}{(1-\gamma)(\lambda-\mu)\lambda |\widehat{k}(\lambda)|}. \end{aligned}$$

Now observe that, for $x \in D(A)$,

$$H_k(\lambda)x - H_k(\lambda; A)x = \frac{1}{\lambda \widehat{k}(\lambda)} (H(\lambda)x - H(\lambda; A)x)$$

and

$$\begin{aligned} H(\lambda)x - H(\lambda; A)x &= \widehat{S}_{a,k}(\lambda)x - \widehat{R}_{a,k}(\lambda)x \\ &= \widehat{R}_{a,k}(\lambda)x + \widehat{S}_{a,k}(\lambda)B\widehat{b}(\lambda)\widehat{R}_{a,k}(\lambda)x - \widehat{R}_{a,k}(\lambda)x \\ &= \widehat{b}(\lambda)H(\lambda)BH(\lambda; A)x. \end{aligned}$$

Then,

$$\begin{aligned} H_k(\lambda)x - H_k(\lambda; A)x &= \frac{\widehat{b}(\lambda)}{\lambda \widehat{k}(\lambda)} H(\lambda)BH(\lambda; A)x = \frac{H(\lambda)}{\lambda \widehat{k}(\lambda)} \widehat{b}(\lambda)BH(\lambda; A)x \\ &= H_k(\lambda)\widehat{b}(\lambda)BH(\lambda; A)x. \end{aligned}$$

So, since $D(A)$ is dense on X , it follows that

$$H_k(\lambda) - H_k(\lambda; A) = H_k(\lambda)\widehat{b}(\lambda)BH(\lambda; A).$$

But $H(\lambda; A) = \widehat{R}_{a,k}(\lambda)$, then, for $x \in D(A)$,

$$\begin{aligned} \|\widehat{b}(\lambda)BH(\lambda; A)x\| &= \|B\widehat{R}_{a,k}(\lambda)\widehat{b}(\lambda)x\| = \left\| B \int_0^\infty e^{-\lambda\tau} \int_0^\tau b(\tau-s)R_{a,k}(s)x ds d\tau \right\| \\ &\leq \int_0^\infty e^{-\mu\tau} \left\| B \int_0^\tau b(\tau-s)R_{a,k}(s)x ds \right\| d\tau \\ &\leq \gamma\|x\|, \quad \gamma < 1. \end{aligned}$$

Thus $(I - \widehat{b}(\lambda)BH(\lambda; A))^{-1}$ exists and is bounded.

So, $H_k(\lambda) = H_k(\lambda; A)(I - \widehat{b}(\lambda)BH(\lambda; A))^{-1}$ implies that

$$\begin{aligned} &[\lambda - \lambda\widehat{a}(\lambda)(A + B)]H_k(\lambda) \\ &= [\lambda - \lambda\widehat{a}(\lambda)(A + B)]H_k(\lambda)(I - \widehat{b}(\lambda)BH(\lambda; A))^{-1} \\ &= [\lambda(I - \widehat{a}(\lambda)A)H_k(\lambda; A) - \lambda\widehat{a}(\lambda)BH_k(\lambda; A)](I - \widehat{b}(\lambda)BH(\lambda; A))^{-1} \\ &= \left[\frac{(I - \widehat{a}(\lambda)A)}{\widehat{k}(\lambda)}H(\lambda; A) - \lambda\widehat{a}(\lambda)BH_k(\lambda; A) \right] (I - \widehat{b}(\lambda)BH(\lambda; A))^{-1} \\ &= \left[\frac{(I - \widehat{a}(\lambda)A)}{\widehat{k}(\lambda)}\widehat{R}_{a,k}(\lambda) - \lambda\widehat{a}(\lambda)BH_k(\lambda; A) \right] \left(I - \frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}BH(\lambda; A) \right)^{-1} \\ &= I - \lambda\widehat{a}(\lambda)BH_k(\lambda; A)^{-1} \\ &= I. \end{aligned}$$

This proves that $(\lambda - \lambda\widehat{a}(\lambda)(A + B))$ is invertible and satisfies

$$(I - \widehat{a}(\lambda)(A + B))^{-1}x = \frac{1}{\widehat{k}(\lambda)} \int_0^\infty e^{-\lambda t} S_{a,k}(t)x dt, \quad x \in X.$$

Using Proposition 2.8 the proof is complete. \square

2.5 Spectral properties

This section presents some spectral properties of the generator of an (a, k) -regularized resolvent family. This result for C_0 -semigroups can be found in [55, Theorem 2.3, p. 45], and for cosine families in [53].

For more results of spectral properties of (a, k) -regularized resolvent families the reader can see [49].

For each $\lambda \in \mathbb{C}$, $s_\lambda(t)$ denotes the unique solution of the scalar valued convolution equation

$$s_\lambda(t) = a(t) + \lambda \int_0^t a(t-\tau)s_\lambda(\tau)d\tau, \quad t \geq 0. \quad (2.17)$$

It is also defined

$$r_\lambda(t) := k(t) + \lambda \int_0^t s_\lambda(t-\tau)k(\tau)d\tau. \quad (2.18)$$

Theorem 2.14. Let $\{R_{a,k}(t)\}_{t \geq 0}$ be an (a, k) -regularized resolvent family with generator A . Then,

$$\sigma(R_{a,k}(t)) \supset r_{\sigma(A)}(t), \quad t \geq 0.$$

Proof. Let $x \in D(A)$. Then (iii) of Definition 2.1 and Lemma 2.3 show that

$$\begin{aligned} (s_\lambda * (\lambda - A)R_{a,k})(t)x &= \lambda(s_\lambda * R_{a,k})(t)x - (s_\lambda * AR_{a,k})(t)x \\ &= \lambda(s_\lambda * R_{a,k})(t)x - ([a + \lambda(a * s_\lambda)] * AR_{a,k})(t)x \\ &= \lambda(s_\lambda * R_{a,k})(t)x - (a * AR_{a,k})(t)x - \lambda(a * s_\lambda * AR_{a,k})(t)x \\ &= \lambda(s_\lambda * R_{a,k})(t)x - [R_{a,k} - k](t)x - \lambda(s_\lambda * [R_{a,k} - k])(t)x \\ &= k(t)x + \lambda(s_\lambda * k)(t)x - R_{a,k}(t)x \\ &= r_\lambda(t)x - R_{a,k}(t)x, \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $t \geq 0$.

From the closedness of A , it follows that

$$(\lambda - A)(s_\lambda * R_{a,k})(t)x = r_\lambda(t)x - R_{a,k}(t)x,$$

for all $x \in X$, $\lambda \in \mathbb{C}$ and $t \geq 0$.

Suppose $r_\lambda(t) \in \rho(R_{a,k}(t))$ for some $\lambda \in \mathbb{C}$ and $t \geq 0$, and denote $L_{\lambda,t} := (r_\lambda(t) - R_{a,k}(t))^{-1}$.

Since $L_{\lambda,t}$ commutes with $R_{a,k}(t)$, and hence also with A , it follows that

$$(\lambda - A) \int_0^t s_\lambda(t - \tau)R_{a,k}(\tau)L_{\lambda,t}x d\tau = x,$$

for all $x \in X$, and

$$\int_0^t s_\lambda(t - \tau)R_{a,k}(\tau)L_{\lambda,t}(\lambda - A)x d\tau = x,$$

for all $x \in D(A)$.

Define $B_\lambda x := \int_0^t s_\lambda(t - \tau)R_{a,k}(\tau)L_{\lambda,t}x d\tau$. Then, B_λ is a bounded operator and is a two-sided inverse of $(\lambda - A)$. Thus, $\lambda \in \rho(A)$.

Observe that if $\theta \in \sigma(A)$, then $\theta \notin \rho(A)$ and from the previous assertions, it follows that $r_\theta(t) \notin \rho(R_{a,k}(t))$, that is, $r_\theta \in \sigma(R_{a,k}(t))$.

Further, $r_{\sigma(A)} \subset \sigma(R_{a,k}(t))$. □

The next results present spectral inclusions for the point and residual spectrum.

Theorem 2.15. Let $\{R_{a,k}(t)\}_{t \geq 0}$ be an (a, k) -regularized resolvent family with generator A . Then,

$$\sigma_p(R_{a,k}(t)) \supset r_{\sigma_p(A)}(t), \quad t \geq 0.$$

Proof. Let $\lambda \in \sigma_p(A)$ and $x \in D(A)$ be an eigenvector corresponding to λ , so

$$\int_0^t s_\lambda(t - \tau)R_{a,k}(\tau)(\lambda - A)x d\tau = r_\lambda(t)x - R_{a,k}(t)x, \quad \forall x \in D(A), \quad (2.19)$$

shows that $R_{a,k}(t)x = r_\lambda(t)x$. Then $r_\lambda(t)$ is an eigenvalue of $R_{a,k}(t)$ with eigenvector $x \in D(A)$.

Thus, $r_{\sigma_p(A)}(t) \subset \sigma_p(R_{a,k}(t))$, $t \geq 0$. □

Theorem 2.16. [49, Theorem 5.6] Let $\{R_{a,k}(t)\}_{t \geq 0}$ be an (a, k) -regularized resolvent family with generator A . Then,

$$\sigma_r(R_{a,k}(t)) \supset r_{\sigma_r(A)}(t), \quad t \geq 0.$$

3 Zero-one law

Let A be a closed linear operator with domain $D(A)$ defined on a complex Banach space X . This chapter studies an intriguing structural property of the class of strongly continuous family of bounded linear operators $\{R_{a,k}\}$ and in particular $\{R_{\alpha,\beta}(t)\}_{t \geq 0}$, with $\alpha > 0$, $\beta > 0$.

This family contains several important classes of well-known subfamilies, as seen in Example 2.2. Although several qualitative properties are well known for the class of semigroups and cosine families, much less has been reported in the setting of integrated semigroups and, specially, α -times resolvent families and (a, k) -regularized resolvent families.

Concerning the differences of structure among the various subfamilies $\{R_{\alpha,\beta}(t)\}_{t \geq 0}$, it has been recently proven that, if the set of all bounded strongly continuous cosine families is treated as a metric space under the metric of the uniform convergence associated with the operator norm on the space $\mathcal{L}(X)$ of all bounded linear operators on X , then the isolated points of this set are precisely the scalar cosine families [11]. By definition, a scalar cosine family is a family whose members are all scalar multiples of the identity operator. Remarkably, this picture changes dramatically considering to semigroups of operators. In such case, the isolated points constitute only a small fraction of the set of all scalar semigroups. The proof of this and related properties relies on the fact that if the distance between cosine families (resp. semigroups) and their scalar counterparts is less than a certain bound, say γ , then the cosine family (resp. semigroup) must be scalar. They are called $0 - \gamma$ Laws. Only recently the problem to determine the optimal bound for cosine families was solved, obtaining $\gamma = \frac{8}{3\sqrt{3}}$ [12, 21, 20, 26, 27, 64]. It is surprising that a corresponding result for integrated semigroups and sine families ($\alpha = \beta = 2$) has just been discovered in 2017 [10]. Motivated by the above earlier works it is natural to ask the following:

(Q) It is possible to find $0 - \gamma$ Laws for other classes of strongly continuous families of bounded operators?

This question is answered in the first section of this chapter at least for the case $\gamma = 1$. As a second contribution, an example is given to show that the optimal bound, say θ , in case of $\beta \geq \alpha$ and $0 < \alpha \leq 1$ is strictly less than 1, which interpretes the fact that, roughly speaking, β -times integrated α -resolvent families have more regularity at $t = 0$ than (α, β) -resolvent families, when they are over the diagonal $\beta = \alpha$, and in this way widely improving a recent result for the very special case $\alpha = 1, \beta = 2$ (see [10]). The results of the first section were published in [29].

The second section presents the zero-one law for (a, k) -regularized resolvent families.

3.1 Zero-one law for (α, β) -resolvent families

The concept of α -times resolvent families, or solution operators, plays an important role in the theory of fractional abstract Cauchy problems, that models several physical phenomena. One example is the fractional diffusion-wave equation

$$D_t^\alpha u(x, t) = k^2 u_{xx}(x, t), \quad -\infty < x < \infty, \quad t > 0, \quad 0 < \alpha < 2, \quad k \in \mathbb{R}, \quad (3.1)$$

with initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = 0$ (the last one only when $1 < \alpha < 2$). For this case, as seen in [9, Example 3.6], the explicit form of the α -times resolvent family is

$$R_{\alpha,1}(t)f(x) = \frac{1}{2|k|t^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} \phi_{\frac{\alpha}{2}} \left(\frac{|s|}{|k|t^{\frac{\alpha}{2}}} \right) f(x-s) ds,$$

where $\phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\gamma n + 1 - \gamma)}$ is a function of Wright type.

The fractional diffusion equation (3.1) ($\alpha \in (0, 1)$) has been introduced in [54] by Nigmatullin to describe diffusion in special types of porous media. Mainardi [50] has shown that the fractional wave equation (3.1) ($\alpha \in (1, 2)$) governs the propagation of mechanical diffusive waves in viscoelastic media.

First a purely algebraic notion of the theory of (α, β) -resolvents of bounded linear operators is presented and more details can be found on [48].

Recalling the functions $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and the item (6) of Example 2.2.

The $(\alpha, 1)$ -resolvent families are called α -resolvent families, or solution operator, or fractional resolvent family/operator in the current literature. For $0 < \alpha = \beta < 1$ the above definition was studied by Li and Peng [43]. This concept was introduced earlier [4], but without reference to the condition near to zero given in (a).

The linear operator A defined by

$$D(A) := \left\{ x \in X : \lim_{t \rightarrow 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ exists} \right\}$$

and

$$Ax := \lim_{t \rightarrow 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ for } x \in D(A)$$

is called the *generator* of the (α, β) -resolvent family $\{R_{\alpha,\beta}(t)\}_{t \geq 0}$.

For example, if A is a bounded operator, then

$$R_{\alpha,\beta}(t) := \sum_{n=0}^{\infty} g_{\alpha n + \beta}(t) A^n = t^{\beta-1} \sum_{n=0}^{\infty} \frac{A^n t^{\alpha n}}{\Gamma(\alpha n + \beta)} = t^{\beta-1} E_{\alpha,\beta}(At^\alpha), \quad t > 0,$$

defines a uniformly continuous (α, β) -resolvent family. Given $\beta > 1$, observe that the family $\{R_{\alpha,\beta}(t)\}_{t > 0}$ is $(\beta - 1)$ -times integrated with respect to $\{R_{\alpha,1}(t)\}_{t \geq 0}$ as the identity

$$R_{\alpha,\beta}(t) = (g_{\beta-1} * R_{\alpha,1})(t) = J_t^{\beta-1} R_{\alpha,1}(t), \quad t > 0, \quad (3.2)$$

holds. The following characterization is often used as definition.

Theorem 3.1. [48, Theorem 3.1 and Theorem 4.3] Let $\alpha > 0$ and $\beta > 0$ be given. A strongly continuous family $\{R_{\alpha,\beta}(t)\}_{t > 0} \subset \mathcal{L}(X)$ of bounded linear operators in X is an (α, β) -resolvent family generated by A if and only if the following conditions are satisfied

- (i) $\lim_{t \rightarrow 0} t^{1-\beta} R_{\alpha, \beta}(t) = \frac{1}{\Gamma(\beta)} I$ if $0 < \beta < 1$, $R_{\alpha, 1}(0) = I$ and $R_{\alpha, \beta}(0) = 0$ if $\beta > 1$.
- (ii) $R_{\alpha, \beta}(t)x \in D(A)$ and $R_{\alpha, \beta}(t)Ax = AR_{\alpha, \beta}(t)x$ for all $x \in D(A)$ and $t \geq 0$;
- (iii) $R_{\alpha, \beta}(t)x = g_{\beta}(t)x + \int_0^t g_{\alpha}(t-s)AR_{\alpha, \beta}(s)x ds$, $t \geq 0$, $x \in D(A)$.

If $\beta > 1$, A is closed, but not necessarily densely defined [19, Proposition 3.10]. In the diagonal case $\alpha = \beta$ this notion appears by the first time in [4, Definition 2.3]. If $0 < \alpha = \beta < 1$ then A must be densely defined [43, Theorem 3.1].

Roughly speaking, the notion of $(\alpha, 1)$ -resolvent families is associated with the Caputo fractional derivative, whereas the notion of (α, α) -resolvent family is linked with the Riemann-Liouville fractional derivative. Other relevant cases are $(\alpha, \gamma + (1 - \gamma)\alpha)$ -resolvent families with $0 < \alpha < 1$, $0 \leq \gamma \leq 1$, see [32], and $(\alpha, \alpha + \gamma(2 - \alpha))$ -resolvent families with $1 < \alpha < 2$, $0 \leq \gamma \leq 1$, see [52], because they are related with the notion of Hilfer fractional derivative that interpolates between the Caputo and Riemann-Liouville fractional derivative (for $0 < \alpha < 1$ take $\gamma = 1$ and $\gamma = 0$, respectively).

Assuming that A is the generator of an γ -times integrated semigroup ($\gamma \geq 0$), i.e. an $(1, \gamma + 1)$ -resolvent family, then $(\alpha, \alpha\gamma + 1)$ -resolvent families and $(\alpha, \alpha(\gamma + 1))$ -resolvent families are important for $0 < \alpha < 1$ because these are the key for the treatment of existence, regularity and representation of fractional diffusion equations (see [40]). The same happens with $(\alpha, \frac{\alpha\gamma}{2} + 1)$ -resolvent families and $(\alpha, \alpha(\frac{\gamma}{2} + 1))$ -resolvent families for $1 \leq \alpha \leq 2$ because these are present in the theoretical analysis of fractional wave equations (see [41]).

The following theorem is the main result of this section.

Theorem 3.2. Let $0 < \alpha \leq 2$ and $\beta > 0$ be given and let $\{R_{\alpha, \beta}(t)\}_{t \geq 0}$ be an (α, β) -resolvent family generated by A . If

$$\sup_{t > 0} \left\| \frac{1}{t^{\beta-1} E_{\alpha, \beta}(\lambda t^{\alpha})} R_{\alpha, \beta}(t) - I \right\| =: \theta < 1, \quad (3.3)$$

then $R_{\alpha, \beta}(t) = t^{\beta-1} E_{\alpha, \beta}(\lambda t^{\alpha}) I$ for all $t > 0$ and $\lambda \geq 0$.

Proof. For all $t \geq 0$, $x \in X$ and $\lambda \geq 0$ as given in the hypothesis, define

$$B(t)x := \int_0^t (t - \tau)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t - \tau)^{\alpha}) R_{\alpha, \beta}(\tau) x d\tau.$$

Proceeding as in the proof of Theorem 2.14, it follows that $B(t)x \in D(A)$ and

$$(\lambda - A) \int_0^t (t - \tau)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t - \tau)^{\alpha}) R_{\alpha, \beta}(\tau) x d\tau = t^{\beta-1} E_{\alpha, \beta}(\lambda t^{\alpha}) x - R_{\alpha, \beta}(t)x. \quad (3.4)$$

Denote

$$b_{\alpha, \beta}(\lambda, t) := \int_0^t (t - \tau)^{\alpha-1} \tau^{\beta-1} E_{\alpha, \alpha}(\lambda(t - \tau)^{\alpha}) E_{\alpha, \beta}(\lambda \tau^{\alpha}) d\tau.$$

Since $\lambda \geq 0$, $b_{\alpha, \beta}(\lambda, t) > 0$ for $t > 0$ can be obtained from (1.3) and it follows the estimate

$$\left\| x - \frac{1}{b_{\alpha, \beta}(\lambda, t)} B(t)x \right\|$$

$$\begin{aligned}
&= \left\| \frac{1}{b_{\alpha, \beta}(\lambda, t)} \int_0^t (t - \tau)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t - \tau)^\alpha) (\tau^{\beta-1} E_{\alpha, \beta}(\lambda \tau^\alpha) x - R_{\alpha, \beta}(\tau) x) d\tau \right\| \\
&= \left\| \frac{1}{b_{\alpha, \beta}(\lambda, t)} \int_0^t (t - \tau)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t - \tau)^\alpha) \tau^{\beta-1} E_{\alpha, \beta}(\lambda \tau^\alpha) \left[x - \frac{R_{\alpha, \beta}(\tau) x}{\tau^{\beta-1} E_{\alpha, \beta}(\lambda \tau^\alpha)} \right] d\tau \right\| \\
&\leq \frac{1}{b_{\alpha, \beta}(\lambda, t)} \sup_{\tau \geq 0} \left\| I - \frac{R_{\alpha, \beta}(\tau)}{\tau^{\beta-1} E_{\alpha, \beta}(\lambda \tau^\alpha)} \right\| \|x\| \int_0^t (t - \tau)^{\alpha-1} \tau^{\beta-1} E_{\alpha, \alpha}(\lambda(t - \tau)^\alpha) E_{\alpha, \beta}(\lambda \tau^\alpha) d\tau \\
&\leq \theta \|x\|.
\end{aligned}$$

Since $\theta < 1$, the operator $\frac{1}{b_{\alpha, \beta}(\lambda, t)} B(t)$ is boundedly invertible for all $t > 0$ and

$$\|b_{\alpha, \beta}(\lambda, t)(B(t))^{-1}\| \leq \sum_{k=0}^{\infty} \left\| I - \frac{1}{b_{\alpha, \beta}(\lambda, t)} B(t) \right\|^k \leq \sum_{k=0}^{\infty} \theta^k = \frac{1}{1 - \theta},$$

which implies that

$$\|(B(t))^{-1}\| \leq \frac{1}{(1 - \theta)b_{\alpha, \beta}(\lambda, t)}, \quad (3.5)$$

in the norm of $\mathcal{L}(X)$. From (3.3) and (3.4) it follows that

$$\begin{aligned}
\|(\lambda - A)B(t)x\| &= \|R_{\alpha, \beta}(t)x - t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha)x\| \\
&= \left\| t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha) \left[\frac{R_{\alpha, \beta}(t)}{t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha)} - I \right] x \right\| \\
&\leq \theta t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha) \|x\|,
\end{aligned}$$

for all $t > 0$ and $x \in X$. Hence, for each $x \in D(A)$

$$\|(\lambda - A)x\| = \|(\lambda - A)B(t)(B(t))^{-1}x\| \leq \frac{\theta t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha)}{(1 - \theta)b_{\alpha, \beta}(\lambda, t)} \|x\|, \quad \forall t > 0, \quad (3.6)$$

can be obtained from (3.5).

Inserting (1.5) in (3.6) implies that

$$\|(\lambda - A)x\| \leq \frac{\theta E_{\alpha, \beta}(\lambda t^\alpha)}{(1 - \theta)t^\alpha E'_{\alpha, \beta}(\lambda t^\alpha)} \|x\|, \quad \forall t > 0. \quad (3.7)$$

From (1.4) and (1.7) it follows that $E'_{\alpha, \beta}(z) \sim \frac{1}{\alpha^2} z^{\frac{1-\beta-\alpha}{\alpha}} e^{z^{\frac{1}{\alpha}}} [(1 - \beta) + z^{\frac{1}{\alpha}}]$. In particular,

$$\frac{E_{\alpha, \beta}(z)}{E'_{\alpha, \beta}(z)} \sim \frac{\alpha z}{(1 - \beta) + z^{\frac{1}{\alpha}}} \quad |z| \rightarrow \infty, \quad |\arg z| < \frac{\alpha\pi}{2}.$$

Consequently,

$$\frac{E_{\alpha, \beta}(\lambda t^\alpha)}{t^\alpha E'_{\alpha, \beta}(\lambda t^\alpha)} \sim \frac{\alpha \lambda}{(1 - \beta) + \lambda^{\frac{1}{\alpha}} t}, \quad t \rightarrow \infty.$$

Taking $t \rightarrow \infty$ in (3.7), it follows that $Ax = \lambda x$ for all $x \in D(A)$. Since $B(t)x \in D(A)$ for each $x \in X$ and $t > 0$, then (3.4) implies that $R_{\alpha, \beta}(t)x = t^{\beta-1} E_{\alpha, \beta}(\lambda t^\alpha)x$, $x \in X, t \geq 0$. \square

Considering the special case $\lambda = 0$, it follows this important consequence, stated as a Theorem.

Theorem 3.3. Let $0 < \alpha \leq 2$ and $\beta > 0$ be given and let $\{R_{\alpha, \beta}(t)\}_{t \geq 0}$ be an (α, β) -resolvent family generated by A . If

$$\sup_{t > 0} \left\| \frac{1}{g_{\beta}(t)} R_{\alpha, \beta}(t) - I \right\| < 1,$$

then $R_{\alpha, \beta}(t) = g_{\beta}(t)I$ for all $t > 0$ and $\lambda \geq 0$.

Observe that the result is dependent of $\beta > 0$, which interpretes the fact that the property takes into account the regularizing effect of the parameter β , and hence of the family of operators under consideration, near to zero.

The zero-one law for C_0 -semigroups is a corollary of Theorem 3.3 (with $\alpha = \beta = 1$) and can be found for example in [64, Theorem 3.2] (see also [65, Remark 3.1.4]).

Corollary 3.4. Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup generated by A . Suppose that

$$\sup_{t > 0} \|T(t) - I\| < 1.$$

Then $T(t) = I$ for all $t \geq 0$.

From the literature, a zero-two law [64] for cosine families, and also a $0 - \frac{3}{2}$ law [5, Theorem 1.1] for cosine families on general Banach spaces was proved without considering strong continuity (see also the reference [20]). In [21], Chojnacki gives an extension of the results from [64] in the case of cosine families, not necessarily continuous, in a normed algebra.

Observe that a zero-one law for strongly continuous cosine families was proved in [63, Theorem 1.1]. Such zero-one law for cosine families is a consequence of Theorem 3.3 (with $\alpha = 2, \beta = 1$) as follows.

Corollary 3.5. Let $\{C(t)\}_{t \geq 0}$ be a strongly continuous cosine family generated by A . Suppose that

$$\sup_{t > 0} \|C(t) - I\| < 1.$$

Then $C(t) = I$ for all $t \geq 0$.

Remark 3.6. The case $\alpha = 1$ and $\beta = 2$ in Theorem 3.3 is of particular interest. In such case the family $\{R_{1,2}(t)\}_{t \geq 0}$ corresponds to an integrated semigroup and hence Theorem 3.3 coincides with recent results of Bobrowski [10, Theorem 2.3]. The findings of Bobrowski are stated in an arbitrary unital Banach Algebra.

It is interesting to observe that for the range $\beta \geq \alpha$ and $0 < \alpha \leq 1$ in Theorem 3.3, the bound $\theta = 1$ is optimal. The following example inspired in [10, Example 2.4] shows this fact.

Example 3.7. For $\beta \geq \alpha, 0 < \alpha \leq 1$ and $\lambda > 0$ consider the scalar (α, β) -resolvent family defined by

$$R_{\alpha, \beta}(t) := \int_0^t g_{\beta - \alpha}(t - s) E_{\alpha, 1}(-\lambda s^{\alpha}) ds,$$

and using (1.6) it is possible to verify the following identity

$$R_{\alpha, \beta}(t) = t^{\beta - 1} E_{\alpha, \beta}(-\lambda t^{\alpha}).$$

In other words, $R_{\alpha, \beta}(t)$ is an (α, β) -resolvent family with generator $-\lambda$. For all $t > 0$ it follows that

$$\begin{aligned} \left| \frac{1}{g_{\beta}(t)} R_{\alpha, \beta}(t) - 1 \right| &= \left| \frac{1}{g_{\beta}(t)} \int_0^t g_{\beta-\alpha}(t-s) E_{\alpha, 1}(-\lambda s^{\alpha}) ds - 1 \right| \\ &= \left| \frac{1}{g_{\beta}(t)} t^{\beta-\alpha} E_{\alpha, \beta}(-\lambda t^{\alpha}) - 1 \right|. \end{aligned}$$

From [62], the function $x \mapsto E_{\alpha, \beta}(-x)$ is completely monotone if and only if $0 < \alpha \leq 1$ and $\beta \geq \alpha$. Therefore $E_{\alpha, \beta}(-\lambda t^{\alpha}) \geq 0$ for all $t \geq 0$. In particular, $R_{\alpha, \beta}(t) \geq 0$.

The identity

$$R_{\alpha, \beta}(t) = g_{\beta}(t) - \lambda \int_0^t g_{\alpha}(t-s) R_{\alpha, \beta}(s) ds,$$

implies that

$$\frac{1}{g_{\beta}(t)} t^{\beta-1} E_{\alpha, \beta}(-\lambda t^{\alpha}) - 1 = -\frac{\lambda}{g_{\beta}(t)} \int_0^t g_{\alpha}(t-s) R_{\alpha, \beta}(s) ds \leq 0, \quad \text{for } t > 0.$$

Consequently

$$\left| \frac{1}{g_{\beta}(t)} R_{\alpha, \beta}(t) - 1 \right| = 1 - \frac{1}{g_{\beta}(t)} t^{\beta-1} E_{\alpha, \beta}(-\lambda t^{\alpha}) \leq 1, \quad \forall t > 0.$$

Then, $\sup_{t>0} \left| \frac{1}{g_{\beta}(t)} R_{\alpha, \beta}(t) - 1 \right| \leq 1$.

Conversely, from Theorem 1.3 for any $0 < \alpha < 2$ and $\beta > 0$ there exists a constant $C > 0$ such that, for each $\frac{\alpha\pi}{2} < \mu < \min\{\pi, \alpha\pi\}$ the estimate

$$|E_{\alpha, \beta-\alpha+1}(z)| \leq \frac{C}{1+|z|} \quad \mu < |\arg(z)| \leq \pi,$$

holds. This shows that $\lim_{t \rightarrow \infty} E_{\alpha, \beta-\alpha+1}(-\lambda t^{\alpha}) = 0$. Hence

$$\begin{aligned} \sup_{t>0} \left| \frac{1}{g_{\beta}(t)} R_{\alpha, \beta}(t) - 1 \right| &= \sup_{t>0} |\Gamma(\beta) E_{\alpha, \beta-\alpha+1}(-\lambda t^{\alpha}) - 1| \\ &\geq \lim_{t \rightarrow \infty} |\Gamma(\beta) E_{\alpha, \beta-\alpha+1}(-\lambda t^{\alpha}) - 1| = 1. \end{aligned}$$

Therefore,

$$\sup_{t>0} \left| \frac{1}{g_{\beta}(t)} R_{\alpha, \beta}(t) - 1 \right| = 1,$$

while $R_{\alpha, \beta}(t) \neq g_{\beta}(t)$, proving that $\theta < 1$ is optimal.

Now consider α -resolvent families $\{S_{\alpha}(t)\}_{t \geq 0}$ (i.e. $0 < \alpha \leq 2, \beta = 1$). The following zero-one law is a simple corollary of Theorem 3.3 and constitutes a completely new result.

Corollary 3.8. Let $0 < \alpha \leq 2$ and $\{S_{\alpha}(t)\}_{t \geq 0}$ be an α -resolvent family generated by A . Suppose that

$$\sup_{t>0} \|S_{\alpha}(t) - I\| < 1.$$

Then $S_{\alpha}(t) = I$ for all $t \geq 0$.

For the next results $\lambda \neq 0$. They are some of the consequences of Theorem 3.2. Beginning with the following extension of Corollary 3.4, which seems to be new in the present form although is also an easy consequence of Wallen's formula, see [11, Lemma 10].

Corollary 3.9. Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup generated by A . Let $\lambda \geq 0$ be given and suppose that

$$\sup_{t > 0} \|e^{-\lambda t} T(t) - I\| < 1.$$

Then $T(t) = e^{\lambda t} I$ for all $t \geq 0$.

The following corollary is a new result about the zero-one law for cosine families (see [63]).

Corollary 3.10. Let $\{C(t)\}_{t \geq 0}$ be a strongly continuous cosine family generated by A , and $\lambda > 0$ be given. Suppose that

$$\sup_{t \geq 0} \left\| \frac{1}{\cosh(\sqrt{\lambda}t)} C(t) - I \right\| < 1.$$

Then $C(t) = \cosh(\sqrt{\lambda}t) I$ for all $t \geq 0$.

The following is an extension of the zero-one law to α -resolvent families. This result is also new.

Corollary 3.11. Let $0 < \alpha \leq 2$ and $\{S_\alpha(t)\}_{t \geq 0}$ be a α -resolvent family generated by A . Given $\lambda > 0$ suppose that

$$\sup_{t \geq 0} \left\| \frac{1}{E_{\alpha,1}(\lambda t^\alpha)} S_\alpha(t) - I \right\| < 1.$$

Then $S_\alpha(t) = E_{\alpha,1}(\lambda t^\alpha) I$ for all $t \geq 0$. Moreover, $\theta < 1$ is optimal in the range $0 < \alpha \leq 1$.

3.2 Zero-one law for (a,k)-regularized resolvent families

This section presents a general version of the zero-one law for (a, k)-regularized resolvent families.

Recalling the functions s_λ and r_λ defined in Section 2.5 of Chapter 2.

For each $\lambda \in \mathbb{C}$, $s_\lambda(t)$ denotes the unique solution of the scalar valued convolution equation

$$s_\lambda(t) = a(t) + \lambda \int_0^t a(t-\tau) s_\lambda(\tau) d\tau, t \geq 0. \quad (3.8)$$

It is also defined

$$r_\lambda(t) := k(t) + \lambda \int_0^t s_\lambda(t-\tau) k(\tau) d\tau. \quad (3.9)$$

The next result is known as zero-one law for (a, k)-regularized resolvent families.

Theorem 3.12. Let $\{R_{a,k}(t)\}_{t \geq 0}$ be an (a, k)-regularized resolvent family generated by A . Let $\lambda \in \mathbb{C}$ be such that $r_\lambda \neq 0$ and

$$\lim_{t \rightarrow \infty} \frac{r_\lambda(t)}{(s_\lambda * r_\lambda)(t)} = 0. \quad (3.10)$$

For each $t > 0$ consider

$$\epsilon_{r,s} = \sup_{t>0} \frac{\int_0^t |s_\lambda(t-\tau)r_\lambda(\tau)|d\tau}{\left| \int_0^t s_\lambda(t-\tau)r_\lambda(\tau)d\tau \right|}.$$

If

$$\sup_{t>0} \left\| \frac{1}{r_\lambda(t)} R_{a,k}(t) - I \right\| =: \theta, \quad (3.11)$$

with $\theta\epsilon_{r,s} < 1$, then $R_{a,k}(t) = r_\lambda(t)I$ for all $t \geq 0$.

Proof. For all $t \geq 0$, $x \in X$ and $\lambda \in \mathbb{C}$ given as in the hypothesis, define

$$B(t)x := \int_0^t s_\lambda(t-\tau)R_{a,k}(\tau)x d\tau.$$

From Theorem 2.14, it follows that $B(t)x \in D(A)$ and

$$(\lambda - A) \int_0^t s_\lambda(t-\tau)R_{a,k}(\tau)x d\tau = r_\lambda(t)x - R_{a,k}(t)x. \quad (3.12)$$

Therefore, for $t > 0$,

$$\begin{aligned} \left\| x - \frac{1}{(s_\lambda * r_\lambda)(t)} B(t)x \right\| &= \left\| \frac{1}{(s_\lambda * r_\lambda)(t)} \int_0^t s_\lambda(t-\tau)(r_\lambda(\tau)x - R_{a,k}(\tau)x) d\tau \right\| \\ &= \left\| \frac{1}{(s_\lambda * r_\lambda)(t)} \int_0^t s_\lambda(t-\tau)r_\lambda(\tau) \left[x - \frac{R_{a,k}(\tau)}{r_\lambda(\tau)} x \right] d\tau \right\| \\ &\leq \frac{1}{|(s_\lambda * r_\lambda)(t)|} \sup_{\tau>0} \left\| I - \frac{R_{a,k}(\tau)}{r_\lambda(\tau)} \right\| \|x\| \int_0^t |s_\lambda(t-\tau)r_\lambda(\tau)| d\tau \\ &\leq \frac{\theta \left(\int_0^t |s_\lambda(t-\tau)r_\lambda(\tau)| d\tau \right)}{|(s_\lambda * r_\lambda)(t)|} \|x\| \\ &\leq \theta\epsilon_{r,s} \|x\|. \end{aligned}$$

Thus, since $\theta\epsilon_{r,s} < 1$, it follows that the operator $\frac{1}{(s_\lambda * r_\lambda)(t)} B(t)$ is boundedly invertible for all $t > 0$ and

$$\|(s_\lambda * r_\lambda)(t)(B(t))^{-1}\| \leq \sum_{k=0}^{\infty} \left\| I - \frac{1}{(s_\lambda * r_\lambda)(t)} B(t) \right\|^k \leq \sum_{k=0}^{\infty} (\epsilon_{r,s}\theta)^k = \frac{1}{1 - \epsilon_{r,s}\theta},$$

which implies that

$$\|(B(t))^{-1}\| \leq \frac{1}{(1 - \epsilon_{r,s}\theta)|(s_\lambda * r_\lambda)(t)|}, \quad (3.13)$$

in the norm of $\mathcal{L}(X)$. From (3.10) and (3.11) it follows that

$$\|(\lambda - A)B(t)x\| = \|R_{a,k}(t)x - r_\lambda(t)x\| = \left\| r_\lambda(t) \left[\frac{R_{a,k}(t)}{r_\lambda(t)} - I \right] x \right\| \leq \theta|r_\lambda(t)|\|x\|,$$

for all $x \in X$. Hence, for any $x \in D(A)$

$$\|(\lambda - A)x\| = \|(\lambda - A)B(t)(B(t))^{-1}x\| \leq \frac{\theta|r_\lambda(t)|}{(1 - \epsilon_{r,s}\theta)|(s_\lambda * r_\lambda)(t)|} \|x\|, \quad \forall t > 0, \quad (3.14)$$

can be obtained by (3.13).

In consequence, by hypothesis (3.10), taking $t \rightarrow \infty$ in (3.14) it follows that $Ax = \lambda x$ for all $x \in D(A)$. Since $B(t)x \in D(A)$ for each $x \in X$ and $t \geq 0$, (3.12) implies that $R_{a,k}(t)x = r_\lambda(t)x$, $x \in X, t \geq 0$. \square

The following theorem is obtained considering $\lambda = 0$.

Theorem 3.13. Let $\{R_{a,k}(t)\}_{t \geq 0}$ be an (a, k) -regularized resolvent family generated by A , with $k(t) \neq 0$ satisfying

$$\lim_{t \rightarrow \infty} \frac{k(t)}{(k * a)(t)} = 0, \quad (3.15)$$

and consider

$$\epsilon_{a,k} = \sup_{t > 0} \frac{\int_0^t |a(t-\tau)k(\tau)| d\tau}{\left| \int_0^t a(t-\tau)k(\tau) d\tau \right|}.$$

If

$$\sup_{t \geq 0} \left\| \frac{1}{k(t)} R_{a,k}(t) - I \right\| = \theta, \quad (3.16)$$

with $\epsilon_{a,k}\theta < 1$, then $R_{a,k}(t) = k(t)I$ for all $t \geq 0$.

The following example shows a family that satisfies condition (3.15) and such that $\epsilon_{a,k} = 1$.

Example 3.14. Given $b, c > 0$, consider the (a, k) -regularized resolvent family generated by $-A$, with $a(t) = b + c^2t$ and $k \equiv 1$.

Note that

$$\lim_{t \rightarrow \infty} \frac{1}{\int_0^t b + c^2s ds} = \lim_{t \rightarrow \infty} \frac{2}{2bt + c^2t} = 0$$

and, since $a(t) \geq 0$, $k(t) > 0$ for $t \geq 0$, it follows that

$$\int_0^t |a(t-\tau)k(\tau)| d\tau = \left| \int_0^t a(t-\tau)k(\tau) d\tau \right|,$$

that is, $\epsilon_{a,k} = 1$.

4 Blackstock-Crighton-Westervelt equation

The classical models in nonlinear acoustics are partial differential equations of second order in time and characterized by the presence of a viscoelastic damping. The most general of these popular models is the Kuznetsov's equation

$$u_{tt} - c^2 \Delta u - b \Delta u_t = \left(\frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right)_t, \quad (4.1)$$

where u denotes the acoustic velocity potential, $c > 0$ is the speed of sound, $b \geq 0$ is the diffusivity of sound and B/A is the parameter of nonlinearity. Neglecting local nonlinear effects one arrives at the Westervelt equation

$$u_{tt} - b \Delta u_t - c^2 \Delta u = \left(\frac{1}{c^2} \left(1 + \frac{B}{2A} \right) (u_t)^2 \right)_t. \quad (4.2)$$

The Kuznetsov equation can be regarded in some sense as a simplification of the following higher order model

$$(a \Delta - \partial_t)(u_{tt} - c^2 \Delta u - b \Delta u_t) = \left(\frac{1}{c^2} \frac{B}{2A} (u_t)^2 + |\nabla u|^2 \right)_{tt}, \quad a > 0, \quad (4.3)$$

which is called the Blackstock-Crighton-Kuznetsov equation. The constant a is the heat conductivity of the fluid. Neglecting local nonlinear effects as its done when reducing the Kuznetsov to the Westervelt equation, one arrives at the Blackstock-Crighton-Westervelt equation

$$(a \Delta - \partial_t)(u_{tt} - c^2 \Delta u - b \Delta u_t) = \left(\frac{1}{c^2} \left(1 + \frac{B}{2A} \right) (u_t)^2 \right)_{tt}. \quad (4.4)$$

The analysis of partial differential equations describing nonlinear sound propagation has been a fruitful source of research in the last decade. For an overview, see the monograph [35]. The main emphasis lies on well-posedness and decay results. The seminal contributions in this area of research are due to Kaltenbacher and Lasiecka [36, 37]. Her research on the analysis of the Kuznetsov and Westervelt equation are an inspiration source of many additional studies, for instance the Jordan-Moore-Gibson-Thompson equation [38, 42].

A further line of research is the study of linearized abstract models, because this constitutes an important preliminary step for the nonlinear analysis. For example, in the reference [39] the authors consider an abstract Moore-Gibson-Thompson equation which is driven by a selfadjoint positive operator defined on a Hilbert space, in [1] the Laplacian is replaced by an arbitrary closed linear operator defined in a Banach space and, in [22], two closed linear operators are considered.

The seminal mathematical study of this equation was initiated in 2014 by Brunnhuber and Kaltenbacher [15]. These authors used the theory of C_0 -semigroups in order to investigate the linearization of the model, proving that the underlying semigroup is analytic. That leads to exponential decay results for the linear homogeneous equation. Moreover, it was proved local in time well-posedness of the model under the assumption that initial data are sufficiently small and a fixed point argument. Global in time well-posedness was also obtained, by performing energy estimates and using the classical barrier method, again for sufficiently small initial data. Additionally, Brunnhuber and Kaltenbacher provided results concerning exponential decay of solutions of the nonlinear equation.

Later, equation (4.3) was studied by Brunnhuber and Meyer in the reference [16] to show optimal regularity and exponential stability in L_p -spaces with Dirichlet and Neumann boundary conditions. In such reference, it was also proved long-time well-posedness and exponential stability for sufficiently small data.

More recently, in the reference [18], Celik and Kyed have considered the Blackstock-Crighton-Westervelt equation in a three-dimensional bounded domain with both nonhomogeneous Dirichlet and Neumann boundary values. Existence of a solution was obtained via a fixed-point argument based on appropriate a priori estimates for the linearized equations.

However, up to date, there is no research on the abstract modeling of equation (4.3), i.e., replacing the Laplace operator $-\Delta$ by a general closed linear operator A defined on a Banach space.

This chapter is concerned with the study of the Blackstock-Crighton-Westervelt equation in a generalized abstract form

$$(-aA - D_t)(u''(t) + c^2 Au(t) + bAu'(t)) = f(t, u, u_t), \quad t \geq 0, \quad (4.5)$$

defined in a Banach space X with initial conditions $x = u(0)$, $y = u'(0)$, $z = u''(0)$, and $A : D(A) \subset X \rightarrow X$ a closed linear densely defined operator that satisfy appropriate conditions described later. Moreover, in (4.5) D_t denotes the differentiation operator of order 1 with respect to the temporal variable t .

As intimate before, this approach in the setting of Banach spaces is completely new and has been not studied until now. The main advantage is that the abstract model can serve as prototype for other common operators A , like e.g. the fractional Laplacian, among others. As said before, the linear problem constitutes an important preparation for the nonlinear one. It should be noted that when $a = 0$ equation (4.5) reduces to the Moore-Gibson-Thompson equation.

One of our main and surprising results obtained using this abstract approach, use the theory of C_0 -semigroups of operators $\{T(t)\}_{t \geq 0}$, combined with the theory of resolvent families $\{S(t)\}_{t \geq 0}$, see [58], to solve *explicitly* the linearized equation (4.5) which provides new insights even in case that $A = -\Delta$, the negative Laplacian. Namely, it is proved that the solution of the linearized equation can be represented as

$$\begin{aligned} u(t) = & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t T(s)x ds + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t R(s)x ds \\ & + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2} \int_0^t S(s)x ds + \frac{c^2}{b} \int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)x ds \\ & - \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s S(\tau)x d\tau ds + R(t)y - \frac{a+b}{a-b} \int_0^t T(s)y ds \\ & + \frac{a+b}{a-b} \int_0^t S(s)y ds - \frac{c^2(a+b)}{a(a-b)} \int_0^t R(s)y ds + \frac{c^2(a+b)}{a(a-b)} \int_0^t \int_0^s S(\tau)y d\tau ds \end{aligned} \quad (4.6)$$

$$+ \int_0^t R(s)z ds + \int_0^t \int_0^s R(s-\tau)f(\tau)d\tau ds,$$

where $R(t) = (S * T)(t)$ is the finite convolution of a resolvent family $\{S(t)\}_{t \geq 0}$ generated by $-A$ with kernel $a(t) = b + c^2t$, and $\{T(t)\}_{t \geq 0}$ is a C_0 -semigroup generated by $-aA$.

Our second contribution in this study is that, assuming the existence of the above representation, and certain hypothesis on a semilinear source $f(\cdot, u(\cdot))$, the existence of at least one mild solution for the corresponding semilinear model (4.5) can be guaranteed.

This chapter is organized as follows: the first section is concerned with the preliminary results of the theory of analytic semigroups and resolvent families. In the second section it is shown an explicit representation of the solution presenting conditions for a mild solution to be strong. A local mild solution for the semilinear Blackstock-Crighton-Westveelt equation is proved in section 3. And section 4 is concerned with the mild solution of the semilinear version of equation (4.5) with nonlocal inicial conditions.

The study of the problem with nonlocal initial conditions is motivated by the observation that this type of conditions is more practical than classical conditions when treating physical problems. For instance, the sum

$$u(x, 0) + \sum_{k=1}^n \beta_k(x)u(x, T_k) \quad (4.7)$$

is more accurate to measurement of a state than $u(x, 0)$ alone. This approach was used by Deng in [23] to describe the diffusion phenomenon of a small amount of gas in a tube. If there is too little gas at the initial time, the measurement (4.7) of the sum of the amounts of the gas is more reliable than the measurement $u(x, 0)$ of the amount of the gas at the instant $t = 0$. For more information (see [2, 17, 67, 66] and references therein).

The results of the first chapter were published in [30].

4.1 Some results of analytic semigroups and resolvent families

This section presents some definitions and results of the theory of analytic semigroups and resolvent families. These results are essential to the work developed in the other sections. Starting with the following definition of resolvent family due to Prüss [58].

Definition 4.1. [58, Definition 1.3 p. 32] A family $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ of bounded linear operators in X is called a resolvent family if the following conditions are satisfied:

- (S1) $S(t)$ is strongly continuous on \mathbb{R}_+ and $S(0) = I$;
- (S2) $S(t)$ commutes with A , which means that $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
- (S3) the resolvent equation holds

$$S(t)x = x + \int_0^t a(t-s)AS(s)x ds, \text{ for all } x \in D(A), t \geq 0,$$

where $a \in L^1_{loc}(\mathbb{R}_+)$ is a scalar kernel $a \not\equiv 0$. In this case A is called the generator of the resolvent family $\{S(t)\}_{t \geq 0}$.

In this chapter, when working with operators, the sector $\Sigma(0, \pi)$ is used. This sector is given by

$$\Sigma(0, \pi) = \{\lambda \in \mathbb{C}; |\arg(\lambda)| < \pi\}.$$

The following definition was introduced by Prüss [58] and have ultimate importance in the development of the results of this chapter.

Definition 4.2. [58, Definition 3.2 p. 68] Let $a \in L_{loc}^1(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The kernel a is called k -regular if there exists a constant $C > 0$ such that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq C |\hat{a}(\lambda)|, \quad \text{for all } \operatorname{Re}(\lambda) > 0, 0 \leq n \leq k. \quad (4.8)$$

The following example are given to illustrate the above definition and they are very useful later.

Example 4.3. Define $a(t) = b + c^2 t$, $b > 0$, $c \in \mathbb{R}$. Then, a is k -regular for all $k \in \mathbb{N}$. Indeed, given $k \in \mathbb{N}$ notice that for each $n \in \{0, 1, \dots, k\}$ it follows that

$$\lambda^n \hat{a}^{(n)}(\lambda) = (-1)^n n! b \lambda^{-1} + (-1)^n (n+1)! c^2 \lambda^{-2},$$

then,

$$\begin{aligned} \frac{\lambda^n \hat{a}^{(n)}(\lambda)}{\hat{a}(\lambda)} &= \frac{(-1)^n n! [b \lambda^{-1} + c^2 \lambda^{-2}] + (-1)^n n(n!) c^2 \lambda^{-2}}{b \lambda^{-1} + c^2 \lambda^{-2}} \\ &= (-1)^n n! + (-1)^n n(n!) \frac{c^2}{b \lambda + c^2}, \end{aligned}$$

which is bounded, for all $\operatorname{Re}(\lambda) > 0$ and $0 \leq n \leq k$. Therefore $a(t)$ is k -regular, for all $k \in \mathbb{N}$.

The following result will be very useful in this work, and its proof is very similar to the proof of [58, Theorem 3.1, p. 73]

Theorem 4.4. Let $a \in L_{loc}^1(\mathbb{R}_+)$ be k -regular and suppose that A is closed and densely defined operator on a Banach space X such that $\hat{a}(\lambda) \neq 0$ and $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\operatorname{Re}(\lambda) > \omega$. Suppose that

$$\|\lambda^{-1}(I - \hat{a}(\lambda)A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \text{for all } \operatorname{Re}(\lambda) > \omega. \quad (4.9)$$

Then there exists a resolvent family $\{S(t)\}_{t \geq 0}$ such that $S \in C^{k-1}((0, \infty), \mathcal{L}(X))$ and $\|S(t)\| \leq M e^{wt}$.

Remark 4.5. It is possible to obtain spatial regularity of the resolvent $\{S(t)\}_{t \geq 0}$, provided a is k -regular for some large enough k . Furthermore, it follows the estimate

$$\|t^k AS(t)\| \leq M e^{wt}, \quad t \geq 0, \quad (4.10)$$

i.e. $S(t)X \subset D(A)$ for all $t > 0$ (see [58, Comment (f) p. 82]).

Example 4.6. The following is an example of a kernel $a \in L_{loc}^1(\mathbb{R}_+)$ which is k -regular and such that $|\arg(\hat{a}(\lambda))| \leq \theta_0$, for all $\operatorname{Re}(\lambda) > \omega := \frac{c^2}{b} \in \mathbb{R}$ for some $\theta_0 < \pi$, and where $c \in \mathbb{R}$, $b > 0$.

Consider $a(t) = -b - c^2t$. Then there exists $\theta_0 < \pi$ such that $|\arg(\hat{a}(\lambda))| < \theta_0$ for all $\operatorname{Re}(\lambda) > \frac{c^2}{b}$. Indeed, for $\lambda = re^{i\theta}$ with $\operatorname{Re}(\lambda) > \frac{c^2}{b}$ and $\theta < \pi$ it follows that

$$\begin{aligned} \arg(\hat{a}(re^{i\theta})) &= \operatorname{Im}(\log(\hat{a}(re^{i\theta}))) = \operatorname{Im}\left(\int_0^\theta \frac{d}{dt} \log(\hat{a}(re^{it})) dt\right) \\ &= \operatorname{Im}\left(\int_0^\theta \frac{\hat{a}'(re^{it})ire^{it}}{\hat{a}(re^{it})} dt\right) \\ &= -\operatorname{Im}\left(\int_0^\theta 1i + \frac{c^2i}{br(\cos t + i \sin t) + c^2} dt\right) \\ &= -\operatorname{Im}(\theta i) - \operatorname{Im}\left(\int_0^\theta \frac{c^2i(c^2 + br \cos t -ibr \sin t)}{(c^2 + br \cos t)^2 + (br \sin t)^2} dt\right) \\ &= -\theta - c^2 \int_0^\theta \frac{c^2 + br \cos t}{2brc^2 \cos t + c^4 + b^2r^2} dt. \end{aligned}$$

Using [33, 2554 2, p. 173], then

$$\begin{aligned} \arg(\hat{a}(re^{i\theta})) &= -\theta - c^2 \left[\frac{brt}{2brc^2} \right]_0^\theta - c^2 \frac{2brc^4 - (b^3r^3 + brc^4)}{2brc^2} \int_0^\theta \frac{dt}{b^2r^2 + c^4 + 2brc^2 \cos t} \\ &= -\theta - \frac{\theta}{2} + \frac{b^2r^2 - c^4}{2} \int_0^\theta \frac{dt}{b^2r^2 + c^4 + 2brc^2 \cos t}. \end{aligned}$$

From [33, 2553 3, p. 172] and the fact that $(b^2r^2 + c^4)^2 > (2brc^2)^2$, it is possible to conclude that

$$\begin{aligned} \arg(\hat{a}(re^{i\theta})) &= -\frac{3\theta}{2} + \frac{b^2r^2 - c^4}{2} \left[\frac{2}{\sqrt{(b^2r^2 - c^4)^2}} \arctan \left(\frac{(br - c^2)^2}{\sqrt{(b^2r^2 - c^4)^2}} \tan \left(\frac{\theta}{2} \right) \right) \right] \\ &= -\frac{3\theta}{2} + \arctan \left(\frac{br - c^2}{br + c^2} \tan \left(\frac{\theta}{2} \right) \right). \end{aligned}$$

Since $\operatorname{Re}(\lambda) > \frac{c^2}{b}$, then $|\theta| < \frac{\pi}{2}$. Let $\theta_1 > 0$ be such that $|\theta| < \frac{\theta_1}{2} < \frac{\pi}{2}$. Then $|\tan(\frac{\theta}{2})| < 1$. Notice that

$$\left| \frac{br - c^2}{br + c^2} \right| \leq 1$$

then

$$\tan\left(-\frac{\pi}{4}\right) = -1 < \frac{br - c^2}{br + c^2} \tan\left(\frac{\theta}{2}\right) < 1 = \tan\left(\frac{\pi}{4}\right),$$

that is,

$$-\frac{\pi}{4} < \arctan\left(\frac{br - c^2}{br + c^2} \tan\left(\frac{\theta}{2}\right)\right) < \frac{\pi}{4},$$

therefore

$$-\frac{3\theta_1}{4} - \frac{\pi}{4} < -\frac{3\theta}{2} + \arctan\left(\frac{br - c^2}{br + c^2} \tan\left(\frac{\theta}{2}\right)\right) < \frac{3\theta_1}{4} + \frac{\pi}{4} =: \theta_0 < \pi.$$

Then, $|\arg \hat{a}(\lambda)| < \pi$ for all $\operatorname{Re}(\lambda) > \frac{c^2}{b}$.

Example 4.7. From Example 4.6, note that $\frac{1}{a(\lambda)} \in \Sigma(0, \pi)$ for all $\operatorname{Re}(\lambda) > \frac{c^2}{b}$, with $b > 0$ and $c \in \mathbb{R}$.

Let $-A$ be a closed and densely defined operator on X . If $\rho(-A) \supset \Sigma(0, \pi)$ for $\operatorname{Re}(\lambda) > \frac{c^2}{b}$ and

$$\|\lambda(\lambda^2 I + (b\lambda + c^2)A)^{-1}\| \leq \frac{M}{|\lambda - \frac{c^2}{b}|}, \quad \text{for all } \operatorname{Re}(\lambda) > \frac{c^2}{b}, \quad (4.11)$$

then, $-A$ satisfies the hypotheses of Theorem 4.4 and there exists a resolvent family $\{S(t)\}_{t \geq 0}$ such that $S \in C^\infty((0, \infty), \mathcal{L}(X))$ and $\|S(t)\| \leq Me^{\frac{c^2}{b}t}$. Moreover, from Remark 4.5, $S(t)X \subset D(-A)$ for all $t > 0$.

4.2 Well-posedness and strong solutions

Let X be a Banach space. This section is concerned with the study of the well-posedness for the abstract equation (4.5), that is rewritten including their initial conditions as follows

$$\begin{cases} u'''(t) + (a+b)Au''(t) + (abA + c^2)Au'(t) + ac^2A^2u(t) = f(t), & t \geq 0 \\ u(0) = x, \quad u'(0) = y \quad u''(0) = z, \end{cases} \quad (4.12)$$

where $x, y, z \in X$, $c \in \mathbb{R}$, $a > 0, b > 0$ and $A : D(A) \subset X \rightarrow X$ is an operator that satisfies appropriate conditions described later. First, the notion of solution used in this chapter is introduced.

Definition 4.8. A function $u : \mathbb{R}_+ \rightarrow X$ is called a strong solution of (4.12) if satisfies:

- (i) $u \in C(\mathbb{R}_+; D(A^2)) \cap C^3(\mathbb{R}_+; X)$;
- (ii) $u' \in C(\mathbb{R}_+; D(A^2))$;
- (iii) $u'' \in C(\mathbb{R}_+; D(A))$;
- (iv) (4.12) holds on \mathbb{R}_+ .

A closed linear densely defined operator A satisfies **hypothesis (H)** if:

- (i) $-A$ is the generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ uniformly bounded, that is, $\|T(t)\| \leq M, \forall t \geq 0$.
- (ii) $-A$ generates a resolvent family $\{S(t)\}_{t \geq 0}$ with kernel $a(t) = b + c^2t$ and satisfying $\|S(t)\| \leq Me^{\omega t}$, $\omega > \frac{c^2}{b} > 0$ and $S(t)X \subset D(A)$.

In such case, $\{T_a(t)\}_{t \geq 0}$ denotes the semigroup generated by $-aA$, and

$$R(t) := (S * T_a)(t) = \int_0^t S(t-s)T_a(s)ds, \quad t \geq 0, \quad (4.13)$$

is the finite convolution. Here the integral is understood in the Bochner sense. Note that $\|R(t)\| \leq Ke^{\omega t}$ for some $K > 0$ and $\omega \in \mathbb{R}$.

Remark 4.9. If $-A$ generates an analytic semigroup uniformly bounded such that $\Sigma(0, \pi) \subset \rho(-A)$ for $\operatorname{Re}(\lambda) > 0$, then A satisfies hypothesis (H). Indeed, item (i) is clear and it follows that

$$\|(\lambda I + A)^{-1}\| \leq M, \quad \forall \operatorname{Re}(\lambda) > 0$$

and then, since $\operatorname{Re}(\lambda^2(b\lambda + c^2)^{-1}) > 0$,

$$\begin{aligned} \|\lambda(\lambda^2 I + (b\lambda + c^2)A)^{-1}\| &= \|\lambda(b\lambda + c^2)^{-1}(\lambda^2(b\lambda + c^2)^{-1}I + A)^{-1}\| \\ &\leq \frac{|\lambda(b\lambda + c^2)|M}{|\lambda^2(b\lambda + c^2)|} = \frac{M}{|\lambda|} \\ &\leq \frac{M}{|\lambda - \frac{c^2}{b}|}, \quad \operatorname{Re}(\lambda) > \frac{c^2}{b}. \end{aligned}$$

From Example 4.7 it is possible to conclude that $-A$ generates a resolvent family $\{S(t)\}_{t \geq 0}$ of type $(M, \frac{c^2}{b})$ with kernel $a(t) = b + c^2 t$, $b > 0$, $c \in \mathbb{R}$, and such that $S(t)X \subset D(-A)$.

Remark 4.10. By (ii) in hypothesis (H), and (S3) in Definition (4.1),

$$S(t)x = x - b \int_0^t S(s)Ax ds - c^2 \int_0^t (t-s)S(s)Ax ds \quad x \in D(A), \quad t \geq 0,$$

and, since A is closed

$$S(t)x = x - bA \int_0^t S(s)x ds - c^2 A \int_0^t (t-s)S(s)x ds \quad x \in X, \quad t \geq 0,$$

(see [58, Proposition 1.1]). In particular, it follows that

$$S'(t)x = -bS(t)Ax - c^2 \int_0^t S(s)Ax ds \quad x \in D(A), \quad t \geq 0,$$

and, since $S(t)X \subset D(A)$,

$$S'(t)x = -bAS(t)x - c^2 A \int_0^t S(s)x ds \quad x \in X, \quad t \geq 0.$$

Moreover,

$$S''(t)x = -bS'(t)Ax - c^2 S(t)Ax, \quad x \in D(A^2), \quad t \geq 0,$$

and

$$S'''(t)x = -bS''(t)Ax - c^2 S'(t)Ax, \quad x \in D(A^3), \quad t \geq 0.$$

Example 4.11. If $1 < p < \infty$, the Laplacian operator Δ in $L^p(\mathbb{R}^n)$, i.e. the operator Δ_p with domain $D(\Delta_p) = \{f \in L^p(\mathbb{R}^n); \Delta f \in L^p(\mathbb{R}^n)\}$ satisfies hypothesis (H). Indeed, $-\Delta_p$ generates an analytic semigroup uniformly bounded and $\Sigma(0, \pi) \subset \rho(-\Delta_p)$ (see [51, Theorem 2.3.3 p. 40 and A.7.6 p. 329]).

The next Lemma concerns the definition of $R(t)$ given in (4.13).

Lemma 4.12. Let A be a closed linear densely defined operator satisfying hypothesis (H). Suppose $a \neq b$, $a > 0$, $b > 0$, $c \in \mathbb{R}$ then $R(t)x \in D(A)$, for all $x \in X$ and satisfies

$$AR(t)x = \frac{1}{a-b} \left[S(t)x - T_a(t)x + \frac{c^2}{a} \int_0^t S(s)x ds - \frac{c^2}{a} R(t)x \right], \quad x \in X. \quad (4.14)$$

Moreover, $AR(t)x \in D(A)$ for all $x \in X$.

Proof. Since A commutes with $T_a(t)$ and $S(t)$, and A is closed, then A commutes with $R(t)$ on $D(A)$.

Since $T_a(t)$ is uniformly bounded and $S(t)$ and $R(t)$ are exponentially bounded the Laplace transform can be applied, and then, for all $\operatorname{Re}(\lambda) > \omega$ and $x \in D(A)$

$$\begin{aligned} A\hat{R}(\lambda)x &= A\hat{S}(\lambda)\hat{T}_a(\lambda)x = \hat{S}(\lambda) \left[-\frac{\lambda}{a}\hat{T}_a(\lambda) + \frac{1}{a}I \right] x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)\lambda\hat{S}(\lambda)x = \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)[\hat{S}'(\lambda) + I]x. \end{aligned}$$

By (S3) of Definition 4.1 with $a(t) = b + c^2t$, after application of the Laplace transform it follows that

$$\lambda\hat{S}(\lambda)x = Ix + b\hat{S}(\lambda)Ax + \frac{c^2}{\lambda}\hat{S}(\lambda)Ax, \quad (4.15)$$

and therefore

$$\begin{aligned} A\hat{R}(\lambda)x &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda) \left[I - bA\hat{S}(\lambda) - \frac{c^2}{\lambda}A\hat{S}(\lambda) \right] x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}A\hat{R}(\lambda)x + \frac{c^2}{a\lambda}A\hat{T}_a(\lambda)\hat{S}(\lambda)x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}A\hat{R}(\lambda)x + \frac{c^2}{a\lambda} \left[-\frac{\lambda}{a}\hat{T}_a(\lambda) + \frac{1}{a}I \right] \hat{S}(\lambda)x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}A\hat{R}(\lambda)x - \frac{c^2}{a^2}\hat{R}(\lambda)x + \frac{c^2}{a^2\lambda}\hat{S}(\lambda)x. \end{aligned}$$

Then

$$\frac{a-b}{a}A\hat{R}(\lambda)x = \frac{1}{a} \left[\hat{S}(\lambda) - \hat{T}_a(\lambda) - \frac{c^2}{a}\hat{R}(\lambda) + \frac{c^2}{a\lambda}\hat{S}(\lambda) \right] x.$$

So, applying the inversion of the Laplace transform, and by the uniqueness theorem, it follows that

$$AR(t)x = R(t)Ax = \frac{1}{a-b} \left[S(t)x - T_a(t)x + \frac{c^2}{a} \int_0^t S(s)x ds - \frac{c^2}{a} R(t)x \right]. \quad (4.16)$$

for all $x \in D(A)$.

Because A is closed and $D(A)$ is dense in X it is deduced from the above identity that $R(t)x \in D(A)$ for all $x \in X$ and (4.14) holds.

For the last conclusion of the Lemma, first notice that by hypothesis, $S(t)X \subset D(A)$, and $T_a(t)X \subset D(A)$ for any $t > 0$, where $\{T_a(t)\}_{t \geq 0}$ is an analytic semigroup generated by $-aA$ (see [24, Theorem 4.6 (c) p. 101]). Next, observe that from (4.15) and defining $e(t) := e^{-\frac{c^2}{b}t}$ it follows that, for all $x \in D(A)$

$$\begin{aligned} A\frac{1}{\lambda}\hat{S}(\lambda)x &= \frac{1}{b} \frac{\lambda}{\lambda + c^2/b} \hat{S}(\lambda)x - \frac{1}{b} \frac{1}{\lambda + c^2/b} x = \frac{1}{b} \lambda \hat{e}(\lambda) \hat{S}(\lambda)x - \frac{1}{b} \hat{e}(\lambda)x \\ &= \frac{1}{b} \hat{e}'(\lambda) \hat{S}(\lambda)x + \frac{1}{b} \hat{S}(\lambda)x - \frac{1}{b} \hat{e}(\lambda)x. \end{aligned}$$

Therefore

$$A \int_0^t S(s)x ds = -\frac{c^2}{b^2} \int_0^t e^{-\frac{c^2}{b}(t-s)} S(s)x ds + \frac{1}{b} S(t)x - \frac{1}{b} e^{-\frac{c^2}{b}t} x, \quad x \in D(A). \quad (4.17)$$

Using that $D(A)$ is dense in X and A is closed, from the above identity it is deduced that $\int_0^t S(s)x ds \in D(A)$ for all $x \in X$ and (4.17) holds on X . It follows that, for each $x \in X$, all the terms in the right-hand side of (4.16) belong to the domain of A , proving the lemma. \square

The following lemma gives an additional property.

Lemma 4.13. Let A be a closed linear densely defined operator satisfying hypothesis (H). Suppose $a > 0, b > 0, a \neq b, c \in \mathbb{R}$. Then,

$$\begin{aligned} A^2 \int_0^t R(s)x ds &= \frac{1}{a(a-b)} T_a(t)x - \frac{1}{b(a-b)} S(t)x - \frac{c^2}{a(a-b)^2} \int_0^t S(s)x ds \\ &\quad + \frac{c^2}{a(a-b)^2} \int_0^t T_a(s)x ds + \frac{c^4}{a^2(a-b)^2} \int_0^t R(s)x ds \\ &\quad - \frac{c^4}{a^2(a-b)^2} \int_0^t \int_0^s S(\tau)x d\tau ds \\ &\quad + \frac{1}{ab} e^{-\frac{c^2}{b}t} x + \frac{c^2}{ab^2} \int_0^t e^{-\frac{c^2}{b}(t-s)} S(s)x ds \end{aligned} \quad (4.18)$$

for all $x \in X$.

Proof. Given $x \in X$, define by $G(t)x$ the right hand side of (4.18). Notice that $G(t)x$ is well defined for all $t \geq 0$. Moreover, it follows the identity

$$-\frac{1}{b(a-b)} S(t)x = -\left[\frac{1}{a(a-b)} + \frac{1}{ab} \right] S(t)x. \quad (4.19)$$

Then, replacing (4.19) in $G(t)x$ and applying the Laplace transform, which is possible since $T_a(t)$ is uniformly bounded, and $S(t)$ and $R(t)$ are exponentially bounded, it follows that

$$\begin{aligned} \hat{G}(\lambda)x &= \frac{1}{a(a-b)} \hat{T}_a(\lambda)x - \frac{1}{a(a-b)} \hat{S}(\lambda)x + \frac{c^2}{a(a-b)^2 \lambda} \hat{T}_a(\lambda)x - \frac{c^2}{a(a-b)^2 \lambda} \hat{S}(\lambda)x \\ &\quad + \frac{c^4}{a^2(a-b)^2 \lambda} \hat{R}(\lambda)x - \frac{c^4}{a^2(a-b)^2 \lambda^2} \hat{S}(\lambda)x + \frac{1}{ab} \frac{1}{\lambda + \frac{c^2}{b}} x \\ &\quad + \frac{c^2}{ab^2} \frac{1}{\lambda + \frac{c^2}{b}} \hat{S}(\lambda)x - \frac{1}{ab} \hat{S}(\lambda)x \\ &= \frac{1}{a(a-b)} [\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{c^2}{a(a-b)^2 \lambda} \hat{T}_a(\lambda)x - \frac{c^2}{a(a-b)^2 \lambda} \hat{S}(\lambda)x \\ &\quad + \frac{c^4}{a^2(a-b)^2 \lambda} \hat{R}(\lambda)x - \frac{c^4}{a^2(a-b)^2 \lambda^2} \hat{S}(\lambda)x + \frac{1}{a(b\lambda + c^2)} [I - \lambda \hat{S}(\lambda)]x \\ &= \frac{1}{a(a-b)} [\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{1}{a(b\lambda + c^2)} [I - \lambda \hat{S}(\lambda)]x \\ &\quad + \frac{c^2}{a(a-b)^2 \lambda} \left[\hat{T}_a(\lambda) - \hat{S}(\lambda) + \frac{c^2}{a} \hat{R}(\lambda) - \frac{c^2}{a\lambda} \hat{S}(\lambda) \right] x. \end{aligned}$$

By Lemma 4.12 and since $\hat{a}(\lambda) = \frac{b}{\lambda} + \frac{c^2}{\lambda^2}$ it follows that

$$\hat{G}(\lambda)x = \frac{1}{a(a-b)} [\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{1}{a\hat{a}(\lambda)} \left[\frac{1}{\lambda^2} I - \frac{1}{\lambda} \hat{S}(\lambda) \right] x - \frac{c^2}{a(a-b)\lambda} A \hat{R}(\lambda)x.$$

Moreover, by definition of $\{S(t)\}_{t \geq 0}$,

$$A\hat{S}(\lambda)x = \frac{1}{\lambda\hat{a}(\lambda)}x - \frac{\hat{S}(\lambda)}{\hat{a}(\lambda)}x.$$

Then,

$$\begin{aligned} \hat{G}(\lambda)x &= \frac{1}{a(a-b)\lambda}[\lambda\hat{T}_a(\lambda) - \lambda\hat{S}(\lambda)]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x \\ &= \frac{1}{a(a-b)\lambda}[\hat{T}'_a(\lambda) - \hat{S}'(\lambda)]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x \\ &= \frac{1}{a(a-b)\lambda} \left[\hat{T}'_a(\lambda) + \frac{c^2}{\lambda}A\hat{S}(\lambda) + bA\hat{S}(\lambda) \right] x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x \\ &= \frac{1}{(a-b)\lambda} \left[\frac{1}{a}\hat{T}'_a(\lambda) + \frac{c^2}{a\lambda}A\hat{S}(\lambda) + A\hat{S}(\lambda) - \frac{c^2}{a}A\hat{R}(\lambda) \right] x \end{aligned}$$

Using Lemma 4.12 and the fact that $-aA$ is the generator of $T_a(t)$, it is concluded that

$$\begin{aligned} \hat{G}(\lambda)x &= \frac{1}{(a-b)\lambda} \left[A\hat{S}(\lambda) - A\hat{T}_a(\lambda) + \frac{c^2}{a\lambda}A\hat{S}(\lambda) - \frac{c^2}{a}A\hat{R}(\lambda) \right] x \\ &= A^2 \frac{1}{\lambda} \hat{R}(\lambda)x. \end{aligned}$$

Notice that from Lemma 4.12, $AR(t)x \in D(A)$ for $x \in X$, and then $A^2R(t)x$ is well defined for $x \in X$. So, applying the inversion of the Laplace transform and by the uniqueness theorem,

$$G(t)x = A^2 \int_0^t R(s)x ds,$$

for all $x \in X$. This finishes the proof. \square

Remark 4.14. Considering

$$R(t)x = (T_a * S)(t)x = \int_0^t T_a(t-s)S(s)x ds, \quad x \in X,$$

then

$$R'(t)x = S(t)x - aAR(t)x,$$

for all $x \in X$;

In particular, under the hypothesis that $S(t)X \subset D(A)$ and in view of Lemma 4.12 it follows that $R'(t)X \subset D(A)$ for all $t \geq 0$. Moreover,

$$\begin{aligned} R''(t)x &= S'(t)x - aAR'(t)x = S'(t)x - aAS(t)x + a^2A^2R(t)x \\ &= -bAS(t)x - c^2A \int_0^t S(s)x ds - aAS(t)x + a^2A^2R(t)x \\ &= -(a+b)AS(t)x - c^2A \int_0^t S(s)x ds + a^2A^2R(t)x \end{aligned}$$

for all $x \in D(A)$ (see Remark 4.10), and

$$\begin{aligned}
R'''(t)x &= -(a+b)AS'(t)x - c^2AS(t)x + a^2A^2R'(t)x \\
&= -(a+b)AS'(t)x - c^2AS(t)x + a^2A^2S(t)x - a^3A^3R(t)x \\
&= -(a+b)AS'(t)x + (a^2A - c^2)AS(t)x - a^3A^3R(t)x,
\end{aligned}$$

for all $x \in D(A^2)$. Moreover

$$\begin{aligned}
\|R(t)\| &\leq \int_0^t \|T_a(t-s)\| \|S(s)\| ds \leq M^2 \int_0^t e^{\omega s} ds \\
&= \frac{M^2}{\omega} [e^{\omega t} - 1] \leq \frac{M^2}{\omega} e^{\omega t}
\end{aligned}$$

Considering $K = \frac{M^2}{\omega}$, then

$$\|R(t)\| \leq Ke^{\omega t}.$$

The next theorem is the main result of this section.

Theorem 4.15. Let A be a closed linear operator satisfying hypothesis (H), $a > 0$, $b > 0$, $a \neq b$, $c \in \mathbb{R}$. If $f \in C(\mathbb{R}_+, D(A))$, $x \in D(A^3)$, $y \in D(A^2)$ and $z \in D(A)$, then u given by

$$\begin{aligned}
u(t) &= \left[S(t) + bAR(t) + (abA + c^2)A \int_0^t R(s)ds \right] x + \left[R(t) + (a+b)A \int_0^t R(s)ds \right] y \\
&\quad + \int_0^t R(s)z ds + \int_0^t (R * f)(s)ds, \quad t \geq 0,
\end{aligned} \tag{4.20}$$

is the unique strong solution of the initial value problem (4.12).

Proof. For $x, y, z \in X$ and $f \in C(\mathbb{R}_+; X)$ consider

$$\begin{aligned}
u(t) &= \left[S(t) + bAR(t) + (abA + c^2)A \int_0^t R(s)ds \right] x + \left[R(t) + (a+b)A \int_0^t R(s)ds \right] y \\
&\quad + \int_0^t R(s)z ds + \int_0^t (R * f)(s)ds.
\end{aligned}$$

Then u is a strong solution of (4.12). Indeed, since $f \in C(\mathbb{R}_+, D(A))$, $x \in D(A^2)$, $y \in D(A^2)$ and $z \in D(A)$ then $u \in C(\mathbb{R}_+; D(A^2))$ and is differentiable. recalling that $S(t)$ commutes with A on $D(A)$ (see (S2) in Definition 4.1), as well as $R(t)$, and using the second part of Lemma (4.12), it follows that $u \in C(\mathbb{R}_+; D(A^2))$. Note that $u \in C^1(\mathbb{R}_+; X)$ by Remark 4.10.

Moreover

$$u(0) = [S(0) + bAR(0)]x + R(0)y + (R * f)(0) = x.$$

Since $AR(t)x \in D(A)$ for all $x \in X$ (Lemma 4.12) and taking into account Remark 4.10 and Remark 4.14 it follows, for $x \in D(A)$, $y \in D(A)$ and $z \in X$,

$$\begin{aligned}
u'(t) &= [S'(t) + bR'(t)A + (abA + c^2)R(t)A]x + [R'(t) + (a+b)AR(t)]y \\
&\quad + R(t)z + (R * f)(t) \\
&= [S'(t) + b[S(t) - aAR(t)]A + (abA + c^2)R(t)A]x \\
&\quad + [S(t) - aAR(t) + (a+b)AR(t)]y + R(t)z + (R * f)(t)
\end{aligned}$$

$$\begin{aligned}
&= [S'(t) + bAS(t) + c^2AR(t)]x \\
&\quad + [S(t) + bAR(t)]y + R(t)z + (R * f)(t).
\end{aligned}$$

From the above identity, and using that $S(t)X \subset D(A)$ and $AR(t)X \subset D(A)$, it is deduced that $u' \in C(\mathbb{R}_+; D(A^2))$. Moreover, by Remark 4.10 and Remark 4.14 it follows that $u' \in C^1(\mathbb{R}_+; X)$ and

$$u'(0) = [S'(0) + bAS(0) + c^2AR(0)]x + [S(0) + bAR(0)]y + R(0)z + (R * f)(0) = y.$$

Further, for $x \in D(A^2)$, $y \in D(A^2)$, $z \in D(A)$ and using again Remark 4.10 and Remark 4.14,

$$\begin{aligned}
u''(t) &= [S''(t) + bS'(t)A + c^2R'(t)A]x + [S'(t) + bR'(t)A]y + R'(t)z + (R' * f)(t) \\
&= [-c^2S(t)A + c^2R'(t)A]x + [S'(t) + bR'(t)A]y + R'(t)z + (R' * f)(t).
\end{aligned}$$

Since $R'(t)X \subset D(A)$ (cf. Remark 4.14) from the above identity it is deduced that $u'' \in C(\mathbb{R}_+; D(A))$ and, since $y \in D(A^2)$ and $z \in D(A)$ that $u'' \in C^1(\mathbb{R}_+; X)$. Therefore $u \in C^3(\mathbb{R}_+; X)$. In addition,

$$u''(0) = [-c^2AS(0) + c^2AR'(0)]x + [S'(0) + bAR'(0)]y + R'(0)z + (R' * f)(0) = z.$$

At last, for $x \in D(A^2)$, $y \in D(A^2)$ and $z \in D(A)$ it follows that

$$\begin{aligned}
u'''(t) &= [-c^2S'(t)A + c^2R''(t)A]x + [S''(t) + bR''(t)A]y + R''(t)z + (R'' * f)(t) + f(t) \\
&= [-c^2aAR'(t)A]x + [-c^2S(t)A - abAR'(t)A]y + R''(t)z + (R'' * f)(t) + f(t).
\end{aligned}$$

Then, the conditions (i), (ii) and (iii) of Definition 4.8 are satisfied.

To verify that (4.12) holds first notice that

$$\begin{aligned}
S'''(t)x &+ (a + b)S''(t)Ax + (abA + c^2)S'(t)Ax + ac^2S(t)A^2x \\
&= \{S'(t)(b^2A - c^2)A + (a + b)[-bS'(t)A - c^2S(t)A]A + ac^2S(t)A^2 \\
&\quad + S'(t)(abA + c^2)A + c^2bS(t)A^2\}x \\
&= 0,
\end{aligned}$$

for all $x \in D(A^3)$. Moreover, it follows that

$$\begin{aligned}
R'''(t)x &+ (a + b)R''(t)Ax + R'(t)(abA + c^2)Ax + ac^2R(t)A^2x \\
&= -(a + b)S'(t)Ax + S(t)(a^2A - c^2)Ax + (a + b)[S'(t) - aS(t)A \\
&\quad + a^2AR(t)A]Ax - a^3AR(t)A^2x + [S(t) - aAR(t)](abA + c^2)Ax \\
&\quad + ac^2R(t)A^2x = 0,
\end{aligned}$$

for all $x \in D(A^2)$. Now, defining

$$h(t)x := R''(t)x + (a + b)AR'(t)x + (abA + c^2)AR(t)x + ac^2A^2 \int_0^t R(s)x ds, \quad x \in X,$$

then $h'(t)x = 0$ as seen above. But, $h(0)x = 0$, then $h \equiv 0$. Thus $u(t)$ given by (4.20) satisfies (4.12) and is the unique strong solution. This concludes the proof. \square

Remark 4.16. By Lemmas 4.12 and 4.13, an equivalent representation of (4.20) is

$$\begin{aligned}
 u(t) = & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t T_a(s)xds + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t R(s)xds \\
 & + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2} \int_0^t S(s)xds + \frac{c^2}{b} \int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds \\
 & - \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s S(\tau)xd\tau ds + R(t)y - \frac{a+b}{a-b} \int_0^t T_a(s)yds \\
 & + \frac{a+b}{a-b} \int_0^t S(s)yds - \frac{c^2(a+b)}{a(a-b)} \int_0^t R(s)yds + \frac{c^2(a+b)}{a(a-b)} \int_0^t \int_0^s S(\tau)y d\tau ds \\
 & + \int_0^t R(s)zds + \int_0^t \int_0^s R(s-\tau)f(\tau)d\tau ds
 \end{aligned} \tag{4.21}$$

4.3 The semilinear problem

Let A be a closed linear operator satisfying hypothesis (H) . For $(x, y, z) \in X \times X \times X$ and $a > 0, b > 0, a \neq b, c \in \mathbb{R}$, consider the semilinear problem

$$\begin{cases} u'''(t) + (a+b)Au''(t) + (abA + c^2)Au'(t) + ac^2A^2u(t) = f(t, u(t)), & t \geq 0 \\ u(0) = x, \quad u'(0) = y, \quad u''(0) = z. \end{cases} \tag{4.22}$$

Note that the problem (4.22) exclude the full nonlinear term in (4.4), because the only interest is in a semilinear form of the Blackstock-Crighton-Westervelt equation.

The purpose of this section is to prove that under certain conditions the semilinear problem (4.22) has a mild solution.

Starting with the following definition.

Definition 4.17. Given $(x, y, z) \in X \times X \times X$, a continuous function $u(\cdot, x, y, z) : [0, \infty] \rightarrow X$ that satisfies

$$\begin{aligned}
 u(t) = & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t T_a(s)xds + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t R(s)xds \\
 & + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2} \int_0^t S(s)xds + \frac{c^2}{b} \int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds \\
 & - \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s S(\tau)xd\tau ds + R(t)y - \frac{a+b}{a-b} \int_0^t T_a(s)yds \\
 & + \frac{a+b}{a-b} \int_0^t S(s)yds - \frac{c^2(a+b)}{a(a-b)} \int_0^t R(s)yds + \frac{c^2(a+b)}{a(a-b)} \int_0^t \int_0^s S(\tau)y d\tau ds \\
 & + \int_0^t R(s)zds + \int_0^t \int_0^s R(s-\tau)f(\tau, u(\tau))d\tau ds
 \end{aligned} \tag{4.23}$$

is called mild solution of the problem (4.22).

The next result shows that there exists a mild solution of the problem (4.22).

Theorem 4.18. Let A be a closed linear operator satisfying hypothesis (H) . If $f : [0, +\infty) \times X \rightarrow X$ satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}$, with Lipschitz constant $L > 0$, then there exists a unique mild solution of (4.22).

Proof. Consider $R(t) = (S * T_a)(t)$. There exist constants $K > 0$ and $\omega > 0$ such that

$$\|R(t)\| \leq K e^{\omega t}, \quad \forall t \in [0, +\infty).$$

Let $T > 0$ be given and consider the space $C([0, T]; X)$ of the continuous functions from $[0, T]$ to X , with the norm

$$\|u\|_{\tilde{L}} = \max_{t \in [0, T]} \{e^{-(\tilde{L}+\omega)t} \|u(t)\|\},$$

where $u \in C([0, T]; X)$ and $\tilde{L} > \frac{-\omega + \sqrt{\omega^2 + 4KL}}{2} > 0$ is arbitrary but fixed.

First note that the norm $\|\cdot\|_{\tilde{L}}$ is equivalent with the standard norm $\|\cdot\|_{\infty}$, which is defined by

$$\|u\|_{\infty} = \max_{t \in [0, T]} \|u(t)\|.$$

Indeed, since the function $e^{-(\tilde{L}+\omega)t}$ is decreasing in $[0, T]$, $\forall t \in [0, T]$ it follows that

$$e^{-(\tilde{L}+\omega)T} \|u(t)\| \leq e^{-(\tilde{L}+\omega)t} \|u(t)\| \leq \|u(t)\|.$$

Then,

$$e^{-(\tilde{L}+\omega)T} \|u\|_{\infty} \leq \|u\|_{\tilde{L}} \leq \|u\|_{\infty}.$$

Therefore the norms are equivalent and $C([0, T]; X)$ is a Banach space with the norm $\|\cdot\|_{\tilde{L}}$. Let $x, y, z \in X$ be fixed and define the operator $\Gamma : C([0, T]; X) \rightarrow C([0, T]; X)$ by

$$\begin{aligned} \Gamma u(t) = & e^{-\frac{c^2}{b}t} x - \frac{bc^2}{a(a-b)} R(t)x + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t T_a(s)x ds + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t R(s)x ds \\ & + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2} \int_0^t S(s)x ds + \frac{c^2}{b} \int_0^t e^{-\frac{c^2}{b}(t-s)} S(s)x ds \\ & - \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s S(\tau)x d\tau ds + R(t)y - \frac{a+b}{a-b} \int_0^t T_a(s)y ds \\ & + \frac{a+b}{a-b} \int_0^t S(s)y ds - \frac{c^2(a+b)}{a(a-b)} \int_0^t R(s)y ds + \frac{c^2(a+b)}{a(a-b)} \int_0^t \int_0^s S(\tau)y d\tau ds \\ & + \int_0^t R(s)z ds + \int_0^t \int_0^s R(s-\tau) f(\tau, u(\tau)) d\tau ds. \end{aligned} \quad (4.24)$$

Notice that given $u \in C([0, T]; X)$, the function $s \mapsto f(s, u(s))$ is continuous in $[0, T]$ and therefore integrable in $[0, t]$ for all $t \in [0, T]$. Then Γu is a continuous function from $[0, T]$ to X , implying that Γ is well defined.

Consider $u, v \in C([0, T]; X)$ and $t \in [0, T]$. thus

$$\begin{aligned} \|\Gamma u(t) - \Gamma v(t)\| &= \left\| \int_0^t \int_0^s R(s-\tau) [f(\tau, u(\tau)) - f(\tau, v(\tau))] d\tau ds \right\| \\ &\leq \int_0^t \int_0^s \|R(s-\tau)\| \|f(\tau, u(\tau)) - f(\tau, v(\tau))\| d\tau ds, \end{aligned}$$

and

$$e^{-(\tilde{L}+\omega)t} \|\Gamma u(t) - \Gamma v(t)\| \leq e^{-(\tilde{L}+\omega)t} \int_0^t \int_0^s \|R(s-\tau)\| \|f(\tau, u(\tau)) - f(\tau, v(\tau))\| d\tau ds$$

$$\begin{aligned}
&\leq e^{-(\tilde{L}+\omega)t} \int_0^t \int_0^s K e^{\omega(s-\tau)} L \|u(\tau) - v(\tau)\| d\tau ds \\
&= e^{-(\tilde{L}+\omega)t} K L \int_0^t e^{\omega s} \int_0^s e^{\tilde{L}\tau} e^{-\omega\tau} e^{-\tilde{L}\tau} \|u(\tau) - v(\tau)\| d\tau ds \\
&\leq K L e^{-(\tilde{L}+\omega)t} \int_0^t e^{\omega s} \int_0^s e^{\tilde{L}\tau} \|u - v\|_{\tilde{L}} d\tau ds \\
&= K L e^{-(\tilde{L}+\omega)t} \|u - v\|_{\tilde{L}} \int_0^t e^{\omega s} \int_0^s e^{\tilde{L}\tau} d\tau ds \\
&= \frac{K L}{\tilde{L}} \|u - v\|_{\tilde{L}} e^{-(\tilde{L}+\omega)t} \int_0^t e^{\omega s} [e^{\tilde{L}s} - 1] ds \\
&\leq \frac{K L}{\tilde{L}} \|u - v\|_{\tilde{L}} e^{-(\tilde{L}+\omega)t} \int_0^t e^{(\omega+\tilde{L})s} ds \\
&= \frac{K L}{\tilde{L}(\omega + \tilde{L})} \|u - v\|_{\tilde{L}} e^{-(\tilde{L}+\omega)t} [e^{(\omega+\tilde{L})t} - 1] \\
&\leq \frac{K L}{\tilde{L}(\omega + \tilde{L})} \|u - v\|_{\tilde{L}}.
\end{aligned}$$

Since $\tilde{L} > \frac{-\omega + \sqrt{\omega^2 + 4KL}}{2} > 0$, then $\frac{KL}{\tilde{L}(\omega + \tilde{L})} < 1$. So Γ is a contraction. By the Banach Fixed-Point Theorem, Γ has a unique fixed point, that is, there exists $u \in C([0, T]; X)$ such that $u(t) = \Gamma u(t)$, $\forall t \in [0, T]$, i.e.

$$\begin{aligned}
u(t) &= e^{-\frac{c^2}{b}t} x - \frac{bc^2}{a(a-b)} R(t)x + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t T_a(s)x ds + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t R(s)x ds \\
&\quad + \frac{c^2(a^2 - ab - b^2)}{a(a-b)^2} \int_0^t S(s)x ds + \frac{c^2}{b} \int_0^t e^{-\frac{c^2}{b}(t-s)} S(s)x ds \\
&\quad - \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s S(\tau)x d\tau ds + R(t)y - \frac{a+b}{a-b} \int_0^t T_a(s)y ds \\
&\quad + \frac{a+b}{a-b} \int_0^t S(s)y ds - \frac{c^2(a+b)}{a(a-b)} \int_0^t R(s)y ds + \frac{c^2(a+b)}{a(a-b)} \int_0^t \int_0^s S(\tau)y d\tau ds \\
&\quad + \int_0^t R(s)z ds + \int_0^t \int_0^s R(s-\tau) f(\tau, u(\tau)) d\tau ds,
\end{aligned}$$

for all $t \in [0, T]$. By the uniqueness of the fixed point, the problem (4.22) has a unique mild solution in $[0, T]$. Since $T > 0$ is arbitrary, the result follows. \square

4.4 Mild solutions with nonlocal initial conditions

In this section, it is considered the semilinear Blackstock-Crighton-Westervelt with non-local initial conditions

$$\begin{cases} u'''(t) + (a+b)Au''(t) + (abA + c^2)Au'(t) + ac^2A^2u(t) = f(t, u(t)), & t \in I \\ u(0) = g_1(u), \quad u'(0) = g_2(u), \quad u''(0) = g_3(u). \end{cases} \quad (4.25)$$

where A is a closed linear operator satisfying hypothesis (H) and $I = [0, 1]$. The functions $f : I \times X \rightarrow X$ and $g_1, g_2, g_3 : C(I; X) \rightarrow X$ are X -valued functions that satisfy appropriate conditions described later.

Consider the function $U : I \rightarrow \mathcal{L}(X)$ given by $U(t) = (1 * R)(t) = \int_0^t R(s)ds$. Notice that U is uniformly continuous in $\mathcal{L}(X)$.

For the next result the following assertions will be assumed.

(H1) The functions $g_1, g_2, g_3 : C(I; X) \rightarrow X$ are compact maps.

(H2) The function $f : I \times X \rightarrow X$ satisfies the Carathéodory type conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in I$.

(H3) There exist a function $m \in L^1(I; \mathbb{R}_+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\|f(t, x)\| \leq m(t)\Phi(\|x\|)$$

for all $x \in X$ and almost all $t \in I$.

(H4) There exists a function $G \in L^1(I; \mathbb{R}_+)$ such that for any bounded $S \subseteq X$

$$\xi(f(t, S)) \leq G(t)\xi(S)$$

for almost all $t \in I$, where ξ is the Haudorff measure of noncompactness defined in X (see Section 1.4 of Chapter 1).

Remark 4.19. Assuming that a function g satisfies hypothesis (H1) it is clear that g takes bounded sets into bounded sets. For this reason, denote $g_J = \sup\{\|g(u)\| : \|u\|_\infty \leq J\}$ for each $J \geq 0$.

For the following result consider

$$L = \max \left\{ 1, \sup_{t \in I} \{\|S(t)\|\}, \sup_{t \in I} \{\|R(t)\|\}, \sup_{t \in I} \{\|T_a(t)\|\}, \sup_{t \in I} \|U(t)\| \right\}. \quad (4.26)$$

Theorem 4.20. Suppose $0 < a < b$ and $c \in \mathbb{R}$. If hypotheses (H1)-(H4) are satisfied and there exists a constant $J \geq 0$ such that

$$\begin{aligned} J \geq & \left(1 + \frac{2c^2(b^2 + ab - a^2)}{a(a-b)^2} + \frac{c^2}{b} + \frac{2c^4(2b-a)}{a(a-b)^2} \right) Lg_{1J} + \left(1 + \frac{4b(a+c^2)}{a(a-b)} \right) Lg_{2J} \\ & + Lg_{3J} + L\Phi(J) \int_0^1 m(s)ds, \end{aligned}$$

where L is given by (4.26), then the problem (4.25) has at least one mild solution.

Proof. Given $x, y, z \in X$, define $F : C(I; X) \rightarrow C(I; X)$ by

$$\begin{aligned} (Fu)(t) = & e^{-\frac{c^2}{b}t}g_1(u) - \frac{bc^2}{a(a-b)}R(t)g_1(u) + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t T_a(s)g_1(u)ds \\ & + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t R(s)g_1(u)ds + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2} \int_0^t S(s)g_1(u)ds \\ & + \frac{c^2}{b} \int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)g_1(u)ds - \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s S(\tau)g_1(u)d\tau ds \end{aligned}$$

$$\begin{aligned}
& +R(t)g_2(u) - \frac{a+b}{a-b} \int_0^t T_a(s)g_2(u)ds + \frac{a+b}{a-b} \int_0^t S(s)g_2(u)ds \\
& - \frac{c^2(a+b)}{a(a-b)} \int_0^t R(s)g_2(u)ds + \frac{c^2(a+b)}{a(a-b)} \int_0^t \int_0^s S(\tau)g_2(u)d\tau ds \\
& + \int_0^t R(s)g_3(u)ds + \int_0^t U(t-s)f(s, u(s))ds,
\end{aligned}$$

for all $u \in C(I; X)$ and $t \in I$.

Note that F is a continuous map. Indeed, let $\{u_n\}_{n=1}^\infty \subseteq C(I; X)$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$ in the standard norm of $C(I; X)$. Since $0 < a < b$, it follows that $a^2 - ab - b^2 < 0$, and then

$$\begin{aligned}
\|F(u_n) - F(u)\| & \leq \left(1 + \frac{2c^2(b^2 + ab - a^2)}{a(a-b)^2} + \frac{c^2}{b} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) L\|g_1(u_n) - g_1(u)\| \\
& + \left(1 + \frac{2(a+b)(a+c^2)}{a(b-a)}\right) L\|g_2(u_n) - g_2(u)\| + L\|g_3(u_n) - g_3(u)\| \\
& + L \int_0^1 \|f(s, u_n(s)) - f(s, u(s))\| ds \\
& \leq \left(1 + \frac{c^2(-a^2 + 2b^2)}{ab(a-b)^2} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) L\|g_1(u_n) - g_1(u)\| \\
& + \left(1 + \frac{4b(a+c^2)}{a(a-b)}\right) L\|g_2(u_n) - g_2(u)\| + L\|g_3(u_n) - g_3(u)\| \\
& + L \int_0^1 \|f(s, u_n(s)) - f(s, u(s))\| ds.
\end{aligned}$$

By hypotheses (H1)-(H3) and by the dominated convergence theorem it follows that $\|F(u_n) - F(u)\| \rightarrow 0$ when $n \rightarrow \infty$.

Denote $B_J = \{u \in C(I; X) : \|u(t)\| \leq J, \forall t \in I\}$ and note that for any $u \in B_J$ and $t \in I$,

$$\begin{aligned}
\|(Fu)(t)\| & \leq \|e^{-\frac{c^2}{b}t}g_1(u)\| + \frac{bc^2}{a(b-a)}\|R(t)g_1(u)\| + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t \|T_a(s)g_1(u)\| ds \\
& + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \|R(s)g_1(u)\| ds + \frac{c^2(-a^2 + ab + b^2)}{a(a-b)^2} \int_0^t \|S(s)g_1(u)\| ds \\
& + \frac{c^2}{b} \int_0^t |e^{-\frac{c^2}{b}(t-s)}| \|S(s)g_1(u)\| ds - \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s \|S(\tau)g_1(u)\| d\tau ds \\
& + \|R(t)g_2(u)\| + \frac{a+b}{b-a} \int_0^t \|T_a(s)g_2(u)\| ds + \frac{a+b}{b-a} \int_0^t \|S(s)g_2(u)\| ds \\
& + \frac{c^2(a+b)}{a(b-a)} \int_0^t \|R(s)g_2(u)\| ds + \frac{c^2(a+b)}{a(b-a)} \int_0^t \int_0^s \|S(\tau)g_2(u)\| d\tau ds \\
& + \int_0^t \|R(s)g_3(u)\| ds + \int_0^t \int_0^s \|R(s-\tau)f(\tau, u(\tau))\| d\tau ds \\
& \leq \left(1 + \frac{2c^2(b^2 + ab - a^2)}{a(a-b)^2} + \frac{c^2}{b} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) Lg_{1J} + \left(1 + \frac{4b(a+c^2)}{a(a-b)}\right) Lg_{2J} \\
& + Lg_{3J} + L\Phi(J) \int_0^1 m(s)ds \leq J.
\end{aligned}$$

Therefore F maps B_J into itself and $F(B_J)$ is a bounded set. Moreover, by continuity of the functions $t \mapsto R(t)$, $t \mapsto T_a(t)$, $t \mapsto S(t)$ and $t \mapsto U(t)$ on $[0, 1]$, it follows that the set $F(B_J)$ is an equicontinuous set of functions.

Define $\mathcal{B} = \overline{\text{co}}(F(B_J))$ the closed convex hull of the set $F(B_J)$. It follows from Lemma 1.6 that the set \mathcal{B} is equicontinuous. In addition, the operator $F : \mathcal{B} \rightarrow \mathcal{B}$ is continuous and $F(\mathcal{B})$ is a bounded set of functions.

Let $\varepsilon > 0$ be given. For $t \in I$, recall the notation $F(\mathcal{B})(t) = \{v(t); v \in F(\mathcal{B})\}$. By Lemma 1.9 with $W = F(\mathcal{B})$ there exists a sequence $\{v_n\}_{n=1}^\infty \subset F(\mathcal{B})$ such that

$$\xi(F(\mathcal{B})(t)) \leq 2\xi(\{v_n(t)\}_{n=1}^\infty) + \varepsilon$$

and since the functions g_1, g_2, g_3 are compact maps, it follows that

$$2\xi(\{v_n(t)\}_{n=1}^\infty) + \varepsilon \leq 2\xi\left(\int_0^t \{U(t-s)f(s, u_n(s))\}_{n=1}^\infty ds\right) + \varepsilon,$$

with $v_n = F(u_n)$ for some $\{u_n\}_{n=1}^\infty \subset \mathcal{B}$.

By hypothesis (H3), $\|U(t-s)f(s, u_n(s))\| \leq L\Phi(J)m(s)$ for each $t \in I$. And then, from Lemma 1.8

$$2\xi\left(\int_0^t \{U(t-s)f(s, u_n(s))\}_{n=1}^\infty ds\right) \leq 4L \int_0^t \xi(\{f(s, u_n(s))\}_{n=1}^\infty) ds.$$

Therefore, by condition (H4) and Lemma 1.10,

$$\begin{aligned} \xi(F(\mathcal{B}(t))) &\leq 4L \int_0^t \xi(\{f(s, u_n(s))\}_{n=1}^\infty) ds + \varepsilon \leq 4L \int_0^t G(s) \xi(\{u_n(s)\}_{n=1}^\infty) ds + \varepsilon \\ &\leq 4L\gamma(\mathcal{B}) \int_0^t G(s) ds + \varepsilon, \end{aligned}$$

where γ denotes the Hausdorff measure of noncompactness on $C(I; X)$.

By hypothesis (H4), $G \in L^1(I; \mathbb{R}_+)$. Then for $\kappa < \frac{1}{4L}$ there exists $\varphi \in C(I; \mathbb{R}_+)$ satisfying $\int_0^1 |G(s) - \varphi(s)| ds < \alpha$, where $\alpha = 4\kappa L$. Hence,

$$\begin{aligned} \xi(F(\mathcal{B})(t)) &\leq 4L\gamma(\mathcal{B}) \left[\int_0^t |G(s) - \varphi(s)| ds + \int_0^t \varphi(s) ds \right] + \varepsilon \\ &\leq 4L\gamma(\mathcal{B})[\kappa + Nt] + \varepsilon, \end{aligned}$$

where $N = \|\varphi\|_\infty$. Since $\varepsilon > 0$ is arbitrary,

$$\xi(F(\mathcal{B})(t)) \leq (\alpha + \beta t)\gamma(\mathcal{B}), \quad \text{where } \beta = 4LN. \quad (4.27)$$

Let $\varepsilon > 0$ be given. Since the functions g_1, g_2, g_3 are compact maps and applying Lemma 1.9 there exists a sequence $\{w_n\}_{n=1}^\infty \subseteq \overline{\text{co}}(F(\mathcal{B}))$ such that

$$\begin{aligned} \xi(F^2(\mathcal{B})(t)) &\leq 2\xi\left(\int_0^t \{U(t-s)f(s, w_n(s))\}_{n=1}^\infty ds\right) + \varepsilon \\ &\leq 4L \int_0^t \xi(\{f(s, w_n(s))\}_{n=1}^\infty) ds + \varepsilon \end{aligned}$$

$$\leq 4L \int_0^t G(s) \xi(\overline{c\bar{o}}(F(\mathcal{B})(s))) ds + \varepsilon$$

and by item (iii) of Lemma 1.5,

$$\xi(F^2(\mathcal{B})(t)) \leq 4L \int_0^t G(s) \xi(F(\mathcal{B})(s)) ds + \varepsilon.$$

Using the inequality (4.27) it follows that

$$\begin{aligned} \xi(F^2(\mathcal{B})(t)) &\leq 4L \int_0^t G(s) (\alpha + \beta s) \gamma(\mathcal{B}) ds + \varepsilon \\ &\leq 4L \int_0^t [|G(s) - \varphi(s)| + |\varphi(s)|] (\alpha + \beta s) \gamma(\mathcal{B}) ds + \varepsilon \\ &\leq 4L(\alpha + \beta t) \gamma(\mathcal{B}) \int_0^t |G(s) - \varphi(s)| ds + 4LN \gamma(\mathcal{B}) \left(\alpha t + \frac{\beta t^2}{2} \right) + \varepsilon \\ &\leq \left[\alpha(\alpha + \beta t) + \beta \left(\alpha t + \frac{\beta t^2}{2} \right) \right] \gamma(\mathcal{B}) + \varepsilon \\ &\leq \left(\alpha^2 + 2\beta t + \frac{(\beta t)^2}{2} \right) \gamma(\mathcal{B}) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\xi(F^2(\mathcal{B})(t)) \leq \left(\alpha^2 + 2\beta t + \frac{(\beta t)^2}{2} \right) \gamma(\mathcal{B}).$$

By an inductive process, for all $n \in \mathbb{N}$, it holds

$$\xi(F^n(\mathcal{B})(t)) \leq \left(\alpha^n + C_1^n \alpha^{n-1} \beta t + C_2^n \alpha^{n-2} \frac{(\beta t)^2}{2!} + \dots + \frac{(\beta t)^n}{n!} \right) \gamma(\mathcal{B}),$$

where, for $0 \leq m \leq n$, the symbol C_m^n denotes the binomial coefficient $\binom{n}{m}$.

In addition, for all $n \in \mathbb{N}$ the set $F^n(\mathcal{B})$ is an equicontinuous set of functions. Therefore, using the Lemma 1.7 it is possible to conclude that

$$\gamma(F^n(\mathcal{B})) \leq \left(\alpha^n + C_1^n \alpha^{n-1} \beta + C_2^n \alpha^{n-2} \frac{\beta^2}{2!} + \dots + \frac{\beta^n}{n!} \right) \gamma(\mathcal{B}).$$

Since $0 \leq \alpha < 1$ and $\beta > 0$, it follows from Lemma 1.11 that there exists $n_0 \in \mathbb{N}$ such that

$$\left(\alpha^{n_0} + C_1^{n_0} \alpha^{n_0-1} \beta + C_2^{n_0} \alpha^{n_0-2} \frac{\beta^2}{2!} + \dots + \frac{\beta^{n_0}}{n_0!} \right) = r < 1.$$

Consequently, $\gamma(F^{n_0}(\mathcal{B})) \leq r \gamma(\mathcal{B})$. It follows from Lemma 1.10 that F has a fixed point in \mathcal{B} , and this fixed point is a mild solution of (4.25). □

Bibliography

- [1] ABADÍAS, L., LIZAMA, C. and MURILLO, M. *Hölder regularity for the Moore-Gibson-Thompson equation with infinite delay*. Comm. Pure Appl. Anal., 17 (1): 243 – 265, 2018.
- [2] AIZICOVICI, S. AND STAICU, V. *Multivalued evolution equations with nonlocal initial conditions in Banach spaces*. Nonlinear Differ. Equ. Appl., 14 (3 – 4): 361 – 376, 2007.
- [3] AKHMEROV, R. R., KAMENSKII, M. I., POTAPOV, A. S., RODKINA, A. E. AND SADOVSKII, B. N. *Measures of noncompactness and Condensing Operators*. Springer, Basel, 1992.
- [4] ARAYA, D. AND LIZAMA, C. *Almost automorphic mild solutions to fractional differential equations*. Nonlinear Anal., 69 (11): 3692 – 3705, 2008.
- [5] ARENDT, W. *A $0 - 3/2$ - Law for Cosine Functions*. Ulmer Seminare, Funktionalanalysis und Evolutionsgleichungen, 17: 349 – 350, 2012.
- [6] ARENDT, W., BATTY, C. J. K., HIEBER, M. AND NEUBRANDER, F. *Vector-valued Laplace transforms and Cauchy problems*, 2nd ed. Birkhäuser/Springer Basel AG, Basel, 2011.
- [7] BANAŚ, J. AND GOEBEL, K. *Measures of noncompactness in Banach spaces (Lecture Notes in Pure and Applied Mathematics 60)*. Marcel Dekker, New York, 1980.
- [8] BANAŚ, J., JLELI, M., MURSALEEN, M., SAMET, B. AND VETRO, C. *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*. Springer, Singapore, 2017.
- [9] BAZHLEKOVA, E. G. *Fractional Evolution Equations in Banach Spaces*, PhD thesis. Eindhoven University of Technology, 2001.
- [10] BOBROWSKI, A. *On Wallen-type formulae for integrated semigroups and sine functions*. Arch. Math., 108 (4): 395 – 404, 2017.
- [11] BOBROWSKI, A. AND CHOJNACKI, W. *Isolated points of some sets of bounded cosine families, bounded semigroups, and bounded groups on a Banach space*. Studia Math., 217 (3): 219 – 241, 2013.
- [12] BOBROWSKI, A., CHOJNACKI, W. AND GREGOSIEWICZ, A. *On close-to-scalar one-parameter cosine families*. J. Math. Anal. Appl., 429 (1): 383 – 394, 2015.

-
- [13] BOTHE, D. *Multivalued perturbation of m -accretive differential inclusions*. Israel J. Math., 108: 109 – 138, 1998.
- [14] BREZIS, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York, 2010.
- [15] BRUNNHUBER, R. AND KALTENBACHER, B. *Well-posedness and asymptotic behaviour of solutions for the Blackstock-Crighton-Westervelt equation*. Discrete Contin. Dyn. Syst., 34 (11): 4515 – 4535, 2014.
- [16] BRUNNHUBER, R. AND MEYER, S. *Optimal regularity and exponential stability for the Blackstock-Crighton equations in L_p -spaces with Dirichlet and Neumann boundary conditions*. J. Evol. Equ., 16: 945 – 981, 2016.
- [17] BYSZEWSKIK, L. *Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation $u_{xt} = F(x, t, u, u_x)$* . J. Appl. Math. Stochastic Anal., 3.3: 163 – 168, 1990.
- [18] CELIK, A. AND KYED, M. *Nonlinear acoustics: Blackstock-Crighton equations with a periodic forcing term*. J. Math. Fluid Mech., 21 (3): 21 – 45, 2019.
- [19] CHEN, C. AND LI, M. *On fractional resolvent operator functions*, Semigroup Forum, 80: 121 – 142, 2010.
- [20] CHOJNACKI, W. *On cosine families close to scalar cosine families*. J. Aust. Math. Soc., 99 (2): 166 – 174, 2015.
- [21] CHOJNACKI, W. *Around Schwenninger and Zwart's zero-two law for cosine families*. Arch. Math., 106 (6): 561 – 571, 2016.
- [22] CONEJERO, A., LIZAMA, C., MURILLO, M. AND SEOANE-SEPÚLVEDA, J. B. *Well posedness for degenerate third order equations with delay and applications to inverse problems*. Israel J. Math., 229 (1): 219 – 254, 2019.
- [23] DENG, K. *Exponential decay of solutions of semilinear parabolic equations with non local initial conditions*. J. Math. Anal. Appl., 179: 630 – 637, 1993.
- [24] ENGEL, K-J. AND NAGEL, R. *One-Parameter Semigroups for Linear Evolution Equations*. Springer, Berlin, 1995.
- [25] ERDELYI, A. *Higher Transcendental Functions Vol. III*, McGraw-Hill, New York, 1953.
- [26] ESTERLE, J. *Bounded cosine functions close to continuous scalar bounded cosine functions*. Integral Equations Operator Theory, 85 (3): 347 – 357, 2016.
- [27] ESTERLE, J. *A short proof of the zero-two law for cosine functions*. Arch. Math. (Basel), 105 (4): 381 – 387, 2015.
- [28] FATTORINI, H. O. *Second order linear differential equations in Banach spaces*. Elsevier, Amsterdam, 1985.

-
- [29] GAMBERA, L. R., LIZAMA, C. AND PROKOPCZYK, A. *On close to scalar families for fractional evolution equations: zero-one law*. Semigroup Forum, 99 (1): 140 – 152, 2019.
- [30] GAMBERA, L. R., LIZAMA, C. AND PROKOPCZYK, A. *Well-posedness for the abstract Blackstock-Crighton-Westervelt equation*, to appear in J. Evol. Equ..
- [31] GORENFLO, R., LOUTCHKO, J. AND LUCHKO, Y. *Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivatives*. Fract. Calc. Appl. Anal., 5 (4): 491 – 518, 2002.
- [32] GOU, H. AND LI, B. *Existence of mild solutions for fractional nonautonomous evolution equations of Sobolev type with delay*. J. Inequal. Appl., 252: 1 – 20, 2017.
- [33] GRADSHTEYN, I. S. AND RYZHIK, I. M. *Table of Integrals, Series and Products*, 7th ed. Elsevier, Amsterdam, 2007.
- [34] GUO, F., LIU, S., WU, C. AND WU, Y. *Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces*. J. Math. Anal. Appl., 309 (2): 638 – 649, 2005.
- [35] KALTENBACHER, B. *Mathematics in nonlinear acoustics*. Evol. Equ. Control Theory, 4 (4): 447 – 491, 2015.
- [36] KALTENBACHER, B. AND LASIECKA, I. *Global existence and exponential decay rates for the Westervelt equation*. Discr. Cont. Dyn. Syst. Ser. S, 2 (3): 503 – 523, 2009.
- [37] KALTENBACHER, B. AND LASIECKA, I. *An analysis of nonhomogeneous Kuznetsov's equation: local and global well-posedness; exponential decay*. Math. Nachr., 285 (2-3): 295 – 321, 2012.
- [38] KALTENBACHER, B., LASIECKA, I. AND MARCHAND, R. *Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound*. Control Cybernet, 4 (4): 971 – 988, 2011.
- [39] KALTENBACHER, B., LASIECKA, I. AND POSPIESZALSKA, M. K. *Well-posedness and exponential decay of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound*. Math. Models Methods Appl. Sci., 22 (11), 2012.
- [40] KEYANTUO, V., LIZAMA, C. AND WARMA, M. *Existence, regularity and representation of solutions of time fractional diffusion equations*. Adv. Differential Equations, 21 (9): 837 - 886, 2016.
- [41] KEYANTUO, V., LIZAMA, C. AND WARMA, M. *Existence, regularity and representation of solutions of time fractional wave equations*. Electron. J. Differential Equations, 222: 1 – 42, 2017.
- [42] LASIECKA, I. AND WANG, X. *Moore-Gibson-Thompson equation with memory, part II: General decay of energy*. J. Diff. Equ. , 259 (12): 7610 – 7635, 2015.
- [43] LI, K. AND PENG, J. *Fractional resolvents and fractional evolution equations*. Appl. Math. Lett., 25 (5): 808 – 812, 2012.

-
- [44] LI, K., PENG, J. AND JIA, J. *Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives*. J. Funct. Anal., 263: 476 – 510, 2012.
- [45] LI, M., CHEN, C. AND LI, F. B. *On fractional powers of generators of fractional resolvent families*. J. Funct. Anal., 259 (10): 2702 – 2726, 2010.
- [46] LIZAMA, C., *On the convergence and approximation of integrated semigroups*. J. Math. Anal. Appl., 181: 89 – 103, 1994.
- [47] LIZAMA, C. *Regularized solutions for abstract Volterra equations*. J. Math. Anal. Appl., 243 (2): 278 – 292, 2000.
- [48] LIZAMA, C. AND POBLETE, F. *On a functional equation associated with (a, k) -regularized resolvent families.*, Abstr. Appl. Anal., 2012 (2), 2012.
- [49] LIZAMA, C. AND PRADO, H. *On duality and spectral properties of (a, k) -regularized resolvents*. Proc. Roy. Soc. Edinburgh Sect. A, 139: 507 – 517, 2009.
- [50] MAINARDI, F. AND PARADISI, P. *Fractional diffusive waves*. J. Comput. Acoust., 9 (4): 1417 – 1436, 2001.
- [51] MARTÍNEZ, C. AND SANZ, M. *The Theory of Fractional Powers of Operators*. Elsevier, North Holland, 2001.
- [52] MEI, Z. D., PENG, J. G. AND GAO, J. H. *General fractional differential equations of order $\alpha \in (1, 2)$ and type $\beta \in [0, 1]$ in Banach spaces*. Semigroup Forum, 94 (3): 712 – 737, 2017.
- [53] NAGY, B. *On cosine operator functions in Banach spaces*. Acta Sci. Math., 36 (3): 281 – 290, 1973.
- [54] NIGMATULLIN, R. R. *The realization of the generalized transfer equation in a medium with fractal geometry*. Physica Status Solidi (b), 133 (1): 425 – 430, 1986.
- [55] PAZY, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New York, 1983.
- [56] PECOTCHE, C. B. G. *Logosofía, Ciencia y Método*. Editorial Logosófica, Ciudad Autónoma de Buenos Aires, 1957.
- [57] PODLUBNY, I. *Fractional Differential Equations*. Academic Press, San Diego, 1999.
- [58] PRÜSS, J. *Evolutionary Integral Equations and Applications*. Birkhäuser Verlag, Switzerland, 1993.
- [59] RHANDI, A. *Dyson-Phillips expansion and unbounded perturbations of linear C_0 -semigroups*. J. Comput. Appl. Math., 44: 339 – 349, 1992.
- [60] RHANDI, A. *Positive perturbations of linear Volterra equations and sine functions of operators*. J. Integral Equations Appl., 4: 409 – 420, 1992.
- [61] ROYDEN, H. L. AND FITZPATRICK, P. M. *Real Analysis*, 4th e. Prentice Hall, 2011.
- [62] SCHNEIDER, W. R. *Completely monotone generalized Mittag-Leffler functions*. Expo. Math., 14 (1): 3 – 16, 1996.

-
- [63] SCHWENNINGER, F. L. AND ZWART, H. *Less than one implies zero*. *Studia Math.*, 229 (2): 181 – 188, 2015.
- [64] SCHWENNINGER, F. L. AND ZWART, H. *Zero-two law for cosine families*. *J. Evol. Equ.*, 15: 559 – 569, 2015.
- [65] STAFFANS, O. *Well-Posed Linear Systems (Encyclopedia of Mathematics and its Applications 103)*. Cambridge University Press, Cambridge, 2005.
- [66] VRABIE, I. I. *Existence in the large for nonlinear delay evolution inclusions with non-local initial conditions*. *J. Funct. Anal.*, 262 (4): 1363 – 1391, 2012.
- [67] WANG, R. N. AND P.X. ZHU, P. X. *Non-autonomous evolution inclusions with non-local history conditions: global integral solutions*. *Nonlinear Anal.*, 85: 180 – 191, 2013.
- [68] ZHOU, Y., WANG, R-N. AND PENG, L. *Topological Structure of the Solution Set for Evolution Inclusions*. Springer, Singapore, 2017.