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Existence and qualitative properties of solutions for abstract problems with nonlocal operators

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Trabajo para optar al grado de Doctora en Ciencia con Mención en Matemáticas

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Trabajo de Titulación presentado a la Facultad de Ciencia, en cumplimiento parcial de los requisitos exigidos para optar al grado de **Doctora en Ciencia con Mención en Matemáticas**. Este trabajo de graduación fue presentado bajo la supervisión del profesor guía Dr. Carlos Lizama Yáñez, y los profesores supervisores del trabajo de tesis Dr. Rodrigo Ponce de la Universidad de Talca, Dr. Alberto Conejero de la Universidad Politécnica de Valencia, Dra. Marina Murillo de la Universidad de Cádiz, y Dr. Edgardo Alvarez de la Universidad del Norte.

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Dedicated to
Zoe Bravo

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Contents

Introduction

Fractional calculus in continuous and discrete time has been studied for several decades. Fractional derivatives in continuous time have been defined by numerous authors, based on some speculations by Leibniz and L' Hôpital. Important contributions originates from Laplace, Fourier, Abel, Liouville, Riemann, Caputo, Grünwald, Letnikov and Fabrizio, among many others.

In the discrete case, fractional powers of the forward difference operator Δ have been defined since many years ago. In 1956, Kutter [70] mentioned for the first time differences of fractional order. In 1974, Diaz and Osler [33] introduced a discrete fractional operator defined as an infinite series. In 1988, Grey and Zhang [57] developed a fractional calculus for the discrete nabla (backward) operator. Miller and Ross [78] defined a fractional sum via the solution of a linear difference equation. Their definition is the discrete analogue of the Riemann-Liouville fractional integral, which can be obtained via the solution of a linear differential equation. In 2007, Atici and Eloe [6, 7, 8] introduced the Riemann-Liouville like fractional difference by using the definition of a fractional sum of Miller and Ross, and developed some of its properties that allow one to obtain solutions of certain fractional difference equations. In 2010, Anastassiou [3] defined the Caputo like fractional difference by using also the notion of a fractional sum from Miller and Ross. At the same year, Ferreira [42] introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations. See also Sengul [86] for related work. In 2011, Holm [61] further developed and applied the tools of discrete fractional calculus to the arena of fractional difference equations [96, 97].

One of the more interesting branches of fractional calculus is the theory of fractional evolution equations. They arise naturally in the mathematical modeling of phenomena in natural sciences. The main objective of this thesis is the study of existence, uniqueness and other qualitative properties of solutions for some evolution equations involving continuous and discrete fractional models.

This thesis is composed of four themed chapters. In the following paragraphs, we provide a brief description of each chapter:

In Chapter 1, we present part of the notation, concepts and the preliminary results that will be necessary throughout the work.

Chapter 2 is developed in the area of discrete fractional calculus. We are interested in the connection between the sign of the discrete fractional operator $\Delta^\alpha u$ and the geometry of the sequence u on which it acts. For example, it is well known that if $\Delta u(n) \geq 0$ for $n \in \mathbb{N}_0$ implies that u is increasing on $\mathbb{N}_0$, where $\Delta u(n) := u(n+1) - u(n)$ is the forward difference operator. But, What can we say about the sequence u if the sign of operator $\Delta^\alpha u$, with $2 \leq \alpha < 4$, is known?

There is an extensive theory in continuous calculus that provides us with different qualitative properties of the solutions of an equation. The discrete analogue is important for the numerical analysis of nonautonomous and nonlinear fractional evolution problems, using Euler's method, where the entire order derivative may be approximated. From a numerical point of view, the discrete fractional operator Δ^α - as defined in [73] - approximates the Riemann-Liouville fractional order operator and coincides with the generalized forward Grünwald-Letnikov derivative [81], defined by

$$\Delta^\alpha u(n) := \Delta^m \sum_{j=0}^{n} k^{m-\alpha}(j)u(j), \quad n \in \mathbb{N}_0,$$

where $m-1 < \alpha < m,\ m \in \mathbb{N}$, $k^\alpha(n) := \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, n \in \mathbb{N}_0$, Γ denotes the Gamma function and for $m \in \mathbb{N}$

$$\Delta^m u(n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} u(n+j), \quad n \in \mathbb{N}_0.$$

We point out that the discrete fractional operator Δ^α coincides up to translation with the more studied definition of discrete fractional operator Δ_a^ν given by Atici and Eloe in 2007, namely

$$\left(\Delta_a^\nu f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),\ t \in \mathbb{N}_{a+N-\nu},$$

where $f \in s(\mathbb{N}_a;\mathbb{R})$, $N \in \mathbb{N}_1$ is the unique integer satisfying $N-1 < \nu < N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. This property, called transference principle, has been proved recently by Goodrich and Lizama [50].

In 2014, the following question was addressed by Dahal and Goodrich [29, 30]:

(P) Is there a connection between the sign of the discrete fractional operator $\Delta^\alpha u$, and the positivity, monotonicity and convexity of the sequence u on which it acts?

While it is trivial to prove that $\Delta u(n) \geq 0$ for $n \in \mathbb{N}_0$ implies that u is increasing on $\mathbb{N}_0$, it is very nontrivial to decide how monotonicity is connected to the positivity or negativity of the discrete fractional operator. Similarly, while it is equally trivial to prove that $\Delta^2 u(n) \geq 0$ for $n \in \mathbb{N}_0$ implies that u is increasing on $\mathbb{N}_0$ and thus that u satisfies a convexity-type property, the analogue of this sort of result in the discrete fractional setting is much more difficult to obtain. This is a very nontrivial program due to the inherent nonlocal nature of the fractional operator, a fact that causes great difficulty when trying to equip the operator with some reasonable geometrical meaning.

In case $2 \leq \alpha < 3$ convexity results for discrete fractional operators as well applications to fractional boundary value problems were studied in [46] and then reviewed in the monograph [47, Section 7.3] by Goodrich and Peterson. We point out that due to a flurry of recent work in the area, the basic convexity and concavity results presented in [47] have been substantively extended in a variety of directions. See, for instance [15, 31, 41, 48, 49, 50, 51] and [88].

Recently, in the reference [50] the authors studied the problem (P) in case $2 \leq \alpha < 3$ and found that if $\Delta^\alpha u(n) \geq 0$ for all $n \in \mathbb{N}_0$ and $u(0) \geq 0, u(1) \geq \alpha u(0), u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$ then u is convex. See [50, Theorem 7.1]. This result has been recently refined in [17] where it was proved that u should be positive and nondecreasing, too.

To understand the behavior between monotonicity and convexity, the concept of α-convex sequence was defined in [17, Definition 6.1]. It refers to the continuous behavior (with respect to the parameter α) between convex and non-decreasing sequences. Roughly speaking, an α-convex sequence should be placed geometrically above sequences of the form $\beta a^n + b$ where $\beta, b \in \mathbb{R}, a > 0$, see [17]. By means of this concept, the connection between the sign of the operator $\Delta^\alpha u$ and the convexity of the sequence u has been studied.

However, even though many authors have studied various generalizations of convexity, see e.g. [68] and its references, as far as we know none of these address the problem of what is between the second and third order powers of the forward difference operator Δ and thus remained an open problem.

On the other hand, we notice that nonlinear equations involving the third temporal derivative, have been widely studied [27, 54, 55, 72, 76, 84, 90]. Consider the third-order differential equation

$$x'''(t) = J(t, x(t), x'(t), x''(t)), \quad x(0) = x_0, \quad x'(0) = x_1, \quad x''(0) = x_2, \quad t \geq 0.$$

The jerk term $x'''(t)$ with $x(t)$ being the displacement appear in the Abraham-Lorentz equation [62], describing the motion of a radiating charged particle. On the other hand, it is known that the requirement for the occurrence of chaos of a nonlinear autonomous system is at least the third-order temporal derivative being involved, like the nonlinear jerk equations [53, 89]. In the last time, third-order differential equations appeared in a variety of dissimilar areas such as elastically deformable matter, in the geometric design of roads and tracks, in motion control and in manufacturing processes. Therefore, the study of nonlinear jerk equations is an interesting issue that deserves to be investigated.

Recently, in the papers [44, 45] the authors show that the jerk dynamics are naturally obtained for electrical circuits using the fractional calculus approach, i.e. replacing the third order derivative x''' by a fractional order operator D_t^α with order $2 < \alpha < 4$. The electrical circuits studied in such papers and their respective analogue mechanical system can be used to analyze the vibration levels of machinery, serial mechanisms, robotics, oscillating circuits modeling, and instability of electrical and mechanical circuits, to evaluate reconfigurable machines or to make mobility analysis or algebraic formulations of motion equations [44].

The previous studies detailed above allow us to better understand the discrete case. Therefore, in the first section of this chapter, we provide for the first time a definition of α-jerk sequence. This notion interpolate between the concepts of convex and positive jerk sequence, where the latter refers to sequences that verify $\Delta^3 u(n) \geq 0$. This concept allows us to answer the problem (P) about the connection between the sign of the operator $\Delta^\alpha u$, when $2 \leq \alpha < 4$ and the property of the sequence u on which it acts. It is worthwhile to mention that this new concept not only allows a continuous transition between the geometry of the sequence u as α moves from 2 to 4 but also a continuous transition between the previous results existing in the literature and ours. After providing our definition of positive α-jerk sequence, we realize that its graph must be placed above sequences of the form $\beta a^n + bn + c$. This is shown in Proposition 2.1.2.

In the next section of this chapter, our main results are proved and, by means of the transference principle, the analogous results corresponding to the operator Δ_a^ν are established. Two additional examples are given. One example refers to the optimality of condition (2) in Theorem 2.2.1 to guarantee the convexity of the sequence u. See the example 2.2.2. The second refers to the optimality of the condition (2) in Theorem 2.2.4 to ensure that the sequence u has positive jerk. All these results provide new insights and propose original concepts to better understand the qualitative behavior of discrete fractional operators in this challenging area of study.

In Chapter 3, we are interested in the problem of existence of periodic solutions of the equation

$$\begin{cases} {}_{GL}D_t^\alpha u(t) + Au(t) &= f(t), \quad t \in (0, 2\pi); \\ \\ u(0) &= u(2\pi), \end{cases} \tag{0.0.1}$$

where A is a closed linear operator defined in a Banach space X and ${}_{GL}D^\alpha$ denotes the Grünwald-Letnikov derivative. Existence of periodic solutions for differential equations of fractional order is a very desirable property for analyzing cyclic (e.g. biological) processes, see [95]. In recent years many papers have appeared on this topic [4, 16, 60, 80] and there are different methods that allow periodic solutions, the Fourier transform being the most common. On the other hand, it is well known that we cannot expect the existence of periodic solutions in time-invariant systems with each definition of fractional order derivative, see e.g. [82, 92, 93].

Regarding the fractional abstract Cauchy problem (0.0.1) the authors in [18, 74] have used a method based on *operator-valued* Fourier multipliers to obtain existence of periodic solutions, in several senses. With this method they have obtained solutions belonging to the periodic Lebesgue space $L^p_{2\pi}(\mathbb{R}, X)$, where X is a Banach space. More precisely, assuming that $1 < \alpha \leq 2$ and X satisfy a geometrical hypothesis, in [74, Theorem 3.15] the authors show that for all $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists a unique $u \in H^{\alpha,p}_{2\pi}(\mathbb{R}, X) \cap L^p_{2\pi}(\mathbb{R}, D(A))$ satisfying (0.0.1) if and only if $\{(im)^\alpha\}_{m\in\mathbb{Z}} \subseteq \rho(A)$ and the set $\{(im)^\alpha(A + (im)^\alpha I)^{-1}\}_{m\in\mathbb{Z}}$ is Rademacher bounded (or R-bounded). See also [18, Theorem 3.3] for the analog result in the case $0 < \alpha \leq 1$ and [20, 21, 22, 63] for extensions of this result to more general models.

However, this method has disadvantages in concrete applications because they require checking the R-boundedness condition on the operator valued-symbol. An additional problem of the characterization cited above is the implicit requirement that $0 \in \rho(A)$ which restricts the applicability of the result. For instance, the case where A is the Laplacian operator defined on unbounded domains cannot be considered by the above characterization.

To avoid these difficulties, some authors [19] have proposed the sum method [28] that was first introduced by Da Prato and Grisvard in the context of sectorial operators. The main idea is to transform the problem into the closedness of the sum of two closed operators related to (0.0.1). However, although the R-boundedness condition is unnecessary with this method, it only allows to establish the existence of periodic solutions in some proper subspaces of $L^p_{2\pi}(\mathbb{R}, X)$, see [19, Theorem 1].

In this chapter we will take a different approach. Starting from the observation that periodic solutions of systems are usually represented by series formed by a set of functions, the present chapter introduces a novel concept of solution that implies the formal representation of the solution by means of normally convergent series. This new concept is general enough to admit periodic forcing terms in the space $L^p_{2\pi}(\mathbb{R}, X)$ without assuming any geometrical conditions in X or R-boundedness of the operator-valued symbol. It allows also avoid assuming a fortiori that $0 \in \rho(A)$. We note that Haraux [59, Chapter B, I] gave a similar approach in the case that X is a Hilbert space, which has been the main motivation in this work.

Using this approach, we can capture the minimum requirements that are needed in a system of 2π-periodic solutions of (0.0.1) in the sense that any solution of (0.0.1) can be represented by a normally convergent series formed by functions of the following set

$$\{u_m e^{imt}\}_{|m|\leq t_0} \cup \{(A+(im)^\alpha I)^{-1}\widehat{f}_m e^{imt}\}_{|m|>t_0},$$

where we assume $\{(im)^\alpha\}_{|m|>t_0} \subseteq \rho(A)$ and $\hat{f}_m = ((A+(im)^\alpha I)^{-1})u_m$ for $|m| \leq t_0$ where $\widehat{f}_m$ are the Fourier coefficients of f. It is notable that the exact value of t_0 can be determined explicitly in some important examples. For example, assuming that $A = B + C$ where C is a bounded operator we consider two situations of interest. First, when X is a Hilbert space and B is self-adjoint, then we have $t_0 = \frac{2^{1/\alpha}||C||^{1/\alpha}}{|\sin(\alpha\pi/2)|^{1/\alpha}}$. Second, when X is a Banach space and B is sectorial with spectral angle $0 < \phi < \alpha\pi/2$, then we have $t_0 = 2^{1/\alpha}\|C\|^{1/\alpha}$. We will shown the validity of this criteria for all pairs (p, α) belonging to the sector

$$\left\{ p \in (1,2] : \frac{1}{p} < \alpha \leq \frac{2}{p} \right\}.$$

The organization of this chapter is the following: In the first section, we introduce the notion of *normal periodic solution* and contain our main result (Theorem 3.1.2). In section 2, two important consequences are shown. The first takes the additive perturbation of the operator A in (0.0.1) as the sum of a selfadjoint operator defined in a Hilbert space and a bounded linear operator. The second takes the additive perturbation again but now in the scenario of a Banach space. In this case, we assume that A can be represented as a sum of a sectorial operator with an angle depending on the fractional parameter α, and a bounded linear operator. In both cases, we can guarantee existence of normal 2π-periodic solutions.

In chapter 4, our concern is the study of existence, uniqueness and qualitative properties for the solutions of the abstract Cauchy problem

$$_{CF}D_t^\alpha u(t) = Au(t) + f(t), \quad t \geq 0, \tag{0.0.2}$$

and semilinear versions of it, i.e. where the term $f(\cdot)$ is replaced by $f(\cdot, u(\cdot))$. In the equation (0.0.2) A is a closed linear operator with domain $D(A)$ defined in a Banach space X and $_{CF}D_t^\alpha$ denotes the Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$.

In 2015, the authors Caputo and Fabrizio proposed a new concept of fractional derivative with a regular kernel [23]. This concept has proven to have valuable properties that make it very useful in various areas of science and engineering (see [1, 2, 10, 11, 12, 13, 38, 39, 71, 75, 79, 91]).

For example, in [1], Abbas, Benchohra and Nieto, provided sufficient conditions to ensure the existence of solutions for functional fractional differential equations with instantaneous impulses, involving the Caputo-Fabrizio derivative. As methods, they used fixed point theory and measure of noncompactness. In [10], Baleanu, Jajarmi, Mohammadi and Rezapour proposed a new fractional model for the human liver involving the Caputo-Fabrizio derivative. In such paper, comparative results with real clinical data indicated the superiority of the new fractional model over the pre-existing integer order model with ordinary time derivatives. A similar study carried out by the aforementioned authors, but for the Rubella disease model, was performed in reference [11], while in [12] the analysis was performed in terms of a differential equation model for the COVID-19. In the paper [14], Baleanu, Sajjadi, Jajarmi and Defterli analyzed the complicated behaviors of a nonlinear suspension system in the framework of the Caputo-Fabrizio derivative. They show that both the chaotic and nonchaotic behaviors of the considered system can be identified by the fractional order mathematical model. Very recently, in the reference [69], Kumar, Das and Ong analyzed tumor cells in the absence and presence of chemotherapeutic treatment by use of the Caputo-Fabrizio derivative. This is one of the few studies, together with the references [25, 91], where presence of partial differential equations with the Caputo-Fabrizio derivative over time is considered.

Although this notion of fractional derivative appears to be very auspicious in a variety of concrete applications, so far an unified analysis in the context of abstract partial differential equations, where there is a wider range of mathematical models, remains undeveloped. In this context, one of the basic problems to be studied corresponds to the so-called abstract Cauchy problem.

One of the motivations for this study is that, to our knowledge, similar work has not been done before in abstract spaces with Caputo-Fabrizio or other fractional derivatives that have *non-singular* kernels. Our goal is to clarify to what extent this type of fractional derivative offers advantages/disadvantages in this abstract scenario.

In the existing literature, the problem (0.0.2) has been studied when A is scalar or even a matrix, but when A is simply a closed linear operator, e.g. partial differential operators like the Laplacian, the problem (0.0.2) remains unsolved.

In the border case $\alpha = 1$, it is well-known that solving the linear problem

$$u'(t) = Au(t) + f(t), \quad t \geq 0, \tag{0.0.3}$$

requires A as the *generator of a C_0-semigroup.*

In the first section of this chapter, we will show that *this requirement is not more necessary to solve* (0.0.2). Which brings us to our main result: solving the problem (0.0.2) is equivalent to solving the following problem:

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \quad t \geq 0; \\ u(0) = -A^{-1}f(0), \end{cases} \tag{0.0.4}$$

where B_α are *bounded linear operators* that behaves like a *Yosida approximation* of A, being $B_\alpha \to A$ and $F_\alpha(t) \to f(t)$ as $\alpha \to 1$, in an appropriate sense. In this way, some qualitative properties for (0.0.2) could be directly deduced from the corresponding ones of (0.0.4) with due care given to the special initial condition $u(0) = -A^{-1}f(0)$ that appears in our new context.

In section 2, we study the important issue of stability. We show that under a simple condition, which depends on α, about the location of the spectrum of the operator A, and a decay condition on f, we can conclude that the unique solution u of the *nonhomogenous* equation (0.0.2) satisfies $\|u(t)\| \to 0$ as $t \to \infty$. A concrete example is shown that illustrates this asymptotic behavior and how the connection between (0.0.2) and (0.0.4) works.

In the next section, if A is a closed linear operator, we show existence and uniqueness of mild solutions for the nonlinear equation

$$_{CF}D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad t \in [0, T], \quad T > 0, \tag{0.0.5}$$

under a Lipschitz type condition on f that also depends on α. In particular, assuming that A is densely defined, we realize that as $\alpha \to 1$ our result matches a classical result for the equation (0.0.3) stated in [83, Theorem 6.1.2], where the condition for A to be the generator of a C_0 semigroup appears. Our studies reveal that this condition turns out to be natural thanks to the property $B_\alpha \to A$ as a Yosida approximation, before mentioned.

It should be noted that one of the keys that was taken into account to carry out this work is that the Caputo-Fabrizio fractional derivative has a non-singular kernel. Therefore, it is natural to ask - and we leave it as an open problem - in what extent the results of this thesis could be reproduced if the Caputo-Fabrizio derivative is replaced by another type of fractional derivatives with non-singular kernel. For example, there are fractional time derivatives by the use of Gaussian kernels [24, Section 8], or Mittag-Leffler kernels [5], the last also known as the Atangana-Baleanu-Caputo derivative.

The results described in Chapters 2, 3 and 4 have been published in mainstream international journals (ISI):

1. J. Bravo, C. Lizama, S. Rueda. Second and third order forward difference operator: What is in between? *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas (RACSAM),* 115(2) (2021), 86.
2. J. Bravo and C. Lizama. Normal periodic solutions for the fractional abstract Cauchy problem. *Boundary Value Problems,* 2021(53) (2021).
3. J. Bravo and C. Lizama. The abstract Cauchy problem with Caputo-Fabrizio fractional derivative. *Mathematics,* 10(19) (2022), 3540.

Chapter 1

Preliminaries

The purpose of this chapter is to introduce certain notations, definitions and theorems used throughout this thesis.

1.1 Discrete fractional calculus

In this section, we introduce the notion of the fractional difference operator that will be used mainly in Chapter 2. In what follows, we denote $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$, for some $a \in \mathbb{R}$, and $\mathbb{N} \equiv \mathbb{N}_1$ as usual. We denote by $s(\mathbb{N}_a;\mathbb{R})$ the vectorial space that consists of all sequences $f : \mathbb{N}_a \to \mathbb{R}$. Recall that given a sequence $f \in s(\mathbb{N}_a;\mathbb{R})$ the first-order forward difference operator, denoted by Δ_a, is defined by

$$\big(\Delta_a f\big)(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

Then one may define iteratively the higher order differences Δ_a^n, for $n \in \mathbb{N}_1$, by writing

$$\big(\Delta_a^n f\big)(t) := \big(\Delta_a \circ \Delta_a^{n-1} f\big)(t).$$

We also denote $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_a;\mathbb{R})$ is the identity operator, $\Delta_a^1 \equiv \Delta_a$, and $\Delta^n \equiv \Delta_0^n$.

Remark 1.1.1. For any $f \in s(\mathbb{N}_0; \mathbb{R})$, $l \in \mathbb{N}_0$ we have

$$\Delta^l f(t) = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} f(t+j), \quad t \in \mathbb{N}_0.$$

We define for $n \in \mathbb{N}_0$, the following sequence

$$k^\alpha(n) := \begin{cases} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} & \text{if } \alpha \in \mathbb{C} \setminus \{0, -1, -2, ...\}, \\ \delta_0(n) & \text{if } \alpha = 0, \end{cases}$$

where $\delta_0(n)$ is the delta function,

$$\delta_0(n) := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The sequence k^α has been introduced in [73]. This special sequence has a number of distinguished properties that are fundamental to understand the behavior or discrete fractional operators. Their importance has been recognized in several papers. For an overview, we refer to the reference [50]. Among others, that will be useful in this paper, we notice the semigroup property:

$$(k^\alpha * k^\beta)(n) = k^{\alpha+\beta}(n), \qquad n \in \mathbb{N}_0, \quad \alpha, \beta > 0, \tag{1.1.1}$$

which will be frequently used. We recall from [50, Lemma 3.2] the following result.

Lemma 1.1.2. *For any $\alpha > 0$ and $n \in \mathbb{N}_0$, the following identities hold:*

1. $\Delta k^\alpha(n) = (\alpha - 1)\dfrac{k^\alpha(n)}{n+1}$.
2. $\Delta^2 k^\alpha(n) = (\alpha - 2)(\alpha - 1)\dfrac{k^\alpha(n)}{(n+1)(n+2)}$.
3. $\Delta^3 k^\alpha(n) = (\alpha - 3)(\alpha - 2)(\alpha - 1)\dfrac{k^\alpha(n)}{(n+1)(n+2)(n+3)}$.

Now we recall from [73] the definition of α-th fractional sum operator on the set $\mathbb{N}_0$:

Definition 1.1.3. For each $\alpha > 0$ and $f \in s(\mathbb{N}_0;\mathbb{R})$, we define the fractional sum of order α as follows:

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^{\alpha}(n-j) f(j), \quad n \in \mathbb{N}_0.$$

The next concept, originally proposed in [73], is analogous to the definition of a fractional derivative in the sense of Riemann-Liouville [78].

Definition 1.1.4. Let $\alpha > 0$ be given and $f \in s(\mathbb{N}_0;\mathbb{R})$. The $\alpha-$th fractional discrete operator is defined by

$$\Delta^{\alpha} f(n) := \Delta^{m} \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m-1 < \alpha \leq m,\ m \in \mathbb{N}$.

We recall that the finite convolution $*$ of two sequences u, v where $u \in s(\mathbb{N}_0;\mathbb{C})$ and $v \in s(\mathbb{N}_0;\mathbb{R})$ is defined by

$$(u * v)(n) = \sum_{j=0}^{n} u(n-j) v(j), \quad n \in \mathbb{N}_0.$$

Given $a, b \in \mathbb{R}$, we define the translation (by $a \in \mathbb{R}$) operator $\tau_a : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_0;\mathbb{R})$ by

$$\tau_a f(n) := f(a+n), \quad n \in \mathbb{N}_0.$$

Note that $\tau_a^{-1} = \tau_{-a}$ and $\tau_{a+b} = \tau_a \circ \tau_b = \tau_b \circ \tau_a$.

Lemma 1.1.5. *[50, Lemma 2.3] Let $f, g \in s(\mathbb{N}_0;\mathbb{R})$ be sequences, then for each $p \in \mathbb{N}$ we have*

$$(f * \tau_p g)(n) = \tau_p (f * g)(n) - \sum_{j=0}^{p-1} \tau_p f(n-j) g(j).$$

We recall the most commonly used fractional difference operator of order $\nu > 0$ as defined by

Atici and Eloe [6, 7, 8]

$$\left(\Delta_a^{\nu} f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),\ t \in \mathbb{N}_{a+N-\nu}, \tag{1.1.2}$$

where $f \in s(\mathbb{N}_a;\mathbb{R})$, $N \in \mathbb{N}_1$ is the unique integer satisfying $N-1<\nu<N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. In the integer cases of $\nu = N$ we have

$$\Delta_a^N f(t) = \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} f(t+j), \quad t \in \mathbb{N}_a. \tag{1.1.3}$$

In [50, Theorem 4.3] the authors related the operator Δ_a^{ν} in (1.1.2) to the operator Δ^{α} in Definition 1.1.4 by means of the operator of translation, which allowed to transfer the properties between both definitions. This is called a *transference principle.* In the following, we have extended the formulation of the transference principle in order to include the integer cases $\alpha = N \in \mathbb{N}$, being the proof immediate taking into account (1.1.3).

Theorem 1.1.6. *(Transference Principle) Let $N-1<\alpha\le N$, $N\in\mathbb{N}$ and $a,\beta\in\mathbb{R}$. For each sequence $f \in s(\mathbb{N}_a;\mathbb{R})$ we have*

$$\tau_{a+N-\alpha} \circ \Delta_a^{\alpha} f = \Delta^{\alpha} \circ \tau_a f,$$

and for each $f \in s(\mathbb{N}_{a+N-\beta};\mathbb{R})$,

$$\tau_{N-\beta} \circ \Delta_{a+N-\beta}^{\alpha} f = \Delta_a^{\alpha} \circ \tau_{N-\beta} f.$$

The next results generalize [50, Proposition 2.9, part (v)]

Proposition 1.1.7. *For any $a,b \in s(\mathbb{N}_0;\mathbb{R})$ and $l \in \mathbb{N}$ we have*

$$\Delta^l (a*b)(n) = (\Delta^l a * b)(n) + \sum_{j=1}^{l} \sum_{i=0}^{j-1} \binom{l}{j} (-1)^{l-j} a(i) b(n+j-i).$$

Proof. Note that, by Remark 1.1.1 and Lemma 1.1.5 we get for $l \in \mathbb{N}$

$$\begin{aligned}(\Delta^l a * b)(n) &= \sum_{i=0}^{n} b(i)\Delta^l a(n-i) = \sum_{i=0}^{n} b(i) \sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j} a(n+j-i) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j} a(n+j-i)b(i) = \sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j} \sum_{i=0}^{n} \tau_j a(n-i)b(i) \\ &= \sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n), \quad n \in \mathbb{N}_0.\end{aligned}$$

Thus, we have that for $n \in \mathbb{N}_0$,

$$\begin{aligned}\Delta^l(a*b)(n) &= \sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j}(a*b)(n+j) = \sum_{j=1}^{l} \binom{l}{j}(-1)^{l-j}\tau_j(a*b)(n) + \binom{l}{0}(-1)^{l-0}(a*b)(n) \\ &= \sum_{j=1}^{l} \binom{l}{j}(-1)^{l-j}\left[(\tau_j a * b)(n) + \sum_{i=0}^{j-1} a(i)\tau_j b(n-i)\right] + (-1)^l(a*b)(n) \\ &= \sum_{j=1}^{l} \binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n) + (-1)^l(a*b)(n) + \sum_{j=1}^{l} \binom{l}{j}(-1)^{l-j} \sum_{i=0}^{j-1} a(i)\tau_j b(n-i) \\ &= \sum_{j=0}^{l} \binom{l}{j}(-1)^{l-j}(\tau_j a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1} \binom{l}{j}(-1)^{l-j} a(i)b(n-i+j) \\ &= (\Delta^l a * b)(n) + \sum_{j=1}^{l}\sum_{i=0}^{j-1} \binom{l}{j}(-1)^{l-j} a(i)b(n+j-i).\end{aligned}$$

which proves the result. □

Remark 1.1.8. In particular, by Proposition 4.1.6, we have for $l = 1$: $\Delta(a*b)(n) = (\Delta a * b)(n) + a(0)b(n+1)$ and $\Delta(a*b)(n) = (a * \Delta b)(n) + a(n+1)b(0)$. Thus,

$$(a * \Delta b)(n) = (\Delta a * b)(n) - a(n+1)b(0) + a(0)b(n+1). \tag{1.1.4}$$

1.2 Continuous fractional calculus

This section is devoted to recall some preliminaries that will be used mainly in Chapter 3 and 4. We introduce definition of the Grünwald-Letnikov derivative and Caputo-Fabrizio derivative. Also, the notions of normally convergent series and sectoriality of a closed linear operator.

Let X be a complex Banach space. Given $1 \leq p < \infty$, we consider the Banach space $L^p_{2\pi}(\mathbb{R}, X)$ of X-valued, 2π-periodic measurable functions f on $\mathbb{R}$ such that

$$\|f\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(t)\|^p dt\right)^{1/p} < \infty.$$

For a function $f \in L^1_{2\pi}(\mathbb{R}, X)$, we denote by $\hat{f}_k$, $k \in \mathbb{Z}$, the kth Fourier coefficient of f :

$$\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt, \quad k \in \mathbb{Z}.$$

Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote the domain by $D(A)$ and its resolvent set by $\rho(A)$, and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1}$. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

We recall the well-known definition of the Grünwald-Letnikov fractional derivative and some of its properties presented in [43, 65, 66, 85], see [85, Section 2.3, p. 6]. Let $\alpha > 0$. Given $f \in L^p_{2\pi}(\mathbb{R}, X)$, $1 \leq p < \infty$ the Riemann difference

$$(\Delta_h^\alpha u)(t) = \sum_{j=0}^{\infty} k^{-\alpha}(j) u(t - jh), \tag{1.2.1}$$

where $k^{-\alpha}(n) = \frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)\Gamma(-\alpha+n)}$, $n \in \mathbb{N}_0$, satisfy

$$\sum_{j=0}^{\infty} k^{-\alpha}(j) z^j = (1 - z)^\alpha, \quad |z| < 1, \tag{1.2.2}$$

exists almost everywhere and

$$\|\Delta_h^\alpha u\|_p \leq \|u\|_p \sum_{j=0}^{\infty} |k^{-\alpha}(j)| < \infty \tag{1.2.3}$$

since $k^{-\alpha}(n) = \frac{1}{n^{1+\alpha}\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right)$. For a detailed study of the sequence k^β and its properties we refer the reader to the recent article [50].

The following definition was proposed in [74, Definition 2.1].

Definition 1.2.1. Let X be a complex Banach space, $\alpha > 0$ and $1 \leq p < \infty$. If for $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists $g \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\lim_{h \to 0^+} \frac{\Delta^\alpha_h f}{h^\alpha} = g$ in the L^p-norm, then g is called the αth Gründwald-Letnikov derivative of f. We use the notation $g = {}_{GL}D^\alpha f$.

Remark 1.2.2. It should be noted that in [93] it was shown that the fractional-order derivative, based on the Grünwald-Letnikov definition, of a periodic function with a specific period cannot be a periodic function with the same period. However, the definition of the Grünwald-Letnikov fractional derivative considered in [93, Formula (1)] differs from ours, which is taken from the book by Samko, Kilbas and Marichev [65, Section 20, p. 371] and which, in turn, coincides with the Marchaud derivative [65, Theorem 20.2]. See also [64] for an approach in the context of periodic distributions.

The next result shows some examples and properties. For a proof see e.g. [43, Propositions 9, 11 and 12] and [74, Proposition 2.3].

Proposition 1.2.3. *Let $f \in L^p_{2\pi}(\mathbb{R}, X)$, $1 \leq p < \infty$. For any $z \in \mathbb{C}$, $Re(z) \geq 0$, $\alpha, \beta > 0$, and $x, t \in \mathbb{R}$ we have*

(i) ${}_{GL}D^\alpha_t e^{zt} = z^\alpha e^{zt}$,

(ii) ${}_{GL}D^\alpha_t \sin(xt) = x^\alpha \sin(xt + \alpha\frac{\pi}{2})$, *and* ${}_{GL}D^\alpha_t \cos(xt) = x^\alpha \cos(xt + \alpha\frac{\pi}{2})$,

(iii) If ${}_{GL}D^\alpha f \in L^p_{2\pi}(\mathbb{R}, X)$ *then* ${}_{GL}D^\beta f \in L^p_{2\pi}(\mathbb{R}, X)$ *for all* $0 < \beta < \alpha$,

(iv) ${}_{GL}D^\alpha_{GL}D^\beta f = {}_{GL}D^{\alpha+\beta} f$ *whenever one of the two sides is well defined.*

Definition 1.2.4. [75, Definition 2] Let $0 < \alpha < 1$ and $u : \mathbb{R}_+ \to X$ be a continuously differentiable function. The Caputo-Fabrizio fractional derivative of u of order α is given by:

$$ {}_{CF}D^\alpha_t u(t) := \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right) u'(s)ds, \quad t \geq 0. $$

We recall two important properties (see [23, Section 2]):

(i) For $\alpha \to 1$ we have that

$$\lim_{\alpha \to 1} {}_{CF}D_t^{\alpha} u(t) = u'(t). \tag{1.2.4}$$

(ii) We denote by $\mathscr{L}[u]$ the Laplace Transform of a function u. The Laplace Transform of the fractional operator ${}_{CF}D_t^{\alpha}$ with $0 < \alpha < 1$ is :

$$\mathscr{L}[{}_{CF}D_t^{\alpha} u](\lambda) = \frac{\lambda \mathscr{L}[u](\lambda) - u(0)}{\lambda(1-\alpha) + \alpha}, \quad \lambda > 0.$$

Remark 1.2.5. Note that the Caputo-Fabrizio fractional derivative has a *non-singular* kernel, namely, $\exp\left(-\frac{\alpha t}{1-\alpha}\right)$. This special feature, when compared with the classical Caputo or Riemann-Liouville fractional derivative that instead has the *singular* kernel $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $0 < \alpha < 1$, allows to obtain distinguished properties of the non-local operator ${}_{CF}D_t^{\alpha}$. One of this properties, which is obvious but important in our analysis, is the following:

$${}_{CF}D_t^{\alpha} u(0) = 0, \tag{1.2.5}$$

whenever $0 < \alpha < 1$. This behavior has been remarked by Diethelm, Garrapa, Giusti and Stynes [34], where the general issue of the use of regular kernels in the theory of fractional calculus is discussed.

We recall the definition of normal convergence for series of functions [52, Definition 3, pag 222][35, 6.19, pag 64] and its relationship with uniform convergence [52, Theorem 1, pag 222][35, Theorem 6.1.10, pag 64].

Definition 1.2.6. Given a set I and bounded functions $u_n : I \to X$, the series $\sum_{n\in\mathbb{Z}} u_n(t)$ is called normally convergent on I if the series $\sum_{n\in\mathbb{Z}} ||u_n||_\infty := \sum_{n\in\mathbb{Z}} \sup_{t\in I} \|u_n(t)\|$ converges.

Remark 1.2.7. Every normal convergent series is uniformly convergent.

Let $\Sigma_\phi \subset \mathbb{C}$ be the open sector $\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$. Finally, we recall the following definition.

Definition 1.2.8. [32] Given a closed linear operator A in X, we say that A is sectorial if A satisfy following conditions (i) $\overline{D(A)} = X, \overline{R(A)} = X, (-\infty, 0) \subset \rho(A)$; (ii) $||t(t+A)^{-1}|| \leq M$ for all $t > 0$ and some $M > 0$. The operator A is called R-sectorial if the set $\{t(t+A)^{-1}\}_{t>0}$ is R-bounded.

If A is sectorial then $\Sigma_\phi \subset \rho(-A)$ for some $\phi > 0$ and $\sup_{|\arg \lambda| < \phi} ||\lambda(\lambda + A)^{-1}|| < \infty$. We denote the *spectral angle* of a sectorial operator A by

$$\phi_A = \inf\{\phi : \Sigma_{\pi-\phi} \subset \rho(-A), \quad \sup_{\lambda \in \Sigma_{\pi-\phi}} ||\lambda(\lambda + A)^{-1}|| < \infty\}.$$

Chapter 2

Second and third order forward difference operator: What is in between?

In this chapter, we present a new geometrical notion for a real-valued function defined in a discrete domain that depends on a parameter $\alpha \geq 2$. We give examples to illustrate connections between convexity and this new concept. We then prove two criteria based on the sign of the discrete fractional operator of a function u, $\Delta^{\alpha} u$ with $2 \leq \alpha < 4$. Two examples show that the given criteria are optimal with respect to the established geometrical notion.

2.1 α-jerk sequence

From a physical point of view, jerk is the rate at which the acceleration of an object changes with respect to time. In a discrete setting (following Euler method of discretization) it corresponds to the third order power of the forward difference operator, namely: Δ^3. The connection between positive jerk, i.e. $\Delta^3 u \geq 0$ and u has been an object of recent interest [67, 68] and can be characterized as follows: A sequence u has positive jerk if and only if

$$u(n) = \frac{1}{2}\sum_{k=0}^{n}(n-k+2)(n-k+1)b(k)$$

with $b(k) \geq 0$ for all $k \in \mathbb{N}_3$. See e.g. [68, Lemma 1.1] and references therein. In order to analyze the intermediate cases between Δ^2 and Δ^3 we propose the following definition.

Definition 2.1.1. Let $\alpha \geq 2$. We say that a sequence $u \in s(\mathbb{N}_0, \mathbb{R})$ has positive α-jerk if

$$u(n+3) - \alpha u(n+2) + (2\alpha - 3)u(n+1) - (\alpha - 2)u(n) \geq 0, \quad n \in \mathbb{N}_0, \tag{2.1.1}$$

When $\alpha = 3$ we recover the notion of positive jerk. Note that when $\alpha = 2$ we retrieve the concept of convex sequence on the set $\mathbb{N}$.

Our first result tell us about the geometrical meaning of a positive α-jerk sequence. Compared with a convex sequence, whose graph lies above sequences of the form: $\beta a^n + b$, in the case of positive α-jerk sequences, the graph is placed above sequences of the form: $\beta a^n + bn + c$. This is the content of the following result.

Proposition 2.1.2. *If $u \in s(\mathbb{N}_0, \mathbb{R})$ has positive α-jerk then we have*

$$u(n) \geq \frac{1}{(3-\alpha)^2}[n(3-\alpha) + (\alpha-2)^n - 1]\Delta^2 u(0) + n\Delta u(0) + u(0)$$

for $\alpha \geq 2, \alpha \neq 3$ and

$$u(n) \geq \frac{n(n-1)}{2}\Delta^2 u(0) + n\Delta u(0) + u(0),$$

in case $\alpha = 3$.

Proof. From the definition, we note that u has positive α-jerk if and only if $\Delta^2 u(n+1) \geq (\alpha - 2)\Delta^2 u(n)$, $n \in \mathbb{N}_0$. Iterating, we obtain

$$\Delta^2 u(n) \geq (\alpha - 2)^n \Delta^2 u(0). \tag{2.1.2}$$

Thus, $\Delta u(n+1) - \Delta u(n) \geq (\alpha-2)^n \Delta^2 u(0)$ and hence $\Delta u(n+1) \geq (\alpha-2)^n \Delta^2 u(0) + \Delta u(n)$. Iterating again this last inequality, we obtain

$$\Delta u(n) \geq \sum_{j=0}^{n-1} (\alpha-2)^j \Delta^2 u(0) + \Delta u(0) = \frac{1-(\alpha-2)^n}{3-\alpha} \Delta^2 u(0) + \Delta u(0).$$

in case $\alpha \neq 3$ and

$$\Delta u(n) \geq n\Delta^2 u(0) + \Delta u(0),$$

in case $\alpha = 3$.

Therefore, if $\alpha \neq 3$ we have

$$u(n+1) \geq \frac{1-(\alpha-2)^n}{3-\alpha} \Delta^2 u(0) + \Delta u(0) + u(n). \tag{2.1.3}$$

Thus, iterating again we arrive at

$$\begin{aligned} u(n) &\geq \sum_{j=0}^{n-1} \frac{1-(\alpha-2)^j}{3-\alpha} \Delta^2 u(0) + n\Delta u(0) + u(0) \\ &= \frac{1}{(3-\alpha)^2}[n(3-\alpha) + (\alpha-2)^n - 1]\Delta^2 u(0) + n\Delta u(0) + u(0) \end{aligned}$$

which finish the proof in this case. In contrast, if $\alpha = 3$, we obtain

$$u(n) \geq \frac{n(n-1)}{2} \Delta^2 u(0) + n\Delta u(0) + u(0)$$

which finishes the proof. □

Remark 2.1.3. If a sequence u has positive $\alpha-$jerk and we assume that $u(0) = u(1) = 0, u(2) = 1$, then their graph lies above the graph of the sequence $J_\alpha(n) := \frac{1}{(3-\alpha)^2}[n(3-\alpha) + (\alpha-2)^n - 1]$. The behavior of the sequence $J_\alpha(n)$ for different values of $\alpha \neq 3$ is drawn in Figure 2.1. In case $\alpha = 3$ the graph of a positive jerk sequence lies above the graph of the sequence $J_3(n) = \frac{n(n-1)}{2}$.

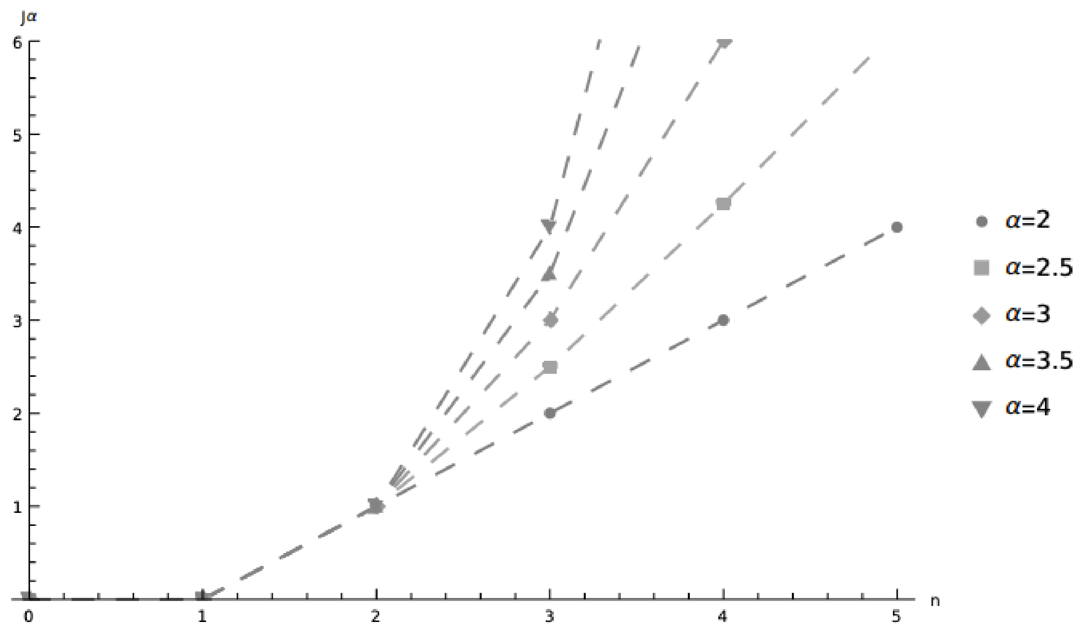

Figure 2.1: J_α with $u(0) = u(1) = 0$ and $u(2) = 1$

2.2 Geometry of the sequence u

In this section, for $2 \leq \alpha < 4$, we assume that $u \in s(\mathbb{N}_0; \mathbb{R})$ and $\Delta^\alpha u(n)$ satisfie suitables conditions, and we conclude properties of positivity, monotonicity, convexity, jerk-positivity and α-jerk-positivity for u. We give new examples shoring the necessity of imposed connditions.

2.2.1 Case: $2 \leq \alpha \leq 3$.

The following is our main result in case $2 \leq \alpha \leq 3$.

Theorem 2.2.1. *Let $2 \leq \alpha \leq 3$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

1. $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
2. $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$;
3. $u(1) \geq \alpha u(0)$;
4. $u(0) \geq 0$.

Then u *is positive, increasing, convex and has positive* α*-jerk on* $\mathbb{N}_0$.

Proof. First, we study the borderline cases. If $\alpha = 2$ then by hypothesis (1) we have $\Delta^2 u(n) \geq 0$, for all $n \in \mathbb{N}_0$, i.e. u is convex (positive 2-jerk) on $\mathbb{N}_0$. Now, using the fact that u is convex on $\mathbb{N}_0$, we get $\Delta u(n+1) \geq \Delta u(n)$. By hypotheses (3) and (4) we also have $u(1) \geq u(0)$, then $\Delta u(0) \geq 0$ and $\Delta u(n) \geq ... \geq \Delta u(0) \geq 0$. Hence, u is monotone increasing and positive on $\mathbb{N}_0$.

On the other hand, if $\alpha = 3$, by hypothesis (1), we have $\Delta^3 u(n) \geq 0$ on $\mathbb{N}_0$, i.e., u has positive jerk. Moreover, by hypotheses (2), (3) and (4) we obtain $\Delta^2 u(0) = u(2) - 2u(1) + u(0) \geq u(1) - 2u(0) \geq 0$, since that positive jerk is equivalent to $\Delta^2 u(n+1) \geq \Delta^2 u(n)$, then $\Delta^2 u(n) \geq ... \geq \Delta^2 u(0) \geq 0$. Thus, u is convex on $\mathbb{N}_0$ and by hypotheses (3) and (4), u is monotone increasing and positive.

Now, we assume $2 < \alpha < 3$. By Proposition 4.1.6, with $a := u$, $b := k^{3-\alpha}$ and $l = 3$, we obtain

$$(k^{3-\alpha} * \Delta^3 u)(n) = \Delta^\alpha u(n) - \sum_{j=1}^{3} \sum_{i=0}^{j-1} \binom{3}{j} (-1)^{3-j} u(i) k^{3-\alpha}(n+j-i). \tag{2.2.1}$$

Since $k^{3-\alpha}(n+j-i) = \tau_{j-i} k^{3-\alpha}(n)$, then convolving (4.1.8) with $k^{\alpha-2}$ we obtain

$$(k^{\alpha-2} * k^{3-\alpha} * \Delta^3 u)(n) = (k^{\alpha-2} * \Delta^\alpha u)(n) - \sum_{j=1}^{3} \sum_{i=0}^{j-1} \binom{3}{j} (-1)^{3-j} u(i) (k^{\alpha-2} * \tau_{j-i} k^{3-\alpha})(n). \tag{2.2.2}$$

Observe that $(k^{\alpha-2} * k^{3-\alpha} * \Delta^3 u)(n) = (k * \Delta^3 u)(n) = \Delta^2 u(n+1) - \Delta^2 u(0)$. Moreover, by Lemma 1.1.5 and the semigroup property of the kernel k^γ, we get

$$\begin{aligned}(k^{\alpha-2} * \tau_{j-i} k^{3-\alpha})(n) &= (k^{\alpha-2} * k^{3-\alpha})(n+j-i) - \sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i) k^{3-\alpha}(l) \\ &= 1 - \sum_{l=0}^{j-i-1} k^{\alpha-2}(n-l+j-i) k^{3-\alpha}(l).\end{aligned}$$

Therefore, replacing the above identity in (4.1.9), we have

$$\begin{aligned}\Delta^2 u(n+1) - \Delta^2 u(0) = (k^{\alpha-2} * \Delta^\alpha u)(n) - \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i) \\ + \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1}k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l).\end{aligned} \tag{2.2.3}$$

Note that,

$$\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i) = 3u(0) - 3u(0) - 3u(1) + u(0) + u(1) + u(2) = \Delta^2 u(0). \tag{2.2.4}$$

Also, since for any $\gamma > 0$, $k^\gamma(0) = 1$, $k^\gamma(1) = \gamma$ and $k^\gamma(2) = \frac{\gamma(\gamma+1)}{2}$, we have

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1}k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l) \\ &= 3u(0)k^{\alpha-2}(n+1) - 3[u(0)(k^{\alpha-2}(n+2) + k^{\alpha-2}(n+1)(3-\alpha)) + u(1)k^{\alpha-2}(n+1)] \\ &+ [u(0)(k^{\alpha-2}(n+3) + k^{\alpha-2}(n+2)(3-\alpha) + k^{\alpha-2}(n+1)\frac{1}{2}(3-\alpha)(4-\alpha)) \\ &+ u(1)(k^{\alpha-2}(n+2) + k^{\alpha-2}(n+1)(3-\alpha)) + u(2)k^{\alpha-2}(n+1)].\end{aligned} \tag{2.2.5}$$

Replacing (2.2.4) and (2.2.5) in (2.2.3) we obtain that for $n \in \mathbb{N}_0$,

$$\begin{aligned}\Delta^2 u(n+1) = (k^{\alpha-2} * \Delta^\alpha u)(n) + k^{\alpha-2}(n+3)u(0) + k^{\alpha-2}(n+2)[u(1) - \alpha u(0)] \\ + k^{\alpha-2}(n+1)\left[u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0)\right].\end{aligned} \tag{2.2.6}$$

Using the hypotheses (1)–(4) we conclude from (4.1.10) that $\Delta^2 u(n) \geq 0$, for all $n \in \mathbb{N}$. On the other hand, using hypothesis (2), we have

$$u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0) = \Delta^2 u(0) - (\alpha-2)u(1) + \frac{(\alpha-2)(\alpha+1)}{2}u(0) \geq 0.$$

Hence, hypotheses (3) and (4) show that $\Delta^2 u(0) \geq 0$. Indeed,

$$\begin{aligned}\Delta^2 u(0) \geq (\alpha-2)u(1) - \frac{(\alpha-2)(\alpha+1)}{2}u(0) \geq \left[(\alpha-2)\alpha - \frac{(\alpha-2)(\alpha+1)}{2}\right]u(0) \\ = \frac{(\alpha-2)(\alpha-1)}{2}u(0).\end{aligned}$$

This proves that $\Delta^2 u(n) \geq 0$ for all $n \in \mathbb{N}_0$ – i.e., u is convex on the set $\mathbb{N}_0$ as claimed. It follows that u is positive and increasing (because it corresponds to the case $\alpha = 2$ proved at the beginning of the proof).

Next, we prove that u has positive α-jerk. Indeed, from (4.1.8) we obtain

$$\begin{aligned}\Delta^\alpha u(n) &= (k^{3-\alpha} * \Delta^3 u)(n) + \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)k^{3-\alpha}(n+j-i)\\ &= (k^{3-\alpha} * \Delta^3 u)(n) + k^{3-\alpha}(n+3)u(0) + k^{3-\alpha}(n+2)[u(1)-3u(0)]\\ &\quad + k^{3-\alpha}(n+1)[u(2)-3u(1)+3u(0)].\end{aligned} \tag{2.2.7}$$

By equation (1.1.4), with $a := k^{3-\alpha}$ and $b := \Delta^2 u$, we get

$$(k^{3-\alpha} * \Delta^3 u)(n) = (\Delta k^{3-\alpha} * \Delta^2 u)(n) - k^{3-\alpha}(n+1)\Delta^2 u(0) + k^{3-\alpha}(0)\Delta^2 u(n+1).$$

Thus, replacing the above identity in (4.1.7) and by hypothesis (1) we have

$$\begin{aligned}0 \le \Delta^\alpha u(n) &= (\Delta k^{3-\alpha} * \Delta^2 u)(n) - k^{3-\alpha}(n+1)\Delta^2 u(0) + k^{3-\alpha}(0)\Delta^2 u(n+1)\\ &\quad + k^{3-\alpha}(n+3)u(0) + k^{3-\alpha}(n+2)[u(1)-3u(0)] + k^{3-\alpha}(n+1)[u(2)-3u(1)+3u(0)]\\ &= \sum_{j=1}^{n}\Delta k^{3-\alpha}(j)\Delta^2 u(n-j) + \Delta k^{3-\alpha}(0)\Delta^2 u(n) + \Delta^2 u(n+1)\\ &\quad + k^{3-\alpha}(n+3)u(0) + k^{3-\alpha}(n+2)[u(1)-3u(0)] - k^{3-\alpha}(n+1)[u(1)-2u(0)]\\ &= \sum_{j=1}^{n}\Delta k^{3-\alpha}(j)\Delta^2 u(n-j) + (2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1)\\ &\quad + \Delta k^{3-\alpha}(n+2)u(0) + \Delta k^{3-\alpha}(n+1)[u(1)-2u(0)],\end{aligned}$$

where we have used $\Delta k^{3-\alpha}(0) = 2-\alpha$ and $k^{3-\alpha}(0) = 1$. By Lemma 1.1.2, part (1), we have $\Delta k^{3-\alpha}(m) \le 0$. Recalling that $\Delta^2 u(m) \ge 0$ we obtain by hypotheses and the above inequality

$$\begin{aligned}(2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1) &\ge -\sum_{j=1}^{n}\Delta k^{3-\alpha}(j)\Delta^2 u(n-j) - \Delta k^{3-\alpha}(n+2)u(0)\\ &\quad - \Delta k^{3-\alpha}(n+1)[u(1)-2u(0)] \ge 0.\end{aligned}$$

for all $n \in \mathbb{N}_0$ – i.e., u has positive α-jerk on $\mathbb{N}_0$. □

The following example show that the condition (2) in Theorem 2.2.1 is necessary in order to guarantee the convexity of the sequence u.

Example 2.2.2. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = \frac{2^n-1}{2^{n-1}}$, and assume that $\frac{4+\sqrt{2}}{2} < \alpha < 3$. The following statements holds.*

(i) $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;

(ii) $u(1) \geq \alpha u(0)$;

(iii) $u(0) \geq 0$;

(iv) u *is positive and increasing.*

(v) u *has positive* α*-jerk and is concave.*

Indeed, first observe that $u(0) = 0$ *and* $u(1) = 1$, *therefore* (ii) *and* (iii) *holds. By their own definition* u *is positive and it is clear that* $\Delta u(n) = u(n+1) - u(n) = \frac{1}{2^n} \geq 0$, *i.e.,* u *is increasing on* $\mathbb{N}_0$. *It proves* (iv).

We now verify (i): *By Proposition 4.1.6, with* $a := k^{3-\alpha}$, $b := u$ *and* $l = 3$, *we obtain for each* $n \in \mathbb{N}_0$

$$\begin{aligned}\Delta^\alpha u(n) &= (\Delta^3 k^{3-\alpha} * u)(n) + \sum_{j=1}^{3} \sum_{i=0}^{j-1} \binom{3}{j} (-1)^{3-j} k^{3-\alpha}(i) u(n+j-i) \\ &= (\Delta^3 k^{3-\alpha} * u)(n) + u(n+3) - \alpha u(n+2) + \frac{\alpha(\alpha-1)}{2} u(n+1).\end{aligned} \tag{2.2.8}$$

Using equation (1.1.4), *with* $a := u$ *and* $b := \Delta^2 k^{3-\alpha}$, *we obtain*

$$(\Delta^3 k^{3-\alpha} * u)(n) = (\Delta^2 k^{3-\alpha} * \Delta u)(n) + \Delta^2 k^{3-\alpha}(n+1) u(0) - \Delta^2 k^{3-\alpha}(0) u(n+1).$$

Now, replacing the above identity in (2.2.8) *we have*

$$\begin{aligned}
\Delta^{\alpha}u(n) &= (\Delta^2 k^{3-\alpha} * \Delta u)(n) + \Delta^2 k^{3-\alpha}(n+1)u(0) - \Delta^2 k^{3-\alpha}(0)u(n+1) + u(n+3) - \alpha u(n+2) \\
&\quad + \frac{\alpha(\alpha-1)}{2}u(n+1) \\
&= (\Delta^2 k^{3-\alpha} * \Delta u)(n) - \frac{(\alpha-1)(\alpha-2)}{2}u(n+1) + u(n+3) - \alpha u(n+2) + \frac{\alpha(\alpha-1)}{2}u(n+1) \\
&= (\Delta^2 k^{3-\alpha} * \Delta u)(n) + u(n+3) - \alpha u(n+2) + (\alpha-1)u(n+1) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \Delta^2 k^{3-\alpha}(0)\Delta u(n) + u(n+3) - \alpha u(n+2) + (\alpha-1)u(n+1) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \frac{(\alpha-1)(\alpha-2)}{2}[u(n+1) - u(n)] + u(n+3) - \alpha u(n+2) \\
&\quad + (\alpha-1)u(n+1) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + u(n+3) - \alpha u(n+2) + [\frac{(\alpha-1)(\alpha-2)}{2} + (\alpha-1)]u(n+1) \\
&\quad - \frac{(\alpha-1)(\alpha-2)}{2}u(n) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + u(n+3) - \alpha u(n+2) + \frac{\alpha(\alpha-1)}{2}u(n+1) \\
&\quad - \frac{(\alpha-1)(\alpha-2)}{2}u(n) \\
&= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \frac{2^{n+3}-1}{2^{n+2}} - \alpha\frac{2^{n+2}-1}{2^{n+1}} + \frac{\alpha(\alpha-1)}{2}\frac{2^{n+1}-1}{2^{n}} \\
&\quad - \frac{(\alpha-1)(\alpha-2)}{2}\frac{2^{n}-1}{2^{n-1}}.
\end{aligned}$$

Note that by Lemma 1.1.2, part (2), *and using the fact that* $\Delta u(n) \geq 0$, *we obtain* $\sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) \geq 0$. *Thus, since* $\alpha \in (\frac{4+\sqrt{2}}{2}, 3)$, *from the above identity we obtain*

$$\begin{aligned}
\Delta^{\alpha}u(n) &\geq \frac{2^{n+3}-1}{2^{n+2}} - \alpha\frac{2^{n+2}-1}{2^{n+1}} + \frac{\alpha(\alpha-1)}{2}\frac{2^{n+1}-1}{2^{n}} - \frac{(\alpha-1)(\alpha-2)}{2}\frac{2^{n}-1}{2^{n-1}} \\
&= \frac{2\alpha^2 - 8\alpha + 7}{2^{n+2}} \geq 0.
\end{aligned}$$

This proves (i) *as claimed.*

Finally, we prove (v). Note that

$$\Delta^2 u(n) = \Delta u(n+1) - \Delta u(n) = -\frac{1}{2^{n+1}} \leq 0.$$

Therefore u is concave. Since u has positive α-jerk if and only if $(2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1) \geq 0$, we obtain after a computation

$$(2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1) = \frac{1}{2^{n+1}}\left[\alpha - \frac{3}{2}\right] \geq 0,$$

since $\alpha > \frac{4+\sqrt{2}}{2} > \frac{3}{2}$. This proves (v).

However, note that $u(2) = \frac{3}{2} < \alpha = \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$. It follows that the condition (2) in Theorem 2.2.1 is necessary in order to ensure the convexity of the sequence u.

From Theorem 2.2.1 and the transference principle (Theorem 1.1.6) we deduce the following corollary.

Corollary 2.2.3. *Let $2 \leq \alpha \leq 3$, $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a; \mathbb{R})$ be given and assume that*

1. *$\Delta_a^\alpha v(t) \geq 0$, for all $t \in \mathbb{N}_{a+3-\alpha}$;*
2. *$v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2}v(a)$;*
3. *$v(a+1) \geq \alpha v(a)$;*
4. *$v(a) \geq 0$.*

Then v is positive, increasing, convex and has positive α-jerk on $\mathbb{N}_a$.

Proof. In case $\alpha = 2$ the conclusion is clear from the hypothesis. Define $u := \tau_a v$. Using the transference principle, we have,

$$\Delta^\alpha u(n) = \tau_{a+3-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u(n) = \tau_{a+3-\alpha} \circ \Delta_a^\alpha \circ v(n) = \Delta_a^\alpha v(t) \geq 0,$$

for each $t := n + a + 3 - \alpha \in \mathbb{N}_{a+3-\alpha}$. Moreover, $u(2) = v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2}v(a) = \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$, $u(1) = v(a+1) \geq \alpha v(a) = \alpha u(0)$, and $u(0) = v(a) \geq 0$. The conclusion follows from Theorem 2.2.1. □

2.2.2 Case: $3 \leq \alpha < 4$.

The following is our main result in case $3 \leq \alpha < 4$.

Theorem 2.2.4. *Let $3 \leq \alpha < 4$, $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

1. $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
2. $u(3) \geq \alpha u(2) - \frac{\alpha(\alpha-1)}{2}u(1) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}u(0)$;
3. $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$;
4. $u(1) \geq \alpha u(0)$;
5. $u(0) \geq 0$.

Then u is positive, increasing, convex and has positive jerk on $\mathbb{N}_0$.

Proof. Observe that in the limit case $\alpha = 3$ all the given hypothesis and conclusions coincides with those of Theorem 2.2.1 (note that (1) equals (2) in such case) and therefore the proof follows as in such theorem. Suppose that $3 < \alpha < 4$. By Proposition 4.1.6, with $a := u$, $b := k^{4-\alpha}$ and $l = 4$, we obtain

$$(k^{4-\alpha} * \Delta^4 u)(n) = \Delta^\alpha u(n) - \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)k^{4-\alpha}(n+j-i). \tag{2.2.9}$$

Since $k^{4-\alpha}(n+j-i) = \tau_{j-i}k^{4-\alpha}(n)$, then convolving (2.2.9) with $k^{\alpha-3}$ we obtain

$$(k^{\alpha-3} * k^{4-\alpha} * \Delta^4 u)(n) = (k^{\alpha-3} * \Delta^\alpha u)(n) - \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)(k^{\alpha-3} * \tau_{j-i}k^{4-\alpha})(n). \tag{2.2.10}$$

By Lemma 1.1.5 and the semigroup property of the kernel k^{γ}, we get

$$\begin{aligned}(k^{\alpha-3} * \tau_{j-i}k^{4-\alpha})(n) &= (k^{\alpha-3} * k^{4-\alpha})(n+j-i) - \sum_{l=0}^{j-i-1} k^{\alpha-3}(n-l+j-i)k^{4-\alpha}(l)\\ &= 1 - \sum_{l=0}^{j-i-1} k^{\alpha-3}(n-l+j-i)k^{4-\alpha}(l).\end{aligned}$$

Therefore, replacing the above identity in (2.2.10), and since $(k^{\alpha-3} * k^{4-\alpha} * \Delta^4 u)(n) = (k * \Delta^4 u)(n) = \Delta^3 u(n+1) - \Delta^3 u(0)$, we obtain

$$\begin{aligned}\Delta^3 u(n+1) - \Delta^3 u(0) = (k^{\alpha-3} * \Delta^{\alpha} u)(n) &- \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)\\ &+ \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)\sum_{l=0}^{j-i-1} k^{\alpha-3}(n-l+j-i)k^{4-\alpha}(l).\end{aligned} \tag{2.2.11}$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i) &= -4u(0) + 6[u(0)+u(1)] - 4[u(0)+u(1)+u(2)]\\ &\quad + [u(0)+u(1)+u(2)+u(3)]\\ &= \Delta^3 u(0).\end{aligned} \tag{2.2.12}$$

Also, since for any $\gamma > 0$, $k^\gamma(0) = 1$, $k^\gamma(1) = \gamma$, $k^\gamma(2) = \frac{\gamma(\gamma+1)}{2}$ and $k^\gamma(3) = \frac{\gamma(\gamma+1)(\gamma+2)}{6}$ we have

$$\begin{aligned}
&\sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)\sum_{l=0}^{j-i-1}k^{\alpha-3}(n-l+j-i)k^{4-\alpha}(l)\\
&\quad= -4u(0)k^{\alpha-3}(n+1) + 6\Big[u(0)\Big(k^{\alpha-3}(n+2) + k^{\alpha-3}(n+1)(4-\alpha)\Big) + u(1)k^{\alpha-3}(n+1)\Big]\\
&\quad - 4\Big[u(0)\Big(k^{\alpha-3}(n+3) + k^{\alpha-3}(n+2)(4-\alpha) + k^{\alpha-3}(n+1)\frac{1}{2}(4-\alpha)(5-\alpha)\Big)\\
&\quad + u(1)\Big(k^{\alpha-3}(n+2) + k^{\alpha-3}(n+1)(4-\alpha)\Big) + u(2)k^{\alpha-3}(n+1)\Big]\\
&\quad + \Big[u(0)\Big(k^{\alpha-3}(n+4) + k^{\alpha-3}(n+3)(4-\alpha) + k^{\alpha-3}(n+2)\frac{1}{2}(4-\alpha)(5-\alpha)\\
&\quad + k^{\alpha-3}(n+1)\frac{1}{6}(4-\alpha)(5-\alpha)(6-\alpha)\Big) + u(1)\Big(k^{\alpha-3}(n+3) + k^{\alpha-3}(n+2)(4-\alpha)\\
&\quad + k^{\alpha-3}(n+1)\frac{1}{2}(4-\alpha)(5-\alpha)\Big) + u(2)\Big(k^{\alpha-3}(n+2) + k^{\alpha-3}(n+1)(4-\alpha)\Big)\\
&\quad + u(3)k^{\alpha-3}(n+1)\Big].
\end{aligned} \tag{2.2.13}$$

Replacing (2.2.12) and (2.2.13) in (2.2.11) we obtain that for $n \in \mathbb{N}_0$,

$$\begin{aligned}
\Delta^3 u(n+1) = (k^{\alpha-3} * \Delta^\alpha u)(n) &+ k^{\alpha-3}(n+4)u(0) + k^{\alpha-3}(n+3)[u(1) - \alpha u(0)]\\
&+ k^{\alpha-3}(n+2)\left[u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0)\right]\\
&+ k^{\alpha-3}(n+1)\left[u(3) - \alpha u(2) + \frac{\alpha(\alpha-1)}{2}u(1) - \frac{\alpha(\alpha-1)(\alpha-2)}{6}u(0)\right].
\end{aligned} \tag{2.2.14}$$

Using the hypotheses (2), (3), (4) and (5) we conclude from (2.2.14) that $\Delta^3 u(n) \geq 0$, for all $n \in \mathbb{N}$.

We claim that $\Delta^3 u(0) \geq 0$. In fact, using (2), we have

$$\begin{aligned}
&u(3) - \alpha u(2) + \frac{\alpha(\alpha-1)}{2}u(1) - \frac{\alpha(\alpha-1)(\alpha-2)}{6}u(0)\\
&\qquad\qquad = \Delta^3 u(0) - (\alpha-3)u(2) + \frac{(\alpha-3)(\alpha+2)}{2}u(1) - \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0) \geq 0.
\end{aligned}$$

Note that, since $3 < \alpha < 4$, then $\alpha^2 - 6\alpha + 11 > 0$. Hence, hypotheses (3), (4) and (5) show that

$$\begin{aligned}
\Delta^3 u(0) &\geq (\alpha-3)u(2) - \frac{(\alpha-3)(\alpha+2)}{2}u(1) + \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0)\\
&\geq (\alpha-3)\left[\alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)\right] - \frac{(\alpha-3)(\alpha+2)}{2}u(1) + \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\alpha-2)(\alpha-3)}{2}u(1) - \frac{\alpha(\alpha-1)(\alpha-3)}{2}u(0) + \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0) \\
&\geq \frac{(\alpha-2)(\alpha-3)}{2}\alpha u(0) - \frac{\alpha(\alpha-1)(\alpha-3)}{2}u(0) + \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0) \\
&= \frac{\alpha^3-6\alpha^2+11\alpha+6}{6}u(0) = \frac{\alpha(\alpha^2-6\alpha+11)+6}{6}u(0) \geq 0.
\end{aligned}
$$

This proves the claim and that $\Delta^3 u(n) \geq 0$ for all $n \in \mathbb{N}_0$ – i.e., the sequence u has positive jerk.

Finally, by hypotheses we obtain $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0) \geq \alpha u(1) - \frac{(\alpha-1)}{2}u(1) = \frac{\alpha+1}{2}u(1) \geq 2u(1)$, then $\Delta^2 u(0) = u(2) - 2u(1) + u(0) \geq 0$. Moreover, $\Delta^2 u(n+1) \geq \Delta^2 u(n) \geq ... \geq \Delta^2 u(0)$, therefore u is convex on $\mathbb{N}_0$. Now, using the fact that u is convex on $\mathbb{N}_0$, we get $\Delta u(n+1) \geq \Delta u(n)$. By hypotheses (4) and (5) we also have $u(1) \geq u(0) \geq 0$, then $\Delta u(0) \geq 0$ and $\Delta u(n) \geq ... \geq \Delta u(0) \geq 0$. Hence, u is monotone increasing and positive on $\mathbb{N}_0$. □

The following example show that the condition (2) in Theorem 2.2.4 is necessary in order to ensure that the sequence u has positive jerk.

Example 2.2.5. *Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = 2n + 2^{2-n} - 4$, and assume that $\frac{6+\sqrt{2}}{2} < \alpha < 4$. We have that the following collection of statements are true.*

(i) $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;

(ii) $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$;

(iii) $u(1) \geq \alpha u(0)$;

(iv) $u(0) \geq 0$;

(v) *u is positive, monotone increasing and convex.*

Indeed, first observe that $u(0) = u(1) = 0$ and $u(2) = 1$. This proves $(ii), (iii)$ and (iv). Also, we have that u is positive. Since $\Delta u(n) = u(n+1) - u(n) = 2 - \frac{1}{2^{n-1}} \geq 0$ and $\Delta^2 u(n) = \Delta u(n+1) - \Delta u(n) = \frac{1}{2^n} \geq 0$, then u is monotone increasing and convex on $\mathbb{N}_0$. This proves (v).

We will prove (i). In fact, by Proposition 4.1.6, with $a := u$, $b := k^{4-\alpha}$ and $l = 4$, we obtain for each $n \in \mathbb{N}_0$

$$\begin{aligned} \Delta^\alpha u(n) &= (k^{4-\alpha} * \Delta^4 u)(n) + \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)k^{4-\alpha}(n+j-i) \\ &= (k^{4-\alpha} * \Delta^4 u)(n) + k^{4-\alpha}(n+2) - \frac{3}{2}k^{4-\alpha}(n+1). \end{aligned} \tag{2.2.15}$$

By equation (1.1.4), with $a := k^{4-\alpha}$ and $b := \Delta^3 u$, we have

$$(k^{4-\alpha} * \Delta^4 u)(n) = (\Delta k^{4-\alpha} * \Delta^3 u)(n) - k^{4-\alpha}(n+1)\Delta^3 u(0) + k^{4-\alpha}(0)\Delta^3 u(n+1).$$

Replacing the above identity in (2.2.15), we obtain

$$\begin{aligned} \Delta^\alpha u(n) &= (\Delta k^{4-\alpha} * \Delta^3 u)(n) - k^{4-\alpha}(n+1)\Delta^3 u(0) + k^{4-\alpha}(0)\Delta^3 u(n+1) + k^{4-\alpha}(n+2) \\ &\quad - \frac{3}{2}k^{4-\alpha}(n+1) \\ &= (\Delta k^{4-\alpha} * \Delta^3 u)(n) + \Delta k^{4-\alpha}(n+1) + \Delta^3 u(n+1). \end{aligned} \tag{2.2.16}$$

On the other hand, using again equation (1.1.4), with $a := \Delta k^{4-\alpha}$ and $b := \Delta^2 u$, we have

$$(\Delta k^{4-\alpha} * \Delta^3 u)(n) = (\Delta^2 k^{4-\alpha} * \Delta^2 u)(n) - \Delta k^{4-\alpha}(n+1)\Delta^2 u(0) + \Delta k^{4-\alpha}(0)\Delta^2 u(n+1).$$

Thus, replacing the above identity in (2.2.16), we have

$$\begin{aligned} \Delta^\alpha u(n) &= (\Delta^2 k^{4-\alpha} * \Delta^2 u)(n) - \Delta k^{4-\alpha}(n+1)\Delta^2 u(0) + \Delta k^{4-\alpha}(0)\Delta^2 u(n+1) \\ &\qquad + \Delta k^{4-\alpha}(n+1) + \Delta^3 u(n+1) \\ &= (\Delta^2 k^{4-\alpha} * \Delta^2 u)(n) + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1) \\ &= \sum_{j=1}^{n}\Delta^2 k^{4-\alpha}(j)\Delta^2 u(n-j) + \Delta^2 k^{4-\alpha}(0)\Delta^2 u(n) + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1) \\ &= \sum_{j=1}^{n}\Delta^2 k^{4-\alpha}(j)\Delta^2 u(n-j) + \frac{(3-\alpha)(2-\alpha)}{2}\Delta^2 u(n) + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1). \end{aligned}$$

By Lemma 1.1.2, part (2), and $\Delta^2 u(n) \geq 0$, we have $\sum_{j=1}^{n}\Delta^2 k^{4-\alpha}(j)\Delta^2 u(n-j) \geq 0$. Thus, since $\alpha \in (\frac{6+\sqrt{2}}{2}, 4)$, and from the above, we have

$$\begin{aligned} \Delta^\alpha u(n) &\geq \frac{(3-\alpha)(2-\alpha)}{2}\Delta^2 u(n) + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1) \\ &= \frac{(3-\alpha)(2-\alpha)}{2}\frac{1}{2^n} + (3-\alpha)\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}} \\ &= \frac{2\alpha^2 - 12\alpha + 17}{2^{n+2}} \geq 0. \end{aligned}$$

This proves (i).

However, u *has negative jerk. In fact, a simple calculation show*

$$\Delta^3 u(n) = \Delta^2 u(n+1) - \Delta^2 u(n) = -\frac{1}{2^{n+1}} \leq 0.$$

Finally, notice that $u(3) = \frac{5}{2} < \alpha = \alpha u(2) - \frac{\alpha(\alpha-1)}{2}u(1) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}u(0)$. *It follows that the condition* (2) *in Theorem 2.2.4 is necessary in order to ensure that the sequence* u *has positive jerk.*

From Theorem 2.2.4 and the transference principle we deduce the following result.

Corollary 2.2.6. *Let* $3 \leq \alpha < 4$, $a \in \mathbb{R}$ *and* $v \in s(\mathbb{N}_a; \mathbb{R})$ *be given and assume that*

1. $\Delta_a^\alpha v(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+4-\alpha}$;
2. $v(a+3) \geq \alpha v(a+2) - \frac{\alpha(\alpha-1)}{2}v(a+1) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}v(a)$;
3. $v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2}v(a)$;
4. $v(a+1) \geq \alpha v(a)$;
5. $v(a) \geq 0$.

Then v *is positive, monotone increasing, convex and has positive jerk on* $\mathbb{N}_a$.

Proof. In case $\alpha = 3$ the conclusion is clear from the hypothesis. Define $u := \tau_a v$. Using the transference principle Theorem 1.1.6, we have,

$$\Delta^\alpha u(n) = \tau_{a+4-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u(n) = \tau_{a+4-\alpha} \circ \Delta_a^\alpha \circ v(n) = \Delta_a^\alpha v(t) \geq 0,$$

for $t := n + a + 4 - \alpha \in \mathbb{N}_{a+3-\alpha}$. The conclusion follows from Theorem 2.2.4. □

We end this chapter, noting that further applications of the results and methods developed in this section are possible in related areas such as, for instance, the study of positive solutions for discrete nonlinear fractional boundary value problems [26].

Chapter 3

Normal periodic solutions for the fractional abstract Cauchy problem.

In this chapter, we show that if A is a closed linear operator defined in a Banach space X and there exist $t_0 \geq 0$ and $M > 0$ such that $\{(im)^\alpha\}_{|m|>t_0} \subset \rho(A)$, the resolvent set of A, and

$$\|(im)^\alpha(A + (im)^\alpha I)^{-1}\| \leq M \quad \text{for all} \quad |m| > t_0, \quad m \in \mathbb{Z},$$

then, for each $\frac{1}{p} < \alpha \leq \frac{2}{p}$ and $1 < p < 2$, the abstract Cauchy problem with periodic boundary conditions

$$\begin{cases} {}_{GL}D_t^\alpha u(t) + Au(t) &= f(t), \quad t \in (0, 2\pi); \\ \\ u(0) &= u(2\pi), \end{cases}$$

where ${}_{GL}D^\alpha$ denotes the Grünwald-Letnikov derivative, admits a *normal* 2π-periodic solution for each $f \in L^p_{2\pi}(\mathbb{R}, X)$ that satisfy appropriate conditions. In particular, this happens if A is a sectorial operator with spectral angle $\phi_A \in (0, \alpha\pi/2)$ and $\int_0^{2\pi} f(t)dt \in Ran(A)$.

3.1 Normal 2π-periodic sequence

Let X be a complex Banach space and $A : D(A) \subset X \to X$ be a closed linear operator on X. Given $\alpha > 0$ and $f \in L^p_{2\pi}(\mathbb{R}, X)$, we are concerned with the problem of existence of periodic solutions of the equation

$$\begin{cases} {}_{GL}D^{\alpha}_t u(t) + Au(t) &= f(t), \quad t \in (0, 2\pi), \\ \\ u(0) &= u(2\pi). \end{cases} \tag{3.1.1}$$

Let $x \in X$ be fixed. Define $f_m(t) := e^{imt}x$, $m \in \mathbb{Z}$. It is clear that $f_m \in L^p_{2\pi}(\mathbb{R}, X)$ for any $p > 1$. Suppose that $x \in Ran(A + (in)^{\alpha} I)$ for some $\alpha > 0$ and some $n \in \mathbb{Z}$ fixed. Here

$$(in)^{\alpha} := |n|^{\alpha} e^{\frac{1}{2} sgn(n) i\pi\alpha}.$$

Then defining $u_n(t) := e^{int}u_n$ where $u_n \in D(A)$ is such that $x = (A + (in)^{\alpha} I)u_n$, and in view of Proposition 1.2.3 (part (i)) we obtain

$${}_{GL}D^{\alpha}_t u_n(t) + Au_n(t) = (in)^{\alpha} e^{int}u_n + e^{int}Au_n = e^{int}((in)^{\alpha} I + A)u_n = f_n(t).$$

In other words, $u_n(t)$ is a strict (or strong) 2π-periodic solution of (3.1.1).

Motivated by the previous example and the concept of a normal convergent series, we present the following definition.

Definition 3.1.1. We say that a sequence of functions $u_m : \mathbb{R} \to X$, $m \in \Lambda \subseteq \mathbb{Z}$ is a *normal 2π-periodic solution* of (3.1.1) if u_m is 2π-periodic, $u_m(t) \in D(A)$ for all $t \in \mathbb{R}$ and satisfies (3.1.1) for each $m \in \Lambda$ and the series $\sum_{n\in\mathbb{Z}} \mathbf{1}_{\Lambda}(n)u_n(t)$ is normally convergent.

In the above definition, we have $\mathbf{1}_{\Lambda}(n) \equiv 1$ if $n \in \Lambda$ and 0 in other case. Our main result is the following theorem.

Theorem 3.1.2. *Suppose there exist $t_0 \geq 0$ and $M > 0$ such that $\{(im)^{\alpha}\}_{|m|>t_0} \subset \rho(A)$ and*

$$\|(im)^{\alpha}(A + (im)^{\alpha} I)^{-1}\| \leq M \quad \textit{for all} \quad |m| > t_0,\ m \in \mathbb{Z}. \tag{3.1.2}$$

If $f \in L^p_{2\pi}(\mathbb{R}, X)$ where $1 < p \leq 2$, and there exists $u_m \in D(A)$ such that $\hat{f}_m = (A + (im)^\alpha I)u_m$ for each $|m| \leq t_0$, then the sequence of partial sums

$$u_N(t) := \sum_{|m|\leq t_0} u_m e^{imt} + \sum_{t_0<|m|\leq N} (A + (im)^\alpha I)^{-1}\widehat{f}_m e^{imt}, \quad N > t_0,$$

is a normal 2π-periodic solution of (3.1.1) *for all $\frac{1}{p} < \alpha \leq \frac{2}{p}$.*

Proof. Let $N > t_0$ be fixed and define

$$f_N(t) := \sum_{m=-N}^{N} \widehat{f}_m e^{imt},$$

and

$$v_{t_0}(t) := \sum_{|m|\leq t_0} u_m e^{imt}.$$

Note that by Proposition 1.2.3 we have

$$\begin{aligned} {}_{GL}D_t^\alpha(e^{imt}u_m) + A(e^{imt}u_m) &= (im)^\alpha e^{imt}u_m + e^{imt}Au_m \\ &= e^{imt}((im)^\alpha I + A)u_m = e^{imt}\widehat{f}_m. \end{aligned}$$

We conclude that v_{t_0} is a 2π-periodic solution of (3.1.1). On the other hand, the identity

$$A(A + (im)^\alpha I)^{-1} + (im)^\alpha(A + (im)^\alpha I)^{-1} = I$$

shows that

$$\begin{aligned} &{}_{GL}D_t^\alpha[(A + (im)^\alpha I)^{-1}\widehat{f}_m e^{imt}] + A[(A + (im)^\alpha I)^{-1}\widehat{f}_m e^{imt}] \\ &\qquad = (im)^\alpha(A + (im)^\alpha I)^{-1}\widehat{f}_m e^{imt} + A(A + (im)^\alpha I)^{-1}\widehat{f}_m e^{imt} \\ &\qquad = [(im)^\alpha(A + (im)^\alpha I)^{-1} + A(A + (im)^\alpha I)^{-1}]\widehat{f}_m e^{imt} \\ &\qquad = \widehat{f}_m e^{imt}. \end{aligned} \tag{3.1.3}$$

Therefore, we can conclude that ${}_{GL}D_t^\alpha u_N(t) + Au_N(t) = f_N(t)$ i.e. u_N is a 2π-periodic solution of (3.1.1).

Now, we study the convergence normal of the series. As a consequence of (3.1.2), we have for $|m| > t_0$:

$$||(A + (im)^\alpha I)^{-1}\widehat{f}_m e^{imt}||_\infty = \sup_{t\in\mathbb{T}}\{||(A + (im)^\alpha I)^{-1}\widehat{f}_m e^{imt}|| \leq \frac{M||\widehat{f}_m||}{|m|^\alpha}. \tag{3.1.4}$$

On the other hand, by a theorem of Hardy-Littlewood (see e.g. [36]) we know that there exists a constant $C > 0$ such that

$$\left(\sum_{m \in \mathbb{Z}} |m|^{p-2} \|\widehat{f}_m\|^p \right)^{1/p} \leq C\|f\|_p,$$

for each $1 < p \leq 2$. Hence, by Hölder's inequality, we obtain for any $1 \leq q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned} \sum_{|m|>t_0} \frac{M||\widehat{f}_m||}{|m|^\alpha} &\leq M \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{|m|^{(p-2)/p}||\widehat{f}_m||}{|m|^{\alpha+(p-2)/p}} \\ &\leq MC \left(\sum_{m \in \mathbb{Z}\setminus\{0\}} |m|^{p-2}||\widehat{f}_m||^p \right)^{1/p} \left(\sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{1}{|m|^{q(\alpha+(p-2)/p)}} \right)^{1/q}. \end{aligned}$$

Since $1 < \alpha p \leq 2$, we obtain $q(\alpha + (p-2)/p) > 1$ and the result follows.

$\square$

For instance, in case $p = 2$ the restriction is : $\frac{1}{2} < \alpha \leq 1$. A complete picture is given in the Figure 3.1, below.

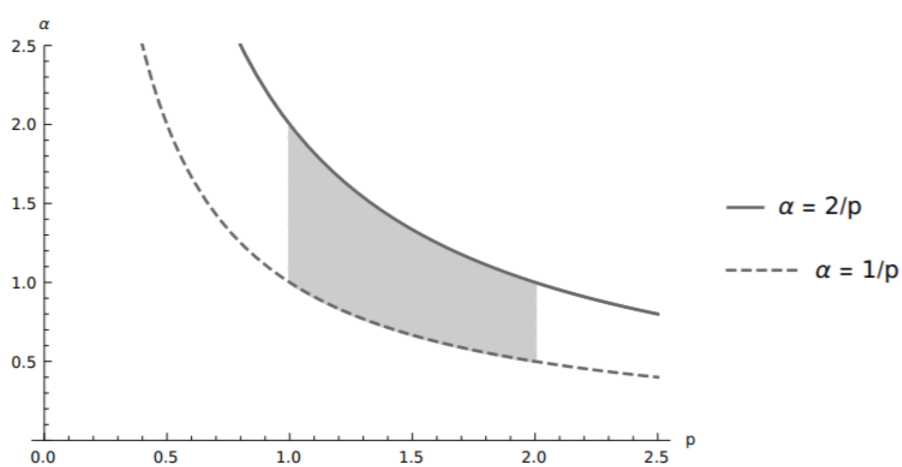

Figure 3.1: $\frac{1}{p} < \alpha \leq \frac{2}{p}$ with $1 < p \leq 2$.

An immediate consequence of Theorem 3.1.2 is the following.

Corollary 3.1.3. *Let $A \in \mathcal{B}(X)$ and $1 < p \leq 2$. If $f \in L^p_{2\pi}(\mathbb{R}, X)$ satisfies that $\widehat{f}_m \in Ran(A + (im)^\alpha I)$ for $|m| \leq \|A\|^{1/\alpha}$, where $\frac{1}{p} < \alpha \leq \frac{2}{p}$, then the conclusions of Theorem 3.1.2 holds.*

Proof. Since A is bounded we have that $\lambda \in \rho(A)$ for all $|\lambda| > \|A\|$ and

$$(\lambda + A)^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}.$$

Choosing $t_0 := \|A\|^{\frac{1}{\alpha}}$ and $\lambda = (it)^\alpha$ we obtain for each $|t| > t_0$ that $\{(it)^\alpha\}_{|t|>t_0} \subset \rho(A)$ and $\|((it)^\alpha + A)^{-1}\| = \|(\lambda + A)^{-1}\| \leq \frac{e}{|\lambda|} = \frac{M}{|t|^\alpha}$ for each $|t| > t_0$, where $M := e$. $\square$

3.2 Additive perturbation

In the case of unbounded operators on Hilbert spaces, we have the following result that generalizes and improves [59, Chapter B, Lecture 20, Corollary 10, pag 157].

Theorem 3.2.1. *Let B a selfadjoint operator with domain $D(B)$ defined on a Hilbert space H and $C \in \mathcal{B}(H)$. Assume that B commutes with C and let $A = B + C$. Let $1 < p \leq 2$. If $f \in L^p_{2\pi}(\mathbb{R}, H)$ satisfies that $\widehat{f}_m \in Ran(A + (im)^\alpha I)$ for $|m| \leq \frac{2^{1/\alpha}||C||^{1/\alpha}}{|\sin(\alpha\pi/2)|^{1/\alpha}}$, $m \in \mathbb{Z}$, where $\frac{1}{p} < \alpha \leq \frac{2}{p}$, then the conclusions of Theorem 3.1.2 holds.*

Proof. Let $s \in \mathbb{R}$ such that $|s| > s_0 := \frac{2^{1/\alpha}||C||^{1/\alpha}}{|\sin(\alpha\pi/2)|^{1/\alpha}}$. Since B is selfadjoint, we have $\sigma(B) \subset \mathbb{R}$ and

$$||(B + \lambda I)^{-1}|| \leq \frac{1}{|Im(\lambda)|}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (see e.g. [58, Proposition C.4.2., pag 321]). In particular, choosing $\lambda = (is)^\alpha$, we obtain that $\{(is)^\alpha\}_{|s|>s_0} \subset \rho(B)$ and

$$||(B + (is)^\alpha I)^{-1}|| \leq \frac{1}{|Im((is)^\alpha)|} = \frac{1}{|\sin(\frac{\pi\alpha}{2})||s|^\alpha}.$$

This yields

$$||C(B + (is)^\alpha I)^{-1}|| \leq \frac{||C||}{|\sin(\frac{\pi\alpha}{2})||s|^\alpha} < \frac{1}{2}.$$

It implies that $(I + C(B + (is)^\alpha I)^{-1})^{-1}$ exists and

$$||(I + C(B + (is)^\alpha I)^{-1})^{-1}|| \leq \frac{1}{1 - ||C(B + (is)^\alpha I)^{-1}||} \leq \frac{1}{1 - \frac{||C||}{|\sin(\frac{\pi\alpha}{2})||s|^\alpha}} \leq 2.$$

Since B commutes with C we have the identity $A + (is)^\alpha I = (B + (is)^\alpha I)(C(B + (is)^\alpha I)^{-1} + I)$. It shows that $\{(is)^\alpha\}_{|s|>s_0} \subset \rho(A)$ and

$$(B + (is)^\alpha I)^{-1}(C(B + (is)^\alpha I)^{-1} + I)^{-1} = (C + (B + (is)^\alpha I))^{-1} = (A + (is)^\alpha I)^{-1}, \qquad (3.2.1)$$

holds. Therefore

$$||(A + (is)^\alpha I)^{-1}|| \leq ||(B + (is)^\alpha I)^{-1}||\, ||(C(B + (is)^\alpha I)^{-1} + I)^{-1}|| \leq \frac{2}{|\sin(\frac{\pi\alpha}{2})||s|^\alpha},$$

for each $|s| > s_0$. □

In case $C \equiv 0$ we obtain the following result that has own interest.

Corollary 3.2.2. *Let A a selfadjoint operator with domain $D(A)$ defined on a Hilbert space H. Let $1 < p \leq 2$. If $f \in L^p_{2\pi}(\mathbb{R}, H)$ satisfies that $\int_0^{2\pi} f(t)dt \in Ran(A)$ then the conclusions of Theorem 3.1.2 holds for all $\frac{1}{p} < \alpha \leq \frac{2}{p}$.*

The case of Banach spaces is considered in the next result.

Theorem 3.2.3. *Let $A = B + C$ where B is sectorial with spectral angle $\phi_B \in (0, \alpha\pi/2)$ where $\frac{1}{p} < \alpha \leq \frac{2}{p}$, $1 < p \leq 2$ and let $C \in \mathcal{B}(X)$ be such that C commutes with B. If $f \in L^p_{2\pi}(\mathbb{R}, X)$ satisfies that $\widehat{f}_m \in Ran(A + (im\omega)^\alpha I)$ for $|m| \leq 2^{1/\alpha}\|C\|^{1/\alpha}$, $m \in \mathbb{Z}$, then the conclusions of Theorem 3.1.2 holds.*

Proof. Let $s \in \mathbb{R}$ be such that $|s| > s_0 := 2^{1/\alpha}\|C\|^{1/\alpha}$. Since B is sectorial, $\Sigma_\phi \subset \rho(-B)$ for some $\phi > 0$ and there exists $M > 0$ such that

$$||(\lambda + B)^{-1}|| \leq \frac{M}{|\lambda|}, \quad \lambda \in \Sigma_\phi.$$

Since $\frac{1}{2} < \alpha < 2$, if $\lambda = (is)^\alpha := |s|^\alpha e^{\frac{1}{2} sgn(s)\pi i\alpha}$ then $\lambda \in \Sigma_\phi$ and

$$||((is)^\alpha I + B)^{-1}|| \leq \frac{M}{|s|^\alpha} \quad \text{for all} \quad 0 \neq s \in \mathbb{R}.$$

Hence,

$$||C(B + (is)^\alpha I)^{-1}|| \leq \frac{M\|C\|}{|s|^\alpha} < \frac{M}{2}. \tag{3.2.2}$$

On the other hand, since B and C commutes, we obtain that $\{(is)^\alpha\}_{|s|>s_0} \subset \rho(A)$ and the identity (3.2.1) as in the proof of Theorem 3.2.1. Therefore

$$\begin{aligned} ||(A + (is)^\alpha I)^{-1}|| &\leq ||(B + (is)^\alpha I)^{-1}|| \, ||(C(B + (is)^\alpha I)^{-1} + I)^{-1}|| \\ &\leq \frac{M}{|s|^\alpha} ||(C(B + (is)^\alpha I)^{-1} + I)^{-1}|| \end{aligned} \tag{3.2.3}$$

for each $|s| > s_0$.

We consider the following two cases:

(i) If $0 < M \leq 1$ then $||C(B + (is)^\alpha I)^{-1}|| < \frac{1}{2}, \quad |s| > s_0$.

(ii) If $M > 1$ then, dividing by M in (3.2.2), we obtain

$$(1/M)||C(B + (is)^\alpha I)^{-1}|| < \frac{1}{2}, \quad |s| > s_0.$$

If we define for $\delta \in \{1, M\}$ the norm $|| \cdot ||_\delta := (1/\delta)|| \cdot ||$, we have

$$||C(B + (is)^\alpha I)^{-1}||_\delta < \frac{1}{2}, \quad |s| > s_0. \tag{3.2.4}$$

Therefore, for all $0 \neq s \in \mathbb{R}$, $||C|| < |s|$, there exists $(I + C(B + (is)^\alpha I)^{-1})^{-1}$ and

$$||(I + C(B + (is)^\alpha I)^{-1})^{-1}||_\delta \leq \frac{1}{1 - ||C(B + (is)^\alpha I)^{-1}||_\delta} \leq 2.$$

Hence

$$||(I + C(B + (is)^\alpha I)^{-1})^{-1}|| \leq 2M.$$

Inserting the above inequality in (3.2.3) we obtain

$$||(A + (is)^\alpha I)^{-1}|| \leq \frac{2M^2}{|s|^\alpha}, \quad |s| > s_0.$$

$\square$

Again, the special case $C \equiv 0$ gives the next corollary.

Corollary 3.2.4. *Let A be a sectorial operator with spectral angle $\phi_A \in (0, \alpha\pi/2)$ where $\frac{1}{p} < \alpha \leq \frac{2}{p}$, $1 < p \leq 2$. Assume that $f \in L^p_{2\pi}(\mathbb{R}, X)$ satisfies that $\int_0^{2\pi} f(t)dt \in Ran(A)$ then the conclusions of Theorem 3.1.2 holds.*

Chapter 4

The abstract Cauchy problem with Caputo-Fabrizio fractional derivative

In this chapter, we consider the following problem: given an injective closed linear operator A defined in a Banach space X, and writting ${}_{CF}D_t^{\alpha}$ the Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$, we show that the unique solution of the abstract Cauchy problem

$$ {}_{CF}D_t^{\alpha} u(t) = Au(t) + f(t), \quad t \geq 0, \tag{4.0.1} $$

where f is continuously differentiable, is given by the unique solution of the first order abstract Cauchy problem

$$ \begin{cases} u'(t) = B_{\alpha} u(t) + F_{\alpha}(t), \quad t \geq 0; \\ u(0) = -A^{-1} f(0), \end{cases} $$

where the family of bounded linear operators B_{α} constitutes a Yosida approximation of A and $F_{\alpha}(t) \to f(t)$ as $\alpha \to 1$. Also, if $\frac{1}{1-\alpha} \in \rho(A)$ and the spectrum of A is contained outside the closed disk of center and radius equal to $\frac{1}{2(1-\alpha)}$ then the solution of (4.0.1) converges to zero as $t \to \infty$, in the norm of X, provided f and f' have exponential decay. Finally, assuming a Lipchitz-type condition on $f = f(t,x)$ (and its time-derivative) that depends on α, we prove the existence and uniqueness of mild solutions for the respective semilinear problem, for all initial conditions in the set $\mathcal{S} := \{x \in D(A) \,:\, x = A^{-1}f(0,x)\}$.

4.1 Well-Posedness

Let X be a complex Banach space and $A : D(A) \subset X \to X$ be a closed linear operator on X. Given $0 < \alpha < 1$ and $f : [0,\infty) \to X$ be a function, in this section we are concerned with the problem of existence of solutions to the equation

$$ {}_{CF}D_t^{\alpha} u(t) = Au(t) + f(t), \quad t \geq 0. \tag{4.1.1}$$

Definition 4.1.1. A function $u : [0,\infty) \to X$ is said a strong solution of (4.1.1) if u is continuously differentiable with $u(t) \in D(A)$, $t \geq 0$, and satisfies (4.1.1).

Remark 4.1.2. Observe that, in equation (4.1.1) when $t = 0$ we have ${}_{CF}D_t^{\alpha} u(0) = Au(0) + f(0)$, i.e., $Au(0) = -f(0)$. Therefore, the value $u(0)$ is implicitly prescribed although it is not given as an initial condition. This condition will be important to show that the solution is unique in the classical sense depending on the properties of the operator A.

Let $0 < \alpha < 1$ be fixed. Assuming that $\frac{1}{1-\alpha} \in \rho(A)$, we define

$$ N_{\alpha} := (I - (1-\alpha)A)^{-1} \in \mathcal{B}(X). $$

Since $B_{\alpha} := \frac{\alpha}{1-\alpha}(N_{\alpha} - I)$ is a bounded operator, it defines the uniformly continuous group $(T_{\alpha}(t))_{t\in\mathbb{R}}$ on X, given by (see [40, Theorem I.3.7]):

$$ T_{\alpha}(t) = \exp(tB_{\alpha}) = \sum_{k=0}^{\infty} \frac{t^k B_{\alpha}^k}{k!}, \; t \in \mathbb{R}. \tag{4.1.2}$$

Let $f : [0, \infty) \to X$ be continuously differentiable and define

$$F_\alpha(t) := (1 - \alpha) N_\alpha f'(t) + \alpha N_\alpha f(t).$$

Note that, $F_\alpha(t) \in D(A)$ for all $t \in [0, \infty)$. For each $x_0 \in X$, we define:

$$u(t) = T_\alpha(t) x_0 + \int_0^t T_\alpha(t - \tau) F_\alpha(\tau) d\tau, \ t \geq 0. \tag{4.1.3}$$

Then is well-known [83, Section 4.2] that u is a strong solution of

$$u'(t) = B_\alpha u(t) + F_\alpha(t), \quad t \geq 0. \tag{4.1.4}$$

Observe from (4.1.3) that $x_0 = u(0)$. If $x_0 \in D(A)$ is such that $Ax_0 = -f(0)$ then u solves the following, particular, initial value problem

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \quad t \geq 0; \\ Au(0) = -f(0). \end{cases} \tag{4.1.5}$$

Note that in the initial condition we are not yet assuming any conditions on the invertibility of A.

We recall the following definitions, applied to (4.1.5).

Definition 4.1.3. Let $x_0 \in X$ be given. The function u defined by (4.1.3) is called *mild* solution of the initial value problem (4.1.5).

Definition 4.1.4. A function $u : [0, \infty) \to X$ is a strong solution of (4.1.5) if u is continuously differentiable with $u(0) \in D(A)$ and satisfies (4.1.5).

The following result is well-known, except for the new necessary condition imposed on the operator A.

Proposition 4.1.5. *Every strong solution of* (4.1.5) *is also a mild solution. And, if A is injective and $u(0) \in D(A)$ then every mild solution of* (4.1.5) *is also strong solution.*

Proof. Suppose u is strong solution of (4.1.5), then it is well know that u is defined by

$$u(t) = T_\alpha(t)u(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ t \geq 0,$$

and is a mild solution (see [83, Section 4.2]). Now, if A is injective then u defined by

$$u(t) = -T_\alpha(t)A^{-1}f(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, ,\ t \geq 0. \tag{4.1.6}$$

verifies (4.1.4) and, since $u(0) \in D(A)$, we have $Au(0) = -AT_\alpha(0)A^{-1}f(0) = -f(0)$. Hence, (4.1.6) is a strong solution of (4.1.5). □

Next, we collect some important properties of the operators previously defined.

Proposition 4.1.6. *Let $0 < \alpha < 1$, $\beta := \frac{\alpha}{1-\alpha}$, A be a closed linear operator on X with domain $D(A)$, $\frac{1}{1-\alpha} \in \rho(A)$ and $T_\alpha(t) = \exp(tB_\alpha)$, $t \in \mathbb{R}$, where $B_\alpha = \beta(N_\alpha - I)$ and $N_\alpha = (I-(1-\alpha)A)^{-1}$. The following statements hold:*

(i) $B_\alpha x = \alpha N_\alpha Ax$ *and* $AN_\alpha x = N_\alpha Ax$, *for* $x \in D(A)$;

(ii) $B_\alpha^k x \in D(A)$ *and* $B_\alpha^k Ax = AB_\alpha^k x$, *for* $x \in D(A)$ *and* $k \in \mathbb{N}_0$;

(iii) $T_\alpha(t)x \in D(A)$ *and* $AT_\alpha(t)x = T_\alpha(t)Ax$, *for* $t \in \mathbb{R}$ *and* $x \in D(A)$;

(iv) $B_\alpha^k N_\alpha x = N_\alpha B_\alpha^k x$, *for* $x \in X$ *and* $k \in \mathbb{N}_0$;

(v) $N_\alpha T_\alpha(t)x = T_\alpha(t)N_\alpha x$, *for* $t \in \mathbb{R}$ *and* $x \in X$.

Proof. (i) For $x \in D(A)$, we have $Ix = N_\alpha N_\alpha^{-1}x = N_\alpha x - (1-\alpha)N_\alpha Ax$ then $N_\alpha x - Ix = (1-\alpha)N_\alpha Ax$, obtaining the claim.

(ii) First, by proceed by induction on k, we prove that $B_\alpha^k x \in D(A)$, for $x \in D(A)$. In fact, for $k = 0$ is trivial and for $k = 1$, by property (i), we have $B_\alpha x = \alpha N_\alpha Ax \in D(A)$, for all $x \in D(A)$. Suppose that for $k \in \mathbb{N}_0$, we have $B_\alpha^k x \in D(A)$, for all $x \in D(A)$. Then, for $k+1$ and $x \in D(A)$ we obtain $B_\alpha^{k+1}x = B_\alpha^k(B_\alpha x) \in D(A)$.

Now, again by proceed by induction on k, we prove that $B_\alpha^k Ax = AB_\alpha^k x$, for $x \in D(A)$. Indeed, for $k = 0$ is trivial and for $k = 1$, by property (i) and the above case, we have for $x \in D(A)$

$$\begin{aligned} B_\alpha Ax &= \beta(N_\alpha - I)Ax = \beta[N_\alpha Ax - Ax] = \beta[AN_\alpha x - Ax] \\ &= A[\beta(N_\alpha - I)]x = AB_\alpha x. \end{aligned}$$

Suppose that for $k \in \mathbb{N}_0$, we have $B_\alpha^k Ax = AB_\alpha^k x$, for all $x \in D(A)$. Then, by property (i), we obtain for $k+1$ and $x \in D(A)$:

$$\begin{aligned} B_\alpha^{k+1} Ax &= B_\alpha^k (B_\alpha A)x = B_\alpha^k (AB_\alpha)x = (B_\alpha^k A)B_\alpha x \\ &= (AB_\alpha^k)B_\alpha x = AB_\alpha^{k+1} x. \end{aligned}$$

(iii) Let $n \in \mathbb{N}_0$ and $x \in D(A)$, define $T_{\alpha,n}(t)x := \sum_{k=0}^n \frac{t^k B_\alpha^k}{k!} x,\ t \in \mathbb{R}$. By property (ii), we have $T_{\alpha,n}(t)x \in D(A)$, for $n \in \mathbb{N}_0$, and

$$\lim_{n\to\infty} T_{\alpha,n}(t)x = T_\alpha(t)x,\ t \in \mathbb{R}.$$

Note that, by property (ii), we obtain for $x \in D(A)$

$$AT_{\alpha,n}(t)x = \sum_{k=0}^n \frac{t^k AB_\alpha^k}{k!} x = \sum_{k=0}^n \frac{t^k B_\alpha^k A}{k!} x = T_{\alpha,n}(t)Ax,\ t \in \mathbb{R}.$$

Hence, by the above identities, we have for $x \in D(A)$

$$\lim_{n\to\infty} AT_{\alpha,n}(t)x = \lim_{n\to\infty} T_{\alpha,n}(t)Ax = T_\alpha(t)Ax,\ t \in \mathbb{R}.$$

Thus, since A is a closed operator, for $x \in D(A)$ we obtain $T_\alpha(t)x \in D(A)$ and

$$AT_\alpha(t)x = T_\alpha(t)Ax,\ t \in \mathbb{R}.$$

(iv) By proceed by induction on k. For $k = 0$ is trivial and for $k = 1$, we have for $x \in X$: $B_\alpha N_\alpha x = \beta(N_\alpha - I)N_\alpha x = \beta[N_\alpha^2 x - N_\alpha x] = N_\alpha[\beta(N_\alpha - I)]x = N_\alpha B_\alpha x$.

Suppose that for $k \in \mathbb{N}_0$, we have $B_\alpha^k N_\alpha x = N_\alpha B_\alpha^k x$, for all $x \in X$. Then, we obtain for $k+1$ and $x \in X$:

$$\begin{aligned} B_\alpha^{k+1} N_\alpha x &= B_\alpha^k (B_\alpha N_\alpha)x = B_\alpha^k (N_\alpha B_\alpha)x = (B_\alpha^k N_\alpha)B_\alpha x \\ &= (N_\alpha B_\alpha^k)B_\alpha x = N_\alpha B_\alpha^{k+1} x. \end{aligned}$$

(v) We define $T_{\alpha,n}(t)x := \sum_{k=0}^{n} \frac{t^k B_\alpha^k}{k!}x,\ t \in \mathbb{R}$, for $n \in \mathbb{N}_0$ and $x \in X$. Since N_α is a bounded operator, we have for $x \in X$

$$N_\alpha T_\alpha(t)x = N_\alpha \lim_{n\to\infty} T_{\alpha,n}(t)x = \lim_{n\to\infty} N_\alpha T_{\alpha,n}(t)x,\ t \in \mathbb{R}.$$

By property (iv), we have for $x \in X$

$$N_\alpha T_{\alpha,n}(t)x = \sum_{k=0}^{n} \frac{t^k N_\alpha B_\alpha^k}{k!}x = \sum_{k=0}^{n} \frac{t^k B_\alpha^k N_\alpha}{k!}x = T_{\alpha,n}(t)N_\alpha x,\ t \in \mathbb{R}.$$

Thus, $N_\alpha T_\alpha(t)x = T_\alpha(t)N_\alpha x$, $t \in \mathbb{R}$, for all $x \in X$.

□

Remark 4.1.7. Let $0 < \alpha < 1$ and A be a closed linear operator on X with $\frac{1}{1-\alpha} \in \rho(A)$. First note that, by Proposition 4.1.6, part (i), we have

$$B_\alpha x = \alpha A N_\alpha x = \alpha A(I - (1-\alpha)A)^{-1}x = \frac{\alpha}{1-\alpha}A(\frac{1}{1-\alpha}I - A)^{-1}x,\ \text{for all}\ \ x \in D(A).$$

Thus, by the above identity, we obtain

$$\lim_{\alpha\to 1} B_\alpha x = Ax, \quad \text{for all}\ \ x \in D(A). \tag{4.1.7}$$

Now, let $s := \frac{1}{1-\alpha}$, we define for every $s > 1$

$$B_s := (s-1)A(sI - A)^{-1} = (1-s)[s(sI-A)^{-1} - I].$$

Note that if A is a densely defined operator on X then we deduce the following: *B_s is a Yosida approximation of A* [83, Theorem 1.3.1]. Moreover, since each B_s is bounded, it generates a uniformly continuous semigroup $(T_s(t))_{t>0}$ on X. Then, there exists $M \geq 1$ such that

$$||T_s(t)||_{\mathcal{B}(X)} \leq M \quad \text{for all}\ \ t \geq 0. \tag{4.1.8}$$

Observe that for $\alpha \to 1$ we have $s \to \infty$. Thus, by (4.1.7), we get

$$\lim_{s\to\infty} B_s x = Ax, \quad \text{for all}\ \ x \in D(A). \tag{4.1.9}$$

Since $\frac{1}{1-\alpha} \in \rho(A)$, we have for $\alpha \to 1$ that

$$(\omega, \infty) \subset \rho(A), \quad \omega > 1. \tag{4.1.10}$$

Therefore, by [9, Corollary 3.6.3], (4.1.8), (4.1.9) and (4.1.10), we have that A generates a C_0-semigroup T and for all $x \in X$,

$$\lim_{s\to\infty} T_s(t)x = T(t)x$$

uniformly for $t \in [0,\tau]$ for all $\tau > 0$.

The following is the main result of this section, and one of the main theorems of this chapter: We show that (4.1.1) is well-posed if and only if (4.1.5) is well-posed.

Theorem 4.1.8. *Let $0 < \alpha < 1$, A be a closed linear operator on X with domain $D(A)$ and $f : [0,\infty) \to X$ continuously differentiable. Assume that $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$. Then, the problem given by*

$$_{CF}D_t^\alpha v(t) = Av(t) + f(t), \quad t \geq 0, \tag{4.1.11}$$

has a unique strong solution if and only if the initial value problem given by

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \quad t \geq 0; \\ u(0) = -A^{-1}f(0), \end{cases} \tag{4.1.12}$$

has a unique strong solution, where $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ and $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$ with $N_\alpha = (I-(1-\alpha)A)^{-1}$.

Proof. Suppose that v is the unique strong solution of (4.1.12). Then, v is continuously differentiable with $v(0) \in D(A)$ and satisfies (4.1.12). In particular, v is mild solution of (4.1.12). Thus, by Definition 4.1.3

$$v(t) = T_\alpha(t)v(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \ t \geq 0, \tag{4.1.13}$$

where $(T_\alpha(t))_{t\in\mathbb{R}}$ is the uniformly continuous group generated by B_α.

We first observe that $v(t) \in D(A), \ t \geq 0$. Indeed, by hypothesis, $v(0) \in D(A)$ and by Proposition 4.1.6, part (iii), we have $T_\alpha(t)v(0) \in D(A)$. Again by Proposition 4.1.6, part (iii), we obtain that $\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau \in D(A)$, because $F_\alpha(t) \in D(A)$ and A is a closed operator. This proves that

$$v(t) \in D(A), \quad t \geq 0.$$

Next, we observe some identities that v verifies. Since v is strong solution we have $v(0) \in D(A)$ and $v(0) = -A^{-1}f(0)$, i.e., $Av(0) = -f(0)$. Then, by Proposition 4.1.6, part (iii), we have

$$AT_\alpha(t)v(0) = T_\alpha(t)Av(0) = -T_\alpha(t)f(0).$$

Thus, applying A to the identity (4.1.13) and using the above identity, we obtain that v verifies

$$Av(t) = -T_\alpha(t)f(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ t \geq 0. \tag{4.1.14}$$

Note that, by Proposition 4.1.6, part (i), we have

$$N_\alpha A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau = AN_\alpha\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ t \geq 0, \tag{4.1.15}$$

since that $\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau \in D(A)$ for all $t \geq 0$. Then, operating by αN_α the identity (4.1.14) and using the identity (4.1.15), we have

$$\alpha N_\alpha Av(t) = -\alpha N_\alpha T_\alpha(t)f(0) + \alpha AN_\alpha\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ t \geq 0.$$

By Proposition 4.1.6, part (i), and the property of group, we have that the previous identity is equivalent to

$$B_\alpha v(t) = -\alpha N_\alpha T_\alpha(t)f(0) + \alpha AN_\alpha\int_0^t T_\alpha(t)T_\alpha(-\tau)F_\alpha(\tau)d\tau,\ t \geq 0. \tag{4.1.16}$$

After this preliminaries, we will show that v satisfies (4.1.11). By definition of Caputo-Fabrizio fractional derivative and the fact that v satisfies equation (4.1.12), we have

$$\begin{aligned}
{}_{CF}D_t^\alpha v(t) &= \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)v'(s)ds \\
&= \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)[B_\alpha v(s) + F_\alpha(s)]ds \\
&= \frac{\exp\left(-\frac{\alpha t}{1-\alpha}\right)}{1-\alpha}\left[\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)B_\alpha v(s)ds + \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)F_\alpha(s)ds\right] \\
&= \frac{\exp\left(-\frac{\alpha t}{1-\alpha}\right)}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)B_\alpha v(s)ds \\
&\quad + \frac{\exp\left(-\frac{\alpha t}{1-\alpha}\right)}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)F_\alpha(s)ds.
\end{aligned} \tag{4.1.17}$$

In the above identity, we will find an equivalent representation of the following expression:

$$\frac{\exp\left(-\frac{\alpha t}{1-\alpha}\right)}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds. \tag{4.1.18}$$

Replacing the identity (4.1.16) in $\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds$ we obtain

$$\begin{aligned}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds = -\alpha N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) T_\alpha(s) f(0) ds\\ + \alpha A N_\alpha \int_0^t \int_0^s \exp\left(\frac{\alpha s}{1-\alpha}\right) T_\alpha(s) T_\alpha(-\tau) F_\alpha(\tau) d\tau ds.\end{aligned} \tag{4.1.19}$$

Observe that, $(\exp\left(\frac{\alpha t}{1-\alpha}\right) T_\alpha(t))_{t\in\mathbb{R}}$ is a group whose generator is $\frac{\alpha}{1-\alpha} I + B_\alpha = \frac{\alpha}{1-\alpha} N_\alpha$. Hence, by [40, Chapter II], we have

$$\alpha N_\alpha \int_\tau^t \exp\left(\frac{\alpha s}{1-\alpha}\right) T_\alpha(s) f(0) ds = (1-\alpha)[\exp\left(\frac{\alpha t}{1-\alpha}\right) T_\alpha(t) - \exp\left(\frac{\alpha \tau}{1-\alpha}\right) T_\alpha(\tau)] f(0). \tag{4.1.20}$$

Thus, using the identity (4.1.20) in (4.1.19), we obtain

$$\begin{aligned}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds = (1-\alpha) f(0) - (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) T_\alpha(t) f(0)\\ + \alpha A N_\alpha \int_0^t \int_0^s \exp\left(\frac{\alpha s}{1-\alpha}\right) T_\alpha(s) T_\alpha(-\tau) F_\alpha(\tau) d\tau ds.\end{aligned} \tag{4.1.21}$$

Note that, using Fubini's theorem, Proposition 4.1.6, part (v), and the identity (4.1.20), we get that

$$\begin{aligned}&\alpha A N_\alpha \int_0^t \int_0^s \exp\left(\frac{\alpha s}{1-\alpha}\right) T_\alpha(s) T_\alpha(-\tau) F_\alpha(\tau) d\tau ds\\ &= \alpha A N_\alpha \int_0^t \int_\tau^t \exp\left(\frac{\alpha s}{1-\alpha}\right) T_\alpha(s) T_\alpha(-\tau) F_\alpha(\tau) ds d\tau\\ &= A \int_0^t T_\alpha(-\tau) \alpha N_\alpha \int_\tau^t \exp\left(\frac{\alpha s}{1-\alpha}\right) T_\alpha(s) F_\alpha(\tau) ds d\tau\\ &= A \int_0^t T_\alpha(-\tau)(1-\alpha)\left[\exp\left(\frac{\alpha t}{1-\alpha}\right) T_\alpha(t) - \exp\left(\frac{\alpha \tau}{1-\alpha}\right) T_\alpha(\tau)\right] F_\alpha(\tau) d\tau\\ &= (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) A \int_0^t T_\alpha(t-\tau) F_\alpha(\tau) d\tau - (1-\alpha) A \int_0^t \exp\left(\frac{\alpha \tau}{1-\alpha}\right) F_\alpha(\tau) d\tau.\end{aligned} \tag{4.1.22}$$

Thus, replacing (4.1.22) in (4.1.21) and using the identity (4.1.14), we obtain

$$\begin{aligned}
&\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds \\
&= (1-\alpha)f(0) - (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) T_\alpha(t)f(0) \\
&\quad + (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau - (1-\alpha)A\int_0^t \exp\left(\frac{\alpha\tau}{1-\alpha}\right) F_\alpha(\tau)d\tau \\
&= (1-\alpha)f(0) + (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right)\left[-T_\alpha(t)f(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau\right] \\
&\quad - (1-\alpha)A\int_0^t \exp\left(\frac{\alpha\tau}{1-\alpha}\right) F_\alpha(\tau)d\tau \\
&= (1-\alpha)f(0) + (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) Av(t) - (1-\alpha)A\int_0^t \exp\left(\frac{\alpha\tau}{1-\alpha}\right) F_\alpha(\tau)d\tau.
\end{aligned} \tag{4.1.23}$$

Therefore, multiplying by $\frac{1}{1-\alpha}\exp\left(-\frac{\alpha t}{1-\alpha}\right)$ the identity (4.1.23), we get finally (4.1.18):

$$\begin{aligned}
&\frac{\exp\left(-\frac{\alpha t}{1-\alpha}\right)}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds \\
&= \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) + Av(t) - \exp\left(\frac{-\alpha t}{1-\alpha}\right) A\int_0^t \exp\left(\frac{\alpha\tau}{1-\alpha}\right) F_\alpha(\tau)d\tau.
\end{aligned} \tag{4.1.24}$$

This gives us the desired representation.

We return to the identity (4.1.17). Replacing (4.1.24) in (4.1.17), we obtain

$$\begin{aligned}
{}_{CF}D_t^\alpha v(t) &= \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) + Av(t) - \exp\left(\frac{-\alpha t}{1-\alpha}\right) A\int_0^t \exp\left(\frac{\alpha\tau}{1-\alpha}\right) F_\alpha(\tau)d\tau \\
&\quad + \frac{\exp\left(\frac{-\alpha t}{1-\alpha}\right)}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s)ds \\
&= Av(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) \\
&\quad + \left[-\exp\left(\frac{-\alpha t}{1-\alpha}\right) A + \frac{\exp\left(\frac{-\alpha t}{1-\alpha}\right)}{1-\alpha} I\right]\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s)ds.
\end{aligned} \tag{4.1.25}$$

Now, we calculate $\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s)ds$. By definition of F_α and integration by parts, we have

$$\begin{aligned}
&\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s)ds \\
&\quad= \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) [(1-\alpha)N_\alpha f'(s) + \alpha N_\alpha f(s)]ds \\
&\quad= (1-\alpha)N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f'(s)ds + \alpha N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f(s)ds \\
&\quad= (1-\alpha)N_\alpha \left[\exp\left(\frac{\alpha t}{1-\alpha}\right) f(t) - f(0) - \frac{\alpha}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f(s)ds\right] \qquad (4.1.26)\\
&\qquad + \alpha N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f(s)ds \\
&\quad= (1-\alpha)N_\alpha \left[\exp\left(\frac{\alpha t}{1-\alpha}\right) f(t) - f(0)\right] \\
&\quad= (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) N_\alpha f(t) - (1-\alpha)N_\alpha f(0).
\end{aligned}$$

Thus, replacing (4.1.26) in (4.1.25) and using the identity $I = N_\alpha - (1-\alpha)AN_\alpha$, we obtain

$$\begin{aligned}
&{}_{CF}D_t^\alpha v(t) \\
&= Av(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) \\
&\quad + \left[-\exp\left(\frac{-\alpha t}{1-\alpha}\right) A + \frac{\exp\left(\frac{-\alpha t}{1-\alpha}\right)}{1-\alpha} I\right] \left[(1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) N_\alpha f(t) - (1-\alpha)N_\alpha f(0)\right] \\
&= Av(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) - (1-\alpha)AN_\alpha f(t) \\
&\quad + (1-\alpha)\exp\left(\frac{-\alpha t}{1-\alpha}\right) AN_\alpha f(0) + N_\alpha f(t) - \exp\left(\frac{-\alpha t}{1-\alpha}\right) N_\alpha f(0) \\
&= Av(t) + [N_\alpha - (1-\alpha)AN_\alpha]f(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right)[I + (1-\alpha)AN_\alpha - N_\alpha]f(0) \\
&= Av(t) + f(t).
\end{aligned}$$

The above show that v is strong solution of (4.1.11) and, by (4.1.13):

$$v(t) = T_\alpha(t)v(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau. \qquad (4.1.27)$$

Finally, we show uniqueness. Assume that w is strong solution of (4.1.11) and set $s := v - w$.

Then, by linearity of the operator ${}_{CF}D_t^{\alpha}$, we have that s is a strong solution of the equation

$$ {}_{CF}D_t^{\alpha}s(t) = As(t),\ t \geq 0. \tag{4.1.28}$$

Since ${}_{CF}D_t^{\alpha}s(0) = 0$, then $As(0) = 0$. Thus, $s(0) = 0$ because A is injective. Using the identity (4.1.27) for the problem (4.1.28) ($f \equiv 0$), we have $s(t) = T_{\alpha}(t)s(0) = 0$. Hence, $v(t) = w(t)$, $t \geq 0$. This proves the first part of the theorem.

Conversely, assume that v is the unique strong solution of (4.1.11). Then, v is continuously differentiable with $v(t) \in D(A)$, $t \geq 0$, and satisfies (4.1.11), i.e.,

$$ {}_{CF}D_t^{\alpha}v(t) = Av(t) + f(t), \quad t \geq 0. \tag{4.1.29}$$

Note that, since B_{α} is bounded operator and f is continuously differentiable, then we can define the function u by

$$u(t) := T_{\alpha}(t)v(0) + \int_0^t T_{\alpha}(t-\tau)F_{\alpha}(\tau)d\tau, \quad t \geq 0, \tag{4.1.30}$$

where $(T_{\alpha}(t))_{t\in\mathbb{R}}$ is the group generated by B_{α}. By identities (4.1.29) and (4.1.30), we have that

$$u(0) = v(0) \in D(A). \tag{4.1.31}$$

We claim that the function u defined by (4.1.30) is continuously differentiable and satisfies (4.1.12). In fact, it is clear that it is continuously differentiable because $(T_{\alpha}(t))_{t\in\mathbb{R}}$ is a uniformly continuous group. We will check that satisfy (4.1.12). First, note that by [40, Section VI.7] it clearly satisfies the identity

$$u'(t) = B_{\alpha}u(t) + F_{\alpha}(t), \quad t \ \geq 0.$$

It remains to check that satisfies the initial condition. In fact, using (4.1.29) and (4.1.31), we have $Au(0) = Av(0) = {}_{CF}D_t^{\alpha}v(0) - f(0) = -f(0)$ since ${}_{CF}D_t^{\alpha}v(0) = 0$ by Remark 1.2.5. Thus, $u(0) = -A^{-1}f(0)$ because A is injective. This proves the claim. Therefore, u is strong solution of (4.1.12).

We show uniqueness. Assume that w is strong solution of (4.1.12) and set $s := u - w$. Then, $s(0) \in D(A)$ and is a strong solution of the problem

$$s'(t) = B_{\alpha}s(t),\ t \geq 0.$$

It is well known that $s(t) = T_\alpha(t)s(0)$, $t \geq 0$, with $T_\alpha(t)$ the group generated by B_α. Note that, $s(0) = u(0) - w(0) = 0$ then $s(t) \equiv 0$, $t \geq 0$. Hence, $u(t) = w(t)$, $t \geq 0$. $\square$

Remark 4.1.9. Examining the previous proof, we deduce that if A is a non-injective operator, then $\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau \in D(A)$ and that a strong solution v of (4.1.11) (and of (4.1.12)) verifies $Av(0) = -f(0)$ and

$$Av(t) = T_\alpha(t)Av(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \quad t \geq 0,$$

where $(T_\alpha(t))_{t\in\mathbb{R}}$ is a uniformly continuous group generated by $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ and $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$.

Remark 4.1.10. Note that, Theorem 4.1.8 does not assume A to be the generator of any one-parameter family of operators, or A to be densely defined, in contrast with the limit case $\alpha = 1$ that requires A to be the generator of a C_0-semigroup. This reveal an important advantage of the fractional abstract Cauchy problem (0.0.2) when compared with the abstract Cauchy problem (0.0.3). However, we have a restriction over the spectrum, namely: $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$. In conclusion, although always the Cauchy problem with the Caputo-Fabrizio fractional derivative can be theoretically reduced to a first order abstract Cauchy problem, the first could be much more flexible when dealing with applications. This is probably the reason why problems with the Caputo-Fabrizio derivative find many applications in the real world.

Remark 4.1.11. If $f \equiv 0$ in Theorem 4.1.8, since $u(0) = -A^{-1}f(0) = 0$, then the unique strong solution u of (4.1.11) (and of (4.1.12)) is $u \equiv 0$.

Remark 4.1.12. If f is not zero in Theorem 4.1.8, then the unique strong solution u of (4.1.11) (and of (4.1.12)) is

$$u(t) = T_\alpha(t)A^{-1}f(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \quad t \geq 0.$$

In order to avoid the hypothesis of injectivity, we introduce now the following definitions.

Definition 4.1.13. Let A be a closed linear operator. A function $u : [0,\infty) \to X$ is called an A-unique strong solution of (4.1.1)-(4.1.5) if and only if any strong solution v of (4.1.1)-(4.1.5) satisfies that $Au(t) = Av(t),\ t \geq 0$.

Remark 4.1.14. Observe that if A is an injective operator then an A-unique strong solution is unique in the classical sense.

With the above preliminaries, we show the following corollary of Theorem 4.1.8.

Corollary 4.1.15. *Let $0 < \alpha < 1$, A be a closed linear operator on X with domain $D(A)$ and $f : [0, \infty) \to X$ continuously differentiable. Assume that $\frac{1}{1-\alpha} \in \rho(A)$. Then, the problem given by*

$$ {}_{CF}D_t^{\alpha} v(t) = Av(t) + f(t), \quad t \geq 0, \tag{4.1.32}$$

has an A-unique strong solution if and only if the initial value problem given by

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \quad t \geq 0; \\ Au(0) = -f(0), \end{cases} \tag{4.1.33}$$

has an A-unique strong solution, where $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ and $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$ with $N_\alpha = (I - (1-\alpha)A)^{-1} \in \mathcal{B}(X)$.

Proof. The proof of the existence of strong solutions for the problems (4.1.32) and (4.1.33) is the same that Theorem 4.1.8. We show A-uniqueness. Let v be a strong solution of (4.1.32), by Remark 4.1.9 we have that v verifies

$$Av(t) = T_\alpha(t)Av(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \quad t \geq 0. \tag{4.1.34}$$

Assume that w is another strong solution of (4.1.32) and set $s := v - w$, then we have that s is a strong solution of the following

$$ {}_{CF}D_t^{\alpha} s(t) = As(t),\ t \geq 0. \tag{4.1.35}$$

Since ${}_{CF}D_t^{\alpha} s(0) = 0$, then $As(0) = 0$. Using the identity (4.1.34) for the problem (4.1.35) ($f \equiv 0$), we have $As(t) = T_\alpha(t)As(0) = 0$. Hence, $Av(t) = Aw(t)$, $t \geq 0$.

Now, let u be a strong solution of (4.1.33) and assume that w is another strong solution of

(4.1.33). We set $s := u - w$, then $s(0) \in D(A)$ and is a strong solution of the following

$$s'(t) = B_\alpha s(t), \ t \geq 0, \quad \text{and} \quad As(0) = 0.$$

Therefore, we get $s(t) = T_\alpha(t)s(0)$, $t \geq 0$, with $T_\alpha(t)$ the group generated by B_α. By Proposition 4.1.6, part (iii), we obtain $s(t) = T_\alpha(t)s(0) \in D(A)$, $t \geq 0$, and $As(t) = AT_\alpha(t)s(0) = T_\alpha(t)As(0) = 0$, since $s(0) \in D(A)$ and $As(0) = 0$. Hence, $Au(t) = Aw(t)$, $t \geq 0$. □

Remark 4.1.16. If $f \equiv 0$ in Corollary 4.1.15, then the A-unique strong solution u of (4.1.11) (and of (4.1.12)) is not zero for an initial condition in the kernel of A.

Next we present an immediate consequence of the above results. Note that the proof requires the use of [40, Definition II.4.1] (see also [83, Section 2.2.5])

Corollary 4.1.17. *Let $0 < \alpha < 1$, A be a sectorial operator on X of angle $\delta \in (0, \pi/2)$ with domain $D(A)$ and $f : [0, \infty) \to X$ continuously differentiable. Then, the problem given by*

$$_{CF}D_t^\alpha v(t) = Av(t) + f(t), \quad t \geq 0, \tag{4.1.36}$$

has a unique strong solution if and only if the initial value problem given by

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \quad t \geq 0; \\ u(0) = -A^{-1}f(0), \end{cases} \tag{4.1.37}$$

has a unique strong solution.

Remark 4.1.18. By [83, Corollary II.4.7], if A is a normal operator on a Hilbert space H satisfying

$$\sigma(A) \subset \{z \in \mathbb{C} : \arg(-z) < \delta\}$$

for some $\delta \in [0, \pi/2)$, then A generates a bounded analytic semigroup, and hence A is sectorial in the sense of [40, Definition II.4.1]. Therefore, Corollary 4.1.17 applies.

4.2 Stability

The stability of the fractional order linear systems has been studied for many years and powerful criteria have been proposed. The best known one is Matignon's stability theorem [77], and is the starting point for several useful and important results in the field. The stability of the linear fractional order systems described by the Caputo-Fabrizio derivative has recently been studied in the reference [71], where the authors gave necessary and sufficient conditions for the stability of the solutions of the problem

$$_{CF}D_t^{\alpha}u(t) = Au(t), \quad t \geq 0,$$

where A *is a matrix.* In what follows we will extended the results of [71] to the case of closed linear operators A.

After recalling some spectral properties, we study the asymptotic behavior of the solutions for the problem

$$_{CF}D_t^{\alpha}u(t) = Au(t) + f(t), \quad t \geq 0, \tag{4.2.1}$$

where A is a closed injective linear operator.

Remark 4.2.1. We recall that the Spectral Mapping Theorem for the resolvent operator ([40, Theorem IV.1.13]) and for polynomials ([58, Proposition A.6.2], [37, Theorem VII.9.10]) says that given $A : D(A) \subset X \to X$ a closed operator with nonempty resolvent set $\rho(A)$, we have

(*i*) $\sigma((\beta - A)^{-1}) \setminus \{0\} = (\beta - \sigma(A))^{-1}$ for each $\beta \in \rho(A)$.

(*ii*) $\sigma(q(A)) = q(\sigma(A))$ for each polynomial $q \in \mathbb{C}[z]$.

Using the previous spectral properties we can prove the following result.

Proposition 4.2.2. *Let* $0 < \alpha < 1$ *and* A *be a closed operator on* X. *Assume that* $\frac{1}{1-\alpha} \in \rho(A)$. *Let* $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$, *where* $N_\alpha = (I - (1-\alpha)A)^{-1}$, *then the following identity holds*

$$\sigma(B_\alpha) = \frac{\alpha}{1-\alpha}\left[\frac{1}{1-(1-\alpha)\sigma(A)} - 1\right].$$

Proof. By Remark 4.2.1, we have

$$\sigma(B_\alpha) = \sigma\left(\frac{\alpha}{1-\alpha}(N_\alpha - I)\right) = \frac{\alpha}{1-\alpha}\sigma(N_\alpha - I) = \frac{\alpha}{1-\alpha}[\sigma(N_\alpha) - 1].$$

Thus, by definition of N_α, we get

$$\sigma(B_\alpha) = \frac{\alpha}{1-\alpha}[\sigma((I-(1-\alpha)A)^{-1}) - 1] = \frac{\alpha}{1-\alpha}\left[\frac{1}{1-(1-\alpha)\sigma(A)} - 1\right],$$

and we obtain the claim. □

Remark 4.2.3. Note that, by Proposition 4.2.2, we obtain

$$\sigma(A) = \frac{1}{1-\alpha}I - \frac{1}{1-\alpha}\frac{\alpha}{(1-\alpha)\sigma(B_\alpha)+\alpha}.$$

We recall that a semigroup $(T(t))_{t>0}$ on a Banach space X is called uniformly exponentially stable if there exist constants $\omega > 0$, $M \geq 1$ such that

$$||T(t)||_{\mathcal{B}(X)} \leq Me^{-\omega t} \ \text{ for all } \ t \geq 0.$$

Remark 4.2.4. Let $0 < \alpha < 1$. Since $(T_\alpha(t))_{t>0}$ is the uniformly continuous semigroup generated by B_α, by [40, Proposition I.3.12 and Theorem I.3.14], the following assertions are equivalent

(*i*) $(T_\alpha(t))_{t>0}$ is uniformly exponentially stable.

(*ii*) $\lim_{t\to\infty} ||T_\alpha(t)||_{\mathcal{B}(X)} = 0$.

(*iii*) $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(B_\alpha)$.

On the other hand, by Proposition 4.2.2, we obtain the following result that will be important for our main result on stability. In what follows we denote by $\overline{\mathbb{D}}(z,r) := \{w \in \mathbb{C} : |w-z| \leq r\}$ the closed disk of center z and radius $r > 0$.

Proposition 4.2.5. *Let $0 < \alpha < 1$ and A be a closed operator on X. Assume that $\frac{1}{1-\alpha} \in \rho(A)$. Let $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$, where $N_\alpha = (I-(1-\alpha)A)^{-1}$, then we have*

$$\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}\left(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)}\right) \iff \mathrm{Re}(\lambda) < 0 \ \textit{ for all } \ \lambda \in \sigma(B_\alpha).$$

Proof. Suppose $\mu \in \sigma(A)$ such that $\mu = a + ib \in \mathbb{C}$. Since $\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)})$, we have that μ verify $\left(a - \frac{1}{2(1-\alpha)}\right)^2 + b^2 > \left(\frac{1}{2(1-\alpha)}\right)^2$ which, after a computation, is equivalent to

$$a - (1-\alpha)(a^2 + b^2) < 0. \tag{4.2.2}$$

By Proposition 4.2.2, we have

$$\lambda = \frac{\alpha}{1-\alpha}\left[\frac{1}{1-(1-\alpha)\mu} - 1\right] \in \sigma(B_\alpha),$$

where after some computations, we obtain the equivalent representation

$$\lambda = \alpha \frac{[a - (1-\alpha)(a^2+b^2)] + ib}{(1-(1-\alpha)a)^2 + ((1-\alpha)b)^2}. \tag{4.2.3}$$

Hence, by identity (4.2.3), we have $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(B_\alpha)$ if and only if $a - (1-\alpha)(a^2+b^2) < 0$. Thus, by equivalence (4.2.2), we conclude the claim. □

Remark 4.2.6. Observe that, the condition $\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)})$ implies that the operator A is injective.

Next, we apply the Theorem 4.1.8 to study stability of the solution to the problem (4.2.1). The following is our main result in this section.

Theorem 4.2.7. *Let $0 < \alpha < 1$, A be a closed operator on X with domain $D(A)$ and $f : [0,\infty) \to X$ continuously differentiable. Assume that*

(i) $\frac{1}{1-\alpha} \in \rho(A)$,

(ii) $\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)})$,

(iii) there exist constants $\beta, M > 0$ such that

$$||f(t)||_X + ||f'(t)||_X \le Me^{-\beta t}, \quad t \ge 0.$$

Then, the problem given by

$$_{CF}D_t^\alpha u(t) = Au(t) + f(t), \quad t \ge 0, \tag{4.2.4}$$

has a unique strong solution u such that

$$\lim_{t\to\infty} ||u(t)||_X = 0.$$

Proof. Suppose $f \equiv 0$, then we have that the problem (4.2.4) has a unique strong solution $u \equiv 0$, by Remark 4.1.11. Thus, the theorem holds.

Now, suppose f is not zero. By Remark 4.1.12, we have that the problem (4.2.4) has a unique strong solution given by

$$u(t) = T_\alpha(t)A^{-1}f(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \quad t \geq 0,$$

where $(T_\alpha(t))_{t\in\mathbb{R}}$ is a uniformly continuous group generated by $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ and $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$ with $N_\alpha = (I-(1-\alpha)A)^{-1} \in \mathcal{B}(X)$. Thus, we have

$$||u(t)||_X \leq ||T_\alpha(t)A^{-1}f(0)||_X + \int_0^t ||T_\alpha(t-\tau)||_{\mathcal{B}(X)}||F_\alpha(\tau)||_X d\tau, \quad t \geq 0. \tag{4.2.5}$$

Note that, by Proposition 4.2.5 and Remark 4.2.4, we have there exist constants $\omega_\alpha > 0$, $M_\alpha \geq 1$ such that $||T_\alpha(t)||_{\mathcal{B}(X)} \leq M_\alpha e^{-\omega_\alpha t}$ for all $t \geq 0$. Hence, from inequality (4.2.5), we get

$$||u(t)||_X \leq ||T_\alpha(t)A^{-1}f(0)||_X + \int_0^t M_\alpha e^{-\omega_\alpha(t-\tau)}||F_\alpha(\tau)||_X d\tau. \tag{4.2.6}$$

On the other hand, by hypothesis (iii), we have

$$\begin{aligned}
||F_\alpha(t)||_X &\leq (1-\alpha)||N_\alpha f'(t)||_X + \alpha||N_\alpha f(t)||_X \\
&\leq (1-\alpha)||N_\alpha||_{\mathcal{B}(X)}||f'(t)||_X + \alpha||N_\alpha||_{\mathcal{B}(X)}||f(t)||_X \\
&\leq (1-\alpha)||N_\alpha||_{\mathcal{B}(X)}Me^{-\beta t} + \alpha||N_\alpha||_{\mathcal{B}(X)}Me^{-\beta t} \\
&= ||N_\alpha||_{\mathcal{B}(X)}Me^{-\beta t},
\end{aligned}$$

where $C_\alpha := ||N_\alpha||_{\mathcal{B}(X)}M$ is a positive constant. Let $\gamma_\alpha := \min\{\beta, \omega_\alpha/2\}$, then we obtain that

$$||F_\alpha(t)||_X \leq C_\alpha e^{-\gamma_\alpha t}, \quad t \geq 0.$$

Therefore, by (4.2.6), we have

$$
\begin{aligned}
||u(t)||_X &\le ||T_\alpha(t)A^{-1}f(0)||_X + \int_0^t M_\alpha e^{-\omega_\alpha(t-\tau)} C_\alpha e^{-\gamma_\alpha \tau} d\tau \\
&= ||T_\alpha(t)A^{-1}f(0)||_X + M_\alpha C_\alpha e^{-\omega_\alpha t} \int_0^t e^{(\omega_\alpha-\gamma_\alpha)\tau} d\tau \\
&= ||T_\alpha(t)A^{-1}f(0)||_X + \frac{M_\alpha C_\alpha}{\omega_\alpha-\gamma_\alpha} e^{-\omega_\alpha t}[e^{(\omega_\alpha-\gamma_\alpha)t}-1] \\
&= ||T_\alpha(t)A^{-1}f(0)||_X + \frac{M_\alpha C_\alpha}{\omega_\alpha-\gamma_\alpha}[e^{-\gamma_\alpha t}-e^{-\omega_\alpha t}].
\end{aligned} \tag{4.2.7}
$$

Observe that, by Proposition 4.2.5 and Remark 4.2.4, we obtain

$$\lim_{t\to\infty} ||T_\alpha(t)A^{-1}f(0)||_X = 0.$$

Finally, by the above identity and (4.2.5), we conclude

$$\lim_{t\to\infty} ||u(t)||_X \le \lim_{t\to\infty} \frac{M_\alpha C_\alpha}{\omega_\alpha-\gamma_\alpha}[e^{-\gamma_\alpha t}-e^{-\omega_\alpha t}] = 0.$$

This proves the claim. □

The following example illustrates how Theorems 4.1.8 and 4.2.7 can be applied to obtain solutions of ${}_{CF}D_t^\alpha u(t) = Au(t) + f(t)$, $t \ge 0$, and know about its behavior.

Example 4.2.8. *Fix $0 < \alpha < 1$ and consider in $X := C([0,1])$ the operator $Au(x) = u''(x)$, $x \in [0,1]$, with $D(A) = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}$. Since $\sigma(A) = \{-\pi^2 k^2 : k \in \mathbb{N}\}$ we have that A is an injective operator and*

$$A^{-1}g(x) = -x\int_0^1 (1-s)g(s)ds + \int_0^x (x-s)g(s)ds, \quad x \in [0,1].$$

Let us now consider the problem

$${}_{CF}D_t^\alpha u(t,x) = u_{xx}(t,x) + \gamma e^{-\beta t}\sin(x),$$

where $\gamma, \beta > 0$, $x \in [0,1]$ and $t > 0$. Since the equation forces the initial condition, we obtain

$$u(0,x) = A^{-1}[\gamma\sin(x)] = \gamma[x\sin(1) - \sin(x)].$$

Our results shows that the solution is given by

$$u(t,x) = T_\alpha(t)\gamma[x\sin(1)-\sin(x)] + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau,x)d\tau, \tag{4.2.8}$$

where $(T_\alpha(t))_{t\geq 0}$ *is the uniformly continuous semigroup generated by* $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ *and* $F_\alpha(t,x) = \gamma[\alpha(1+\beta)-\beta]e^{-\beta t}N_\alpha \sin(x)$ *with* $N_\alpha = (I-(1-\alpha)A)^{-1} \in \mathcal{B}(X)$.

By Theorem 4.1.8, we have that (4.2.8) *is also the solution of the problem*

$$u_t(t,x) = B_\alpha u(t,x) + F_\alpha(t,x), \quad t \geq 0, \quad x \in [0,1].$$

A computation shows that for $f : [0,\infty) \times X \to X$ *we have*

$$\begin{aligned} N_\alpha f(x) = & \frac{\sqrt{1-\alpha}}{1-\alpha}\frac{\sinh(\frac{x}{\sqrt{1-\alpha}})}{\sin(\frac{1}{\sqrt{1-\alpha}})}\int_0^1 \sinh(\frac{1-y}{\sqrt{1-\alpha}})f(y)dy \\ & -\frac{\sqrt{1-\alpha}}{1-\alpha}\int_0^x \sinh(\frac{x-y}{\sqrt{1-\alpha}})f(y)dy. \end{aligned}$$

In particular, using [56, Formula 2.671.1], we have

$$N_\alpha \sin(x) = \frac{\sin(x)}{2-\alpha} - \frac{\sin(1)}{(2-\alpha)\sinh(\frac{1}{\sqrt{1-\alpha}})}\sinh(\frac{x}{\sqrt{1-\alpha}})$$

and hence we conclude that

$$F_\alpha(t,x) = \gamma e^{-\beta t}[\alpha(1+\beta)-\beta][\frac{\sin(x)}{2-\alpha} - \frac{\sin(1)}{(2-\alpha)\sinh(\frac{1}{\sqrt{1-\alpha}})}\sinh(\frac{x}{\sqrt{1-\alpha}})].$$

On the other hand, by Theorem 4.2.7, we conclude that the solution (4.2.8) *satisfies*

$$\lim_{t\to\infty}\sup_{x\in[0,1]}|u(t,x)| = 0.$$

Remark 4.2.9. Let $f \equiv 0$ in Example 4.2.8, i.e. $\gamma = 0$, then from (4.2.8) we see that the solution is $u(t,x) \equiv 0$.

4.3 The Semilinear Problem

Let $0 < \alpha < 1$ and assume that $A : D(A) \subset X \to X$ is a closed linear operator defined on a complex Banach space X and $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$. We recall that $B_\alpha := \frac{\alpha}{1-\alpha}(N_\alpha - I)$, where $N_\alpha := (I-(1-\alpha)A)^{-1} \in \mathcal{B}(X)$, defines the uniformly continuous group $(T_\alpha(t))_{t\in\mathbb{R}}$ on X, given by (4.1.2).

Let $f : [0, \infty) \times X \to X$ be continuously differentiable in $t \in [0, \infty)$. We define

$$F_\alpha(t, x) := (1 - \alpha)N_\alpha f_t(t, x) + \alpha N_\alpha f(t, x).$$

For each $u_0 \in X$ we consider the integral equation:

$$u(t) = T_\alpha(t)u_0 + \int_0^t T_\alpha(t - \tau)F_\alpha(\tau, u(\tau))d\tau,\ t \geq 0. \tag{4.3.1}$$

Let $u_0 \in X$ be given, it is well known (see [83, Chapter 6]) that a continuous solution u of the integral equation (4.3.1) is called a *mild solution* of the following semilinear initial value problem:

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t, u(t)), \quad t \geq 0; \\ u(0) = u_0. \end{cases}$$

For our purposes, we use the equivalence given by Theorem 4.1.8 to extend the previous terminology as follows.

Definition 4.3.1. Assume that $0 \in \rho(A)$ and let $\mathcal{S} := \{x \in D(A) : x = A^{-1}f(0, x)\}$. A continuous solution u of the integral equation

$$u(t) = -T_\alpha(t)A^{-1}f(0, u_0) + \int_0^t T_\alpha(t - \tau)F_\alpha(\tau, u(\tau))d\tau,\ t \geq 0 \tag{4.3.2}$$

is called a mild solution of the initial value problem

$$\begin{cases} {}_{CF}D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad t \geq 0; \\ u(0) = u_0 \in \mathcal{S}. \end{cases} \tag{4.3.3}$$

We finish with the following result that assures the existence and uniqueness of mild solutions of (4.3.3) for Lipschitz continuous functions f and A an injective operator. The proof is relatively standard, but we give it for completeness.

Theorem 4.3.2. *Let $0 < \alpha < 1$ and A be a closed linear operator on a complex Banach space X. Suppose $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$ and $f : [0, T] \times X \to X$ is continuously differentiable in t on $[0, T]$ and* *satisfies the following Lipschitz type condition with constant L on X:*

$$(1-\alpha)||f_t(t,x)-f_t(t,y)||_X+\alpha||f(t,x)-f(t,y)||_X\leq L||x-y||_X$$

for all $x,y\in X$. Then, the problem given by

$$\begin{cases} {}_{CF}D_t^{\alpha}u(t)=Au(t)+f(t,u(t)), \quad 0\leq t\leq T; \\ u(0)=u_0\in\mathcal{S}, \end{cases} \tag{4.3.4}$$

has a unique mild solution in $C([0,T]:X)$.

Proof. By hypothesis, we define $N_\alpha=(I-(1-\alpha)A)^{-1}\in\mathcal{B}(X)$. Then, we obtain

$$\begin{aligned} &||F_\alpha(t,x)-F_\alpha(t,y)||_X \\ &\quad=||N_\alpha[(1-\alpha)f_t(t,x)+\alpha f(t,x)]-N_\alpha[(1-\alpha)f_t(t,y)+\alpha f(t,y)]||_X \\ &\quad\leq||N_\alpha||_{\mathcal{B}(X)}||[(1-\alpha)f_t(t,x)+\alpha f(t,x)]-[(1-\alpha)f_t(t,y)+\alpha f(t,y)]||_X \\ &\quad\leq||N_\alpha||_{\mathcal{B}(X)}[(1-\alpha)||f_t(t,x)-f_t(t,y)||+\alpha||f(t,x)-f(t,y)||_X] \\ &\quad\leq||N_\alpha||_{\mathcal{B}(X)}L||x-y||_X. \end{aligned} \tag{4.3.5}$$

Hence, F_α is uniformly Lipschitz continuous with constant $L||N_\alpha||_{\mathcal{B}(X)}$. Moreover, we recall that $(T_\alpha(t))_{t\geq0}$ is the uniformly continuous semigroup generated by $B_\alpha=\frac{\alpha}{1-\alpha}(N_\alpha-I)$. In particular, there exists $M=M(\alpha,T)\geq1$ such that

$$||T_\alpha(t)||_{\mathcal{B}(X)}\leq M \quad \text{for all} \quad t\in[0,T]. \tag{4.3.6}$$

For a given $u_0\in\mathcal{S}$ we define a mapping $G:C([0,T]:X)\to C([0,T]:X)$ by

$$(G_\alpha u)(t):=-T_\alpha(t)A^{-1}f(0,u_0)+\int_0^t T_\alpha(t-s)F_\alpha(s,u(s))ds, \quad t\in[0,T]. \tag{4.3.7}$$

Denoting by $||u||_\infty$ the norm of u as an element of $C([0,T]:X)$ it follows readily from the definition of G and (4.3.5)-(4.3.6) that

$$\begin{aligned} ||(G_\alpha u)(t)-(G_\alpha v)(t)|| &= ||\int_0^t T_\alpha(t-s)[F_\alpha(s,u(s))-F_\alpha(s,v(s))]ds|| \\ &\leq\int_0^t ||T_\alpha(t-s)||_{\mathcal{B}(X)}||F_\alpha(s,u(s))-F_\alpha(s,v(s))||_X ds \\ &\leq\int_0^t M||N_\alpha||_{\mathcal{B}(X)}L||u(s)-v(s)||_X ds \\ &\leq M||N_\alpha||_{\mathcal{B}(X)}Lt||u-v||_\infty. \end{aligned} \tag{4.3.8}$$

In general we get using (4.3.7), (4.3.8) and induction on n that

$$||(G_\alpha^n u)(t) - (G_\alpha^n v)(t)|| \leq \frac{(M||N_\alpha||_{\mathcal{B}(X)} Lt)^n}{n!} ||u - v||_\infty$$

whence

$$||G_\alpha^n u - G_\alpha^n v|| \leq \frac{(M||N_\alpha||_{\mathcal{B}(X)} LT)^n}{n!} ||u - v||_\infty.$$

Since $\frac{(M||N_\alpha||_{\mathcal{B}(X)} LT)^n}{n!} < 1$ for n sufficiently large, aplying the contraction principle we conclude that G_α has a unique fixed point u in $C([0,T] : X)$. This fixed point is the desired solution of the integral equation (4.3.3). Thus, by Definition 4.3.1, we have that (4.3.4) has a mild solution.

Now, we show the uniqueness. Assume that v is a mild solution of (4.3.4) on $[0,T]$ with the initial value $v_0 \in \mathcal{S}$. Then

$$\begin{aligned}
||u(t) - v(t)||_X &\leq || - T_\alpha(t) A^{-1} f(0, u_0) + T_\alpha(t) A^{-1} f(0, v_0)||_X \\
&\quad + \int_0^t ||T_\alpha(t-s)[F_\alpha(s, u(s)) - F_\alpha(s, v(s))]||_X ds \\
&\leq M||A^{-1} f(0, u_0) - A^{-1} f(0, v_0)||_X \\
&\quad + M||N_\alpha||_{\mathcal{B}(X)} L \int_0^t ||u(s) - v(s)||_X ds.
\end{aligned}$$

which implies, by Gronwall's inequality, that

$$||u(t) - v(t)||_X \leq M||A^{-1} f(0, u_0) - A^{-1} f(0, v_0)||_X e^{M||N_\alpha||_{\mathcal{B}(X)} LT},$$

and therefore

$$||u - v||_\infty \leq M||A^{-1} f(0, u_0) - A^{-1} f(0, v_0)||_X e^{M||N_\alpha||_{\mathcal{B}(X)} LT}$$

which yield the uniqueness of u (with $v_0 = u_0$). □

Remark 4.3.3. Theorem 4.3.2 does not assume that A is the generator of a C_0-semigroup in contrast to the case of the first order abstract Cauchy problem ([83, Theorem 6.1.2]).

Remark 4.3.4. Note that, if $\alpha \to 1$ in Theorem 4.3.2 then we obtain that the Lipschitz condition with respect to f and f_t simplifies to a Lipschitz condition with respect to f only. On the other hand, by identity (1.2.4) and Remark 4.1.7, and assuming that A is a densely defined, we obtain by Theorem 4.3.2 that, when $\alpha \to 1$, A is the infinitesimal generator of a C_0-semigroup. It shows that Theorem 4.3.2 extends the case $\alpha = 1$ proved in [83, Theorem 6.1.2] to the case $0 < \alpha < 1$.

4.4 Conclusions

In this chapter we study the abstract Cauchy problem with the fractional derivative of order $\alpha \in (0,1)$ of Caputo-Fabrizio and compare its performance from a mathematical point of view. As advantage, and in contrast to the finite dimensional case, i.e. A being a matrix, we observe that being A an unbounded closed linear operator (e.g. a differential operator like the Laplacian), the abstract Cauchy problem with operator A turns out to be equivalent to a first order abstract Cauchy problem with a family of *bounded* operators B_α - that behave like a Yosida's approximation of A - and that makes unnecessary any previous assumptions about A, to solve it, such as a generator of a C_0-semigroup or cosine family of operators, for example. As disadvantage, the non-singular character of the kernel that defines the Caputo-Fabrizio derivative, forces an initial condition (somewhat artificial) that involves the operator A itself, condition that can be overcome if we assume certain conditions of invertibility of the operator A that hold for certain classes of differential operators, for example, the Dirichlet Laplacian operator on a smooth bounded domain. We leave similar studies for other classes of fractional derivatives with non-singular (or regular) kernels as an open problem.

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