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ON THE GROUPS OF DIFFEOMORPHISMS OF INTERMEDIATE REGULARITY

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Abstract

In this work we study the groups of orientation preserving diffeomorphisms of the closed interval whose derivative is α-Hölder continuous. We are interested in how these groups change with respect to the parameter $\alpha \in [0, 1)$. Our specific contributions are the following works.

On the critical regularity of nilpotent groups acting on the interval: the metabelian case [**6**]. Here, for a finitely-generated, torsion-free, nilpotent and metabelian group G, we build an embedding into the group of orientation preserving $C^{1+\alpha}$-diffeomorphisms of the closed interval, for all $\alpha < 1/k$ where k is the torsion-free rank of G/A and A is a maximal abelian subgroup of G. We show that in many situations, this embedding has critical regularity in the sense that there is no embedding of G with higher regularity. A particularly nice family where this critical regularity is achieved, is the family of $(2n+1)$-dimensional Heisenberg groups, where we can show that its critical regularity equals $1 + 1/n$.

Examples of distorted interval diffeomorphisms of intermediate regularity [**5**]. In this joint work with L. Dinamarca, we improve a construction of Andrés Navas to produce the first examples of C^2-undistorted diffeomorphisms of the interval that are $C^{1+\alpha}$-distorted (for every $\alpha \in [0, 1)$).

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Introduction

Due to the works of J. Whittaker [**42**] and M. Rubin [**38**], we know that two differentiable manifolds have different groups of homeomorphisms. That is, if M and N are manifolds whose groups of homeomorphisms are isomorphic, say $\mathrm{Homeo}(M) \simeq \mathrm{Homeo}(N)$, then the manifolds M and N are homeomorphic. Furthermore, R. Filipkiewicz [**17**], showed that in fact the group of C^r-diffeomorphisms of a manifold M is not isomorphic to the group of C^s-diffeomorphisms of a manifold N, unless $s = r$ and $M = N$, and in this case, the group isomorphism is a conjugation by a C^r-diffeomorphism between M and N. Presumably these groups of diffeomorphisms also have a very different local algebraic structure, for different manifolds and differentiability parameters. As we will comment below, this has been extensively investigated in dimension one, also including the so-called intermediate regularity groups.

Associated with a differentiable manifold M there is the filtration of C^n-diffeomorphisms groups ($n \in \mathbb{N}$) which are isotopic to the identity, say $\mathrm{Diff}^n_+(M)$. This filtration can be refined to include the groups of diffeomorphisms whose n-th derivative is α-Hölder continuous. That is, for $n \geqslant 1$ and $\alpha \in (0,1)$, we let $\mathrm{Diff}^{n+\alpha}_+(M)$ be the group of C^n-diffeomorphisms satisfying that

$$\left|D^{(n)}f(x) - D^{(n)}f(y)\right| \leqslant C\,|x-y|^{\alpha},$$

for some constant $C > 0$. And so we get the filtration

$$\mathrm{Homeo}_+(M) = \mathrm{Diff}^0_+(M) \geqslant \mathrm{Diff}^1_+(M) \geqslant \mathrm{Diff}^{1+\alpha}_+(M) \geqslant \mathrm{Diff}^2_+(M) \geqslant \mathrm{Diff}^{2+\alpha}_+(M) \geqslant \cdots$$

In this context, roughly speaking, we want to know more about the finitely generated subgroups of these groups. In particular we are interested in the case in which the manifold is the compact interval.

Problem 1. It is quite natural to try to understand which groups $G \leqslant \mathrm{Diff}^0_+(M)$ can be conjugated deep inside the above filtration and also to determine how deep can the given group be realized. To be more precise, we define the algebraic *critical regularity* of a group G at M as

$$\mathrm{Crit}_M(G) = \sup\{\, \alpha \in \mathbb{R} \,:\, G \text{ embeds into } \mathrm{Diff}^{\alpha}_+(M)\,\},$$

where we set $\mathrm{Crit}_M(G) = -1$ if G does not embed in $\mathrm{Diff}^0_+(M)$.

The problem of computing the critical regularity of a group G turns out to be very interesting in the case that G is finitely-generated (the reader may wish to consult [**23**] for an introduction). For instance, we know from a theorem of Deroin, Klepstyn, Navas [**12**] (see also [**13**]) that every countable subgroup of $\mathrm{Diff}^0_+([0,1])$ is conjugated to a group of bilipchitz transformations, and hence $1 \leqslant \mathrm{Crit}_{[0,1]}(G)$ for every countable subgroup of $\mathrm{Diff}^0_+([0,1])$ (for uncountable subgroups $\mathrm{Diff}^0_+([0,1])$ this is no longer true, see [**9**]). However, the celebrated Stability Theorem of Thurston [**40**] implies that every finitely-generated group of $\mathrm{Diff}^1_+([0,1])$ admits a surjective homomorphisms onto the integers, and so not every group of homeomorphisms of the interval can be realized as a group of diffeomorphisms[1]. Further obstructions appears in higher regularity: for C^2 there is the important Kopell's obstruction

[1]Concrete examples of finitely-generated subgroups of $\mathrm{Diff}^0_+([0,1])$ having trivial abelianization can be found in [**40, 2, 37**]. However, Thuston's obstruction is not the only obstruction for C^1 smoothability as there

[**26**], and between C^1 and C^2 there are the generalized Kopell's obstruction from [**12**]. In a related spirit, Kim and Koberda [**22**], and later Mann and Wolff [**27**], have shown that for every $n \geqslant 1$ and every α in $[0, 1)$, there is a finitely-generated group whose critical regularity on $[0, 1]$ is exactly $n + \alpha$.

Here we focus on actions on the interval of finitely-generated and torsion-free nilpotent groups. Let G be one such group. It follows from the work of Mal'cev that G embeds into $\mathrm{Diff}^0_+([0, 1])$ (see, for instance, [**11**, §5.2] and [**14**, §1.2]), and we know from the work of Farb and Franks [**16**], that every action of G on $[0, 1]$ by homeomorphisms can be conjugated inside $\mathrm{Diff}^1_+([0, 1])$ (see also the universal contruction from E. Jorquera [**20**]). This was further refined by Parkhe [**35**] who showed that actually G can be conjugated inside $\mathrm{Diff}^{1+\alpha}_+([0, 1])$ as long as $\alpha < 1/\tau$, where τ is the degree of the polinomial growth of the nilpotent group G. On the other hand, Plante and Thurston [**39**], have shown that every nilpotent subgroup of $\mathrm{Diff}^2_+([0, 1])$ must be abelian. So, if G is a torsion-free, finitely-generated and nilpotent group which is non-abelian, then

$$1 + 1/\tau \leqslant \mathrm{Crit}_{[0,1]}(G) \leqslant 2.$$

The exact critical regularity of concrete nilpotent groups has been computed only in few cases and one important goal of this work is to provide new explicit computations of critical regularity for certain groups. Castro, Jorquera and Navas [**10**], build a family of nilpotent abelian-by-cyclic groups whose critical regularity is 2. These examples can be made of arbitrarily large nilpotency degree, yet they are all metabelian (i.e. its commutator subgroup is abelian). Jorquera, Navas and Rivas showed in [**21**] that the critical regularity of N_4 (the group of 4 by 4 upper triangular matrices with 1's in the diagonal) is $1 + 1/2$. We point out that at the time of this writing, N_4 is the only torsion-free nilpotent group whose critical regularity is computed and turns out not to be an integer. Remark that N_4 is also a metabelian group.

One of our main purposes is to exhibit many other nilpotent groups whose critical regularity is strictly between 1 and 2. Our main technical result is an improvement of Parkhe's lower bound for the critical regularity in the class of torsion-free nilpotent groups which are metabelian (see Remark 0.1). For the statement recall that the torsion-free rank of an abelian group A is the dimension of the $\mathbb{Q}$-vector space $A \otimes \mathbb{Q}$. We denote this rank by $\mathrm{rank}(A)$.

Theorem A. *Let G be a torsion-free, finitely-generated nilpotent group which is metabelian, and let A be a maximal abelian subgroup containing $[G, G]$. If $k = \mathrm{rank}(G/A)$, then*

$$G \text{ embeds into } \mathrm{Diff}^{1+\alpha}_+([0, 1]), \text{ for all } \alpha < 1/k.$$

In particular $1 + 1/k \leqslant \mathrm{Crit}_{[0,1]}(G)$.

Remark 0.1. *By Bass-Guivarc'h formula* [**1, 19**]*, the degree of the polynomial growth of a nilpotent group G is $\tau = \sum_{i \geqslant 1} i\, \mathrm{rank}(\gamma_i/\gamma_{i+1})$, where $G = \gamma_1 \geqslant \gamma_2 \geqslant ...$ is the lower central series of G. In particular, for a nilpotent group G as in Theorem A with maximal abelian subgroup A, we have that* $\mathrm{rank}(G/A) < \tau$. *Hence the lower bound for* $\mathrm{Crit}_{[0,1]}(G)$ *in Theorem A is (strictly) greater than Parkhe's lower bound (in the non-abelian case).*

are also known examples of finitely-generated and locally indicable groups having no *faithful* C^1 action on the interval, see [**9, 34, 4, 24**].

The proof of Theorem A is given in Chapter 2. In §2.1.2 we build, for a metabelian and torsion-free nilpotent group G, a family of actions of G on the interval $[0,1]$ by orientation preserving homeomorphisms. This is done by first building actions of G on $\mathbb{Z}^k$ which preserves a lexicographic order and then *projecting* them into the interval. In §2.1.3, we use the Pixton-Tsuboi technique [**36, 41**] to show that these actions can be smoothed to actions by $C^{1+\alpha}$-diffeomorphisms for any $\alpha < 1/k$. This section closely follows the works [**10**] and [**21**] and the main difference is that we don't have explicit polinomials but only bounds on them (see Proposition 2.2). Although these actions may not be faithful, in §2.1.4 we explain how to glue some of these actions in order to obtain an *embedding* of G into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for any $\alpha < 1/k$.

In some situations even the lower bound in Theorem A is not sharp in the sense that there are groups for which the theorem applies yet its critical regularity is strictly greater than the corresponding $1 + 1/k$. This is related to the possibility of splitting the group as a product of two groups each allowing an embedding with higher regularity. We provide an easy example of this phenomenon in §2.2.3. However, in many cases we can ensure that the inequality in Theorem A is indeed optimal, and in §2.2.1 and §2.2.2 we provide two families of examples were we can obtain upper bounds for the regularity and hence compute its critical regularity.

The first family of examples are the $(2n + 1)$-dimensional discrete Heisenberg groups, that we denote $\mathscr{H}_n$. Recall that by definition

$$\mathscr{H}_n := \left\{ \begin{pmatrix} 1 & \vec{x} & c \\ \vec{0}^{\,t} & I_n & \vec{y}^{\,t} \\ 0 & \vec{0} & 1 \end{pmatrix} : \vec{x}, \vec{y} \in \mathbb{Z}^n \text{ and } c \in \mathbb{Z} \right\},$$

where I_n denotes the $(2n-1)$ identity matrix and $\vec{0}^{\,t}$, $\vec{y}^{\,t}$ are the transposes of $\vec{0}$, $\vec{y}$ respectively. It is easy to see that these groups are nilpotent of degree two and hence they are metabelian. Moreover, a maximal abelian subgroup A of $\mathscr{H}_n$ is given by the set of matrices whose corresponding vector $\vec{x} = 0$. In particular $\mathscr{H}_n/A$ has torsion-free rank equals to n. For this family we show in §2.2.1 that there is no embedding of $\mathscr{H}_n$ into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for $\alpha > 1/n$. In particular we obtain

Theorem B. *Let $\mathscr{H}_n$ be the $(2n + 1)$-dimensional discrete Heisenberg group. Then*

$$\mathrm{Crit}_{[0,1]}(\mathscr{H}_n) = 1 + \frac{1}{n}.$$

Finally, in §2.2.2 we produce examples of metabelian and torsion-free nilpotent groups where we can compute its critical regularity but its nilpotency degree can be choose to be arbitrarily large. More precisely we show

Theorem C. *For any integers k and d with $d > k$, there is a nilpotent group G and a maximal abelian subgroup A containing $[G, G]$ such that d is the nilpotency degree of G, k is the torsion-free rank of G/A and*

$$\mathrm{Crit}_{[0,1]}(G) = 1 + \frac{1}{k}.$$

In both cases, the key to obtain an upper bound for the regularity is to use the internal algebraic structure of the groups in order to be able to apply the generalized Kopell lemma from [**12**].

Before moving on to the next problem, we leave two open questions regarding this problem. Given a torsion-free and finitely-generated nilpotent group G (not necessarily metabelian):

QUESTION 1. *Is there a natural number k such that* $\mathrm{Crit}_{[0,1]}(G) = 1 + 1/k$ *?*

QUESTION 2. *Let A be a maximal abelian subgroup of G which is normal, and let k be the Hirsch length of G/A. Is it true that for* $\alpha < 1/k$*, the group G embeds into* $\mathrm{Diff}_+^{1+\alpha}([0,1])$ *?*

Problem 2. To present the second problem let us recall the terminology introduced by Michail Gromov [**18**]. Given a finitely generated group Γ, we fix a finite system of generators, and we denote $\|\cdot\|$ the corresponding word-length. An element $f \in \Gamma$ is said to be distorted if

$$\lim_{n\to\infty} \frac{\|f^n\|}{n} = 0.$$

(Notice that this condition does not depend on the choice of the finite generating system.) Given an arbitrary group G, an element $f \in G$ is said to be distorted if there exists a finitely generated subgroup $\Gamma \subset G$ containing f so that f is distorted in Γ in the sense above.

Examples of "large" groups for which this notion becomes interesting are groups of diffeomorphisms of compact manifolds M. Very little is known about distorted elements therein. In particular, the following question from [**33**] is widely open:

QUESTION 3. *Given* $r < s$*, does there exist an undistorted element* $f \in \mathrm{Diff}_+^s(M)$ *that is distorted when considered as an element of* $\mathrm{Diff}_+^r(M)$ *?*

In [**33**], Andrés Navas proves that this is the case for M the closed interval, $r = 1$ and $s = 2$. Actually, undistortion holds in the larger group $\mathrm{Diff}_+^{1+bv}([0,1])$ of C^1 diffeomorphisms with derivative of bounded variation.

Here, in Chapter 3, we give an extension of this result from C^1 to $C^{1+\alpha}$ regularity.

THEOREM D. *There exist* C^∞ *diffeomorphisms of* $[0,1]$ *that are distorted in* $\mathrm{Diff}_+^{1+\alpha}([0,1])$ *for all* $\alpha > 0$ *yet undistorted in* $\mathrm{Diff}_+^{1+bv}([0,1])$.

The groups we consider are variations of those introduced in [**33**]. One of the new contribution consists in improving the regularity of some elements, which is not at all straightforward. Indeed, the construction of [**33**] uses a well-known lemma that ensures C^1 regularity of maps built by pasting together infinitely many diffeomorphisms that are defined on disjoint intervals and satisfy certain equivariance relations. This idea comes from the thesis of Nancy Kopell [**26**], and has been systematically used in the study of codimension-1 foliations [**15**] and centralizers of diffeomorphisms [**3**]. Nevertheless, such a lemma is unavailable in $C^{1+\alpha}$ regularity and, as we show in §1.4.3, it cannot hold without imposing extra hypothesis. We are hence forced to go into more explicit constructions and very long computations, which are however interesting by themselves. To do this, we use a classical technique of Dennis Pixton (later extended by Takashi Tsuboi [**41**]) to produce commuting diffeomorphisms and control their $C^{1+\alpha}$ norms.

Contents

CHAPTER 1

Preliminars

Let us start by presenting many results and tools related to group theory and group dynamics. They will be essential in the development of our work.

1.1. Nilpotent Groups

Given a group G and two elements $f, g \in G$, we let $[f, g] = fgf^{-1}g^{-1}$ denotes the commutator of f and g. Further, if G is finitely-generated and S is a finite generating set, an element of the form $[s_1, s_2]$ with $s_1, s_2 \in S$ is called a simple commutator of weight 2. Inductively, a *simple commutator of weight n* is defined as an element of the form

$$[s_1, ..., s_n] := [s_1, [s_2, ..., s_n]], \quad s_1, ..., s_n \in S.$$

Note that given n, there exist only a finite number of simple commutators of weight n.

The following lemma collects some commutator identities that, although their proof is direct, are very useful.

Lemma 1.1. *Let a, b, and c be elements in a group G.*

(i) $[a, b]^{-1} = [b, a]$.
(ii) $a[b, c]a^{-1} = [aba^{-1}, aca^{-1}]$.
(iii) $[ab, c] = a[b, c]a^{-1}[a, c] = [a, b, c][b, c][a, c]$.
(iv) $[a, bc] = [a, b]b[a, c]b^{-1} = [a, b][b, a, c][a, c]$.
(v) $[a, b^{-1}] = (b^{-1}[a, b]b)^{-1}$.
(vi) $[a^{-1}, b] = (a^{-1}[a, b]a)^{-1}$.

Proof. i, ii, iii and iv are direct equalities. v and vi follow from applying iii and iv to the identities $e = [a^{-1}a, b]$ and $e = [a, bb^{-1}]$ respectively. □

Let H and K be subgroups of G. $[K, H]$ denotes the subgroup of G generated by all commutators $[g, h]$ with $g \in K$ and $h \in H$. The subgroup $[G, G]$ is called the *commutator subgroup* and we say that G is *metabelian* if $[G, G]$ is abelian.

Remember that the *lower central series* of G is

$$G = \gamma_0 \geqslant \gamma_1 \geqslant \gamma_2 \geqslant \cdots$$

where $\gamma_1 = [G, G]$ and $\gamma_i = [G, \gamma_{i-1}]$; and the *upper central series* of G is

$$\{e\} = \zeta_0 \leqslant \zeta_1 \leqslant \zeta_2 \leqslant \cdots$$

where $\zeta_i/\zeta_{i-1} = Z(G/\zeta_{i-1})$, and $Z(G)$ denotes the center of G.

A group G is said to be *nilpotent of degree n* if $\zeta_n = G$ but $\zeta_{n-1} \neq G$. In what follows we will see some elementary facts about these groups, see [**11**] for a more in-depth introduction.

Proposition 1.2. *If G is a nilpotent group and $\{e\} \neq N \lhd G$, then $N \cap Z(G) \neq \{e\}$*

Proof. Assume that G has nilpotency degree equals n. Since $G = \zeta_n$, there is a least positive integer i such that $N \cap \zeta_i \neq \{e\}$. Now $[N \cap \zeta_i, G] \leqslant N \cap \zeta_{i-1} = \{e\}$ and $N \cap \zeta_i \leqslant N \cap \zeta_1$. Hence $N \cap \zeta_1 = N \cap \zeta_i \neq \{e\}$. □

An immediate consequence of the above proposition is the following useful result.

Proposition 1.3. *Let G be a nilpotent group and $\varphi : G \to H$ be a group homomorphism. Then φ is injective if and only if $\varphi\,|_{Z(G)}$ (the restriction of φ to $Z(G)$) is injective.*

If G has nilpotency degree n, it also happens that $\gamma_n = \{e\}$ but $\gamma_{n-1} \neq \{e\}$. This is a consequence of the following proposition.

Proposition 1.4. *If G is a nilpotent group of degree n we have that*

$$\gamma_i \leqslant \zeta_{n-i}.$$

Proof. We do induction on i. Since G/ζ_{n-1} is abelian we have that $\gamma_1 \leqslant \zeta_{n-1}$. Now assume that $\gamma_i \leqslant \zeta_{n-(i-1)}$, then

$$\gamma_i = [G, \gamma_{i-1}] \leqslant [G, \zeta_{n-(i-1)}] \leqslant \zeta_{n-i},$$

where the last inequality follows from the fact that $\zeta_{n-(i-1)}/\zeta_{n-i}$ is the center of G/ζ_{n-i}. □

The previous proposition shows that if G has degree equal to n, then $\gamma_n = \{e\}$. And the proof of it implies that $\gamma_{n-1} \neq \{e\}$, since if we assume $\gamma_{n-1} = \{e\}$ we have that $\gamma_{n-2} \leqslant \zeta_1$, and the same argument yields $\gamma_{n-i} \leqslant \zeta_{i-1}$ which implies that $\zeta_{n-1} = G$ contradicting the nilpotency degree of G.

Another consequence is the fact that in a finitely-generated nilpotent group of degree n, we only have a finite number of simple commutators (for a fixed generator), this is because all simple commutators of weight n are trivial.

The following proposition, which is a consequence of lemma 1.1, will allow us to find finitely many generators for the subgroups γ_i of the lower central chain of a nilpotent group. The proof is long and tedious, so we are going to demonstrate the main idea with a simple example.

Proposition 1.5. *Let G be a group generated by a symmetric set S, and let $n \in \mathbb{N}$. Then γ_n is generated by all simple commutators of weight n or more in the elements of S. In particular, if G is a finitely-generated nilpotent group, then all subgroups γ_n are finitely generated.*

Example 1.6. *Let G be a group generated by a set $S = \{a, b, c\}$, and consider the element $[a^{-1}b^2, c] \in \gamma_2$. We want to express this element as a product of simple commutators of weight 2 or more (over the generator S). For this we successively apply the item iii of lemma 1.1, this yields*

$$\begin{aligned}
[b^2a^{-1}, c] &= [b^2, a^{-1}, c][a^{-1}, c][b^2, c] \\
&= [b, b, a^{-1}, c][b, a^{-1}, c]^2[b^2, c] \\
&= [b, b, a^{-1}, c][b, a^{-1}, c]^2[b, b, c][b, c]^2,
\end{aligned}$$

therefore we express the element $[b^2a^{-1}, c]$ *as a product of simple commutators of weights 2, 3 and 4.*

Note that every subgroup of a finitely-generated nilpotent group is finitely generated as well. This is clear for finitely-generated abelian groups, and this fact is preserved under group extension. So, by proposition 1.5 it also holds for nilpotent groups.

Theorem 1.7. *(Mal'cev) If the center of a group G is torsion-free, each upper central factor ζ_i/ζ_{i-1} is torsion-free, where $i \in \{1, ..., n\}$.*

Proof. Let $Z(G) = \zeta_1$ be torsion-free; it is enough to prove that ζ_2/ζ_1 is torsion-free. Suppose that $x \in \zeta_2$ and $x^m \in \zeta_1$ for some $m > 0$. Then we have that $[x, g]^m = [x^m, g] = e$ because $[x, g] \in \zeta_1$. Therefore $[x, g] = e$ for all $g \in G$, so $x \in \zeta_1$. □

For $g \in G$, $\mathrm{Centr}(g) = \{h \in G \mid gh = hg\}$ denotes the centralizer of G. The following proposition, although elementary, will be very important to build actions of G in $\mathbb{Z}^k$ in §2.1.2.

Proposition 1.8. *Let G be a torsion-free and finitely-generated nilpotent metabelian group. Then:*

- *Given $g \in G$ and $0 \neq m \in \mathbb{Z}$ we have that* $\mathrm{Centr}(g^m) = \mathrm{Centr}(g)$.
- *Let $A \leqslant G$ be a maximal abelian subgroup. If A is normal in G, then G/A is torsion-free.*

Proof. Assume that there exist $f, g \in G$, $m \in \mathbb{Z}$ such that $[f, g^m] = e$ but $[f, g] \neq e$, and define $H := \langle f, g\rangle$, the subgroup generated by f and g. Since $[g, g^m] = [f, g^m] = e$ we have that $g^m \in Z(H)$, and since $H/Z(H)$ is torsion-free then $g \in Z(H)$, which is a contradiction because f and g do not commute. Then $\mathrm{Centr}(g^m) \subseteq \mathrm{Centr}(g)$ (the other direction is obvious).

The second point follows from the first. Let A be a maximal abelian subgroup which is also normal, and assume G/A is not torsion-free. Say $g \in G$ is such that $g \notin A$ but $g^m \in A$ for some $m \neq 0$. Then, since A is abelian, we have that $A \subseteq \mathrm{Centr}(g^m) = \mathrm{Centr}(g)$. In particular $\langle A, g\rangle$, the group generated by A and g, is an abelian subgroup larger than A, contradicting our assumption. □

1.2. Orders on groups

A group G is *left-orderable* if it admits a total order relation, say $\preceq$, which is invariant under multiplication from the left, that is

$$\text{if } f \preceq g \text{ then } hf \preceq hg \text{ for all } h \in G.$$

If additionally, $\preceq$ is also invariant under multiplication from the right, we say that $\preceq$ is a bi-invariant order (bi-order for short).

Given a left-order $\preceq$ on a group G, we say that a subset $S \subseteq G$ is *convex* if for all $h \in G$ and $f, g \in S$ satisfying that $f \preceq h \preceq g$, we have that $h \in S$. In other words, S is an interval. The notion of convexity is useful to build orders on group and to build set carrying invariant orders. This is the content of the following two lemmas.

Lemma 1.9 (Convex extension). *If G contains a normal subgroup A such that both A and G/A are left-orderable. Then G admits a left-order $\preceq$ for which A is convex.*

Proof. Indeed, letting $\preceq_0$ and $\preceq_1$ be left-orders on A and G/A, respectively, we may define $\preceq$ on G by

$$f \prec g \text{ if either } fA \prec_1 gA \text{ or } fA = gA \text{ and } e \prec_0 f^{-1}g.$$

□

Lemma 1.10 (Induced order on cosets). *Let $(G, \preceq)$ be a left-ordered group, and let H be a convex subgroup of G. Then the set G/H carries a natural total order which is invariant under left-multiplication by G.*

Proof. Indeed, $\leqslant$ is given by

$$fH < gH \quad \text{if} \quad fh_1 \prec gh_2 \quad \text{for some} \quad h_1, h_2 \in H.$$

First note that the definition does not depend on the choice of h_1 and h_2: choose two cosets $fH \neq gH$, and assume that there are elements $h_1, h_2, h_3, h_4 \in H$ such that $fh_1 \prec gh_2$ and $gh_3 \prec fh_4$. From here it follows that either $h_1 \prec f^{-1}gh_i \prec h_4$ for some $i \in \{2, 3\}$, or $f^{-1}gh_3 \prec h_i \prec f^{-1}gh_2$ for some $i \in \{1, 4\}$. In both cases, due to the convexity of H we conclude $f^{-1}g \in H$, which is a contradiction since $fH \neq gH$.

Now it is easy to see that the order is well defined. Let $fH < gH$, that is, $fh_1 \prec gh_2$ for some $h_1, h_2 \in H$. If we consider other representatives of the cosets, namely $fH = f'H$ and $gH = g'H$, then there exist h_3 and h_4 such that $f = f'h_3$ and $g = g'h_4$, therefore $f'h_3h_1 \prec g'h_4h_2$, so we conclude $f'H < g'H$.

The left-invariance follows directly from the definition since the order $\preceq$ of G is left-invariant.

□

An important family of examples of left-orderable groups are finitely-generated and torsion-free abelian groups. Indeed we will repetitively use the *lexicographic order* of $\mathbb{Z}^n$: we say that

(1) $$(i_1, \dots, i_n) \prec (i'_1, \dots, i'_n) \Leftrightarrow \exists\, k \in \{1, \dots, n\} \text{ such that } i_k < i'_k \text{ and } i_s = i'_s \text{ for } s < k.$$

Remark 1.11. *From this point it is easy to see that finitely-generated and torsion-free nilpotent groups are left-orderable. Indeed, using the upper central series (which has torsion-free abelian successive quotient) and applying Lemma 1.9 successively, one obtains a total left-order on the nilpotent group.*

Further, it is well known that a group is left-orderable if and only if all its finitely-generated subgroups are left-orderable, see [**14**, §1.1.2]. *Hence any torsion-free nilpontent group admits a left-order.*

1.3. Dynamical realization of an action on a ordered set

What is important for this work is that a countable group is left-orderable if and only if it embeds into $\mathrm{Diff}^0_+([0,1])$ (see [**29**, §2] or [**14**, §1.1.3] for details). More generally, given a group G that acts on a countable and totally order set $(\Omega, \preceq)$ by order preserving bijections, say $\omega \mapsto g(\omega)$, for $g \in G$ and $\omega \in \Omega$, then there is a *dynamical realization* of this action. This means that there is an order preserving map $i : (\Omega, \preceq) \to ([0,1], \leqslant)$ and a homomorphism $\upsilon : G \to \mathrm{Diff}^0_+([0,1])$ satisfying that

$$\upsilon(g)(i(\omega)) = i(g(\omega))$$

for every $\omega \in \Omega$ and every $g \in G$. See [**8**, Lemma 2.40] for a proof. Clearly υ is an embedding whenever the G action on Ω is faithful.

1.3.1. Using equivariant families to build embeddings. Let $(\Omega, \leqslant)$ be a totally ordered and countable set. If we choose a family of intervals $\{I_i \,:\, i \in \Omega\}$ such that $\sum_{i\in\Omega} |I_i| = 1$, then we can place them in the interval $[0,1]$ respecting the order. More precisely, if $i \leqslant j$ in the order of Ω then I_i has disjoint interior to I_j and is located to its left.

Assume that a group G acts on Ω preserving the order and consider a family of C^∞-diffeomorphisms $\{\varphi_i^j : I_i \to I_j\}$. We can try to define a map

$$\upsilon : G \to \mathrm{Diff}^0_+([0,1]),$$

by letting

$$\upsilon(g)|_{I_i} := \varphi_i^{g(i)},$$

but this map is not always well defined or turns out to be a group homomorphism. But in the case that the family of diffeomorphisms is an *equivariant family*, the map turns out to be a group homomorphism. Furthermore, as we will see below, some families have special properties that allow us to define homomorphisms into more regular spaces than C^0.

Definition 1.1. *A family $\{\varphi_I^J : I \to J \mid I$ and J vary over all compact intervals$\}$ of homeomorphisms is said to be equivariant if*

$$\varphi_J^K \circ \varphi_I^J = \varphi_I^K, \tag{2}$$

for all intervals $I, J, K \subset \mathbb{R}$.

Note that to obtain an equivariant family it is enough to define it over the intervals of the form $[0,a]$. Indeed, consider a family of homeomorphisms of the form $\{\varphi_a^b : [0,a] \to [0,b]\,;\, a > 0\,,\, b > 0\}$ that satisfy (2), and let $I = [x_1, x_2]$, $J = [y_1, y_2]$. Then just define

$$\varphi_I^J(x) = \varphi_{x_2-x_1}^{y_2-y_1}(x - x_1) + y_1.$$

Example 1.12. *(linear maps) for a and b positive real numbers define*

$$\varphi_a^b(x) = \frac{b}{a}\,x.$$

Example 1.13. *(Yoccos's family) For any $a > 0$ we define $\varphi_a : \mathbb{R} \to (-a/2, a/2)$ by*

$$\varphi_a(x) = \frac{a}{\pi}\arctan(ax).$$

Then we define $\varphi_a^b : (-a/2, a/2) \to (-b/2, b/2)$ by

$$\varphi_a^b = \varphi_a \circ \varphi_b^{-1}.$$

*It is immediate from this definition that the diffeomorphisms φ_a^b satisfy (2). Moving the intervals with translations an equivariant family is obtained. This family also has special characteristics such as derivative 1 on the end points and nice bounds for the Hölder norms (see [**29**]).*

The family that is important for our work (since it is the one we use to prove the Theorem A) is usually called the Pixton-Tsuboi family. We present this in the following lemma whose proof we take from [**41**] (see also [**36**]).

Lemma 1.14. *There exists a family of C^∞ diffeomorphisms $\varphi_{I',I}^{J',J} : I \to J$ between intervals I, J, where I' (resp. J') is an interval adjacent to I (resp. J) by the left, such that:*

1) For all I', I, J', J, K', K as above,

$$\varphi_{J',J}^{K',K} \circ \varphi_{I',I}^{J',J} = \varphi_{I',I}^{K',K}.$$

2) For all I', I, J', J,

$$D\varphi_{I',I}^{J',J}(x_-) = \frac{|J'|}{|I'|} \text{ and } D\varphi_{I',I}^{J',J}(x_+) = \frac{|J|}{|I|},$$

where x_- (resp. x_+) is the left (resp. right) endpoint of I.

3) There is a constant M such that for all $x \in I$, we have

$$D\log(D\varphi_{I',I}^{J',J})(x) \leqslant \frac{M}{|I|}\left|\frac{|I|\,|J'|}{|J|\,|I'|} - 1\right|.$$

4) Given I', I, J', J, K', K, L', L, as above, then

$$\left|\log(D\varphi_{I',I}^{K',K})(x) - \log(D\varphi_{J',J}^{L',L})(y)\right| \leqslant \left|\log\frac{|K|\,|J|}{|I|\,|L|}\right| + \left|\log\frac{|K'|\,|I|}{|I'|\,|K|}\right| + \left|\log\frac{|L'|\,|J|}{|J'|\,|L|}\right|,$$

for all $x \in I$, $y \in J$.

Proof. Let V be a C^∞ vector field on $[0,1]$ such that

- $V(x) = x$ near 0,
- $V(x) = 0$ on $[\frac{1}{2}, 1]$ and
- $\|DV(\cdot)\| \leqslant 1$.

Let ϕ_t be the flow associated with the differential equation

$$\begin{cases} \dfrac{d\phi_t(x)}{dt} = V(\phi_t(x)) \\ \\ \phi_0(x) = x. \end{cases}$$

Consider the diffeomorphism $x \mapsto b\,\phi_t(x/a)$ which sends the interval $[0,a]$ onto the interval $[0,b]$. The derivative of this diffeomorphism is equal to b/a on $[a/2,a]$ and is equal to $(b/a)e^t$ at 0. For real numbers a', a, b' and b such that $a' < 0 < a$ and $b' < 0 < b$, let $\varphi_{a',a}^{b',b} : [0,a] \to [0,b]$ be the diffeomorphism defined by

$$\varphi_{a',a}^{b',b}(x) = b\,\phi_{\log\left(\frac{b'a}{a'b}\right)}\left(\frac{x}{a}\right).$$

Then it is easy to check that for real numbers a', a, b', b, c and c' such that $a' < 0 < a$, $b' < 0 < b$ and $c' < 0 < c$,

$$\varphi_{b',b}^{c',c} \circ \varphi_{a',a}^{b',b} = \varphi_{a',a}^{c',c},$$

so we have that 1) and 2) holds. To Show 3), we use the following estimates. First note that

$$\left(\frac{d}{dt}\frac{\partial \phi_t}{\partial x}\right)(x) = DV(\phi_t(x))\frac{\partial \phi_t}{\partial x}(x).$$

Hence

$$\log\left(\frac{\partial \phi_t}{\partial x}\right)(x) = \int_0^t \frac{d}{ds}\log\left(\frac{\partial \phi_s}{\partial x}\right)(x)ds = \int_0^t DV(\phi_s(x))ds.$$

Since $\|DV(x)\| \leqslant 1$, we see that

$$\frac{\partial \phi_t}{\partial x}(x) \leqslant e^t \text{ and } |\phi_t(x) - \phi_t(y)| \leqslant e^t|x-y|.$$

Now

$$\log D\varphi_{a',a}^{b',b}(x) = \log\left(\frac{b}{a}\right) + \log D\phi_{\log\left(\frac{b'a}{a'b}\right)}\left(\frac{x}{a}\right) \tag{3}$$

and

$$\left|\log D\phi_{\log\left(\frac{b'a}{a'b}\right)}\right| \leqslant \left|\log\frac{b'a}{a'b}\right| = \left|\log\frac{b'}{a'} - \log\frac{b}{a}\right|. \tag{4}$$

Suppose that $\|D^2V\| < C$ for a positive real number C. Then we have

$$\begin{aligned}
\left|D\log D\varphi_{a',a}^{b',b}(x)\right| &= \left|D\log D\phi_{\log\left(\frac{b'a}{a'b}\right)}\left(\frac{x}{a}\right)\right| \\
&= \frac{1}{a}\left|\int_0^{\log(b'a/a'b)} D^2V\left(\phi_s\left(\frac{x}{a}\right)\right)\frac{\partial \phi_s}{\partial x}\left(\frac{x}{a}\right)ds\right| \\
&\leqslant \frac{C}{a}\left|\int_0^{\log(b'a/a'b)} e^s ds\right| \\
&= \frac{C}{a}\left|\frac{b'a}{a'b} - 1\right|.
\end{aligned}$$

So we conclude 3). From (3) and (4) we immediately obtain 4). □

1.4. Obstructions to regularity

In the search for the critical regularity of a group, it is of crucial importance to know which algebraic characteristics of the group impose restrictions on the regularity, for certain types of actions. The interested reader may like to see [**30**], where it is proved that the groups $\mathrm{Diff}_+^{1+\alpha}([0,1])$ do not contain subgroups of intermediate growth. Or [**28**] where A.Navas shows that infinite groups having Kazhdan property (T) does not embed into $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S}^1)$ for $\alpha > 1/2$.

1.4.1. Classical and generalized Kopell's lemma. We will start with the so-called Kopell 's lemma [**26**], which roughly states that two non-trivial C^2-diffeomorphisms of the closed interval cannot commute if one of them has fixed points in the interior of the interval and the other does not. Here we will present a shorter proof than Kopell's original (see [**29**]).

For the statement let us denote by $\mathrm{Diff}_+^{1+bv}([0,1])$ the group of C^1-diffeomorphisms with derivatives of bounded variation. More precisely, for an element $f \in \mathrm{Diff}_+^{1+bv}([0,1])$ we have that

$$\mathrm{var}(Df) = \sup_{0=x_0<x_1<\cdots<x_n=1} \sum_{i=1}^{n} \left|\log(Df(x_i)) - \log(Df(x_{i-1}))\right| < \infty.$$

Theorem 1.15. *(Kopell's lemma) Let f and g be commuting diffeomorphisms of the interval $[0,1]$. Suppose that f is of class C^{1+bv} and g of class C^1. If f has no fixed point in $(0,1)$ and g has at least one fixed point in $(0,1)$, then g is the identity.*

Proof. Replacing f by f^{-1} if necessary, we may suppose that $f(x) < x$ for all $x \in (0,1)$. Since g commutes with f and has a fixed point in $(0,1)$, it is easy to see that it does, in fact, have infinitely many fixed points that converge to 0 and 1. Therefore $Dg(0) = Dg(1) = 1$.

Now, consider a fixed point y of g and let $x \in I := [f(y), y]$. By the equality $g(x) = f^n g f^{-n}(x)$ we have that

$$Dg(x) = \frac{Df^n(g(f^{-n}(x)))}{Df^n(f^{-n}(x))}\, Dg(f^{-n}(x)).$$

Therefore, if we use the fact that the expression

$$\log\left(\frac{Df^n(g(f^{-n}(x)))}{Df^n(f^{-n}(x))}\right) \leqslant \sum_{i=1}^{n} \left|\log(Df(f^{-i}(g(x)))) - \log(Df(f^{-i}(x)))\right| \tag{5}$$

is bounded by the variation of Df, we conclude that $Dg(x)$ is uniformly bounded on the interval I. Note that the above argument can be repeated by replacing g with a positive power g^k. So, calling V the variation of f we have

$$\sup_{x\in I} Dg^k(x) \leqslant e^V.$$

This implies that the restriction of g to I is the identity. It is easy to choose I to get a contradiction. □

Remark 1.16. *If G is a non-abelian nilpotent group, the last subgroup of the lower central series provide us with non-trivial central commutators. Combining Kopell's lemma with lemma 2.6, it automatically follows that these groups do not act faithfully on $[0,1]$ by diffeomorphisms of class C^2.*

To prove Theorems B and C we will use a generalized version of Kopell's lemma. This is due to B. Deroin, V. Kleptsyn and A. Navas [**12**], and follows from estimates similar to those made above.

Theorem 1.17. *Let $f_1, \ldots, f_k$ be C^1-diffeomorphisms of the interval $[0,1]$ that commute with a C^1-diffeomorphism g. Assume that g fixes a subinterval I of $[0,1]$ and its restriction* *to I is non-trivial. Assume moreover that for a certain $0 < \alpha < 1$ and a sequence of indexes $i_j \in \{1, \ldots, k\}$, the sum*

$$L_\alpha := \sum_{j \geqslant 0} \left| f_{i_j} \cdots f_{i_1}(I) \right|^\alpha < \infty.$$

Then $f_1, \ldots, f_k$ cannot be all of class $C^{1+\alpha}$.

Proof. Let x_0 be such that $g(x_0) \neq x_0$. Denote by $[a, b]$ the shortest interval containing x_0 that is fixed by g. For each $j \geqslant 1$, $n \geqslant 1$ and $z \in [a, b]$, the equality $g^n = (f_{i_j} \cdots f_{i_1})^{-1} \circ g^n \circ (f_{i_j} \cdots f_{i_1})$ yields

$$\log(Dg^n(z)) = \log(D(f_{i_j} \cdots f_{i_1})(z)) + \log(Dg^n(f_{i_j} \cdots f_{i_1}(z))) - \log(D(f_{i_j} \cdots f_{i_1})(g^n(z))).$$

Assume that $f_1, \ldots, f_k$ are $C^{1+\alpha}$ diffeomorphisms, and fix an α-Holder constant M for the functions $Df_1, \ldots, Df_k$. Letting $z_n := g^n(z)$ and noticing that z_n belongs to $[a, b] \subset I$ for all $n \geqslant 1$, we obtain that $|\log(Dg^n(z))|$ is bounded above by

$$\left|\log(Dg^n(f_{i_j} \cdots f_{i_1}(z)))\right| + \sum_{m=1}^{j} \left|\log(Df_{i_m}(f_{i_{m-1}} \cdots f_{i_1}(z))) - \log(Df_{i_m}(f_{i_{m-1}} \cdots f_{i_1}(z_n)))\right|.$$

So, using the α-Hölder condition we have

$$|\log(Dg^n(z))| \leqslant \left|\log(Dg^n(f_{i_j} \cdots f_{i_1}(z)))\right| + M\, L_\alpha.$$

Now, since g^n fixes the intervals $f_{i_j} \cdots f_{i_1}(I)$, in each of them there must be a point in which the derivative of g^n is equal to 1. Therefore $Dg^n(f_{i_j} \cdots f_{i_1}(z))$ converges to 1 as j goes to infinity. Hence we conclude that $Dg^n(z) \leqslant e^{M\, L_\alpha}$, which is a contradiction because the restriction on g to $[a, b]$ is non-trivial.

□

The following lemma is useful to get into the hypotheses of Theorem 1.17. Although it is stated in a slightly different way, the proof is the same as that of [**12**, Lemma 3.3].

Lemma 1.18. *Let $f_1, \ldots, f_k$ be C^1-diffeomorphisms of $[0, 1]$, and I subinterval of $[0, 1]$ such that $\mathbb{Z}^k \simeq \langle f_1, \ldots, f_k \rangle / \mathrm{Stab}(I)$, where $\mathrm{Stab}(I)$ is the stabilizer of I (which is assumed to be a normal subgroup). Then, if $\alpha > 1/k$ there exists a sequence $(f_{i_j})_{j \in \mathbb{N}}$ of elements in $\{f_1, \ldots, f_k\}$ such that*

$$\sum_{j \geqslant 0} \left| f_{i_j} \cdots f_{i_1}(I) \right|^\alpha < \infty.$$

Before giving the proof we will explain the main idea. It is easy to see that the set

$$\{f_1^{n_1} \cdots f_k^{n_k}(I) : (n_1, \ldots, n_k) \in \mathbb{N}^k \text{ and } \sum_{i=1}^{k} n_i \leqslant n\}$$

contains exactly

$$\frac{(n+1)(n+2)\cdots(n+k)}{k!} \sim n^k$$

disjoint intervals. So, we should expect that, "typically", their length has order $1/n^k$. Hence, for a "generic" random sequence $(f_{i_j})_{j\in\mathbb{N}}$ of elements in $\{f_1,\ldots,f_k\}$, we should have that for $\alpha > 1/k$

$$\sum_{n\geqslant 1} |f_{i_n}\cdots f_{i_1}(I)|^{\alpha} \leqslant C \sum_{n\geqslant 0} \frac{1}{n^{k\alpha}} < \infty.$$

To formalize the idea, let $\{e_1,\ldots,e_k\}$ be the canonical generators of $\mathbb{Z}^k$; and consider the Markov process on $\mathbb{N}^k$ with transition probabilities

$$p\left[\left(\sum_{j=1}^{k} n_j e_j\right) \to \left(e_i + \sum_{j=1}^{k} n_j e_j\right)\right] = \frac{n_i+1}{n_1+\cdots+n_k+k}.$$

This process induces a probability measure $\mathbb{P}$ on the space of infinite paths

$$\Omega = \{e_1,\ldots,e_k\}^{\mathbb{N}}.$$

Now, let $\alpha > 1/k$. Given a sequence $\omega = (e_{i_j})_{j\in\mathbb{N}} \in \Omega$, we identify the elements e_{i_j} with f_{i_j}, and define the functions $X_n : \Omega \to \mathbb{R}$ as follows

$$X_n(\omega) := |f_{i_n}\cdots f_{i_1}(I)|^{\alpha}.$$

Define also $X : \Omega \to \mathbb{R}\cup\{\infty\}$ as

$$X(\omega) := \sum_{n\geqslant 0} X_n(\omega).$$

Proof. We are going to prove the lemma showing that the expectation of the function X is finite $\mathbb{P}$-almost everywhere. Since the probability of reaching $(n_1,\ldots,n_k)$ in $n = n_1+\cdots+n_k$ steps is equal to

$$\frac{(k-1)!}{(n+1)(n+2)+\cdots+(n+k-1)} \sim \frac{(k-1)!}{n^{k-1}},$$

we have that

$$\mathbb{E}(X_n) \leqslant (k-1)! \sum_{n_1+\cdots+n_k=n} \left|f_1^{n_1}\cdots f_k^{n_k}(I)\right|^{\alpha} \frac{1}{n^{k-1}}.$$

By Holder's inequality,

$$\mathbb{E}(X_n) \leqslant (k-1)! \left(\sum_{n_1+\cdots+n_k=n} \left|f_1^{n_1}\cdots f_k^{n_k}(I)\right|\right)^{\alpha} \left(\sum_{n_1+\cdots+n_k=n} \frac{1}{n^{(k-1)/(1-\alpha)}}\right)^{1-\alpha}.$$

So, since the number of elements that satisfy $n_1+\cdots+n_k = n$, has order n^{k-1}, we have that there exists a positive constant C such that

$$\mathbb{E}(X_n) \leqslant C \left(\sum_{n_1+\cdots+n_k=n} \left|f_1^{n_1}\cdots f_k^{n_k}(I)\right|\right)^{\alpha} \frac{1}{n^{\alpha(k-1)}}.$$

Again Holder's inequality yields

$$\mathbb{E}(X) = \sum_{n\geqslant 0}\mathbb{E}(X_n) \leqslant C \left(\sum_{n_1,\ldots,n_k\in\mathbb{N}} \left|f_1^{n_1}\cdots f_k^{n_k}(I)\right|\right)^{\alpha} \left(\sum_{n\geqslant 0} \frac{1}{n^{\alpha(k-1)/(1-\alpha)}}\right)^{1-\alpha},$$

and since $\alpha > 1/k$ we conclude that $\mathbb{E}(X)$ is finite, as desired. □

1.4.2. Positive version of Kopell's lemma. The next lemma is strongly inspired by the work of Kopell (see the appendix of [**33**]). Below, by $\mathrm{Diff}_+^{1+Lip}([0,1])$ we refer to the group of orientation preserving diffeomorphisms of $[0,1]$ with Lipschitz derivative.

Lemma 1.19. *Let $f \in \mathrm{Diff}_+^{1+Lip}([0,1])$ be such that $f(x) \neq x$ for all $x \in (0,1)$. Fix $x_0 \in (0,1)$, and let $x_n := f^n(x_0)$. Let $(g_n)_{n\in\mathbb{Z}}$ be a sequence of C^1 diffeomorphisms of the interval of endpoints x_0, x_1 such that $Dg_n(x_0) = Dg_n(x_1) = 1$ for all $n \in \mathbb{Z}$. Let $g : (0,1) \to (0,1)$ be the diffeomorphism whose restriction to each interval of endpoints x_n, x_{n+1} coincides with $f^n g_n f^{-n}$. If $g_n \to Id$ in C^1 topology as $|n| \to \infty$, then g extends to a C^1 diffeomorphism of $[0,1]$ by letting $f(0) = 0$ and $f(1) = 1$.*

Proof. Assume $n \geqslant 0$ and $f(x) < x$ for all $x \in (0,1)$. Define $I_n := f^n([x_1, x_0])$, take a point $z \in I_n$ and put $z_n = f^{-n}(z)$. We only need to check the differentiability of g at point 0 (at 1 is analogous). Arguing as above we have

$$Dg(z) = \frac{Df^n(g_n(z_n))}{Df^n(z_n)}\, Dg_n(z_n).$$

By hypothesis we know that $Dg_n(z_n) \to 1$ as $|n| \to \infty$. And on the other hand

$$\left|\log\left(\frac{Df^n(g_n(z_n))}{Df^n(z_n)}\right)\right| \leqslant \sum_{i=0}^{n-1} \left|\log Df(f^i(g_n(z_n))) - \log Df(f^i(z_n))\right| \tag{6}$$

$$\leqslant M \sum_{i=0}^{n-1} \left|f^i(g_n(z_n)) - f^i(z_n)\right|, \tag{7}$$

where M is a Lipschitz constant for Df. We claim that this last expression goes to zero as n goes to infinity. Indeed, by the mean value theorem we know that $(g_n(z_n) - x_1)/(z_n - x_1) = Dg_n(\xi_n)$ for some $\xi_n \in \mathrm{conv}\{x_1, z_n\}$. And since g_n converges to the identity in the C^1 topology we have

$$\frac{g_n(z_n) - z_n}{z_n - x_1} = (Dg_n(\xi_n) - 1) \sim 0.$$

Now, let $\varepsilon > 0$. By the previous part and the mean value theorem we eventually have (for certain points $c_n \in \mathrm{conv}\{g_n(z_n), z_n\}$ and $\tilde{c_n} \in (x_1, x_0)$) that

$$\left|\frac{f^i(g_n(z_n)) - f^i(z_n)}{f^i(x_0) - f^i(x_1)}\right| = \left|\frac{g_n(z_n) - z_n}{x_1 - x_0}\right| \left|\frac{Df^i(c_n)}{Df^i(\tilde{c_n})}\right| \leqslant \varepsilon\, e^V,$$

where the last inequality of follows from the estimates made in (5) (and we are calling V the variation of Df). Now we return to the expression (7) and we will see that it goes to zero. For this we simply use the above and conclude that

$$\sum_{i=0}^{n-1} \left|f^i(g_n(z_n)) - f^i(z_n)\right| \leqslant \varepsilon\, e^V \sum_{i=0}^{n-1} \left|f^i(x_1) - f^i(x_0)\right| \leqslant \varepsilon\, e^V$$

as desired. □

1.4.3. No strong Kopell's lemma in class $C^{1+\alpha}$. We end this section by showing that lemma 1.19 is not valid in class $C^{1+\alpha}$ (at least not without adding some extra hypothesis). Namely, there exist:

- a C^∞ diffeomorphism of $[0,1]$ such that $f(x) > x$ for all $x \in (0,1)$ with a fundamental domain $[x_0, x_1]$ (where $x_1 := f(x_0)$),
  and
- a sequence $(g_n)_{n\in\mathbb{Z}}$ of $C^{1+\alpha}$ diffeomorphisms of $[x_0, x_1]$,
  such that $Dg_n(x_0) = Dg_n(x_1) = 1$ for all n, one has the convergence $g_n \to Id$ in C^∞ topology, but the C^1 diffeomorphism $g\colon [0,1] \to [0,1]$ defined as

$$g|_{f^n([x_0,x_1])} := f^n g_n f^{-n}|_{f^n([x_0,x_1])} \tag{8}$$

  is not $C^{1+\alpha}$ for any $\alpha \in (0,1)$.

Our diffeomorphism f is

$$f(x) = \frac{2x}{x+1}.$$

Notice that

$$f^n(x) = \frac{2^n x}{(2^n-1)x+1}.$$

Hence, for all x, y,

$$\left|f^n(x) - f^n(y)\right| = \left|\frac{2^n x}{(2^n-1)x+1} - \frac{2^n y}{(2^n-1)y+1}\right| = \frac{2^n|x-y|}{[(2^n-1)x+1]\cdot[(2^n-1)y+1]},$$

which yields

$$\left|f^n(x) - f^n(y)\right| \leqslant \frac{C}{2^n} \tag{9}$$

for a certain universal constant C provided x, y both belong to a compact subinterval of $(0,1)$ (say $[1/2, 2/3]$).

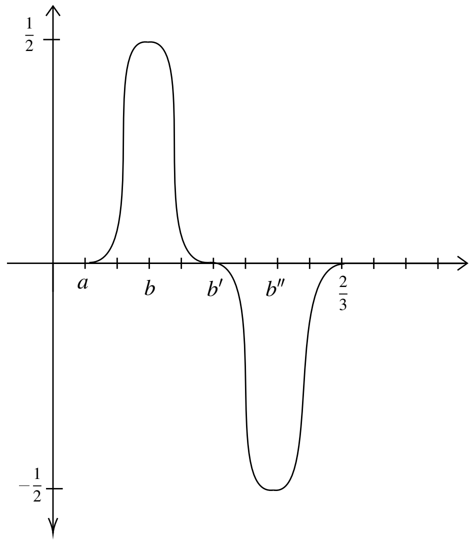
Now consider the points $x_0 := 1/2$ and $x_1 := f(x_0) = 2/3$. Notice that $x_1 - x_0 = 1/6$. Then let

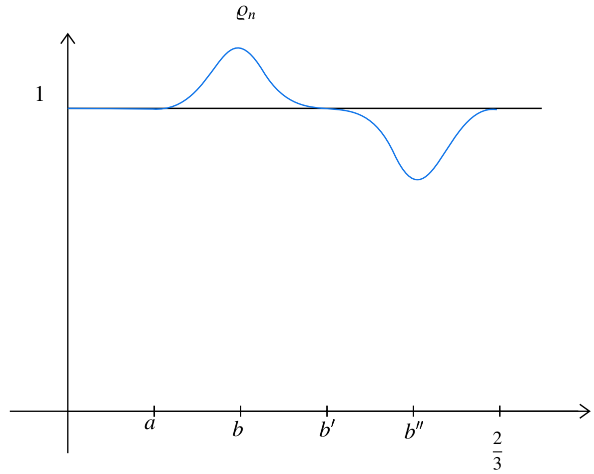
$$a := \frac{1}{2} + \frac{1}{30} = \frac{8}{15}, \quad b := \frac{1}{2} + \frac{2}{30} = \frac{17}{30}, \quad b' := \frac{1}{2} + \frac{3}{30} = \frac{3}{5}, \quad b'' := \frac{1}{2} + \frac{4}{30} = \frac{19}{30}.$$

Let $\varrho\colon [a, 2/3] \to [-1/2, 1/2]$ be a C^∞ function such that

$$\varrho(a) = \varrho(b') = \varrho(2/3) = 0, \qquad \varrho(b) = \frac{1}{2}, \qquad \varrho(b'') = -\frac{1}{2}.$$

Assume also that ϱ is strictly increasing on $[a,b]$ and $[b'', 3/2]$, strictly decreasing on $[b, b'']$, infinitely flat at a and $3/2$, and its graph is symmetric with respect to the point $(b', 0)$. Let $\rho_n\colon [1/2, 2/3]$ be the function that is identically equal to 1 on $[1/2, a]$ and whose restriction to $[a, 2/3]$ coincides with $1 + \varrho/n$.





By the symmetry property of ϱ, we have

$$\int_0^1 \rho_n(s)\, ds = b - a,$$

hence ρ_n is the derivative of a diffeomorphism g_n of $[a, b]$. Since ρ_n is C^∞, the diffeomorphism g_n is of class C^∞. We claim that g_n converges to the identity in the C^k topology for every integer k. (Hence, by definition, the convergence holds in C^∞ topology.) Indeed, for $k \geqslant 2$, as n goes to infinite, we have

$$\|D^k(g_n)\|_{C^0} = \|D^{k-1}(\rho_n)\|_{C^0} = \frac{1}{n}\|D^{k-1}\varrho\|_{C^0} \longrightarrow 0.$$

We will next show that the corresponding diffeomorphism g obtained via (8) is not $C^{1+\alpha}$ for any $\alpha > 0$.

Since $g_n(a) = a$ and $Dg_n(a) = 1$, we have

$$Dg(f^n(a)) = D(f^n g_n f^{-n})(f^n(a)) = \frac{Df^n(g_n(a))}{Df^n(a)} \cdot Dg_n(a) = 1.$$

To compute $Dg(f^n(b))$, first notice that

$$Df^n(x) = \frac{2^n}{\left[(2^n-1)x+1\right]^2}.$$

We compute:

$$Dg(f^n(b)) = \frac{Df^n(g_n(b))}{Df^n(b)} \cdot Dg_n(b) = \left[\frac{(2^n-1)b+1}{(2^n-1)g_n(b)+1}\right] \cdot \left(1 + \frac{1}{n}\right).$$

Since

$$\begin{aligned} g_n(b) &= g_n(a) + \int_a^b \rho_n(s)\, ds = g_n(a) + \int_a^b \left(1 + \frac{\varrho(s)}{n}\right)(s)\, ds \\ &= a + (b-a) + \int_a^b \frac{\varrho(s)}{n}\, ds = b + \frac{C}{n}, \end{aligned}$$

where

$$I := \int_a^b \varrho(s)\, ds > 0,$$

this yields

$$Dg(f^n(b)) = \left[\frac{(2^n-1)b+1}{(2^n-1)(b+I/n)+1}\right] \cdot \left(1+\frac{1}{n}\right),$$

hence

$$|Dg(f^n(b)) - Dg(f^n(a))| = \frac{1}{n}\left[\frac{(2^n-1)b+1}{(2^n-1)(b+I/n)+1}\right] - \left[\frac{I}{n\,(2^n-1)(b+I/n)+1}\right].$$

Therefore, for a certain constant C',

$$\big|Dg(f^n(b)) - Dg(f^n(a))\big| \geqslant \frac{C'}{n}. \tag{10}$$

Finally, putting (9) and (10) together, we obtain

$$\frac{|Dg(f^n(b) - Dg(f^n(a)))|}{|f^n(b) - f^n(a)|^\alpha} \geqslant \frac{C'}{C^\alpha} \cdot \frac{2^{n\alpha}}{n},$$

which diverges to infinite as $n \to \infty$ provided $\alpha > 0$. This shows that g is not of class $C^{1+\alpha}$.

1.5. Distortion

We close this preliminars with the basic definitions and examples that we will need for Chapter 3. Remember that if G is a group generated by a finite and symmetric set S, the word metric $\|\cdot\| : G \to \mathbb{R}$ is defined as

$$\ell_S(f) = \|f\| := \min\{\, n \,:\, f = s_1 \cdots s_n, \text{ with } s_1, \ldots, s_n \in S\,\}.$$

Since the sequence $n \mapsto \|f^n\|$ satisfy $\|f^{n+m}\| \leqslant \|f^n\| + \|f^m\|$, we have by Fekete's lemma (see [**33**] lemma 2.2.1) that the limit

$$\lim_{n\to\infty} \frac{\|f^n\|}{n} \tag{11}$$

exists. It is not difficult to see that if S and T are finite generating sets of the group G, then there exists a positive constant M such that $\ell_S(f) \leqslant M\,\ell_T(f)$ for all $f \in G$. Therefore, if the above limit is zero with the metric associated with one generator, then the limit will also be zero with the metric associated with any other generator. So we can do the following definition.

Definition 1.2. *Let G be a finitely generated group. An element f is distorted in G if the respective limit (11) is zero, for some finite and symmetric generating set S.*

Definition 1.3. *Let G be a group. An element f is distorted in G if it is distorted in some finitely generated subgroup.*

EXAMPLE 1.20. *The Baumslag-Solitar group contains the canonical example of a distorted element. Remember that this group has the presentation*

$$BS(1,2) = \langle f, g \,:\, gfg^{-1} = f^2 \rangle.$$

The relation of the group yields that for all $n \in \mathbb{N}$ we have $g^n f g^{-n} = f^{2^n}$. Therefore

$$\frac{\|f^{2^n}\|}{2^n} \sim \frac{2n+1}{2^n} \xrightarrow[n\to\infty]{} 0.$$

So, since we know that the sequence $(\|f^n\|/n)_{n\in\mathbb{N}}$ is convergent, its limit must be zero.

EXAMPLE 1.21. *(Nilpotent Groups) Let G be a non-abelian nilpotent group of nilpotency degree n. We know that $\gamma_n = [G, \gamma_{n-1}] = \{e\}$, therefore the subgroup γ_{n-1} is non-trivial an central. So, we can choose $z \in \gamma_{n-1}$ of the form $z = [x, y]$, and since z is central, it is easy to see that*

$$z^{n^2} = [x^n, y^n],$$

this yields

$$\frac{\|z^{n^2}\|}{n^2} \sim \frac{4n}{n^2} \xrightarrow[n\to\infty]{} 0.$$

Therefore, the same argument used in the previous example tells us that element z is distorted in G.

We now introduce a tool to identify undistorted elements in a group.

DEFINITION 1.4. *Let G be a group. A length function on G is a function $\ell : G \to \mathbb{R}_+$ satisfying $\ell(1) = 0$ which is symmetric and subadditive.*

Note that if $\ell : G \to \mathbb{R}_+$ is a length function and

$$\lim_{n\to\infty} \frac{\ell(f^n)}{n} > 0,$$

then, the element f is non-distorted in G. Indeed, if f is distorted in a group generated by a finite and symmetric set S, we have

$$\lim_{n\to\infty} C\,\frac{\|f^n\|}{n} \geqslant \lim_{n\to\infty} \frac{\ell(f^n)}{n} > 0,$$

where $C = \max\{\ell(s) \,:\, s \in S\}$ and $\|\cdot\|$ is the metric asociated to S.

As we will see in Chapter 3, the variation of the derivative

$$f \mapsto \mathrm{var}(\log Df)$$

will be our length function and it will allow us to find undistorted elements in the group $\mathrm{Diff}_+^{1+bv}([0,1])$.

CHAPTER 2

On the critical regularity of metabelian nilpotent groups

2.1. Proof of Theorem A

Throughout this chapter, G will denote a torsion-free nilpotent group of degree n which is also metabelian and A a maximal abelian subgroup which we assume that contains $[G,G]$ (in particular it is normal).

2.1.1. On the conjugacy action of G/A on A. In view of Proposition 1.8, we have that G is an extension of $\mathbb{Z}^k$ by $\mathbb{Z}^d$,

$$1 \longrightarrow \mathbb{Z}^d \longrightarrow G \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

where $A \simeq \mathbb{Z}^d$ and $G/A \simeq \mathbb{Z}^k$. We begin by studying the natural action of G/A on A coming from the conjugacy action of G on A.

Let $\{g_1, \ldots, g_d\}$ and $\{f_1A, \ldots, f_kA\}$ be generators of A and G/A respectively. Since A is normal, the subgroup of G generated by $f_1, \ldots, f_k$, acts on A by automorphisms

$$\langle f_1, \ldots, f_k\rangle \longrightarrow \mathrm{Aut}(\mathbb{Z}^d).$$

Therefore, the action of each $f \in \langle f_1, \ldots, f_k\rangle$ is given by a matrix $A_f \in GL_d(\mathbb{Z})$, which depends on the set $\{g_1, \ldots, g_d\}$. We call A_f the *conjugacy matrix* of f. In the special case of the generators $f_1, \ldots, f_k$, we denote the conjugacy matrix of f_i simply by A_i.

In the next lemma, we will see that we can always choose a generator of A, such that the conjugacy matrices of the elements $f_1, \ldots, f_k$ belong to $U_d(\mathbb{Z})$, the group of upper triangular matrices with 1's in the diagonal. This is known to Mal'cev in the case where the matrix entries belong to a field. We write a direct proof in our special case. For the statement we will say that a generating set of a group is *minimal*, if it has least possible cardinality.

Lemma 2.1. *Let $A \leqslant G$ be a maximal abelian subgroup satisfying that $[G,G] \subseteq A$. Suppose that $\mathbb{Z}^d \simeq A$ and $\mathbb{Z}^k \simeq G/A = \langle f_1A, \ldots, f_kA\rangle$. Then, there exists a generating set $\{g_1, \ldots, g_d\}$ of A, such that the conjugacy matrices of the elements $f_1, \ldots, f_k$ belong to $U_d(\mathbb{Z})$. In particular, the nilpotency degree of G is bounded by $d+1$.*

Proof. Since G is nilpotent of degree n, the upper central series

$$\{e\} = \zeta_0 \leqslant \zeta_1 \leqslant \cdots \leqslant \zeta_n = G,$$

is finite. Remember that all the factors ζ_i/ζ_{i-1} are torsion-free. Combining this with the fact that G/A is also torsion-free, we have , for $g \in G$, that

$$g^j \in \zeta_i \cap A \Rightarrow g \in \zeta_i \cap A \quad \forall i \in \{0, \ldots, n\}, j \in \mathbb{Z}. \tag{12}$$

Define $\Gamma_i := \zeta_i \cap A$ and let m be the smallest element in $\{1, \ldots, n\}$ such that $\Gamma_m = A$. This yields the filtration

$$\{e\} = \Gamma_0 \leqslant \Gamma_1 \leqslant \cdots \leqslant \Gamma_m = A,$$

such that

$$[G,\Gamma_i] \subseteq \Gamma_{i-1}, \tag{13}$$

and, by (12), also enjoys the property that each factor Γ_{i+1}/Γ_i is torsion-free abelian.

Note that if $\Gamma_{m-1} \simeq \mathbb{Z}^{n_{m-1}}$ and $\Gamma_m/\Gamma_{m-1} \simeq \mathbb{Z}^{n_m}$ then, since $\Gamma_m = A \simeq \mathbb{Z}^d$ is abelian, we have that $d = n_{m-1} + n_m$. Therefore, if $\{g_1, \dots, g_{n_{m-1}}\}$ and $\{g_{n_{m-1}+1}\Gamma_{m-1}, \dots, g_{n_{m-1}+n_m}\Gamma_{m-1}\}$ are minimal generating sets of Γ_{m-1} and Γ_m/Γ_{m-1} respectively, then $\{g_1, \dots, g_{n_{m-1}}, g_{n_{m-1}+1}, \dots, g_d\}$ is a minimal generating set of $\Gamma_m = A$.

Recursively, we obtain a minimal generating set $\{g_1, \dots, g_d\}$ of A which, by (13), has the property that for $g_s \in \{g_1, ..., g_d\} \cap \Gamma_i$, it holds that $[f_j, g_s] \in \Gamma_{i-1} \subseteq \langle g_1, ..., g_{s-1}\rangle$ for all $j \in \{1, ..., k\}$. In other words the conjugacy matrices of each $f \in \{f_1, \dots, f_k\}$ belong to $U_d(\mathbb{Z})$.

The fact that G has nilpotency degree bounded by $d+1$ follows from the fact that $U_d(\mathbb{Z})$ has nilpotency degree $d+1$. □

2.1.2. An action of G on a totally ordered set.

Proposition 2.2. *Let $A \leqslant G$ be a maximal abelian subgroup satisfying $[G,G] \subseteq A$. Fix a generating set $\{g_1, \dots, g_d, f_1, \dots, f_k\}$ of G, such that, $\{g_1, \dots, g_d\}$ is the generating set of A given by Lemma 2.1 and $\langle f_1A, \dots, f_kA\rangle = G/A \simeq \mathbb{Z}^k$. Then, for a fixed $s \in \{1, \dots, d\}$ there is an action of G on $\mathbb{Z}^{k+1}$ satisfying:*

1) *The action of G on $\mathbb{Z}^{k+1}$ preserves the lexicographic order.*
2) *There exist functions $\ell_t, r_m : \mathbb{Z}^k \to \mathbb{Z}$, such that*
$$f_t(i_1, .., i_t, .., i_k, j) = (i_1, .., i_t + 1, .., i_k, j + \ell_t(i_1, \dots, i_k)),$$
$$g_m(i_1, \dots, i_k, j) = (i_1, \dots, i_k, j + r_m(i_1, \dots, i_k)),$$
for all $m \in \{1, \dots, d\}$ and $t \in \{1, \dots, k\}$. Besides, $r_s \equiv 1$ and $r_1 = \cdots = r_{s-1} \equiv 0$.
3) *There exists a positive constant M, such that for all $t \in \{1, \dots, k\}$, $m \in \{1, \dots, d\}$ and $(i_1, \dots, i_k) \neq (0, \dots, 0)$ we have*
$$|\ell_t(i_1, \dots, i_k)| \leqslant M(|i_1| + \cdots + |i_k|)^d, \qquad |r_m(i_1, \dots, i_k)| \leqslant M(|i_1| + \cdots + |i_k|)^d.$$

Proof. We first show item 2. To this end, fix $s \in \{1, \dots, d\}$ and consider the subgroup $H_s = \langle \{g_1, \dots, g_d\} \setminus \{g_s\}\rangle$. Since the sets $\{f_1^{i_1} \cdots f_k^{i_k} A : i_1, \dots, i_k \in \mathbb{Z}\}$ and $\{g_s^j H_s : j \in \mathbb{Z}\}$ are partitions of G and A respectively, the coset space can be described by the *normal forms*

$$G/H_s = \{f_1^{i_1} \cdots f_k^{i_k} g_s^j H_s \ : \ i_1, \dots, i_k, j \in \mathbb{Z}\}. \tag{14}$$

Hence we can identify G/H_s with $\mathbb{Z}^{k+1}$ by identifying $f_1^{i_1} \cdots f_k^{i_k} g_s^j H_s$ with $(i_1, \dots, i_k, j)$. In particular, the left-multiplication action of G on G/H_s provides an action of G on $\mathbb{Z}^{k+1}$. This is the action we want to consider.

Now, by Lemma 2.1, we have that for all $i, j \in \{1, \dots, k\}$ and $l \in \{1, \dots, d\}$ it holds that

$$f_i f_j \in f_j f_i \langle g_1, \dots, g_d\rangle \ \text{ and } \ g_l f_j \in f_j g_l \langle g_1, \dots, g_{l-1}\rangle.$$

Therefore, for $t \in \{1, \dots, k\}$, the action of f_t is addition by 1 on the t coordinate and the and the action on the $k+1$ coordinate depends on previous k coordinates, hence the function ℓ_t. The function r_m, for $m \in \{1, \dots, d\}$, can be found analogously.

We are now in position to check item 1. This follows from the fact that the lexicographic order on the coset space $\mathbb{Z}^{k+1} \simeq G/H_s$ given by (1), can recovered as a convex extension of the lexicographic order on $\mathbb{Z}^k \simeq G/A$ by the (lexicographic) order on $A/H_s \simeq \mathbb{Z}$. See Lemma 1.9 and Lemma 1.10.

Finally we check item 3. Let $t \in \{1, ..., k\}$. Recall that the action of f_t on $\mathbb{Z}^{k+1}$ is nothing but the left-multiplication action of f_t on G/H_s. Hence, in order to compute the image of $f_1^{i_1} \cdots f_t^{i_t} \cdots f_k^{i_k} g_s^j H_s$ under f_t, we need to multiply and find the representative in normal form (14). To do this, observe that $f_t f_j = [f_t, f_j] f_j f_t$. Hence, bringing f_t to the t-th position, generate at most $|i_1| + \cdots + |i_k|$ simple commutators of weight 2, which we now need to move to the right most place. Since G is metabelian, the commutators commute with each other. So, moving them all to the rightmost place generates at most $(|i_1| + \cdots + |i_k|)^2$ simple commutators of weight 3. Analogously, moving them all to the rightmost place, we have at most $(|i_1| + \cdots + |i_k|)^3$ simple commutators of weight 4, and so on. Since G has nilpotency degree bounded by $d + 1$, all simple commutators of this weight are trivial (see Lemma 2.1). Therefore, repeating the previous argument $d + 1$ times, we have

$$f_t.(f_1^{i_1} \cdots f_t^{i_t} \cdots f_k^{i_k} g_s^j H_s) = f_1^{i_1} \cdots f_t^{i_t+1} \cdots f_k^{i_k} g g_s^j H_s,$$

where $g \in A$ is the product of at most

$$\sum_{i=1}^{d} (|i_1| + \cdots + |i_k|)^i \leqslant d(|i_1| + \cdots + |i_k|)^d$$

simple commutators. Now note that

$$g.g_s^j H_s = g_s^{\ell_t(i_1,...,i_k)} g_s^j H_s,$$

since $\ell_t(i_1, ..., i_k)$ agrees with the exponent of g_s in the expression of g over the generators $\{g_1, ..., g_d\}$. Therefore, letting $\mathscr{S} \subseteq A$ be the set of all simple commutators of G (which is finite), and defining

$$\lambda := \max\{|m_s| : \exists\, m_1, ..., m_d \text{ for which } (g_1^{m_1} \cdots g_s^{m_s} \cdots g_d^{m_d}) \in \mathscr{S}\},$$

we see that $\ell_t(i_1, \ldots, i_k)$ is bounded by λ times the number of simple commutators that were used to write g. Hence

$$|\ell_t(i_1, ..., i_k)| \leqslant \lambda d(|i_1| + \cdots + |i_k|)^d.$$

Analogous computations give the inequality for the functions r_m. □

REMARK 2.3. *Note that the action built in the Proposition 2.2 is not necessarily faithful. However, given* $g_s \in \{g_1, ..., g_d\}$ *we have that the action* $G \curvearrowright \mathbb{Z}^{k+1}$ *is such that the elements* $g_1, ..., g_{s-1}$ *act trivially and* $g_s(i_1, ..., i_k, j) = (i_1, ..., i_k, j + 1)$. *This will be used in §2.1.4 in order to build a faithful action.*

2.1.3. Action by diffeomorphisms of $[0, 1]$. Above we built an action of G on $\mathbb{Z}^{k+1}$ which preserves the lexicographic order, and hence we can consider the dynamical realization of this action to get a G-action by orientation preserving homeomorphisms of $[0, 1]$.

However, since the group is nilpotent and we have good control from the polynomials appearing in Proposition 2.2, we will use Lemma 1.14 to see that this action can actually be projected to an action by diffeomorphisms of $[0, 1]$.

Let $\{I_{i_1,\ldots,i_k,j} : (i_1,\ldots,i_k,j) \in \mathbb{Z}^{k+1}\}$ be a family of intervals whose union is dense in $[0,1]$ and that are disposed preserving the lexicographic order of $\mathbb{Z}^{k+1}$. We identify the generators $g_1,\ldots,g_d,f_1,\ldots,f_k$ from Lemma 2.1 with elements in $\mathrm{Diff}^0_+([0,1])$ as follows: f_t and g_s will be homeomorphisms of $[0,1]$, such that their restriction to $I_{i_1,\ldots,i_k,j}$ coincides, respectively, with

$$\varphi_{I_{i_1,\ldots,i_k,j-1},\,I_{i_1,\ldots,i_k,j}}^{I_{i_1,\ldots,i_t+1,\ldots,i_k,j+\ell_t(i_1,\ldots,i_k)-1},\,I_{i_1,\ldots,i_t+1,\ldots,i_k,j+\ell_t(i_1,\ldots,i_k)}} \quad \text{and} \quad \varphi_{I_{i_1,\ldots,i_k,j-1},\,I_{i_1,\ldots,i_k,j}}^{I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_k)-1},\,I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_k)}},$$

for $t \in \{1,\ldots,k\}$, and $s \in \{1,\ldots,d\}$. Thus, by 1) in Lemma 1.14, we have a group homomorphism $G \to \mathrm{Diff}^0_+([0,1])$. The main technical step for proving Theorem A is the following proposition.

Proposition 2.4. *For an appropriate choice of length of the intervals $|I_{i_1,\ldots,i_k,j}|$, the homeomorphisms $f_1,\ldots,f_k,g_1,\ldots,g_d$ are simultaneously diffeomorphisms of class $C^{1+\alpha}$, for any $\alpha < 1/k$.*

The rest of §2.1.3 is devoted to give the proof of Proposition 2.4. We will assume that $k \geqslant 2$ since, after Condition 3 in Proposition 2.2, we can use the estimates from [**10**, §4] to ensure that, when $k=1$, the action is by $C^{1+\alpha}$ diffeomorphisms for any $\alpha < 1$.

So let $k \geqslant 2$ and consider $\alpha < 1/k$. Choose positive real numbers $p_1,\ldots,p_k,r$ such that for all $n \in \{1,\ldots,k\}$ they satisfy the following conditions:

I) $\alpha + r \leqslant 2$,
II) $d(r-1) \leqslant (1-\alpha)$,
III) $2dr \leqslant p_n$,
IV) $2d \leqslant p_n(1-\alpha)$,
V) $\frac{1}{p_1} + \cdots + \frac{1}{p_k} + \frac{1}{r} < 1$,
VI) $\alpha \leqslant \frac{1}{p_n} + \frac{1}{r}$ and $\alpha \leqslant \frac{r}{p_n(r-1)}$.

For instance, one can take $p_1 = \cdots = p_k = 3d/\alpha$ and $r = 3d/(3d-1)$.

Now define the lengths of the intervals $I_{i_1,\ldots,i_k,j}$ as

$$|I_{i_1,\ldots,i_k,j}| = \frac{1}{|i_1|^{p_1} + \cdots + |i_k|^{p_k} + |j|^r + 1}.$$

From condition V) it follows that $\sum |I_{i_1,\ldots,i_k,j}| < \infty$, hence this family of intervals can be disposed on a finite interval respecting the lexicographic order. After renormalization, we can assume that this interval is $[0,1]$.

Following [**21**], we say that two real-valued functions f and g satisfy $f \prec g$ if there is a constant $M > 0$ such that $|f(x)| \leqslant Mg(x)$ for all x. We also write $f \asymp g$ if $f \prec g$ and $g \prec f$.

Let θ be a C^2 real-valued function satisfying $\theta(\xi) = |\xi|^r$ for $|\xi| \geqslant 1$, and $\theta(0) = 0$. Consider the auxiliary functions (C^2 with respect to ξ):

- $\psi(i_1,\ldots,i_k,\xi) := 1 + |i_1|^{p_1} + \cdots + |i_k|^{p_k} + \theta(\xi)$,
- $\Psi_{i_1,\ldots,i_k}(\xi) := \log(\psi(i_1,\ldots,i_k,\xi))$.

Lemma 2.5. *Let $S = 1 + |i_1|^{p_1} + \cdots + |i_k|^{p_k}$, and suppose $|\xi - j| \leqslant C(S^{1/r} + (|i_1| + \cdots + |i_k|)^d)$ for some positive constant C. Then*

$$\psi(i_1,\ldots,i_k,j) \asymp \psi(i_1,\ldots,i_k,\xi).$$

Proof. By symmetry, it is enough to show that $\frac{\psi(i_1,\ldots,i_k,\xi)}{\psi(i_1,\ldots,i_k,j)}$ is bounded above. For this we note that

$$\begin{aligned}\frac{\psi(i_1,\ldots,i_k,\xi)}{\psi(i_1,\ldots,i_k,j)} &\prec \frac{1+|i_1|^{p_1}+\cdots+|i_k|^{p_k}+|j|^r+|\xi-j|^r}{1+|i_1|^{p_1}+\cdots+|i_k|^{p_k}+|j|^r}\\ &\prec 1+\frac{S+(|i_1|+\cdots+|i_k|)^{dr}}{\psi(i_1,\ldots,i_k,j)}\\ &\prec 2+\frac{(|i_1|+\cdots+|i_k|)^{dr}}{\psi(i_1,\ldots,i_k,j)}.\end{aligned}$$

Where we repeatedly use the inequality $|x+y|^a \prec |x|^a+|y|^a$, which holds for any $a>0$. Now just notice that the last expression is bounded. Indeed, since $(|i_1|+\cdots+|i_k|)^{dr} \prec |i_1|^{dr}+\cdots+|i_k|^{dr}$, it is enough to observe that for each $n\in\{1,\ldots,k\}$,

$$|i_n|^{dr} \leqslant (\psi(i_1,\ldots,i_k,j))^{\frac{dr}{p_n}} \leqslant \psi(i_1,\ldots,i_k,j),$$

which holds thanks to condition III). □

2.1.3.1. *The maps g_s are $C^{1+\alpha}$-diffeomorphisms.* We start the proof of Proposition 2.4 by showing that the maps g_s, for $s\in\{1,\ldots,d\}$, are of class $C^{1+\alpha}$. That is, we want to show that there is constant $C>0$ such that

$$\frac{|\log Dg_s(x)-\log Dg_s(y)|}{|x-y|^\alpha} \leqslant C \text{ for all } x,y\in[0,1]\,.$$

To check this, it is enough to consider points x,y in intervals $I_{i_1,\ldots,i_k,j}$ and $I_{i_1,\ldots,i_k,j'}$. Indeed, after condition 2) in Lemma 1.14 and the definition of g_s, it follows that g_s has derivative 1 at the end points of the intervals $\cup_j I_{i_1,\ldots,i_k,j}$, so this case follows by applying triangular inequality.

Case 1: The points x,y belong to the same $I := I_{i_1,\ldots,i_k,j}$.

Condition 3) in Lemma 1.14 provides a Lipschitz constant for $\log(Dg_s)$. So it is enough to bound

$$\frac{1}{|I|^\alpha}\left|\frac{|I|\,|J'|}{|J|\,|I'|}-1\right|,$$

where $I'=I_{i_1,\ldots,i_k,j-1}$, $J=I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_d)}$ and $J'=I_{i_1,\ldots,i_k,j+r_s(i_1,\ldots,i_k)-1}$.

We will in fact bound the following asymptotically equivalent expression

$$\frac{1}{|I|^\alpha}\log\frac{|I|\,|J'|}{|J|\,|I'|}.$$

For this notice that $\log\frac{|I||J'|}{|J||I'|}$ equals to

$$\Psi_{i_1,\ldots,i_k}(j+r_s(i_1,\ldots,i_k))-\Psi_{i_1,\ldots,i_k}(j+r_s(i_1,\ldots,i_k)-1)-(\Psi_{i_1,\ldots,i_k}(j)-\Psi_{i_1,\ldots,i_k}(j-1)).$$

So applying twice the Mean Value Theorem to the function $x\mapsto \Psi_{i_1,\ldots,i_k}(j+1+x)-\Psi_{i_1,\ldots,i_k}(j+x)$, we have

$$\left|\log\frac{|I|\,|J'|}{|J|\,|I'|}\right| = |r_s(i_1,\ldots,i_k)|\left|D^2(\Psi_{i_1,\ldots,i_k})(\xi)\right|, \tag{15}$$

where ξ is a point in the convex hull of $\{j-1, j, j-1+r_s, j+r_s\}$. Let us find an upper bound for $|D^2(\Psi_{i_1,\dots,i_k})(\xi)|$. Since $D\theta$ and $D^2\theta$ are bounded in $[-1,1]$, and

$$D^2(\Psi_{i_1,\dots,i_k})(\xi) = \frac{D^2\theta(\xi)}{\psi(i_1,\dots,i_k,\xi)} - \frac{(D\theta(\xi))^2}{(\psi(i_1,\dots,i_k,\xi))^2},$$

we have that

$$D^2(\Psi_{i_1,\dots,i_k})(\xi) \prec \frac{1}{\psi(i_1,\dots,i_k,\xi)}$$

for all $\xi \in [-1,1]$. On the other hand for $\xi \notin [-1,1]$ we have that $\theta(\xi) = |\xi|^r$. So, observing $|\xi|^{r-2} < 1$ and $|\xi|^r / \psi(i_1,\dots,i_k,\xi) < 1$, it follows that

$$D^2(\Psi_{i_1,\dots,i_k})(\xi) \prec \frac{|\xi|^{r-2}}{\psi(i_1,\dots,i_k,\xi)} \prec \frac{1}{\psi(i_1,\dots,i_k,\xi)}. \tag{16}$$

Now going back to equation (15) and using 3) of Proposition 2.2 , we have

$$\log\frac{|I|\,|J'|}{|J|\,|I'|} \prec \frac{|i_1|^d + \cdots + |i_k|^d}{\psi(i_1,\dots,i_k,\xi)}.$$

Note that for all $n \in \{1,\dots,k\}$ the condition IV) yields

$$|i_n|^d \leqslant (\psi(i_1,\dots,i_k,\xi))^{\frac{d}{p_n}} \leqslant (\psi(i_1,\dots,i_k,\xi))^{(1-\alpha)}.$$

Finally by Lemma 2.5 we conclude

$$\frac{1}{|I|^\alpha}\log\frac{|I|\,|J'|}{|J|\,|I'|} \prec \frac{(\psi(i_1,\dots,i_k,\xi))^{-\alpha}}{|I|^\alpha} \prec \frac{(\psi(i_1,\dots,i_k,j))^{-\alpha}}{|I|^\alpha} = 1,$$

as desired.

Case 2: The point x belongs to $I_{i_1,\dots,i_k,j}$ and y belongs to $I_{i_1,\dots,i_k,j'}$.

We assume without loss of generality that $j < j'$. The condition 4) of Lemma 1.14 tells us that $|\log Dg_s(x) - \log Dg_s(y)|$ is bounded above by

$$\left|\log\frac{|I_{i_1,\dots,i_k,j+r_s}|\,|I_{i_1,\dots,i_k,j'}|}{|I_{i_1,\dots,i_k,j}|\,|I_{i_1,\dots,i_k,j'+r_s}|}\right| + \left|\log\frac{|I_{i_1,\dots,i_k,j+r_s-1}|\,|I_{i_1,\dots,i_k,j}|}{|I_{i_1,\dots,i_k,j-1}|\,|I_{i_1,\dots,i_k,j+r_s}|}\right| + \left|\log\frac{|I_{i_1,\dots,i_k,j'+r_s-1}|\,|I_{i_1,\dots,i_k,j'}|}{|I_{i_1,\dots,i_k,j'-1}|\,|I_{i_1,\dots,i_k,j'+r_s}|}\right|.$$

The estimates in Case 1 allow us to estimate the last two terms (divided by $|x-y|^\alpha$), thus we only need to bound the first term. So we look for a uniform bound for

$$\frac{1}{|x-y|^\alpha}\left|\log\frac{|I|\,|J'|}{|I'|\,|J|}\right|, \tag{17}$$

where $I = I_{i_1,\dots,i_k,j}$, $I' = I_{i_1,\dots,i_k,j'}$, $J = I_{i_1,\dots,i_k,j+r_s}$ and $J' = I_{i_1,\dots,i_k,j'+r_s}$. Assume that j, j' are positive (the case where both are negative follows by symmetry, and if they have different sign, it suffices to consider an intermediate comparison with the term corresponding to $j'' = 0$). Assume further that $j' - j \geqslant 2$ (the case where $j' - j = 1$ follows from the previous one, passing through the point that separates the intervals and using triangular inequality). Again, applying twice the Mean Value Theorem to the function

$$x \mapsto \Psi_{i_1,\dots,i_k}(j - r_s + x) - \Psi_{i_1,\dots,i_k}(j + x)$$

yields

$$\left|\log\frac{|I|\,|J'|}{|I'|\,|J|}\right| = |j-j'|\,|r_s(i_1,\dots,i_k)|\,\left|D^2(\Psi_{i_1,\dots,i_k})(\xi)\right|, \tag{18}$$

for a certain ξ in the convex hull of $\{j, j', j+r_s, j'+r_s\}$.

We start by bounding $|x-y|^{-\alpha}$. For this note that by Case 1 and the triangle inequality, we can (and will) assume that x is the left endpoint of I and y is the right endpoint of I'. This yields

$$\frac{1}{|x-y|^{\alpha}} = \left(\frac{1}{\sum_{\ell=j}^{j'} |I_{i_1,\dots,i_k,\ell}|}\right)^{\alpha} \leqslant \left(\frac{1}{|j-j'|\,|I_{i_1,\dots,i_k,j'}|}\right)^{\alpha},$$

where the last inequality holds because $|I_{i_1,\dots,i_k,j'}| < |I_{i_1,\dots,i_k,\ell}|$ for $\ell < j'$. Note that if in addition $|j'-j| \leqslant C(S^{1/r} + (|i_1|+\cdots+|i_k|)^d)$, for some $C>0$, we can use Lemma 2.5 to compare $|I|$ with $|I'|$, and eventually obtain the inequality

$$\frac{1}{|x-y|^{\alpha}} \prec \left(\frac{1}{|j-j'|\,|I_{i_1,\dots,i_k,j}|}\right)^{\alpha}. \tag{19}$$

We now exhibit a bound for (17). We consider three separate cases. Let M be the constant in Proposition 2.2 and let $S = 1 + |i_1|^{p_1} + \cdots + |i_k|^{p_k}$.

i) The integers j, j' belong to $[0, 2M(|i_1|+\cdots+|i_k|)^d]$. Since $\xi \in \mathrm{conv}\{j, j', j+r_s, j'+r_s\}$ it follows from Lemma 2.5 that

$$\left|D^2(\Psi_{i_1,\dots,i_k})(\xi)\right| \prec \frac{1}{\psi(i_1,\dots,i_k,\xi)} \asymp \frac{1}{\psi(i_1,\dots,i_k,j)}.$$

Furthermore, we have that

$$|j-j'|\,|r_s(i_1,\dots,i_k)| \prec (|i_1|+\cdots+|i_k|)^{2d} \prec (\psi(i_1,\dots,i_k,j))^{1-\alpha},$$

where the last inequality holds from condition IV). If we combine this with (18), (19), we conclude that

$$\frac{1}{|x-y|^{\alpha}}\left|\log\frac{|I|\,|J'|}{|I'|\,|J|}\right| \prec \frac{1}{|I|^{\alpha}}\frac{(\psi(i_1,\dots,i_k,j))^{1-\alpha}}{\psi(i_1,\dots,i_k,j)} = 1.$$

ii) The integers j, j' belong to[1] $[2M(|i_1|+\cdots+|i_k|)^d, 2Mk^dS^{1/r}]$. Similarly to *i*), the reader can check that we are in the hypotheses of Lemma 2.5 and that $|\xi| \geqslant M(|i_1|+\cdots+|i_k|)^d$. Therefore, by (16), (18) and (19), we get

$$\begin{aligned}\frac{1}{|x-y|^{\alpha}}\left|\log\frac{|I|\,|J'|}{|I'|\,|J|}\right| &\prec \left(\frac{1}{|j-j'|\,|I_{i_1,\dots,i_k,j}|}\right)^{\alpha}|j'-j|\,\frac{(|i_1|+\cdots+|i_k|)^d\,|\xi|^{r-2}}{\psi(i_1,\dots,i_k,\xi)}\\ &\prec |j'-j|^{1-\alpha}\,\frac{(|i_1|+\cdots+|i_k|)^{d(r-1)}}{\psi(i_1,\dots,i_k,j)^{1-\alpha}}.\end{aligned}$$

To prove that this last expression is bounded, it is enough to show that

$$|j'-j|^{1-\alpha}\,(|i_1|+\cdots+|i_k|)^{d(r-1)} \prec \psi(i_1,\dots,i_k,j)^{1-\alpha}.$$

[1] the constant k^d is just to ensure that the interval is non-empty.

Since $j' - j \leqslant 2Mk^d S^{1/r}$, it follows that

$$|j' - j|^{1-\alpha} (|i_1| + \cdots + |i_k|)^{d(r-1)} \prec (1 + |i_1|^{p_1} + \cdots + |i_k|^{p_k})^{\frac{(1-\alpha)}{r}} (|i_1| + \cdots + |i_k|)^{d(r-1)},$$

so it suffices to prove that, given $n, m \in \{1, ..., k\}$ we have

$$|i_n|^{\frac{p_n(1-\alpha)}{r}} |i_m|^{d(r-1)} \prec (\psi(i_1, ..., i_k, j))^{1-\alpha}. \tag{20}$$

But note that

$$|i_n|^{\frac{p_n(1-\alpha)}{r}} |i_m|^{d(r-1)} \leqslant (\psi(i_1, \ldots, i_k, j))^{\frac{(1-\alpha)}{r} + \frac{d(r-1)}{p_m}},$$

and that conditions II) and V) guarantee $\frac{(1-\alpha)}{r} + \frac{d(r-1)}{p_m} \leqslant (1 - \alpha)$, which implies (20).

iii) Finally suppose that the integers j, j' belong to $[2Mk^d S^{1/r}, \infty]$.
If $j' \leqslant 2j$, then

$$\frac{\psi(i_1, \ldots, i_k, j')}{\psi(i_1, \ldots, i_k, j)} \prec 1 + \frac{|j - j'|^r}{\psi(i_1, \ldots, i_k, j)} \prec 1 + \frac{|j|^r}{\psi(i_1, \ldots, i_k, j)} \leqslant 2. \tag{21}$$

In particular, the intervals $|I'|$ and $|I|$ have comparable size and hence we conclude that (19) still holds. Also note that $j' \leqslant 2j$ implies $|\xi - j| \leqslant |j| + M(|i_1| + \cdots + |i_k|)^d$. Then, proceeding as in *ii*), we have that

$$\frac{1}{|x-y|^{\alpha}} \left| \log \frac{|I|\,|J'|}{|I'|\,|J|} \right| \prec \frac{|j|^{1-\alpha} (|i_1| + \cdots + |i_k|)^{d(r-1)}}{\psi(i_1, \ldots, i_k, j)^{1-\alpha}}$$

The reader can check, again as in *ii*), that this last expression is bounded.

For the case $j' > 2j$. We have

$$|x - y| = \sum_{\ell=j}^{j'} |I_{i_1,\ldots,i_k,\ell}| = \sum_{\ell=j}^{j'} \frac{1}{|i_1|^{p_1} + \cdots + |i_k|^{p_k} + |\ell|^r}$$
$$\succ \sum_{\ell=j}^{j'} \frac{1}{|\ell|^r} \succ \int_{\ell=j}^{j'} \frac{1}{x^r} dx \succ \frac{1}{|j|^{r-1}},$$

where the last inequality holds because $j' > 2j$. On the other hand, applying the Mean Value Theorem, it follows that

$$\log \frac{|I|\,|J'|}{|I'|\,|J|} = |r_s(i_1, \ldots, i_k)| \left| D(\Psi_{i_1,\ldots,i_k})(\xi) - D(\Psi_{i_1,\ldots,i_k})(\tilde{\xi}) \right|,$$

with $\xi \in \operatorname{conv}\{j, j + r_s\}$ and $\tilde{\xi} \in \operatorname{conv}\{j', j' + r_s\}$. Therefore, observing that the function $\xi \mapsto D(\Psi_{i_1,\ldots,i_k})(\xi) = r\,|\xi|^{r-1} / \psi(i_1, \ldots, i_k, \xi)$ is decreasing, we have

$$\frac{1}{|x-y|^{\alpha}} \left| \log \frac{|I|\,|J'|}{|I'|\,|J|} \right| \prec (|i_1| + \cdots + |i_k|)^d \frac{|j|^{(\alpha+1)(r-1)}}{\psi(i_1, \ldots, i_k, j)}.$$

Now we want to see that this last expression is bounded, in other words that the inequality $|j|^{(\alpha+1)(r-1)}\left(|i_1|+\cdots+|i_k|\right)^d < \psi(i_1,\ldots,i_k,j)$ holds. For this, arguing as in (20), it is enough to check that for all $n \in \{1,...,k\}$ the inequality

$$\frac{(\alpha+1)(r-1)}{r}+\frac{d}{p_n}\leqslant 1$$

holds. To see this, note that from IV) it follows that $\frac{d}{p_n} \leqslant \frac{1-\alpha}{2}$. Finally notice that $\frac{(\alpha+1)(r-1)}{r}+\frac{1-\alpha}{2} \leqslant 1 \Leftrightarrow r \leqslant 2$, which is ensured by condition I).

2.1.3.2. *The maps f_t are $C^{1+\alpha}$-diffeomorphisms.* In the same way that for the maps g_s, we want to see that for all $x, y \in [0,1]$

$$\frac{|Df_t(x)-Df_t(y)|}{|x-y|^\alpha}\leqslant C \text{ holds for some constant } C>0.$$

To simplify notation we will only work with $t=1$ as the other cases are analogous. As for the case of the maps g_s, we only have two cases to analyze.

Case 1: The points x, y belongs to the same interval $I_{i_1,\ldots,i_k,j}$.

By Lemma 1.14 it is enough to show that the following expression is uniformly bounded

$$\frac{1}{|I_{i_1,\ldots,i_k,j}|^\alpha}\log\frac{|I_{i_1,\ldots,i_k,j}|\,|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1-1}|}{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}|\,|I_{i_1,\ldots,i_k,j-1}|}.$$

To see the this, simply note that the above expression is equal to

$$\frac{1}{|I_{i_1,\ldots,i_k,j}|^\alpha}\log\frac{|I_{i_1,\ldots,i_k,j}|\,|I_{i_1+1,i_2,\ldots,i_k,j-1}|}{|I_{i_1+1,i_2,\ldots,i_k,j}|\,|I_{i_1,\ldots,i_k,j-1}|}+\frac{1}{|I_{i_1,\ldots,i_k,j}|^\alpha}\log\frac{|I_{i_1+1,i_2,\ldots,i_k,j}|\,|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1-1}|}{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}|\,|I_{i_1+1,i_2,\ldots,i_k,j-1}|}.$$

By condition VI) we know from [**10**, §3.3] that the first term is uniformly bounded. The second term is bounded as well since it is the same that we bound when dealing with g_s (changing i_1 for i_1+1).

Case 2: The point $x \in I = I_{i_1,\ldots,i_k,j}$ and $y \in J = I_{i_1,\ldots,i_k,j'}$, with $j < j'$.

Here we can use 4) from Lemma 1.14 to bound $|\log Df_1(x) - \log Df_1(y)|$ by

$$\left|\log\frac{|I_{i_1+1,\ldots,i_k,j+\ell_1}|\,|I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j}|\,|I_{i_1+1,\ldots,i_k,j'+\ell_1}|}\right|+\left|\log\frac{|I_{i_1+1,\ldots,i_k,j+\ell_1-1}|\,|I_{i_1,\ldots,i_k,j}|}{|I_{i_1,\ldots,i_k,j-1}|\,|I_{i_1+1,\ldots,i_k,j+\ell_1}|}\right|+\left|\log\frac{|I_{i_1+1,\ldots,i_k,j'+\ell_1-1}|\,|I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j'-1}|\,|I_{i_1+1,\ldots,i_k,j'+\ell_1}|}\right|,$$

and then work in the same way as for the functions g_s. For example, we express the term

$$\frac{1}{|x-y|^\alpha}\log\frac{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}|\,|I_{i_1,\ldots,i_k,j'}|}{|I_{i_1,\ldots,i_k,j}|\,|I_{i_1+1,i_2,\ldots,i_k,j'+\ell_1}|}$$

as

$$\frac{1}{|x-y|^\alpha}\log\frac{|I_{i_1+1,i_2,\ldots,i_k,j}|\,|I_{i_1,\ldots,i_k,j'}|}{|I_{i_1+1,i_2,\ldots,i_k,j'}|\,|I_{i_1,\ldots,i_k,j}|}+\frac{1}{|x-y|^\alpha}\log\frac{|I_{i_1+1,i_2,\ldots,i_k,j+\ell_1}|\,|I_{i_1+1,i_2,\ldots,i_k,j'}|}{|I_{i_1+1,i_2,\ldots,i_k,j}|\,|I_{i_1+1,i_2,\ldots,i_k,j'+\ell_1}|}.$$

The first term is bounded by [**10**, §3.3] and the second is also bounded by the same argument used for the functions g_s.

2.1.4. Faithful action. Given $s \in \{1, \ldots, d\}$ and a compact interval I_s, we have seen how to produce certain actions

$$\phi_s : G \to \mathrm{Diff}_+^{1+\alpha}(I_s),$$

where the subgroup $\langle g_1, \ldots, g_{s-1} \rangle$ acts trivial, while $\langle g_s \rangle$ acts faithful (recall that we are using the generating set from Lemma 2.1).

To obtain a faithful action we do the following: consider $I_1, \ldots, I_d$ compact intervals such that for all $s \in \{1, \ldots, d-1\}$, I_{s+1} is contiguous to I_s by the right. Then define on $I := I_1 \cup \cdots \cup I_d$ the action $\phi : G \to \mathrm{Diff}_+^{1+\alpha}(I)$ as

$$\phi \mid_{I_s} = \phi_s.$$

We claim that ϕ is injective. Indeed, since $Z(G) \leqslant A$, by the Proposition 1.3 we only need to check that $\phi \mid_A$ is inyective. Let $g \in A$ be an element that acts trivially on I. Using the generator of A, we have that there exist $j_1, \ldots, j_d \in \mathbb{Z}$ such that $g = g_1^{j_1} \cdots g_d^{j_d}$. Now, since $\phi(g) = id$, it follows that

$$\phi_s(g) = id, \ \forall s \in \{1, \ldots, d\}.$$

This yields that $j_d = \cdots = j_1 = 0$ and hence g is the trivial element. This finishes the proof of Theorem A.

2.2. Examples

In this section we give examples of nilpotent groups for which we can compute their critical regularity. In each case we use Theorem A to obtain a lower bound for the critical regularity and we argue that in our examples this is also an upper bound for the regularity.

We begin by recalling that if G is a finitely-generated nilpotent group of homeomorphisms of $(0,1)$ that has no global fixed points (or more generally a group acting without crossings, see [**29, 14**]), then there is a well-defined group homomorphism $\rho : G \to \mathbb{R}$, which is usually called the *translation number* of the action. This map characterizes the elements of G that have fixed points, in the sense that $\rho(g) = 0$ if and only if g has a fixed point in $(0,1)$. We further note that the action of G on the interval has *no crossings*. By this we mean that if an element $f \in G$ fixes an open subinterval I of $(0,1)$ and satisfies that $f(x) \neq x$ for all x in I, then for any other $g \in G$ we have that $g(I) = I$ or $g(I) \cap I = \varnothing$. See [**29**, §2.2.5] for background. With this, it is easy to prove the following result that we will repeatedly use.

Lemma 2.6. *Let* $G \leqslant \mathrm{Diff}_+^0(0,1)$ *be a nilpotent group, and let* $c \in G$ *be a non-trivial element such that* $c = [a,b]$ *for some elements* $a, b \in G$. *If* c *fixes an open interval* I *and has no fixed point inside, then either* a *or* b *move* I *(disjointly).*

Proof. Looking for a contradiction, assume that a and b fix I. Then we have the translation number homomorphism for the group $\langle a, b, c \rangle \leqslant \mathrm{Diff}_+^0(I)$. Since c is a commutator, it is in the kernel of this morphism. Hence we conclude that c has a fixed point inside I, which is contrary to our assumptions. ◻

2.2.1. Heisenberg Groups. For a natural number $n \geqslant 1$, the discrete $(2n+1)$-dimensional *Heisenberg group*, is defined as the set of matrices

$$\mathscr{H}_n := \left\{ \begin{pmatrix} 1 & \vec{x} & c \\ \vec{0}^t & I_n & \vec{y}^t \\ 0 & \vec{0} & 1 \end{pmatrix} : \vec{x}, \vec{y} \in \mathbb{Z}^n, c \in \mathbb{Z} \text{ and } I_n \text{ is the identity matrix of size } n \right\},$$

with the usual matrix product. Note that the center of $\mathscr{H}_n$ coincides with the commutator subgroup and is generated by the matrix

$$\mathbf{C} := \begin{pmatrix} 1 & \vec{0} & 1 \\ \vec{0}^t & I_n & \vec{0}^t \\ 0 & \vec{0} & 1 \end{pmatrix}.$$

We want to prove Theorem B, before this, it will be useful for us to bound the rank of maximal abelian subgroups. Assume that there exists a maximal abelian subgroup of $\mathscr{H}_n$ (note that it contains $\mathbf{C}$) of rank m. Then we can choose elements

$$\mathbf{A}_i := \begin{pmatrix} 1 & \vec{a}_i & c_i \\ \vec{0}^t & I_n & \vec{b}_i^t \\ 0 & \vec{0} & 1 \end{pmatrix} \in \mathscr{H}_n \text{ for } i \in \{1, \dots, m-1\},$$

such that $\langle \mathbf{A}_1, \dots \mathbf{A}_{m-1}, \mathbf{C} \rangle \simeq \mathbb{Z}^m$. Note that the commutativity of these matrices is equivalent to the equations

$$\vec{a}_i \cdot \vec{b}_j = \vec{a}_j \cdot \vec{b}_i \quad \forall\, i, j \in \{1, ..., m-1\}. \tag{22}$$

Note also that $\{\mathbf{A}_1, \dots, \mathbf{A}_{m-1}, \mathbf{C}\}$ generates a free abelian subgroup of rank m if and only if the set of vectors $\mathscr{B} := \{(\vec{b}_i, \vec{a}_i) \in \mathbb{Z}^n \times \mathbb{Z}^n : 1 \leqslant i \leqslant m-1\}$ is linearly independent over $\mathbb{Z}$. Indeed, if we have a dependency relation, say $r(\vec{b}_1, \vec{a}_1) \in \langle (\vec{b}_2, \vec{a}_2), \dots, (\vec{b}_{m-1}, \vec{a}_{m-1}) \rangle$ for some $0 \neq r \in \mathbb{Z}$, then $\mathbf{A}_1^r \in \langle \mathbf{A}_2, \dots, \mathbf{A}_{m-1}, \mathbf{C} \rangle$, which contradicts that the abelian group has rank m.

Having said this, we claim that $m \leqslant n+1$. To see this, note that, by equations (22) any vector of the form $(\vec{a}_i, -\vec{b}_i)$, with $1 \leqslant i \leqslant m-1$, is perpendicular to $\langle \mathscr{B} \rangle$. Hence we have two orthogonal subgroups of rank $m-1$, and thus $m-1 \leqslant n$, which proves our claim.

Realization. Consider the abelian subgroup

$$A := \left\{ \begin{pmatrix} 1 & \vec{x} & c \\ \vec{0}^t & I_n & \vec{0}^t \\ 0 & \vec{0} & 1 \end{pmatrix} : \vec{x} \in \mathbb{Z}^n \text{ and } c \in \mathbb{Z} \right\},$$

it has rank equal to $n+1$, which is the largest we can expect. Since the rank of $\mathscr{H}_n / A$ is n, we have that Theorem A provides an injective group homomorphism

$$\mathscr{H}_n \hookrightarrow \mathrm{Diff}_+^{1+\alpha}([0,1]) \text{ for } \alpha < 1/n.$$

Bounding the regularity. Now we consider a faithful action $\phi : \mathscr{H}_n \hookrightarrow \mathrm{Diff}_+^1([0,1])$. Making a little abuse of notation, we can think $\mathscr{H}_n \leqslant \mathrm{Diff}_+^1([0,1])$.

Since the commutator subgroup of $\mathscr{H}_n$ is generated by $\mathbf{C}$, we deduce from Lemma 2.6 that $\mathbf{C}$ has fixed points inside $(0,1)$. Therefore, we can find an interval $I \subsetneq [0,1]$ such that

$\mathbf{C}(I) = I$ and $\mathbf{C}(x) \neq x$ for all x in the interior of I. Let $\mathrm{Stab}(I)$ be the stabilizer of I. It is easy to see that this is an abelian subgroup. Indeed, if we take $\mathbf{A}, \mathbf{B} \in \mathrm{Stab}(I)$ and assume that they do not commute, then there must exist $m \in \mathbb{Z}$ such that $[\mathbf{A}, \mathbf{B}] = \mathbf{C}^m$. Since $\mathbf{C}$ has no fixed points inside I, Lemma 2.6 tell us that either $\mathbf{A}$ or $\mathbf{B}$ moves I , which is a contradiction. Note that $\mathrm{Stab}(I)$ is a normal subgroup as it contains the commutator subgroup.

Further, we know that there is a natural number k and elements $\mathbf{B}_1, \ldots, \mathbf{B}_k \in \mathscr{H}_n$ such that

$$\mathbb{Z}^k \simeq \frac{\mathscr{H}_n}{\mathrm{Stab}(I)} = \frac{\langle \mathbf{B}_1, \ldots, \mathbf{B}_k \rangle}{\mathrm{Stab}(I)}.$$

So, given $\alpha > 1/k$, we can find by Lemma 1.18 a sequence $(\mathbf{B}_{i_j})_{j \in \mathbb{N}}$ of elements in $\{\mathbf{B}_1, \ldots, \mathbf{B}_k\}$ such that

$$\sum_{j \geqslant 0} \left| \mathbf{B}_{i_j} \cdots \mathbf{B}_{i_1}(I) \right|^{\alpha} < \infty,$$

and hence Theorem 1.17 yields that ϕ is not an action by $C^{1+\alpha}$-diffeomorphisms.

Now since the rank of $\mathrm{Stab}(I)$ is bounded by $n - 1 = \mathrm{rank}(A)$, we have that

$$k = \mathrm{rank}\left(\frac{\mathscr{H}_n}{\mathrm{Stab}(I)} \right) \geqslant \mathrm{rank}\left(\frac{\mathscr{H}_n}{A} \right) = n,$$

which tells us that the regularity of the action ϕ is bounded by $1 + 1/n$. So we conclude that

$$\mathrm{Crit}_{[0,1]}(\mathscr{H}_n) = 1 + \frac{1}{n}.$$

The reader can easily check the following corollary of the proof of Theorem B.

Corollary 2.7. *Let G be a finitely-generated nilpotent group with cyclic center and such that $[G, G] \leqslant Z(G)$. Then, the lower bound for* $\mathrm{Crit}(G)$ *obtain in Theorem A is also an upper bound.*

2.2.2. Examples with large nilpotency degree. Theorem B gives us the critical regularity for the Heisenberg groups, which are groups having nilpotency degree 2. Here we provide more examples of nilpotent groups where we can compute its critical regularity, but their nilpotency degree can be arbitrarily large. As for the Heisenberg groups, in these examples we will show that the lower bound provided by Theorem A is also an upper bound.

Fix $d, k \in \mathbb{N}$, assume $d \geqslant k$ and consider a matrix $(m_{i,s}) \in M_k(\mathbb{Z})$ with non-zero determinant and positive entries. We let G be the group generated by the set

$$\{g_0\} \cup \{g_{i,j} : (i,j) \in \{1, \ldots, k\} \times \{1, \ldots, d\}\} \cup \{f_1, \ldots, f_k\},$$

subject to the relations

- $[g_0, g_{i,j}] = [g_0, f_i] = [f_s, f_i] = [g_{i,j}, g_{l,m}] = e, \forall\ s, i, l \in \{1, \ldots, k\},\ j, m \in \{1, \ldots, d\}$,
- $[f_s, g_{i,j}] = g_{i,j-1}^{m_{i,s}}\ \forall s, i \in \{1, \ldots, k\}$ y $j \in \{2, \ldots, d\}$,
- $[f_s, g_{i,1}] = g_0^{m_{i,s}}\ \forall s, i \in \{1, \ldots, k\}$.

Note that from the identities $[ab, c] = a[b, c]a^{-1}[a, c]$ and $[a, bc] = [a, b]b[a, c]b^{-1}$, we immediately have the following additional relations

- $[f_s^{-1}, g_{i,j}] \in \langle g_0, g_{i,1}, \ldots, g_{i,j-2} \rangle g_{i,j-1}^{-m_{i,s}}\ \forall s, i \in \{1, \ldots, k\}, j \in \{2, \ldots, d\}$,
- $[f_s^{-1}, g_{i,1}] = g_0^{-m_{i,s}}\ \forall s, i \in \{1, \ldots, k\}$.

It is easy to see that G is a nilpotent group of degree $d+1$, and $A = \langle \{g_0\} \cup \{g_{i,j} : (i,j) \in \{1,...,k\} \times \{1,\dots,d\}\}\rangle$ is a maximal abelian subgroup containing the commutator of G (see Lemma 2.8 below). Moreover k is the torsion-free rank of G/A, therefore in view of Theorem A we know that G embeds in $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for $\alpha < 1/k$. To show that $1+1/k$ is actually an upper bound for the regularity we are going to need the following elementary lemma.

Lemma 2.8. *For all $(i,j) \in \{1,\dots,k\} \times \{2,\dots,d\}$ and $n_1,\dots,n_k \in \mathbb{Z}$ we have*

(1) $[f_1^{n_1}\cdots f_k^{n_k}, g_{i,j}] \in \langle g_0, g_{i,1},\dots,g_{i,j-2}\rangle g_{i,j-1}^{\lambda_i}$,

(2) $[f_1^{n_1}\cdots f_k^{n_k}, g_{i,1}] = g_0^{\lambda_i}$,

where $\lambda_i = \sum_{s=1}^k n_s m_{i,s}$. In particular, the subgroup A is a maximal abelian subgroup.

Proof. To show 1. we do induction on $n = \sum_{s=1}^k |n_s|$.

Note that when $n=1$ we have the result by the relations of G. So, consider an arbitrary natural number $n = \sum_{j=1}^k |n_j|$ and assume that $n_k < 0$ (the other case is similar). For all $i \in \{1,\dots,k\}$ and $j \in \{2,...,d\}$ we have that

$$[f_1^{n_1}\cdots f_k^{n_k}, g_{i,j}] = [f_1^{n_1}\cdots f_k^{n_k+1}, [f_k^{-1}, g_{i,j}]]\, [f_k^{-1}, g_{i,j}]\, [f_1^{n_1}\cdots f_k^{n_k+1}, g_{i,j}],$$

and since $[f_k^{-1}, g_{i,j}]$ belongs to $\langle g_0, g_{i,1},\dots,g_{i,j-2}\rangle g_{i,j-1}^{-m_{i,k}}$, it follows that $[f_1^{n_1}\cdots f_k^{n_k+1}, [f_k^{-1}, g_{i,j}]] \in \langle g_0, g_{i,1},\dots,g_{i,j-2}\rangle$. Also, by induction hypothesis we have

$$[f_1^{n_1}\cdots f_k^{n_k+1}, g_{i,j}] \in \langle g_0, g_{i,1},\dots,g_{i,j-2}\rangle g_{i,j-1}^{(\sum_{s=1}^{k-1} n_s m_{i,s} + (n_k+1)m_{i,k})}.$$

Plugging these into the previous equation yields assertion 1. The proof of assertion 2 is analogous. □

Remark 2.9. *The most useful part of Lemma 2.8 is the explicit expression for the integers λ_i appearing. These will be used in the proof of Theorem C.*

Proof of Theorem C:

Proof. Suppose that G embeds into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for some $\alpha > 1/k$. Let x_0 be a point in $(0,1)$ such that $g_0(x_0) \neq x_0$ and define the intervals

$$I_0 := (\inf_n g_0^n(x_0), \sup_n g_0^n(x_0)) \text{ and } I_{i,j} := (\inf_n g_{i,j}^n(x_0), \sup_n g_{i,j}^n(x_0)).$$

Case 1: $f(I_0) \cap I_0 = \emptyset$ for all $f \in \langle f_1,\dots,f_k\rangle \simeq \mathbb{Z}^k$.

In this case I_0 is a wandering interval for the dynamics of $\langle f_1,\dots,f_k\rangle$. A contradiction is provided by Lemma 1.18 followed by Theorem 1.17 since the central element g_0 acts non-trivially I_0.

Case 2: There is a non-trivial element $f \in \langle f_1,\dots,f_k\rangle$ such that $f(I_0) = I_0$.

Let us put $f = f_1^{n_1}\cdots f_k^{n_k}$. Given $i \in \{1,\dots,k\}$, by the Lemma 2.8 we have that

$$[f, g_{i,1}] = g_0^{\lambda_i} \text{ and } [f, g_{i,j}] \in \langle g_0, g_{i,1},\dots,g_{i,j-2}\rangle g_{i,j-1}^{\lambda_i} \quad \text{for all } j \in \{2,\dots,d\}, \tag{23}$$

where $\lambda_i = \sum_{j=1}^k n_j m_{i,j}$. Since the vectors $(m_{i,1},\dots,m_{i,k})$ are linearly independent in $\mathbb{R}^k$, we can choose i to obtain $\lambda_i \neq 0$. Then, the relations (23) and Lemma 2.6 implies that $g_{i,1}(I_0) \cap I_0 = \emptyset$. Since the action has *no crossings*, the element f also fixes the intervals $I_{i,j}$

and hence the same argument also yields that $g_{i,j}(I_{i,j-1}) \cap I_{i,j-1} = \varnothing$ for all $j > 2$. Therefore, I_0 is a wandering interval for the action of $\langle g_{i,1}, \ldots, g_{i,k} \rangle \simeq \mathbb{Z}^k$, So, a contradiction is reached using Lemma 1.18 and Theorem 1.17 as before. □

2.2.3. An example with even higher regularity. It is easy to see that in some situations the regularity given by Theorem A is not critical. In the examples that we know of, this is related to the fact that the group can be splitted as a direct product of groups each of which allows an embedding with better regularity. Take for example the groups of [**10**, §4]. These are given by the presentation

$$G_d := \langle f, g_1, \ldots, g_d : [g_i, g_j] = id, [f, g_0] = id, [f, g_i] = g_{i-1} \ \forall j \geqslant 0, i \geqslant 1 \rangle.$$

Note that G_d is isomorphic to a non-trivial semidirect product of the form $\mathbb{Z}^d \rtimes \mathbb{Z}$. Now define the group $G := G_d \times G_d$. On one hand, it is easy to see that

$$G \simeq \mathbb{Z}^{2d} \rtimes \mathbb{Z}^2,$$

and $\mathbb{Z}^{2d} \times \{0\}$ is a maximal abelian subgroup of G. Therefore, if we apply the Theorem A, we obtain an embedding of G into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for all $\alpha < 1/2$. However, on the other hand, the critical regularity of G es 2. Indeed, we can apply Theorem A to each factor of G to obtain an embedding of the factor into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for all $\alpha < 1$. If we put these two actions together acting on disjoint intervals (as we did in Section 2.1.4), we end up with an embedding if G into $\mathrm{Diff}_+^{1+\alpha}([0,1])$ for all $\alpha < 1$.

CHAPTER 3

Examples of distorted interval diffeomorphisms of intermediate regularity

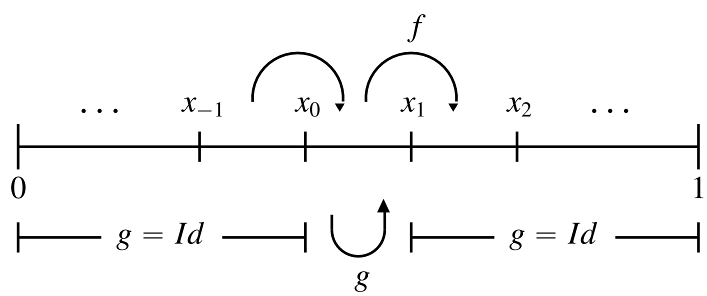
3.1. On a family of C^{1+bv}-undistorted diffeomorphisms

Recall that for a C^{1+bv} diffeomorphism f of a compact 1-manifold, the *asymptotic distortion* was defined by Navas in [**32**] as

$$\mathrm{dist}_\infty(f) := \lim_{n\to\infty} \frac{\mathrm{var}(\log Df^n)}{n}.$$

By the subaditivity of $\mathrm{var}(\log D(\cdot))$, if f is a distorted element of the group of C^{1+bv} diffeomorphisms, then $\mathrm{dist}_\infty(f) = 0$.

The family of diffeomorphisms with positive asymptotic distortion studied in [**33**] is as follows: Start with a C^{1+bv} diffeomorphism of $[0,1]$ with vanishing asymptotic distortion and no fixed point in $]0,1[$. Let I be a *fundamental domain* for the action of f, that is, an open interval with endpoints x_0 and $x_1 := f(x_0)$ for a certain $x_0 \in]0,1[$. Let g be any nontrivial C^{1+bv} diffeomorphism of $]0,1[$ supported on I. Then the diffeomorphism $\bar{f} := fg$ has positive asymptotic distortion and, in particular, it is undistorted in $\mathrm{Diff}_+^{1+bv}([0,1])$ (hence in $\mathrm{Diff}_+^2([0,1])$). This fact follows from [**7**] (see Lemmas 2.2 and 7.2 therein) by using the relation between the asymptotic distortion and the Mather invariant. For the reader's convenience, below we present a short and direct argument based on Kopell's like estimates [**26, 29**].



Let us consider the product $\bar{f}^n f^{-n}$. Since f has vanishing asymptotic distortion, if we show that $\mathrm{var}(\log D(\bar{f}^n f^{-n}))$ has linear growth, the same will hold for $\mathrm{var}(\log D\bar{f}^n)$. Now, notice that

$$\bar{f}^n f^{-n} = (fgf^{-1})(f^2gf^{-2})(f^3gf^{-3})\cdots(f^ngf^{-n})$$

has support in the union of the intervals $f(I), f^2(I), \dots, f^n(I)$, and equals f^kgf^{-k} on each such interval $f^k(I)$. In particular, its derivative at the endpoints x_k, x_{k+1} of each of these intervals equals 1. We claim that there is a constant $\lambda > 1$ such that, for all $k \geqslant 1$, there is a point $y_k \in f^k(I)$ satisfying $D(f^k g f^{-k})(y_k) \geqslant \lambda$. Assuming this, we conclude

$$\mathrm{var}(\log D(\bar{f}^n f^{-n})) \geqslant \sum_{k=1}^{n} \left|\log D(f^k g f^{-k})(y_k) - \log D(f^k g f^{-k})(x_k)\right| \geqslant n \log(\lambda),$$

which yields the desired linear growth.

Now, to check the existence of λ and the points y_k, let $V := \mathrm{var}(\log Df)$, and let $N \geqslant 1$ be such that $Dg^N(z) > e^{2V}$ holds for some $z \in I$. We claim that $\lambda := e^{V/N}$ works. Assume otherwise. Then, for a certain $K \geqslant 0$, one would have $\|D(f^K g f^{-K})\|_\infty \leqslant e^{V/N}$, which by the chain rule would yield $\|D(f^K g^N f^{-K})\|_\infty \leqslant e^V$. However, at the point $z_K := f^K(z)$, we have $D(f^K g^N f^{-K})(z_K) > e^V$. Indeed,

$$\begin{aligned}\log(D(f^K g^N f^{-K})(z_K)) &= \log Df^K(g^N(z)) + \log Dg^N(z) - \log Df^K(z)\\ &\geqslant \log Dg^N(z) - \left|\log Df^K(g^N(z)) - \log Df^K(z)\right|\\ &> 2V - \sum_{k=0}^{K-1} \left|\log Df(f^k(g^N(z))) - \log Df(f^k(z))\right|.\end{aligned}$$

Since both z and $g^N(z)$ lie in the fundamental domain I of f,

$$\sum_{k=0}^{K-1} \left|\log Df(f^k(g^N(z))) - \log Df(f^k(z))\right| \leqslant \mathrm{var}(\log Df) \leqslant V.$$

We thus conclude that $\log(D(f^K g^N f^{-K})(z_K)) > V$, as announced.

3.2. Distortion in class $C^{1+\alpha}$ for $\alpha < 1/2$

In this section, we start by briefly recalling the construction of the group Γ with a distorted element $\bar{f}$ considered in [**33**]. Next, we proceed to smooth the action of Γ in order to achieve any differentiability class $C^{1+\alpha}$ for $\alpha < 1/2$. Upgrading α to any number less than 1 will require the introduction of an extra element plus a tricky new computation, and will be carried out in the next section.

Start with the vector fields $\hat{\mathscr{X}}$ and $\mathscr{X}$ on the real line whose time-1 maps are, respectively,

$$\hat{F} := \hat{\mathscr{X}}^1 : x \mapsto 2x \qquad \text{and} \qquad F := \mathscr{X}^1 : x \mapsto x+1.$$

Let $\varphi : \mathbb{R} \to]0,1[$ be a C^∞ diffeomorphism such that $\hat{\mathscr{Y}} := \varphi_*(\hat{\mathscr{X}})$ and $\mathscr{Y} := \varphi_*(\mathscr{X})$ extend to the endpoints of $[0,1]$ as infinitely flat vector fields. Denote $\hat{f} := \hat{\mathscr{Y}}^1$ and $f := \mathscr{Y}^1$, which we view as diffeomorphisms of $[-1,2]$ that coincide with the identity outside $[0,1]$. The affine relation $\hat{f} f \hat{f}^{-1} = f^2$ yields that $\|f^n\| = O(\log(n))$; in particular, f has vanishing asymptotic distortion.

Let $x_0 := \varphi(0)$ and, for each $k \in \mathbb{Z}$, let $x_k := f^k(x_0) = \varphi(k)$. Denote also $x_{-1/2} := \varphi(-1/2)$ and $x_{-3/4} := \varphi(-3/4)$. Let φ_0 the affine diffeomorphism sending $I := [x_0, x_1]$ onto $[0,1]$, and let $g := \varphi_0^{-1} f \varphi_0$. This can be extended to $[-1,2]$ by the identity outside I.

We next define two diffeomorphisms $\hat{h}$ and h as follows:

(i) They act by the identity outside $[0,1]$.

(ii) On each interval $I_k := f^k(I)$, the diffeomorphism $\hat{h}$ (resp. h) coincides with the s_k-time map (resp. t_k-time map) of the flow of the vector field $f_*^k\big(\varphi_0^*(\hat{\mathscr{Y}})\big)$ (resp. $f_*^k\big(\varphi_0^*(\mathscr{Y})\big)$).

Here, s_k and t_k are sequences of real numbers such that:

(iii) If $2^{i-1} \leqslant k < 2^i$ for a certain positive even integer i, then

$$s_k := \log_2\left(1 - \frac{1}{\sqrt{\ell_{i/2}}}\right) \qquad \text{and} \qquad t_k := \frac{1}{\sqrt{\ell_{i/2}}},$$

where ℓ_j is a prescribed sequence of positive integers diverging to infinity to be fixed below.

(iv) Otherwise, $s_k = t_k := 0$.

Finally, we let ψ be a C^∞ diffeomorphism of $[-1,2]$ such that:

(v) ψ coincides with the identity on $[x_{-1/2}, x_0]$,

(vi) $\psi(x_{-3/4}) = 0$ and $\psi(x_1) = 1$.

The group we consider is $\Gamma := \langle \hat{f}, f, g, \hat{h}, h, \psi\rangle$. The computations of [**33**] show the following relation for certain powers of $\bar{f} := fg$ (which justifies the construction):

$$(\bar{f})^{2^{i-1}} = f^{1+n/2}\hat{f}^i[\hat{f}^{-i}f^{-n}hf^n\hat{f}^i, \psi\hat{f}^{-i}f^{-n}\hat{h}f^n\hat{f}^i\psi^{-1}]^{\ell_{i/2}}\hat{f}^{-i}f^{-1} \tag{24}$$

where i is an even (positive) integer and $n := 2^i$. Roughly, this works as follows: set

$$a_n := \hat{f}^{-i}f^{-n}hf^n\hat{f}^i \quad \text{and} \quad b_n := \psi\hat{f}^{-i}f^{-n}\hat{h}f^n\hat{f}^i\psi^{-1}.$$

One easily checks that

$$\mathrm{supp}(a_n) \subset [0, x_{-3/2}] \cup [x_{-1/2}, x_0] \cup [x_1, 1] \quad \text{and} \quad \mathrm{supp}(b_n) \subset [-1, 0] \cup [x_{-1/2}, x_0] \cup [1, 2].$$

Thus, the commutator $c_n := [a_n, b_n] = a_n b_n a_n^{-1} b_n^{-1}$ is supported on $[x_{-1/2}, x_0]$, hence the conjugate $\hat{f}^i c_n \hat{f}^{-i}$ is supported on $[x_{-2^{i-1}}, x_0]$. Besides, on each $[x_k, x_{k+1}] \subset [x_{-2^{i-1}}, x_0]$, this conjugate $\hat{f}^i c_n \hat{f}^{-i}$ coincides with the time-$\frac{1}{\ell_{i/2}}$ map of the flow of $f_*^k(\varphi_0^*(\mathscr{Y}))$. Moreover, the restriction of the map

$$h_{n/2} := (f^{-n/2}gf^{n/2}) \cdots (f^{-2}gf^2)(f^{-1}gf) = f^{-n/2}(f^{-1}\bar{f}^{n/2}f)$$

to each $[x_k, x_{k+1}] \subset [x_{-2^{i-1}}, x_0]$ equals the time-1 map of the flow of $f_*^k(\varphi_0^*(\mathscr{Y}))$. This implies that

$$h_{n/2} = (\hat{f}^i c_n \hat{f}^{-i})^{\ell_{i/2}},$$

which corresponds to (24).

Since $\|f^n\| = O(\log(n))$, identity (24) implies that $\|(\bar{f})^{2^{i-1}}\| = O(i\,\ell_{i/2})$. Therefore, $\bar{f}$ is distorted provided ℓ_j grows to infinite in such a way that

$$\lim_{n\to\infty} \frac{\log(n)\,\ell_i}{n} = \lim_{i\to\infty} \frac{i\,\ell_i}{2^i} = 0 \tag{25}$$

The maps $\hat{f}, f, g, \psi$ above are obviously smooth. However, regularity for $\hat{h}, h$ is more subtler. Indeed, their C^1 smoothness is ensured by the conditions $s_n \to 0$ and $t_n \to 0$ as $|n| \to \infty$ (which are equivalent to $\ell_j \to \infty$ as $j \to \infty$) together with the lemma 1.19. Unfortunately, as we see in §1.4.3, this lemma fails to extend to class $C^{1+\alpha}$. Because of this, we need to go into more explicit computations for our example. Although these are difficult to handle, the following key elementary lemma taken from the work of Pixton [**36**] and Tsuboi [**41**] will be enough for us.

LEMMA 3.1. Given a C^2 vector field $\mathscr{X}$ on an interval $[0,a]$, denote $C_1 := \|D\mathscr{X}\|$ and $C_2 := \|D^2\mathscr{X}\|$. If f^t denotes its flow, then, for all $t \geqslant 0$,

$$\left\|D \log Df^t\right\| \leqslant \frac{C_2}{C_1}\left(e^{C_1 t} - 1\right).$$

PROOF. Taking derivatives on the equality $df^t/dt = \mathscr{X} \circ f^t$, we deduce

$$\frac{d}{dt}Df^t = D\mathscr{X}(f^t) \cdot Df^t,$$

hence

$$\frac{d}{dt}\log Df^t = \frac{\frac{d}{dt}Df^t}{Df^t} = D\mathscr{X}(f^t).$$

Since $f^0 = Id$, we conclude

$$\log Df^t = \int_0^t D\mathscr{X}(f^s)\,ds.$$

Since $|D\mathscr{X}| \leqslant C_1$, this yields $|\log Df^t| \leqslant C_1 t$, hence $|Df^t| \leqslant e^{C_1 t}$. Moreover,

$$D\log Df^t = D\left(\int_0^t D\mathscr{X}(f^s)\,ds\right) = \int_0^t D^2\mathscr{X}(f^s)\cdot Df^s\,ds.$$

Since $|D^2\mathscr{X}| \leqslant C_2$, we conclude

$$|D\log Df^t| \leqslant C_2\int_0^t |Df^s|\,ds \leqslant C_2\int_0^t e^{C_1 s}ds = \frac{C_2}{C_1}\left(e^{C_1 t}-1\right),$$

as announced. □

We now turn into very long computations that will allow us to ensure that the resulting maps $h, \hat{h}$ built via the procedure above are $C^{1+\alpha}$ diffeomorphisms for well chosen φ and ℓ_i that respect all the properties we have imposed. This will close the proof of our theorem.

In order to simplify these computations, let us remind the *chain rules* for different derivatives of maps between 1-dimensional spaces, namely logarithmic (L), affine (A), and Schwarzian (S):

$$L(f) := \log Df, \quad A(f) := DL(f) = \frac{D^2 f}{Df},$$

$$S(f) := DA(f) - \frac{A(f)^2}{2} = \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2.$$

These are listed below:

$$L(fg) = L(g) + L(f)\circ g, \tag{26}$$

$$A(fg) = A(g) + A(f)\circ g\cdot Dg, \tag{27}$$

$$S(fg) = S(g) + S(f)\circ g\cdot (Dg)^2. \tag{28}$$

We let $\varphi\colon (0,1) \to \mathbb{R}$ be a C^∞ diffeomorphism such that, for a small-enough $\delta > 0$

$$\varphi(x) = \begin{cases} -\exp(\exp(1/x)) & \text{if} \quad 0 < x \leqslant \delta \\ \exp(\exp(1/(1-x))) & \text{if} \quad 1-\delta \leqslant x < 1 \end{cases}$$

If we denote $\mathscr{Z} := \varphi_0^*(\hat{\mathscr{Y}})$, then we need to control $f_*^n(\mathscr{Z})$.

An estimate for lengths of fundamental domains. Let us come back to the group $\Gamma = \langle \hat{f}, f, g, \hat{h}, h, \psi \rangle$. Recall that I denotes the interval $[x_0, x_1]$. We claim that, for a certain constant $C > 0$,

$$|f^n(I)| = O\left(\frac{C}{n\log(n)\,(\log(\log(n)))^2}\right). \tag{29}$$

This is checked via a direct computation. Namely, for a large-enough n,

$$|f^n(I)| = \varphi^{-1}(n+1) - \varphi^{-1}(n) = \frac{1}{\log\log(n)} - \frac{1}{\log\log(n+1)}.$$

Since the derivative of $x \mapsto 1/\log\log(x)$ is $1/[x\,\log(x)\,(\log\log(x))^2]$, a direct application of the Mean Value Theorem yields the desired estimate (29).

Estimates for the vector field and its derivative. Notice that

$$f_*^n(\mathscr{Z})(x) = [Df^n(f^{-n}(x))]\,\mathscr{Z}(f^{-n}(x)), \qquad x \in I_n := f^n(I).$$

Taking derivatives, we obtain

$$D(f_*^n(\mathscr{Z}))(x) = \frac{D^2f^n(f^{-n}(x))}{Df^n(f^{-n}(x))}\,\mathscr{Z}(f^{-n}(x)) + D\mathscr{Z}(f^{-n}(x)).$$

Now, using the chain rule (27), this yields

$$D(f_*^n(\mathscr{Z}))(x) = \sum_{i=0}^{n-1}\left(\frac{D^2f(f^{i-n}(x))}{Df(f^{i-n}(x))}\right)Df^i(f^{-n}(x))\,\mathscr{Z}(f^{-n}(x)) + D\mathscr{Z}(f^{-n}(x)).$$

Thus, letting

$$C' := \left\|\frac{D^2f}{Df}\right\|\,\|\mathscr{Z}\|, \qquad C'' := \|D\mathscr{Z}\|,$$

we obtain

$$|D(f_*^n(\mathscr{Z}))(x)| \leqslant C'\sum_{i=0}^{n-1}Df^i(f^{-n}(x)) + C''.$$

We claim that the sum above is uniformly bounded (independently of n and $x \in I_n$), so that

$$|D(f_*^n(\mathscr{Z}))(x)| \leqslant C \tag{30}$$

for a certain constant C. Indeed, a standard control of distortion argument yields that $Df^i(f^{-n}(x))$ is of the order of $|I_i|/|I_0|$, hence

$$\sum_{i=0}^{n-1}Df^i(f^{-n}(x)) \sim \sum_{i=0}^{n-1}|I_i| \leqslant 1.$$

Estimates for the second derivative. We now claim that, for a certain constant $C > 0$ and all $x \in I_n$,

$$\big|D^2(f_*^n(\mathscr{Z}))(x)\big| \leqslant C\, n \log(n)\, (\log\log(n))^2. \tag{31}$$

To show this, notice that, from

$$D(f_*^n(\mathscr{Z}))(x) = A(f^n)(f^{-n}(x))\, \mathscr{Z}(f^{-n}(x)) + D\mathscr{Z}(f^{-n}(x)),$$

we obtain

$$D^2(f_*^n(\mathscr{Z}))(x) = \big[DA(f^n)\cdot \mathscr{Z} + A(f^n)\, D\mathscr{Z} + D^2\mathscr{Z}\big] \circ f^{-n}(x)\cdot Df^{-n}(x).$$

which is equal to

$$\Big[\Big(S(f^n) + \frac{1}{2}A(f^n)^2\Big)\cdot \mathscr{Z} + A(f^n)\, D\mathscr{Z} + D^2\mathscr{Z}\Big] \circ f^{-n}(x)\cdot Df^{-n}(x).$$

Let us analise each term entering in this expression. First, by (29) and the control of distortion argument above,

$$Df^{-n}(x) = 1/Df^n(f^{-n}(x)) = O\big(n\log(n)\,(\log\log(n))^2\big).$$

We next claim that $A(f^n)$ is uniformly bounded on I_0. Indeed, letting $C := \|A(f)\|$, the chain rule (27) yields

$$A(f^n) = \sum_{i=0}^{n-1} A(f)\circ f^i \cdot Df^i \leqslant C\sum_{i=0}^{n-1} Df^i.$$

The control of distortion argument above shows that the last sum is bounded from above by a constant, hence the claim.

Since $\mathscr{Z}$, $D\mathscr{Z}$ and $D^2\mathscr{Z}$ are obviously uniformly bounded, to show (31) it remains to check that $S(f^n)(f^{-n}(x))$ is uniformly bounded. To see this, we use the chain rule (28):

$$S f^n(f^{-n}(x)) = \sum_{i=0}^{n-1} S f(f^{i-n}(x))(Df^i(f^{-n}(x)))^2.$$

This implies

$$\big|S f^n(f^{-n}(x))\big| \leqslant C\sum_{i=0}^{n-1}\big(Df^i(f^{-n}(x))\big)^2,$$

and the last sum can be estimated as it was done before. (The sum here is even smaller since it involves the squares of the derivatives.)

Estimates for the maps. We are now in position to check that the group Γ is made of $C^{1+\alpha}$ diffeomorphisms for $\alpha < 1/2$ and ℓ_i of order $n/\log(n)^2$. (Notice that, according to (25), the element $\bar{f}$ is distorted in Γ for this choice.) Notice that this is obvious for all the generators except h and $\hat{h}$. The estimates for these two elements are similar, so that we only deal with h. Besides, we may deal with $\log Dh$ instead of Dh, since the condition "Dh is of class C^α" is equivalent to that "$\log Dh$ is of class C^α".

We need to check that there exists a uniform bound B for expressions of type

$$\frac{|\log Dh(y) - \log Dh(x)|}{|y-x|^\alpha}$$

for all points $x < y$ in the same interval I_n. Indeed, having such an estimate, one can easily treat the case of arbitrary pairs $x < y$ just by noticing that at each endpoint of an interval of the form above, the derivative of h equals 1. Namely, letting z_1 (resp. z_2) be such an endpoint that is immediately to the right of x (resp. to the left of y), one has

$$\begin{aligned}\frac{|Dh(y)-Dh(x)|}{|y-x|^\alpha} &\leqslant \frac{|Dh(y)-Dh(z)|}{|y-x|^\alpha}+\frac{|Dh(z)-Dh(x)|}{|y-x|^\alpha}\\ &\leqslant \frac{|Dh(y)-Dh(z)|}{|y-z|^\alpha}+\frac{|Dh(z)-Dh(x)|}{|z-x|^\alpha}\\ &\leqslant 2\,B.\end{aligned}$$

Now, for all $z \in I_n$ (with $n \geqslant 0$), Lemma 3.1 and estimate (30) yield, for t_n small enough,

$$D(\log Dh)(z) \leqslant 2\,\|D^2 f_*^n(\mathscr{X})\|\,t_n.$$

By estimate (31), this implies, for a certain constant $C > 0$,

$$D(\log Dh)(z) \leqslant 2\,C\,n\log(n)\,(\log(\log(n)))^2\,t_n. \tag{32}$$

Moreover, for x, y in I_n,

$$\frac{|\log Dh(y)-\log Dh(x)|}{|y-x|^\alpha}=\frac{|\log Dh(y)-\log Dh(x)|}{|y-x|}\,|y-x|^{1-\alpha}=D(\log Dh(z))\,|y-x|^{1-\alpha}$$

for a certain point $z \in I_n$. By (32), this yields

$$\frac{|\log Dh(y)-\log Dh(x)|}{|y-x|^\alpha}\leqslant 2\,C\,n\log(n)\,(\log(\log(n)))^2\,t_n\left[\frac{C}{n\log(n)(\log(\log(n)))^2},\right]^{1-\alpha}. \tag{33}$$

Since $t_n = 1/\sqrt{\ell_{i/2}} \leqslant C\,\log(n)/\sqrt{n}$, we finally obtain

$$\frac{|\log Dh(y)-\log Dh(x)|}{|y-x|^\alpha}\leqslant 2C'n\log(n)(\log(\log(n)))^2\frac{\log(n)}{n^{1/2}}\frac{1}{[n\log(n)(\log(\log(n)))^2]^{1-\alpha}}.$$

To get the desired upper bound B, it suffices that the total exponent of n in the expression above is negative. Since this exponent equals $1-1/2-(1-\alpha)=\alpha-1/2$, this condition reduces to $\alpha < 1/2$, which is our hypothesis.

3.3. Distortion in class $C^{1+\alpha}$ for $\alpha < 1$

It is unclear whether the previous action can be smoothed beyond the class $C^{3/2}$ (compare [**10, 12, 25, 31**]). To achieve a larger differentiability class, we will need to accelerate the distorted behavior of $\bar{f}$, which will allow us to consider smaller integration times for the flows of vector fields (in concrete terms, we will increase the sequence ℓ_i). This will be crucial to improve the regularity from $\alpha < 1/2$ to any $\alpha < 1$.

Adding an extra element. We consider the map $\tilde{h}$ acts by the identity outside the intervals I_k, and that on each such interval coincides with the r_k-time of the time flow of the vector field $f_*^k(\varphi_0^*(\hat{\mathscr{Y}}))$, where $r_k := 1/\sqrt{\ell_{i/2}}$ for $2^{i-1} \leqslant k < 2^i$ and $r_k := 0$ otherwise. Notice that $\tilde{h}$ is very similar to $\hat{h}$. (Actually, we could perform the computations that follow using $\hat{h}$ instead of $\tilde{h}$, but this would become much harder.)

Then we let $d_n := \hat{f}^{-i} f^{-n} \tilde{h} f^n \hat{f}^i$ for $n = 2^i$, where i is an even integer. We have $\mathrm{supp}(d_n) \subset [0, x_{-3/2}] \cup [x_{-1/2}, x_0] \cup [x_1, 1]$. Since $\mathrm{supp}(c_n) \subset [x_{-1/2}, x_0]$, for every integer $L_i \geqslant 1$, the support of $d_n^{L_i} c_n d_n^{-L_i}$ is also contained in $[x_{-1/2}, x_0]$, thus the support of

$$\hat{f}^{-i} d_n^{L_i} c_n d_n^{-L_i} \hat{f}^i = (\hat{f}^{-i} d_n^{L_i} \hat{f}^i)(\hat{f}^{-i} c_n \hat{f}^i)(\hat{f}^{-i} d_n^{-L_i} \hat{f}^i)$$

is contained in $[x_{-2^{i-1}}, x_0]$.

Now recall that, on each $[x_k, x_{k+1}] \subset [x_{-2^{i-1}}, x_0]$, the conjugate $\hat{f}^i c_n \hat{f}^{-i}$ coincides with the time-$\frac{1}{\ell_{i/2}}$ map of the flow of $f_*^k(\varphi_0^*(\mathscr{Y}))$. Moreover, by construction, on the same interval, the conjugate $\hat{f}^i d_n \hat{f}^{-i}$ coincides with the time-$\frac{1}{\sqrt{\ell_{i/2}}}$ map of the flow of $f_*^k(\varphi_0^*(\hat{\mathscr{Y}}))$. By the affine relation, still on the same interval, the map $\hat{f}^{-i} d_n^{L_i} c_n d_n^{-L_i} \hat{f}^i$ lies in the flow of $f_*^k(\varphi_0^*(\mathscr{Y}))$, and arises at time

$$\frac{2^{\frac{L_i}{\sqrt{\ell_{i/2}}}}}{\ell_{i/2}}.$$

If $L_i := \sqrt{\ell_{i/2}}\, \log_2(\ell_{i/2})$ (which will be chosen to be an integer number), then this quantity equals 1. Therefore, for this choice, $\hat{f}^{-i} d_n^{L_i} c_n d_n^{-L_i} \hat{f}^i$ coincides with $h_{n/2}$.

The distortion estimate. The identity $h_{n/2} = \hat{f}^{-i} d_n^{L_i} c_n d_n^{-L_i} \hat{f}^i$ implies that, in the new group $\tilde{\Gamma} := \langle \hat{f}, f, g, \hat{h}, h, \tilde{h}, \psi \rangle$, we have the estimate

$$\|h_{n/2}\| \leqslant 2\|\hat{f}^i\| + 2\, L_i \|d_n\| + \|c_n\| \leqslant 2i + 2\, L_i\,(2i + 1 + 2\|f^n\|) + 8\,(1 + i + \|f^n\|).$$

Since $\|f^n\| = O(\log(n)) = O(i)$, we conclude that

$$\|h_{n/2}\| = O\left(i\, \sqrt{\ell_{i/2}}\, \log(\ell_{i/2})\right).$$

Since $h_{n/2} = f^{-n/2}(f^{-1} \bar{f}^{n/2} f)$ and $\|f^{n/2}\| = O(\log(n))$, this yields

$$\|\bar{f}^{n/2}\| = 2 + \|f^{n/2}\| + \|h_{n/2}\| = O\left(i\, \sqrt{\ell_{i/2}}\, \log(\ell_{i/2})\right).$$

Notice that the last estimate is much better than what we had in the group Γ of the previous section. In there, $\|\bar{f}^{n/2}\|$ was of the order $O(i\, \ell_{i/2})$, hence, $\bar{f}$ was distorted provided the growth of ℓ_j was smaller than exponential. In the new setting, that is, in the modified group $\tilde{\Gamma}$, the diffeomorphism $\bar{f}$ is distorted whenever the condition below is satisfied (recall that $n = 2^i$):

$$\lim_{n\to\infty} \frac{i\, \sqrt{\ell_{i/2}}\, \log(\ell_{i/2})}{2^i} = 0. \tag{34}$$

Checking regularity. We thus choose a new sequence ℓ_i so that condition (34) holds and $\sqrt{\ell_{i/2}}\, \log_2(\ell_{i/2})$ is an integer number. This can be achieved for a sequence of type

$$\sqrt{\ell_{i/2}} \sim \frac{n}{\log(n)^3},$$

that we fix from now on. With such a choice, we claim that $\tilde{\Gamma}$ is a group of $C^{1+\alpha}$ diffeomorphisms. Again, this is obvious for all generators except $h, \hat{h}, \tilde{h}$, and for these three elements the computations are the exact same, because each of the sequences r_n, s_n, t_n is equivalent to $1/\sqrt{\ell_{i/2}}$. We thus write everything only for h. Remind estimate (33):

$$\frac{|\log Dh(y) - \log Dh(x)|}{|y-x|^{\alpha}} \leqslant 2\,C\,n\log(n)\,(\log(\log(n)))^2\,t_n\,\left[\frac{C}{n\log(n)(\log(\log(n)))^2},\right]^{1-\alpha}.$$

With the new estimate for t_n, this becomes

$$\frac{|\log Dh(y) - \log Dh(x)|}{|y-x|^{\alpha}} \leqslant 2\,C\,n\,(\log(\log(n)))^2\,\frac{\log(n)^4}{n}\,\left[\frac{C}{n\log(n)(\log(\log(n)))^2},\right]^{1-\alpha}.$$

The expression on the right is of order

$$O\left(\frac{\log(n)^{3+\alpha}\,(\log\log(n))^{2\alpha}}{n^{1-\alpha}}\right),$$

which converges to 0 as n goes to infinite. This allows showing that h is a $C^{1+\alpha}$ diffeomorphism as it was done in the previous section.

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